Matrices Revisited, Part I

Chris Wymant

Department of Infectious Disease Epidemiology,
Imperial College London

February 2015

Acknowledgement: I wrote this while funded by ERC Advanced Grant PBDR-339251.

Recap: Linear Dependence for Single Numbers

One of the simplest ways a single number y can depend on a single number x is *linearly*: $y = mx + c$, with m and c constant. This means that if we increase x by a certain amount Δx and then increase it by the same amount again, y will increase by a certain amount Δy and then increase by the same amount again.

$$\begin{aligned} & x \to x + \Delta x \to x + 2\Delta x \\ \implies & y \to y + \Delta y \to y + 2\Delta y \end{aligned}$$

(Specifically, $\Delta y = m\Delta x$.) In other words, a given change in x always produces the same change in y. Note that this is not true for any non-linear relationship between x and y, for example $y = x^2$ or $y = e^x$.
NB a linear relationship, when plotted on a graph, is a straight line; m is the gradient and c is the y-intercept.

Why We Need Matrices: Abstract, part I

Matrices are what allow us to have the same situation when x and y are no longer single numbers but *collections* of numbers, i.e. vectors: $\mathbf{x} = (x_1, x_2, \ldots)$, $\mathbf{y} = (y_1, y_2, \ldots)$. A naïve generalisation of $y = mx + c$ would be $\mathbf{y} = m\mathbf{x} + \mathbf{c}$ (recall that a vector is multiplied by a single number elementwise, thus: $c\mathbf{x} = (cx_1, cx_2, \ldots)$). In fact, each element of $\mathbf{x}$ can affect each element of $\mathbf{y}$ independently. So, if there are M elements in $\mathbf{x}$ and N in $\mathbf{y}$, the equivalent of what was the gradient now consists of $M \times N$ numbers: it's a matrix. The definition of how matrix multiplication works, matching each row in the matrix with the vector in turn and adding as you go, e.g.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 + cx_3 \\ dx_1 + ex_2 + fx_3 \\ gx_1 + hx_2 + ix_3 \end{pmatrix}$$

guarantees the property of linearity we are looking for. Let's see this in the simplest case of a 2×2 matrix...

Why We Need Matrices: Abstract, part II

Let $\mathbf{y} = \mathbf{M} \cdot \mathbf{x} + \mathbf{c}$:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 + c_1 \\ cx_1 + dx_2 + c_2 \end{pmatrix}$$

When we add the same vector $\mathbf{g}$ twice to $\mathbf{x}$,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} x_1 + g_1 \\ x_2 + g_2 \end{pmatrix} \to \begin{pmatrix} x_1 + 2g_1 \\ x_2 + 2g_2 \end{pmatrix},$$

$\mathbf{y}$ increases twice thus:

$$\begin{aligned} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &\to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 + g_1 \\ x_2 + g_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a(x_1 + g_1) + b(x_2 + g_2) + c_1 \\ c(x_1 + g_1) + d(x_2 + g_2) + c_2 \end{pmatrix} \\ &\to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 + 2g_1 \\ x_2 + 2g_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a(x_1 + 2g_1) + b(x_2 + 2g_2) + c_1 \\ c(x_1 + 2g_1) + d(x_2 + 2g_2) + c_2 \end{pmatrix} \end{aligned}$$

Why We Need Matrices: Abstract, part III

What just happened? $\mathbf{y}$ increased by the same vector twice. That vector is

$$\Delta\mathbf{y} = \begin{pmatrix} ag_1 + bg_2 \\ cg_1 + dg_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

So the same change in $\mathbf{x}$ produces the same change in $\mathbf{y}$ (the behaviour we were looking for). The reason this happened is because $\mathbf{M} \cdot (\mathbf{x} + \mathbf{g}) = \mathbf{M} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{g}$. This is the same as saying that matrix multiplication is a *linear operation*, like differentation (see the calculus notes at http://www.imperial.ac.uk/people/c.wymant).

Why We Need Matrices: Practical

Because in public health and epidemiology, one collection of numbers very often affects another collection of numbers.

The Next-Generation Matrix $\mathbf{K}$, part I

Definition: K_{ij} is the total number of i-type people (or people in group i, if you like) infected by one j-type person throughout the course of their infection.

i and j run over the same set of possible types (type '1', type '2' etc.) so $\mathbf{K}$ is square. The diagonal entries of $\mathbf{K}$ are the numbers of infections people transmit to the same type as themselves; off-diagonal entries enumerate transmissions between different types. A *diagonal matrix* (all off-diagonal elements are zero) describes no mixing: people only transmit to their own group.

$\sum_i K_{ij}$ (i.e. the sum of the jth column) is the total number of people infected by one j-type person throughout the course of their infection.

e.g. $\quad \mathbf{K} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies$ a person in group 1 infects a people in group 1 and c people in group 2; a person in group 2 infects b people in group 1 and d people in group 2.

An Aside: Understanding a *Generation* of Infecteds

Consider the exact analogy between infection as a process of 'epidemiological birth' (producing a new infected individual) and the process of actual birth, i.e. population growth. Date of infection/transmission corresponds to date of birth. Just as the individuals in one generation of a family do not all have the same age, one generation of individuals has a variety of dates of infection. Individuals in one generation of a family may even have non-overlapping lives: a child may die before his/her sibling is born, giving non-overlap within generation one; death of a grandchild before his/her cousin is born gives non-overlap in generation two; etc. Thus, infecteds in the same generation can have infections that do not overlap in time.

As the Nth generation of a family can be connected to the 'root' of the family through N processes of birth but no fewer, the Nth generation of infecteds in an epidemic can be connected to the founder through N transmission contacts but no fewer.

The connection between a generational view and a real-time view of an epidemic is a mathematically challenging problem, considered in e.g. doi:10.1016/j.mbs.2008.08.009 by Neil and Christophe.

The Next-Generation Matrix $\mathbf{K}$, part II

If we write the numbers of people of each type, in a given generation of infecteds, in a vector $\mathbf{N} = (N_1, N_2, \ldots)$, then the numbers in the next generation are given by the vector $\mathbf{K} \cdot \mathbf{N}$. The numbers in the generation after that are given by the vector $\mathbf{K} \cdot (\mathbf{K} \cdot \mathbf{N}) = \mathbf{K}^2 \cdot \mathbf{N}$. Multiplication by $\mathbf{K}$ gives the next generation, hence the name.

Eigenvalues and Eigenvectors

Vectors can be proportional to each other or not.
Proportional $\iff$ in the same direction in space $\iff$ differ only by a factor.
e.g. $\binom{2}{3} \propto \binom{1}{1.5}$ because $\binom{2}{3} = 2 \times \binom{1}{1.5}$, and they point in the same direction. (Try drawing them both on a 2D plot: the vector $\binom{2}{3}$ can be shown by an arrow from the origin to the point $x = 2, y = 3$.) A closely related idea is the fact that $6'' \times 4''$ photos have the same proportion as $9'' \times 6''$, but this is different from $7'' \times 5''$.

The *eigenvectors* of a matrix are those vectors that point in the same direction when multiplied by the matrix; they are a merely multiplied by a constant, called the *eigenvalue*.
e.g. $\binom{1}{1}$ is an eigenvector of $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ because $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\binom{1}{1} = \binom{3}{3} = 3 \times \binom{1}{1}$, which is in the same direction as the original vector. We see that the eigenvalue is 3. $\binom{1}{2}$ is *not* an eigenvector of $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ because $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\binom{1}{2} = \binom{4}{5}$, which is in a different direction.