Quiz for the *Core Mathematics* Course
(Part of the MPH and MSc in Epidemiology Courses, Imperial College London)

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Very important note: don't panic, you are <u>not</u> expected to know how to answer these questions already! The point of this quiz is to help you establish, individually, whether this course contains some material that you are already so familiar with that it's not a priority for you to recap it. Our default expectation is that you cannot answer any of these questions – the sessions this term will teach you how; any question you *can* answer is an unexpected bonus. Question 1 corresponds to session 1, question 2 to session 2 etc. If you can answer question n easily and correctly, consider skipping session n; otherwise you should attend. Answers will go up at http://www.imperial.ac.uk/people/c.wymant

Calculators should not be used unless explicitly noted to the contrary. 'ln' means '$\log_e$', as is conventional.

Basic Manipulation of Numbers, Oct 14th

1. (a) Evaluate $\Big((3+2)\times 3+2\Big) - \Big((3+2)\times(3+2)\Big)$
   (b) Express $a \times \frac{b}{c} - \frac{e}{f} \div g$ as a single fraction.
   (c) Evaluate
      i. $\frac{11^{11}}{11^9}$
      ii. $(49)^{\frac{1}{2}}$
      iii. $(-27)^{\frac{2}{3}}$
   (d) $a = 70$; $b = 3,000,000,000$; $c = 0.00035$.
      i. Quote each number to one significant figure, and also to one decimal place.
      ii. Put each number in scientific notation, also known as standard form.
      iii. Calculate $(ab/c)^2$. Remember, no calculators! It's easier when a, b and c are already in scientific notation.

Functions, Inequalities and Units, Oct 21st

2. (a) Explain what is meant by a *function*, and why functions are useful in science.
   (b) Simplify
      i. $x + 7 - 2\sin x + \frac{1}{x} - 7 + \frac{3x}{2} + \frac{2}{3x} + \sin x$
      ii. $(2+x)^3 - (3 + 2x^2 - 5x^3)$
   (c) x can be any number greater than or equal to -2, and less than 3: $-2 \leq x < 3$. What inequalities do
      i. $x - 3$, and
      ii. $-2x$

      satisfy? Say as much as you can: for example it is true that $x - 3$ must be less than a million, but you can say more than that – you can give a 'more constraining' inequality.
   (d) An interesting organism (a tiny parasite perhaps) is found x times in one cm^2.
      i. Assuming constant number per unit area, how many would there be in an area that is a square, with each side 2 km long?
      ii. Can you say how many there would be in a 2cm × 2cm × 2cm cube of water?
   (e) *You may use a calculator for this question.* Erythropoiesis – production of red blood cells – occurs at a rate of two million cells per second in humans. There are 270 million haemoglobin molecules in each red blood cell; each haemoglobin molecule contains four iron atoms. 1g of iron contains 1.1×10^{21} atoms. The iron content of grilled sirloin steak, by mass, is one part in 10^5; a typical serving is eight ounces. 100g is 3.5 ounces. Iron is lost from the body in various ways at a total rate of 1.5mg per day. Estimate:

i. the number of milligrams of iron used by the body each day for erythropoiesis[1]. Note that much of this is obtained by recycling old red blood cells.

ii. the number of grilled sirloin steaks one should eat a week to get enough iron, assuming such steaks are one's only source of iron and that the body consumes iron only for erythropoiesis (both dubious assumptions of course).

(Disclaimer: the purpose of this question is to illustrate multiplying together real-world quantities with different units, *not* to inform a healthy balanced diet – the authors possess very limited knowledge of nutrition! All figures are rounded, and we have roughly considered an average human – ignoring differences due to age, gender, socio-economic status etc. for simplicity in combining figures obtained from different sources.)

Important Functions, Oct 28th

3. (a) What are radians? How are they related to degrees (as a measure of angle)?

(b) Define $\cos(\theta)$ and $\sin(\theta)$ for the triangle in Fig. 1. (A is the length of the adjacent side, O of the opposite side, and H of the hypotenuse.)

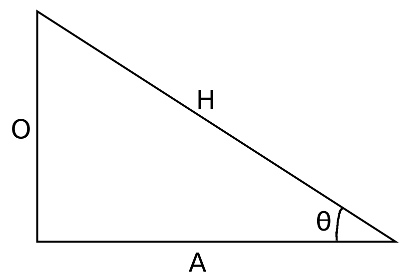

Figure 1: A right-angled triangle.

(c) The biting rate for a parasite vector, $\beta(t)$, varies with time as $\beta(t) = \beta_0 + \beta_1 \cos(\pi t)$, where t is in years. What is the period of oscillation?

(d) For each of the following say whether the answer is defined, and if so where it is relative to the values -1, 0 and $+1$ (i.e. whether it is equal to one of these values, between two of them, less than -1 or greater than $+1$). Remember, no calculators!

i. e^{-1}

ii. e^0

iii. e^1

iv. $\ln(-1)$

v. $\ln(0)$

vi. $\ln(0.5)$ (Hint: $e = 2.7$ to one decimal place.)

vii. $\ln(1)$

(e) *You may use a calculator for this question.* The incidence for a particular epidemic, I, grows exponentially with time t as $I(t) = 1 \text{ per day} \times 10^{t\times 2 \text{ per month}}$. What is

i. the incidence after one week (starting from $t = 0$)?

ii. the time taken to reach an incidence of 1000 per day?

iii. the doubling time (time taken to double)?

iv. the exponential growth rate (i.e. the factor r when incidence is expressed as a constant times e^{rt})?

[1] Your result, if you use the figures provided here as intended, should be significantly different from the value quoted without reference at the top of http://en.wikipedia.org/wiki/Human_iron_metabolism#Systemic_iron_regulation (in the figure caption). We don't know why; do you?

Differentiation, Nov 4th

4. (a) Describe in a sentence or two what the derivative of a function is.
   (b) If $S(t)$ is the probability for an individual to survival to time t, is its derivative $\frac{dS}{dt}$ positive, negative or zero?
   (c) One health centre can provide care for N people, and costs C to build. A new long-term government grant provides a constant source of funding, at a rate F, for building more health centres. What is the rate at which the number of people with access to health care centres increases?
   (d) What are the derivatives of
      i. $\frac{3}{8}x^4 + 12$, with respect to x, evaluated at $x = 2$?
      ii. $e^{2\ \text{day}^{-1} t}$ with respect to t?
      iii. $10°\text{C} + 3°\text{C} \sin\left(\frac{2\pi t}{1\ \text{week}}\right)$ with respect to t? (By °C we mean degrees Celsius.)

What Functions Look Like, Nov 11th

5. (a) Sketch the following functions, labelling the point at which they cross the x and y axes, and showing what they do as $x \to \infty$ and also as $x \to -\infty$:
      i. $y = mx + c$ (m and c are positive constants)
      ii. $y = \ln(x)$
      iii. $y = x^3 - x$
      iv. $y = \frac{1+x}{1-x}$
      v. $\frac{e^x}{1+e^x}$
   (b) What does the function $A \times f(x - c)$ look like, compared to $f(x)$? (A and c are constants. We are deliberately not specifying what the function $f(x)$ is.)
   (c) How does the function $\dfrac{e^{ax} - e^{bx}}{e^{cx} + e^{dx}}$ behave for x very close to zero? (Give an approximate expression involving no exponentials.)

Integration, Nov 18th

6. (a) Describe in a sentence or two what integration is.
   (b) Where $I(t)$ is the rate of new infections occurring in a population at time t, describe in words what the quantity $\int_{t=0}^{T} I(t)dt$ is.
   (c) Find the indefinite integral of:
      i. x
      ii. x^2
      iii. $e^{x/2}$
      iv. $\frac{1}{x}$
   (d) What is the area under
      i. $\sin x$ between 0 and $\pi/2$,
      ii. e^{-x} between 0 and ∞?

Probability I, Nov 25th

7. (a) $P(A)$ and $P(B)$ are the probabilities of A and B respectively.
      i. Give an expression for the probability of 'A or B'.
      ii. Is P(A and B) equal to $P(A)P(B)$? Discuss.
   (b) In a study population, 60% of men are married and 40% are not. In the context of HIV transmission, 10% of the population are in a high-risk category (the rest are considered low-risk); 70% of these are found to be unmarried. If we choose one man at random from the population, what is the probability that he is

i. married and high-risk,

ii. unmarried and low-risk? (Assume that risk status is independent of marital status.)

(c) The expected number of new infections for an endemic disease in a given period of time is n; this can be modelled as a Poissonian process. What is the probability that during this time there are $2n$ or more infections? Leave your answer as an unevaluated sum.

(d) On a given day, N patients must independently choose between two hospitals to receive care. The probability that any given patient chooses hospital 1 is p. What is the probability that at least two patients choose hospital 1? (Hint: binomial probability is relevant here.)

Probability II, Dec 2nd

8. (a) The probability distribution function $p(x)$ for a variable x which is *normally* distributed is given by

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

How can μ and σ be interpreted in this expression? What is the probability that x is somewhere in the range from $\mu - \sigma$ to $\mu + \sigma$? Leave your answer as an unevaluated integral.

(b) When there is a constant hazard λ for an individual to leave a particular state/compartment, the time spent therein, T, is a continuous random variable that is exponentially distributed with mean $\frac{1}{\lambda}$.

i. Calculate the probability that T is greater than its mean.

ii. What is the variance of T? Leave your answer as an unevaluated integral.

(c) State Bayes' Theorem. Explain its relevance when distinguishing between scientific models based on data.

Matrices, Dec 9th

9. (a) When is it helpful to use matrices in the description of a problem? Explain what the eigenvectors and eigenvalues of a matrix are. When a vector is multiplied by the same matrix a very large number of times, which direction in space does it end up pointing in?

(b) Evaluate the following, where the dot denotes matrix multiplication:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \cdot \begin{pmatrix} j \\ k \\ l \end{pmatrix}$$

(c) Find the determinants of the following matrices:

$$\text{i)}\ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},\ \text{ii)}\ \begin{pmatrix} 6 & 2 \\ 7 & 4 \end{pmatrix},\ \text{iii)}\ \begin{pmatrix} 2 & 8 \\ 1 & 4 \end{pmatrix}.$$

(d) Express the following set of equations in matrix form:

$$\begin{aligned} x + y + z &= 9, \\ x - y &= 7, \\ z + y &= 5 \end{aligned}$$

Numerical Methods, Dec 16th

10. This lecture is an odd one out, in that it will not cover mathematical tools necessary for understanding the content of your other courses; rather it will explain how the computational methods used to solve compartmental model equations actually work, how they can go wrong, and why it's important to understand this. (We put this "question" here merely as a reminder for the date.)