Core Mathematics in the MPH and MSc in Epidemiology Courses, Imperial College London

Session 2: Inequalities, Polynomials and Units

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1 Inequalities

Inequalities are based on the following four symbols, recalled from the table in Session 1.

Symbol	Meaning	Example
$<$	less than	$1 < 2$ (true); $2 < 2$ (false); $3 < 2$ (false)
$>$	greater than	$1 > 2$ (false); $2 > 2$ (false); $3 > 2$ (true)
$\leq, \leqslant$	less than or equal to	$1 \leq 2$ (true); $2 \leq 2$ (true); $3 \leq 2$ (false)
$\geq, \geqslant$	greater than or equal to	$1 \geq 2$ (false); $2 \geq 2$ (true); $3 \geq 2$ (true)

Table 1: The inequality symbols, their meanings, and examples.

Inequalities can be chained together to form composite statements. The arrows should all point in the same direction to avoid confusion.

$$1 < 2 < 10 \quad \text{true;} \tag{1}$$
$$1 < 3 < 2 < 5 \quad \text{false.} \tag{2}$$

Inequalities can be manipulated in a similar way to equations, though depending on the manipulation to be performed, more care might be required. Adding to and subtracting from inequalities works just as with equations:

$$a = b \implies a + c = b + c \tag{3}$$
$$a < b \implies a + c < b + c \tag{4}$$

However, when it comes to multiplication, inequalities are trickier than equations:

$$a = b \implies ac = bc \tag{5}$$
$$a < b \implies \begin{cases} ac < bc & \text{if } c > 0 \\ ac > bc & \text{if } c < 0 \end{cases} \tag{6}$$

In other words, you need to know whether the thing you're multiplying by is negative, and if so, you reverse the direction of the arrow. This is true because 'less than' does not mean 'closer to zero', it means 'less positive' which is equivalent to 'more negative', i.e. further to the left on the number line. An example is helpful: $2 < 3$ but $-2 > -3$. Other manipulations can also be tricky: if $x < y$, one does not know in general which is larger out of x^2 and y^2, nor which is larger out of $\frac{1}{x}$ and $\frac{1}{y}$, because x or y could be negative. However, restricting them both to be positive:

$$\text{if} \quad x > 0 \quad \text{and} \quad y > 0, \quad \text{then} \tag{7}$$

$$x < y \implies x^2 < y^2, \quad \text{and} \tag{8}$$

$$x < y \implies \frac{1}{x} > \frac{1}{y}. \tag{9}$$

Again, note the reversal of the arrow in the last case: this is because for positive numbers, the greater the number is, the smaller its reciprocal is. $2 < 3$ but $\frac{1}{2} > \frac{1}{3}$. I think the only subtlety to manipulating inequalities you need to know for your other courses is that when multiplying by a negative number, you reverse the arrow.

Inequalities are how we compare quantities, and therefore are often useful when we describe things quantitatively, such as here in your masters. In a number of your other courses you will be exposed to the quantity R_0 which, for a given infectious disease, is the average number of new infections caused by a typical infected individual, in a totally susceptible population. As will be explained elsewhere, if $R_0 > 1$, the number of people infected will increase; if $R_0 < 1$, the number of people infected will decrease. What we want to achieve is the reduction of R_0 to less than 1 through a public health intervention. Let's say we're able to prevent a fraction σ of transmissions, so that a typically infected individual now only infects $(1-\sigma)R_0$ other individuals in a totally susceptible population. We want σ to be sufficiently large that $(1-\sigma)R_0 < 1$ – that way we reverse the epidemic and cause it to shrink. Rearranging this inequality by multiplying by $-1/R_0$ (which is negative, so the arrow flips) then adding 1, we find that we need

$$\sigma > 1 - \frac{1}{R_0} = \frac{R_0 - 1}{R_0} \tag{10}$$

The right-hand side is less than 1, but gets closer to 1 as R_0 increases. This quantifies how it gets harder to control the spread of a disease the more infectious the disease is.

DISCUSS EXAMPLES IN CLASS

2 Polynomials

Polynomials are fundamental in mathematics, and crop up all over the place in applied mathematics. Just one example is binomial probability, which we will look at later in the course. Briefly, this arises when we have a situation in which one of two things can happen, the situation occurs several (perhaps many) times, and we're interested in the total number of times each of the two things happens. For example a disease may take one of two distinct courses (death or not, paralysis or not, etc.) to which we asign probabilities, and we want to know how likely it is that a given number of patients end up following a particular course.

A polynomial of a variable, x say, is a sum of different non-negative powers of x:

$$\sum_{i=0}^{n} a_i x^i = a_0 x^0 + a_1 x^1 + a_2 x^2 + \ldots + a_n x^n; \tag{11}$$

each x^i is multiplied by a constant a_i called the *coefficient* of x^n. n in this expression – the highest power of x – is called the *degree* of the polynomial. A degree-one polynomial is also called linear, a degree-two polynomial quadratic, and a degree-three polynomial cubic; a degree-n polynomial is generally called an nth order polynomial.

For an nth order polynomial, the number of roots (points where the polynomial equals zero) can be anywhere between 0 and n. (This illustrates a general principle in maths: depending on the question, there may not be one answer.)

Polynomials are added and subtracted by simply adding or subtracting the coefficients for the same powers of x, thus:

$$\sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{m} b_i x^i = \sum_{i=0}^{\max(m,n)} (a_i + b_i) x^i \tag{12}$$

$$\sum_{i=0}^{n} a_i x^i - \sum_{i=0}^{m} b_i x^i = \sum_{i=0}^{\max(m,n)} (a_i - b_i) x^i \tag{13}$$

where '$\max(m,n)$' means whichever is larger out of m and n. Imagine n is 2 and m is 3; you might be confused that the only coefficients we have defined in the first polynomial on the left-hand side are a_0, a_1 and a_2, and yet the right-hand side involves a_3 which we have not defined. Worry not, a_3 is simply zero; any coefficients not appearing explicitly can be considered to be there, but equal to zero. For example, writing

$$2 + 5x^2 + 3x^5 = \sum_{i=0}^{n} a_i x^i, \tag{14}$$

$$\text{we have} \quad a_0 = 2,\ a_1 = 0,\ a_2 = 5,\ a_3 = a_4 = 0,\ a_5 = 3. \tag{15}$$

Before considering the product of two polynomials, let us clarify how the sum of some numbers is multiplied by a constant:

$$a(b + c) = ab + ac \tag{16}$$

Then define a to be equal to $d + e$, and consider Eq. (16) again:

$$(d + e)(b + c) = b(d + e) + c(d + e) = bd + be + cd + ce \tag{17}$$

In words, everything in one bracket gets multiplied by everything in the other, then everything is added together. This remains true when more than two things are added together inside one bracket, and we use this to multiply together polynomials. Consider this example:

$$\begin{aligned}(2 + x + 4x^2)(1 + 6x + 3x^2) &= 2(1 + 6x + 3x^2) + x(1 + 6x + 3x^2) + 4x^2(1 + 6x + 3x^2) && (18)\\ &= 2 + 12x + 6x^2 \quad + x + 6x^2 + 3x^3 \quad + 4x^2 + 24x^3 + 12x^4 && (19)\\ &= 2 + 13x + 16x^2 + 27x^3 + 12x^4 && (20)\end{aligned}$$

To *factorise* something means to split it into factors, i.e. things that multiply together to give what one started with. When factorising polynomials it is usually understood that we only want to deal with whole numbers, so that, for example, one would not write $3 = 2 \times \frac{3}{2}$ (it's true but unhelpful here); this means that some polynomials cannot be factorised. Others can be factorised in different ways. Some examples:

$$2 + 3x \quad \text{cannot be factorised} \tag{21}$$

$$2 + 4x = 2(1 + 2x) \tag{22}$$

$$8x + 4x^2 + 12x^4 = 4x(2 + x + 3x^2) \tag{23}$$

$$-6 + x + x^2 = (x - 2)(x + 3) \tag{24}$$

Most of you won't need to know how to factorise polynomials; however it's worth knowing what it means.

3 Understanding Units

3.1 Introduction

The *dimension* of a quantity is the kind of units it is measured in. The quantities '3 hours', 'seven days' and 'one decade' all have dimension time (note that it doesn't matter which specific unit of time has been chosen to express the quantity); we may write this as [3 hours] = [time]. The quantities 'sixty miles per hour' and $10m/s$ have dimension length per time; [sixty miles per hour] = [length][time]$^{-1}$. Note that *per* means divided by. (This is why sixty miles per hour equals one mile per minute.)

The dimension of the product of two quantities is equal to the product of the dimensions. For example, recall the following basic equation from physics: distance travelled = speed × travelling time. The dimension of 'distance travelled' is length, the dimension of speed is length per time, and the dimension of 'travelling time' = time: multiplying the dimensions of the latter two gives the dimension of the first. Note that when specifying the dimension of a quantity, people usually say 'per unit' instead of simply 'per', with exactly the same meaning.

A fundamental distinction that can be made between all quantities is between those that are *dimensionless* and those that are *dimensionful*. Dimensionless quantities can be expressed without any kind of unit; dimensionful quantities cannot – they require a unit to make sense. A ratio of two quantities with the same dimension will always be dimensionless: e.g. 2 weeks / 1 day = 14. Similarly, the product of two quantities with inverse dimensions will always be dimensionless; a particularly important class of examples for our purposes is that a rate (in the sense of 'per unit time') multiplied by a length of time gives a dimensionless number. For example, (one per day) × (one week) = 7.

Note that the units on one side of an equation do not have to equal the units of the other side: 'one week = seven days' is a perfectly good equation, though the units are weeks on one side and and days on the other. However, the dimension of the two sides of an equation must be identical (in the previous example, time) otherwise the equation is pure nonsense. Even stronger than this, every term appearing in an equation must have the same dimension: if $a + b = c - d + e$, then a, b, c, d and e must all have the same dimension. This is because, while multiplication and division of quantities of different dimenionsion is fine (recall one per day times seven days equals seven), adding and subtracting quantities of different dimension is meaningless. To some extent this is intuitive – you would all think me mad if I wrote down 'three miles + 2 days'.

EXERCISE: which of the following quantities are dimensionless and which are dimensionful? For those that are dimensionful, state the dimension and give an example of an appropriate unit. Do think about it before reading the answers – this is a fundamental concept to understand.

1. the mass of a particular drug required,
2. the number of people infected with a given disease,

3. the length of time between being infected and becoming infectious,
4. the number of days between being infected and becoming infectious,
5. the power to which we raise some quantity,
6. an amount of money,
7. a fractional increase in an amount of money,
8. a rate of fractional increase in an amount of money (e.g. the interest rate on a bank account),
9. the concentration of parasites in a patient's blood,
10. the rate of change in the (population average) time from HIV infection to AIDS.

(Material continues on the next page.)

Answers:

1. [the mass of a particular drug required] = [mass]. e.g. kg (kilogram), mg (milligram)

2. [the number of people infected with a given disease] = [1] i.e. it is dimensionless (has the same dimension as the number 1).

3. [the length of time between being infected and becoming infectious] = [time]. e.g. years, hours

4. [the number of days between being infected and becoming infectious] = [1]. This is a subtle one. Interpreting the words with their precise meanings, 'the number of days between…' means 'the length of time between…' divided by 1 day. Compare 'the number of days in a week = 7' and 'the length of one week is 7 days'. If you're constructing your own equation, *never do this kind of thing*: avoid saying 't is the number of days', 't is the number of hours' etc. and just let t be a length of time. You don't want to separate the units and the number that goes with them: it's unnatural and more likely to make you go wrong. If you're working with someone else's equation, obviously you have to start with what they've given you, and if one of their variables is 'number of days' etc. then you have to work with it and be more careful. People – even smart people in your field – are prone to misuse this, so beware…

5. [the power to which we raise some quantity] = [1]. This is because of what raising to a power really means: e.g. x^3 means three factors of x get multiplied together. This still makes sense if x is dimensionful, but makes no sense if we replace '3' by something with dimensions. For example the quantities $x^{3 \text{ days}}, x^{1 \text{ mile}}$ are nonsensical regardless of what x is. Equivalently, any time you see e^{rt} and t is a time, r must have the dimensions of time^{-1}, because rt is a power and so must be dimensionless.

6. [an amount of money] = [currency], e.g. pounds, pence, dollars

7. [a fractional increase in an amount of money] = [1]. A fractional change in anything is dimensionless, because a fractional change in a quantity x means $(x_{\text{new}} - x_{\text{old}})/x_{\text{old}}$.

8. [a rate of fractional increase in an amount of money] = [time]$^{-1}$. Interest rates are so frequently quoted simply as percentages that it's easy to forget that the passage of time is relevant – 'per year' is usually implicit. Note also that the % symbol itself is dimensionless – it represents the number 0.01 (see also the discussion of percentages in the notes for session 1).

9. [the concentration of parasites in a patient's blood] = [length]$^{-3}$. The number of parasites is just a number. The extent to which something occupies space in 3D – 'volume' – has dimensions of length^3. (For example the volume of a cuboid is the product of the lengths of the three different sides.) Therefore [per volume] = [length]$^{-3}$. Note that one litre is a thousand cubic centimetres: $1\text{L} = 1000\text{cm}^3$, and so 'per litre' $= 1/(1000\text{cm}^3) = 10^{-3}\text{cm}^{-3}$; and 'per cm^3' $= 1/(1\text{cm}^3) = 1/(10^{-3}\text{L}) = 10^3$ per litre. This last coversion is perhaps more intuitive – if you find something once per cubic centimetre of water, you'll find it a thousand times per litre.

10. [the rate of change in the (population average) time from HIV infection to AIDS] = [1]. Here I've deliberately tried to trip you up. If you're getting the hang of this, you might be expecting to see 'per time' associated with anything that is a rate. Note that a careless statement about the dimension of rates would be 'rates have dimension of

time^{-1}'. A careful statement would be 'the rate at which x changes with respect to time has the dimension of time^{-1} multiplied by whatever the dimension of x is'. Here, x has the dimension of time, and so the rate is dimensionless! To elaborate, in case this seems paradoxical, take the purely hypothetical example of a country slowly increasing the availability of anti-retroviral treatment (ART) for HIV positive people, with the result that each passing (calendar) year, the average time taken to reach AIDS is prolonged by 6 months. Equivalently, in each two-month period the average time to AIDS increases by one month; in each day it increases by 12 hours, etc. In summary for this (fictional!) example, the rate of change of the average time to AIDS is 0.5.

3.2 Two Involved Examples

Macdonald's equation for R_0 for malaria,

$$R_0 = \frac{ma^2bc}{gr}\, e^{-gv}, \tag{25}$$

involves these quantities:

- The mosquito death rate, g. $[g] = [\text{time}]^{-1}$
- The rate at which a mosquito feeds on humans, a. $[a] = [\text{time}]^{-1}$
- The probability a mosquito becomes infected after biting an infected human, c, which is dimensionless.
- The proportion of bites by infectious mosquitoes that infect a human, b, which is dimensionless.
- The rate each human recovers from infection, r. $[r] = [\text{time}]^{-1}$
- The time from infection to infectiousness in the mosquito, v. $[v] = [\text{time}]$
- The ratio of mosquitoes to humans, m, which is dimensionless.

On a completely unrelated subject, imagine that fundraising in preparation for a vaccination project resulted in N people steadily donating at an average rate r_d over a time T. It costs P to buy an amount D of the vaccine. A single vaccination requires an amount d of the vaccine, and one (disposable) needle which costs p. In the project, n nurses each administer the vaccine at a rate r_v, and are each paid at a rate r_p. A fraction f of all money raised must be used indirectly (for administration etc.). How many people can this project vaccinate? Hint: considering the dimensions of all these quantities helps you to see which ones should be multiplied together. The answer is at the end of these notes.

3.3 A Subtlety: Optional Units

Units are effectively optional for variables which *count* things; truly, such variables are dimensionless, but a unit can be bolted on provided this is done consistently. Imagine I have £5 and one apple costs 50p. Both of these are acceptable ways of calculating the number of apples I can afford, n:

$$n = \frac{\text{budget}}{\text{price per apple}} = \frac{£5}{50p \text{ per apple}} = \frac{£5}{50p}\text{apples} = 10 \text{ apples}; \tag{26}$$

alternatively,
$$n = \frac{\text{budget}}{\text{price of one apple}} = \frac{£5}{50p} = 10 \tag{27}$$

They're both acceptable because since n is a number of apples, it doesn't need a unit; however no-one is going to get confused if you say 10 apples instead of just 10. The slight difference in interpretation is simply whether you consider 50p as 'the price of one apple', or whether you think of apples of being units in their own right, so that we have '50p per apple = £1 / (two apples)' etc. Consistency didn't seem to be important in this example, however consider the following rearrangement of the equation:

$$\text{budget} = \text{price per apple} \times \text{number of apples} = 50p \text{ per apple} \times 10 \text{ apples} = £5 \tag{28}$$

$$\text{or,} \quad \text{budget} = \text{price of one apple} \times \text{number of apples} = 50p \times 10 = £5 \tag{29}$$

both of which are fine; however if I'd have mixed up the two distinct ways of thinking about it and multiplied '50p per apple' by 10, I would have got £5 per apple as my answer and been thoroughly confused.

In an exactly analogous way, if n is a number of people, you can choose to consider a person to be a unit or not. In the UK in 2011 there were 32.2 million women and the per capita birth rate was 0.0225 per year. Both of the following are acceptable ways of calculating and presenting the number of births that took place in one year, n:

$$\begin{aligned}
n &= \text{number of women} \times \text{per capita birth rate} \times \text{length of time} && (30)\\
&= 32.2 \times 10^6 \times 0.0225 \text{ year}^{-1} \times 1 \text{ year} && (31)\\
&= 725,000; && (32)\\
\text{alternatively,} \quad n &= \text{number of women} \times \text{per capita birth rate} \times \text{length of time} && (33)\\
&= 32.2 \times 10^6 \text{ women} \times 0.0225 \text{ births per woman per year} \times 1 \text{ year} && (34)\\
&= 725,000 \text{ births}; && (35)
\end{aligned}$$

Another example: let's say an infectious person infects others at a rate 0.2 people per day for seven days. The standard way to calculate and present R_0 is as follows: $R_0 = 0.2 \text{ day}^{-1} \times 7 \text{ days} = 1.4$. Alternatively you could explicitly say that we are counting the number of infections caused: $R_0 = 0.2$ infections caused/day $\times$ 7 days = 1.4 infections caused. Even more explicit (and even more non-standard) would be to incorporate the fact that R_0 is a quantity defined for a single infected individual – twice as many infected individuals would cause twice as many new infections between them – and talk about the quantity '0.2 infections caused per day per infected individual'. This quantity needs to be multiplied by both a time and a number of infected individuals, to get the number of infections caused.

In summary, if your equation involves counting *something* (e.g. *something* = people), you can either (a) consider *something* to be a unit in its own right, in which case quantities defined for just one of the things you're counting should have their units set to be 'per *something*'; or (b) not consider *something* to be a unit, then quantities defined for just one of the things you're counting should just be remembered to be defined thus. The second option is more standard, the first is more pedagogical.

3.4 A Word on Unit Prefixes

Some common unit prefixes, abbreviating multiplication by various powers of ten, are shown in Table 2. Other prefixes exist for even greater and even smaller powers ten; see for example http://en.wikipedia.org/wiki/Metric_prefix (my table is a fragment of the one found there).

Viewed as simply representing multiplication by their associated power of ten, these prefixes appear to break the operator precedence rule for powers: $1\text{km}^2 = 1(\text{km})^2$; $1\text{km}^2 \neq 1\text{k}(\text{m})^2$. One can try to rationalise this as follows: these unit prefixes are so common that we subconsciously

Name	Prefix	Meaning
giga	G	10^9
mega	M	10^6
kilo	k	10^3
deci	d	10^{-1}
centi	c	10^{-2}
milli	m	10^{-3}
micro	μ	10^{-6}

Table 2: Some common unit prefixes.

think of the unit absorbing them to define a new unit – one 'km' is thought of as one thing in its own right, rather than $10^3\times$m. Whether that makes sense or not, that's what everyone means so get used to it.

Answer to the Vaccination Fundraising Problem

The money available is $r_d NT(1-f)$: the rate of donating per person, times the number of people donating, times the length of time over which donations continued, times the fraction which can actually be used. In time t, (ntr_v) vaccines are given; the cost per vaccine $p + dP/D$ just for the drug and the needle, but the wages of the nurses must be added. Vaccinating for a time t therefore costs $(ntr_v)(p + dP/D) + ntr_p$. The cost per vaccine, incorporating nurses' wages, is equal to the cost of vaccinating for a time t divided by the number of vaccines given in this time: this is $p + dP/D + r_p/r_v$ (note that time cancels, as does the number of nurses). The total number of vaccines that can be given is equal to the budget for vaccines divided by the cost per vaccine: $(Nr_dT(1-f))/(p + dP/D + r_p/r_v)$.

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