Core Mathematics,

Part of the MPH and MSc in Epidemiology Courses, Imperial College London

Session 1: Basic Manipulation of Numbers

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Motivation I

"As a matter of fact all epidemiology, concerned as it is with variation of disease from time to time or from place to place, must be considered mathematically, however many variables are implicated, if it is to be considered scientifically at all. *To say that a disease depends on certain factors is not to say much, until we can also form an estimate as to how largely each factor influences the whole result.* And the mathematical method of treatment is really nothing but the application of careful reasoning to the problems at hand."

Sir Ronald Ross, 1911. (My italics.)

Motivation II

Public health questions we can investigate with mathematical modelling, courtesy of Christophe Fraser:

- Why can we eradicate smallpox but not obesity?
- How much more do we need to do to eradicate polio?
- Why is it that measles epidemics are biannual, norovirus epidemics annual, and *Staphylococcus aureus* is stably endemic?
- To mitigate malaria should we prioritise bed nets, insecticides, ...?
- How much should we expect a 30% efficacious vaccine against malaria or HIV to achieve?
- Should we close schools in the event of an influenza pandemic? Should we implement border controls? When should we start?
- How do we expect annual incidence of measles infection to vary as a function of MMR vaccination coverage?

Motivation III

- To mitigate mortality from seasonal influenza, should we vaccinate children (who are most likely to get infected) or the elderly (who are most likely to die if infected)?
- Why is HIV resurgent amongst MSM (men who have sex with men) even though more men are being tested and treated than ever before?
- Why, despite constant coverage, did the pertussis vaccine induce herd immunity at first, but not any longer?
- What is the likely long term impact of strain-specific vaccination against pneumococcus?
- How often do people need to get tested for HIV for 'Treatment as Prevention' to work?
- Which combinations of interventions need testing in expensive cluster randomised prevention trials?
- Which infectious disease interventions should we prioritise to meet the millennium development goals?

Before We Get Started: Some Notation

Symbol	Meaning	Examples
$<$	less than	$1 < 2$ (true); $2 < 2$ (false); $3 < 2$ (false)
$>$	greater than	$1 > 2$ (false); $2 > 2$ (false); $3 > 2$ (true)
$\leq, \leqslant$	less than or equal to	$1 \leq 2$ (true); $2 \leq 2$ (true); $3 \leq 2$ (false)
$\geq, \geqslant$	greater than or equal to	$1 \geq 2$ (false); $2 \geq 2$ (true); $3 \geq 2$ (true)
$\ll$	much less than (subjective, context dependent). It's usually based on the relative difference, rather than the absolute difference.	$0.001 \ll 1$ (usually true); $9.001 \ll 10$ (usually false)
$\gg$	much greater than (subjective, context dependent)	$1 \gg 0.001$ (usually true); $10 \gg 9.001$ (usually false)

$\pm$	plus or minus	e.g. $\sqrt{4} = \pm 2$, i.e. $2^2 = 4$ and $(-2)^2 = 4$; e.g. death toll $= 120 \pm 10$, i.e. an indication of uncertainty
$\simeq, \approx$	approximately equal to (subjective, context dependent)	For a *total* budget for a national health emergency, \$1 $\approx$ \$2 $\approx$ \$0. For an average donation per person, the three values are very different!
$\propto$	proportional to. This means if one thing increases by a given factor, the other increases by that same factor (e.g. if one doubles, the other doubles). Equivalently, their ratio is constant. y is proportional to x means $y = mx$ for some constant c.	Without a multi-buy special offer (e.g. buy two get one free), the total price for multiple identically priced items is proportional to the number of items bought.

$\ldots$	Used to replace something the author hopes is obvious.	$1+2+3+\ldots+8 = 1+2+3+4+5+6+7+8$
$\sum$	'sum over', 'the sum of'	$\sum_{i=1}^{4} i = 1+2+3+4$
$\prod$	'multiply together', 'the product of'	$\prod_{i=3}^{5} i = 3 \times 4 \times 5$
!	factorial: the product of all numbers between 1 and the number in question, inclusive	$9! = 9 \times 8 \times 7 \ldots \times 2 \times 1$
%	per cent, literally 'one per hundred'	$f = 5\%$ means $f = 0.05$; x decreased by 25% means $x_{\mathrm{new}} = 0.75 \times x_{\mathrm{old}}$
$\implies$	implies. $A \implies B$ means 'B is true if A is true', or 'A is a sufficient condition for B'.	$x = 2 \implies x^2 = 4$

$\iff$	implies and is implied by. $A \iff B$ means 'B is true if and only if A is true', or 'A is a necessary and sufficient condition for B'.	$x^2 = 4 \iff x = \pm 2$
$\therefore$	'therefore'. Based on everything that has been said so far, we can draw the conclusion that follows.	$x < -2$, $y = -1$, $z > 3$, $\therefore x < y < z$. I think $\therefore$ I am.
$\mid$	'given that'	The probability of wet pavements in the morning $\mid$ we're in a drought $<$ the probability of wet pavements in the morning $\mid$ it rained in the night.
$\lvert \ldots \rvert$	the 'modulus' or 'absolute value' of something. $\lvert x \rvert = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$	$\lvert 2 \rvert = 2; \quad \lvert -2 \rvert = 2$

Note that some symbols exist with slashes through them to negate the symbol's meaning: $\neq$ means not equal to, $\not\Rightarrow$ means 'does not imply' (which is not the same as 'implies that the following is false').

A Guiding Principle in Mathematics

Starting from an equation that is correct,

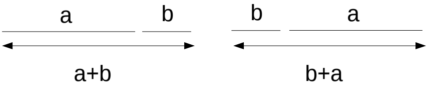
- provided you do exactly the same thing to both sides of the equation (e.g. add something, multiply by something), and
- provided what you do makes sense (i.e. pretty much anything you might think of except dividing by zero),

you'll end up with something that's correct. The trick is figuring out what to do to both sides to get something *useful*.

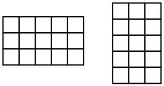
e.g. to solve the equation $2 \times x + 1 = 7$ for x, subtract 1 from both sides and divide by 2 to see that $x = 3$.

Basics: Adding, Multiplying, Subtracting

$a + b = b + a$

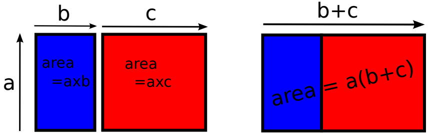


$a \times b = b \times a$ e.g. $3 \times 5 = 5 \times 3 = 15$



$a - b \neq b - a$ e.g. $1 - 2 = -1$, but $2 - 1 = 1$

$$a(b + c) = a \times b + a \times c$$



Basics: Fractions I

It's best to avoid using the $\div$ symbol and write out fractions as one thing on top of another thing, for clarity.

For basic rules regarding fractions it helps to think of $\dfrac{a}{b}$ as the amount of cake one person gets when a cakes are divided equally between b people.

$\frac{a}{a} = 1$, because you have as many cakes as people: they get one each.

$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$ because if I divide $a + b$ cakes between c people, it makes no difference whether I put the a and b cakes together first and then divide them all up, or I divide the a cakes up first and then the b cakes up afterwards – each person gets the same amount of cake either way.

Basics: Fractions II

$a \times \frac{b}{c} = \frac{a \times b}{c}$ because the left-hand side means sharing b cakes between c people then increasing by a factor a; the right-hand side means you had a times as much cake in the first place. It's the same. (Note that choosing b to be equal to 1, this rule tells us that $\frac{a}{c} = a \times \frac{1}{c}$.)

$\frac{a}{b} \div c = \frac{a}{b \times c}$ because the left-hand side means first sharing a cakes between b people, and then each person's share actually has to go to c people; the right-hand side means you had c times as people in the first place. It's the same.

Combining the last two rules: $\dfrac{a}{b} \times \dfrac{c}{d} = \dfrac{a \times c}{b \times d}$

You can get more practise working with fractions at https://www.khanacademy.org/math/arithmetic/fractions

Operator Precedence

To evaluate $3 \times 2 + 2$ does one do the multiplication first (giving 8) or the addition first (giving 12)? Operators with *higher precedence* act first. Precedence is determined by *BIDMAS*:

- Brackets
- Indices (raising things to powers)
- Division & Multiplication
- Addition & Subtraction

NB addition and subtraction have the same precedence as each other, so between these things one evaluates from left to right. e.g. $3 - 2 + 1$ is 2, not 0, because the subtraction is done before the addition.
Likewise, division and multiplication have the same precedence.
Indices have higher precedence still: $2 + 3^2 \times 2 = 2 + 9 \times 2 = 2 + 18 = 20$.
Brackets have the highest precedence of all because their raison d'être is to allow manual specification of the intended order of evaluation, overriding the other rules. e.g. $(2 + 3)^2 = 5^2 = 25$.

Problems

- Without using a calculator, evaluate
  - $\frac{3}{4} + \frac{2}{5}$
  - $1.4 - \frac{3}{8}$
  - $-0.2 \times 3 + \frac{2}{7}$
  - $\Big((3+2) \times 3 + 2\Big) - \Big((3+2) \times (3+2)\Big)$
  - 4^2
  - 0.4^2
  - $2.5^2 - 1.5^2$
  - $1/(1/3)$
- $I = 3t + K$. Is $I \propto t$?
- Where n is an integer and $n \geq 2$, what is $\frac{n!}{(n-1)!}$?
- What is $150\% - \frac{3}{4} + 0.25$?

Playing with Powers I

Raising a number to a power a means a factors of that number all multiplied together:
$x^2 = x \times x$
$x^a = \underbrace{x \times x \times \ldots x}_{a \text{ factors}}$

Raising a product to a power:

$$\begin{aligned}(x \times y)^a &= \underbrace{(x \times y) \times (x \times y) \times \ldots (x \times y)}_{a \text{ factors of } (x \times y)} \\ &= \underbrace{x \times x \times \ldots x}_{a \text{ factors}} \times \underbrace{y \times y \times \ldots y}_{a \text{ factors}} \\ &= x^a \times y^a\end{aligned}$$

Repeatedly applying this rule one can see that
$(x \times y \times z \times \ldots)^a = x^a \times y^a \times z^a \times \ldots$

Playing with Powers II

Multiplying two different powers of the same number:

$$
\begin{aligned}
x^a \times x^b &= \underbrace{x \times x \times \ldots x}_{a \text{ factors}} \times \underbrace{x \times x \times \ldots x}_{b \text{ factors}}, \\
&= \underbrace{x \times x \times \ldots x}_{(a+b) \text{ factors}} \\
&= x^{a+b}
\end{aligned}
$$

Playing with Powers III

Now, multiplying by a number then dividing by that same number does nothing. Therefore when multiplying by x a times and dividing by x b times, b of the a multiplications get cancelled out:

$$
\begin{aligned}
\frac{x^a}{x^b} &= \frac{\overbrace{x \times x \times \ldots x}^{a \text{ factors}}}{\underbrace{x \times x \times \ldots x}_{b \text{ factors}}}, \\
&= \frac{\overbrace{x \times x \times \ldots x}^{b \text{ factors}} \times \overbrace{x \times x \times \ldots x}^{(a-b) \text{ factors}}}{\underbrace{x \times x \times \ldots x}_{b \text{ factors}}}, \\
&= \underbrace{x \times x \times \ldots x}_{(a-b) \text{ factors}} \\
&= x^{a-b}
\end{aligned}
$$

Playing with Powers IV

We know that $x^a/x^b = x^{a-b}$. Choosing $b = a$ gives $x^a/x^a = x^0$; this should clearly equal 1, since any number divided by itself is 1. This tells us that any number raised to the power 0 gives 1:

$$x^0 = 1$$

We can define what is meant by a *negative* power using our understanding that $x^0 = 1$ and $x^a \times x^b = x^{a+b}$. Together, these imply that $x^a \times x^{-a} = x^0 = 1$, which means that x^{-a} is the *reciprocal* of x^a, i.e. 1 divided by it:

$$x^{-a} = \frac{1}{\underbrace{x \times x \times \ldots x}_{a \text{ factors}}}$$

Playing with Powers V

Powers of powers:

$$
\begin{aligned}
(x^a)^b &= \underbrace{\underbrace{(x \times \ldots x)}_{a \text{ factors}} \times \underbrace{(x \times \ldots x)}_{a \text{ factors}} \times \ldots \underbrace{(x \times \ldots x)}_{a \text{ factors}}}_{b \text{ factors}}, \\
&= \underbrace{x \times \ldots x}_{(a \times b) \text{ factors}}
\end{aligned}
$$

$$
\begin{aligned}
\text{Similarly} \quad (x^b)^a &= \underbrace{\underbrace{(x \times \ldots x)}_{b \text{ factors}} \times \underbrace{(x \times \ldots x)}_{b \text{ factors}} \times \ldots \underbrace{(x \times \ldots x)}_{b \text{ factors}}}_{a \text{ factors}}, \\
&= \underbrace{x \times \ldots x}_{(b \times a) \text{ factors}}
\end{aligned}
$$

So

$$(x^a)^b = (x^b)^a = x^{a \times b}$$

Playing with Powers VI

From the definition of x^a, x^1 is simply x. Then since $(x^a)^b = x^{a \times b}$, we know that $(x^{1/n})^n = x^1 = x$, which defines

$$x^{1/n} = \sqrt[n]{x}$$

as the number that gives x when multiplied together n times. For $n = 2$ we say the *square* root, for $n = 3$ the *cube* root, for greater n simply the nth root. For even n there are always two solutions – plus-or-minus the same thing – because the product of an even number of negative numbers is positive.

e.g. $4^{1/2} = \pm 2$ because $2 \times 2 = 4$ and $(-2) \times (-2) = 4$.
c.f. $8^{1/3} = 2$, because $2 \times 2 \times 2 = 8$ but $(-2) \times (-2) \times (-2) \neq 8$.

Problems

Without using a calculator, evaluate

- $2^2 \times 2^3 \times \frac{1}{4}$
- $\frac{11^{11}}{11^9}$
- $\frac{1}{3^{-1}}$
- $(2^{-2})^2$
- $(49)^{\frac{1}{2}}$
- $(-27)^{\frac{2}{3}}$ (hint: $x^{ab} = (x^a)^b = (x^b)^a$)

Simplify $\frac{\sqrt{xyz}}{xy^{-2}z^3} \times (x^{0.5}z)^2$

As x increases, does this expression increase or decrease?

Scientific Notation a.k.a. Standard Form

A figure is in standard form when it is expressed in the following way: $a \times 10^b$,
where $1 \leq a < 10$ and b is an integer (a whole number). b is the number of positions one has to move the left-most non-zero digit to the right to get it on the left of the decimal point sign. e.g.

- $568,000,000 = 5.68 \times 10^8$
- $0.00014 = 1.4 \times 10^{-4}$

The following expressions are all in standard form and equal:
1 year $= 3.6525 \times 10^2$ days $= 8.766 \times 10^3$ hours $= 3.15576 \times 10^7$ s

Using the previous *playing with powers* rules, numbers in standard form are easily multiplied, divided and raised to powers. To add or subtract numbers in standard form it's easiest to take one of them *out* of standard form if necessary, so that they're both multiplied by the same power of ten.

Decimal Places and Significant Figures

How to round a digit: if it's followed by a 0, 1, 2, 3 or 4 we round down (i.e. leave it as is); if it's followed by a 5, 6, 7, 8 or 9 we round up (i.e. increase it by 1). To round the digit 9 upwards, set it to zero and increase the number to the left by 1.

Rounding to n decimal places: the nth digit after the decimal place is rounded. You should explicitly show the n decimal places even if the last ones are zero. Avoid rounding a figure twice: round the original.
e.g. $1.456 = 1.46$ (2 d.p.) $= 1.5$ (1 d.p.) $= 1$ (0 d.p.)
e.g. $2.9995 = 3.000$ (3 d.p.)

Rounding to n significant figures: the nth non-zero digit is rounded. This is useful because it allows you to keep an amount of precision relative to the value itself. e.g. rounding to the nearest whole number: the official population of Greater London in 2013 was $8,416,535$ (0 d.p.) and my height is $2\,\mathrm{m}$ (0 d.p.); c.f. 8.42 million (3 s.f.) and $1.73\,\mathrm{m}$ (3 s.f.) respectively.

Implied Precision

Getting an idea of the implied level of precision from the way a figure is quoted:

- '180 minutes' could mean (a) anything between 175 and 185 minutes, or (b) anything between 179.5 and 180.5 minutes; it depends on the context (we can't tell if it's 2 s.f. or 3 s.f.).
- 'Three hours' should mean anything between two-and-a-half and three-and-a-half hours.
- 'Half a day' probably means anything from a quarter of a day to three-quarters of a day.

There are no guarantees that this usage is shared by everyone...

Multiplication Shorthand

For brevity we usually suppress the $\times$ symbol between two letters and also between a letter and a number (but not between two numbers). For example,
$a \times b = ab$,
$3 \times x = 3x$,
but $2 \times 3 \neq 23$.

Exceptions: a number of functions have a name with multiple letters: $\sin(x)$, $\cos(x)$, $\tan(x)$, $\log_a(x)$, $\ln(x)$, $\exp(x)$; this does not mean those letters are being multiplied together – it's just one function. Sometimes you'll see the brackets dropped with these functions: $\sin(x)$ as $\sin x$, and this doesn't mean the sine function multiplied by x, it still means x is the argument of the sine function.

Problems

Without using a calculator,

- $a = 70$; $b = 3,000,000,000$; $c = 0.00035$.
  - Quote each number to one significant figure, and also to one decimal place.
  - Put each number in scientific notation.
  - Calculate $(ab/c)^2$. (Hint: it's easier when a, b and c are already in scientific notation.)
- Calculate $3.4 \times 10^6 - 4.3 \times 10^5 + 8 \times 10^4$. Give your answer exactly, to two decimal places, and to two significant figures.

Extra Slides

Subscript Indices and Vectors

Sometimes subscripts are added to letters to distinguish between different instances of the same kind of quantity. e.g. if I is incidence, you might see I_{UK} and I_{France}.

Other times the subscript has a more precise role: it is an index (commonly i, j or k) that runs over certain values, allowing different numbers to be collected together into a single object. For example we might split a population into N groups that we label $1, 2, \ldots, N$, and call the incidence in the ith group I_i. In this way the incidences are collected into a single object – the vector $\mathbf{I} = (I_1, I_2, \ldots, I_N)$.
We'll revisit vectors towards the end of the course.

Some Less Common Symbols You Might See

Symbol	Meaning	Examples
‰	per mille; literally one per thousand.	5‰ = five per mille = 0.005
$\Longleftarrow$	is implied by. $A \Longleftarrow B$ means 'A is true if B is true', or 'B is a sufficient condition for A'	$x^2 = 4 \Longleftarrow x = 2$
$\because$	because	$2x^2 \geq x^2$ $\because$ for all $x \neq 0$, $2x^2 > x^2$; but for $x = 0$, $2x^2 = x^2$
$(\ldots, \ldots)$	All numbers between, but not including, the two numbers specified.	$(1, 2)$ means all numbers between 1 and 2 not inclusive.
$[\ldots, \ldots]$	All numbers between and including the two numbers specified.	$[1, 2]$ means all numbers between 1 and 2 inclusive.
$\{\ldots\}$	The set of numbers specified in between the curly braces.	$\{1, 2, 4, 7\}$ the set of numbers 1, 2, 4, and 7.

$\in$	is in, is an element of	p is a probability $\implies p \in [0,1]$
$\mathbb{Z}$	The set of all integers (whole numbers): $\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$	$x \in \mathbb{Z}$ means x is a whole number.
$\mathbb{R}$	The set of all real numbers, including all possible numbers in between any two integers.	$x \in \mathbb{R}$ means x is a real number.
$:=$	is defined to be. Used for things that are true because we define them to be so, not because we have derived them from other results.	For a triangle with a side O opposite an angle θ, and hypotenuse H, $\sin\theta := O/H$.
$\equiv$	Two meanings. The first is identical to ':='. The second is 'is identical to', denoting an equality which is always true and not just in the current context (though many people just use the regular equals sign for this).	$2(x+1) \equiv 2x+2$ (this is true regardless of the value of x).

$\subset$	is a subset of, is contained by	$\{1,2\} \subset \{1,2,3\}$
$\supset$	is a superset of, contains	$(1,2) \supset (1.1, 1.9)$
$\cup$	the union of	$\{1,2,3\} \cup \{3,4,5\} = \{1,2,3,4,5\}$
$\cap$	the intersection of	$\{1,2,3\} \cap \{3,4,5\} = \{3\}$
$\forall$	for all	$2x^2 > x^2 \ \forall \ x \neq 0$ means multiplying x^2 by 2 makes it larger for all x except $x = 0$.
$\mp$	Minus or plus. It's only used when the $\pm$ symbol also appears in the same equation; the meaning is that when one of them is $+$, the other is $-$, and vice versa.	$-1 \times (\pm 1) = \mp 1$

Footnote on Real Numbers

Specifying that one is dealing with *real* numbers explicitly excludes the possibility of *imaginary* numbers. Since no number on the real number line (the one-dimensional continuous line stretching from negative infinity to positive infinity) gives -1 when squared, the imaginary number i is simply defined to be the square root of -1. This is beyond the scope of this course, and indeed your masters – you should take for granted that all numbers you're dealing with are real. It's totally fine to know simply these two points: all real numbers give something greater than or equal to zero when squared, and so square-roots (and fourth-roots, and sixth roots, etc.) should only be used on positive numbers or zero, provided you want a real answer, which you always will here. i is also used to mean other things in other contexts, so if you do see it, assume it means something else and not $\sqrt{-1}$.