Problem Session 1

1. (a) $-5 \leq (x-3) < 0$
   (b) $0 \leq (x+2) < 5$,
   (c) $-6 \leq 3x < 9$, and
   (d) $4 \geq (-2x) > -6 \quad$ or $\quad -6 < (-2x) \leq 4$
2. $(1-\sigma)R_0 < 1$ so $\sigma > \frac{R_0-1}{R_0}$

Problem Session 2

1. $\frac{5x}{2} + \frac{5}{3x} - \sin x$
2. $5 + 12x + 4x^2 + 6x^3$

Problem Session 3, I

State the dimension, and give an example of an appropriate unit, for the following quantities:

1. [the mass of a particular drug required] = [mass]. e.g. kg (kilogram), mg (milligram)
2. [the number of people infected with a given disease] = [1] i.e. it is dimensionless (has the same dimension as the number 1).
3. [the length of time between being infected and becoming infectious] = [time]. e.g. years, hours
4. [the power to which we raise some quantity] = [1]. This is because of what raising to a power really means: e.g. x^3 means three factors of x get multiplied together. This still makes sense if x is dimensionful, but makes no sense if we replace '3' by something with dimensions. For example the quantities $x^{3\text{ days}}, x^{1\text{ mile}}$ are nonsensical regardless of what x is. Equivalently, any time you see e^{rt} and t is a time, r must have the dimensions of time^{-1}, because rt is a power and so must be dimensionless.
5. [an amount of money] = [currency], e.g. pounds, pence, dollars
6. [a fractional increase in an amount of money] = [1]. A fractional change in anything is dimensionless, because a fractional change in a quantity x means $(x_{\text{new}} - x_{\text{old}})/x_{\text{old}}$.
7. [a rate of fractional increase in an amount of money] = [time]$^{-1}$. Interest rates are so frequently quoted simply as percentages that it's easy to forget that the passage of time is relevant – 'per year' is usually implicit. Note also that the % symbol itself is dimensionless – it represents the number 0.01 (see also the discussion of percentages in the notes for session 1).
8. [the concentration of parasites in a patient's blood] = [length]$^{-3}$. The number of parasites is just a number. The extent to which something occupies space in 3D – 'volume' – has dimensions of length^3. (For example the volume of a cuboid is the product of the lengths of the three different sides.) Therefore [per volume] = [length]$^{-3}$. Note that one litre is a thousand cubic centimetres: $1\text{L} = 1000\text{cm}^3$, and so 'per litre' $= 1/(1000\text{cm}^3) = 10^{-3}\text{cm}^{-3}$; and 'per cm^3' $= 1/(1\text{cm}^3) = 1/(10^{-3}\text{L}) = 10^3$ per litre. This last coversion is perhaps more intuitive – if you find something once per cubic centimetre of water, you'll find it a thousand times per litre.
9. [the rate of change in the (population average) time from HIV infection to AIDS] = [1]. Here I've deliberately tried to trip you up. If you're getting the hang of this, you might be expecting to see 'per time' associated with anything that is a rate. Note that a careless statement about the dimension of rates would be 'rates have dimension of time^{-1}'. A careful statement would be 'the rate at which x changes with respect to time has the dimension of time^{-1} multiplied by whatever the dimension of x is'. Here, x has the dimension of time, and so the rate is dimensionless! To elaborate, in case this seems paradoxical, take the purely hypothetical example of a country slowly increasing the availability of anti-retroviral treatment (ART) for HIV positive people, with the result that each passing (calendar) year, the average time taken to reach AIDS is prolonged by 6 months. Equivalently, in each two-month period the average time to AIDS increases by one month; in each day it increases by 12 hours, etc. In summary for this (fictional!) example, the rate of change of the average time to AIDS is 0.5.

Problem Session 3, II

1. (a) 1 per week $\times$ 2 years $= \frac{2 \text{ years}}{1 \text{ week}} = \frac{2\times 365.25 \text{ days}}{7 \text{ days}} = 104$ (3 s.f.)
   NB because of leap years an average year has 365.25 days.

   (b) £21. The structure suggests that this is the total cost of treating a certain number of people for a given length of time.

2. $4 \times 10^{10}\, x$

3. The volume of alcohol in one bottle is 75cl $\times 0.125$.
   The mass of alcohol in one bottle is 75cl $\times 0.125 \times 789\text{kg/m}^{-3}$.
   1 litre = $1\text{dm}^3 = (0.1\text{m})^3$ so $1\text{m}^3 = 1000$l.
   The total mass of alcohol in the bloodstream at the limit is 5l of blood $\times$80mg per 100ml of blood. The number of bottles containing that mass is

$$\begin{aligned}\frac{5\text{l} \times 80\text{mg}/100\text{ml}}{75\text{cl} \times 0.125 \times 789\text{kg/m}^{-3}} &= \frac{5\text{l} \times 80\text{mg} \times 1\text{m}^3}{75\text{cl} \times 0.125 \times 789\text{kg} \times 100\text{ml}} \\ &= \frac{5\text{l} \times 80 \times 10^{-3}\text{g} \times 10^3\text{l}}{75 \times 10^{-2}\text{l} \times 0.125 \times 789 \times 10^3\text{g} \times 100 \times 10^{-3}\text{l}} \\ &= \frac{5 \times 80}{75 \times 0.125 \times 789} \times 10^{-3+3+2-3-2+3} \\ &= 0.05 \text{ (2 d.p.)}\end{aligned}$$

   or 0.11 bottles (2 d.p) between the pair of them. The approximation that the rate of entry into the bloodstream far exceeds the rate of removal is presumably poor. The rate at which the alcohol makes it through the digestive system into the bloodstream and then circulates round the body is not infinite (i.e. it's not instantaneous); as soon as this process has started it is counteracted by metabolism in the liver (at a rate of one standard drink per hour). Alcohol can also collect in body tissues, diluting the concentration in the bloodstream. NB I have no expertise in this area – I'm merely trying to rationalise the surprisingly small number...

Problem Session 3, III

1. x is dimensionless; $[y] = [\text{length}]^{-1}[\text{temperature}]^{-2}[\text{time}]^{-2}$

2. The money available is $r_d NT(1-f)$: the rate of donating per person, times the number of people donating, times the length of time over which donations continued, times the fraction which can actually be used. In time t, (ntr_v) vaccines are given; the cost per vaccine is $p + dP/D$ just for the drug and the needle, but the wages of the nurses must be added. Vaccinating for a time t therefore costs $(ntr_v)(p + dP/D) + ntr_p$. The cost per vaccine, incorporating nurses' wages, is equal to the cost of vaccinating for a time t divided by the number of vaccines given in this time: this is $p + dP/D + r_p/r_v$ (note that time cancels, as does the number of nurses). The total number of vaccines that can be given is equal to the budget for vaccines divided by the cost per vaccine: $(Nr_dT(1-f))/(p + dP/D + r_p/r_v)$.

Problem Session 3, IV

1. $\frac{2\times 10^6 \text{cells s}^{-1} \times 1 \text{ day} \times 270 \times 10^6 \text{molecules cell}^{-1} \times 4 \text{ atoms molecule}^{-1}}{1.1\times 10^{21} \text{ atoms g}^{-1}} = 170\text{mg}$ (3 s.f.)

2. The stated dubious assumption that the body consumes iron only for erythropoiesis means the rate at which iron is needed in one's diet equals the rate at which iron is lost – 1.5mg per day – and all the rest must come from recycling. The mass of iron needed in one week is 1.5mg per day $\times$7 days. One serving of grilled sirloin steak is 8 ounces = 8 ounces $\times$ 100g/3.5 ounces. That amount times 10^{-5} is the mass of iron therein. So the number of servings of grilled sirloin steak needed per week is

$$\frac{1.5\text{mg per day} \times 7 \text{ days}}{8 \text{ ounces} \times 100\text{g}/3.5 \text{ ounces} \times 10^{-5}} = 4.6\ (2\ s.f.)$$