Answers for the Quiz for the *Core Mathematics* Course (Part of the MPH and MSc in Epidemiology Courses, Imperial College London)

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We give answers only, not worked solutions. If you weren't able to get the correct answer before checking the answers, please come to the associated lecture! Please report any typos to c.wymant@imperial.ac.uk

Basic Manipulation of Numbers, Oct 14th

1. (a) -8
   (b) $\frac{abfg-ec}{cfg}$
   (c) i. 121
       ii. ± 7
       iii. 9
   (d) i. $a = 70$ (1 s.f.), $a = 70.0$ (1 d.p.);
       $b = 3,000,000,000$ (1 s.f.), $b = 3,000,000,000.0$ (1 d.p.);
       $c = 0.0004$ (1 s.f.) $c = 0.0$ (1 d.p.)
       ii. $a = 7 \times 10$; $b = 3 \times 10^9$; $c = 3.5 \times 10^{-4}$.
       iii. 3.6×10^{29}

Functions, Inequalities and Units, Oct 21st

2. (a) A function can be described as a 'mapping' between numbers: it takes a number as input and returns a (usually different) number as output. Functions are useful in science because we are often interested in quantifying the relationship between dependent quantities: *how* they depend on each other and *how much*. This can be described by considering one quantity to be a function of another.
   (b) i. $\frac{5x}{2} + \frac{5}{3x} - \sin x$
       ii. $5 + 12x + 4x^2 + 6x^3$
   (c) i. $-5 \leq (x - 3) < 0$
       ii. $4 \geq (-2x) > -6$, or equivalently $-6 < (-2x) \leq 4$
   (d) i. $4 \times 10^{10}\, x$
       ii. No, you cannot. A number per unit area has been defined but not a number per unit volume, and one cannot be deduced from the other – they are fundamentally different things.
   (e) i. 170mg (3 s.f.)

ii. 4.6 (2 s.f.) (The subtlety here is that most of the required iron comes from recycling old iron. The stated dubious assumption that the body consumes iron only for erythropoiesis means the rate at which iron is needed in one's diet equals the rate at which iron is lost – 1.5mg per day – and all the rest must come from recycling.)

Important Functions, Oct 28th

3. (a) Dividing the length of an arc around the circumference of a circle by the radius of the circle gives the angle subtended by the arc, in radians. (In more pedestrian language, dividing the length of the curved edge of a slice of pie by the length of one of the straight edges gives the angle at the point of the slice, in radians.) Considering the whole circle (or pie) shows that 2π radians $= 360°$.
   (b) $\sin(\theta) = O/H$ and $\cos(\theta) = A/H$. For reference, $\tan(\theta) = O/A$, leading to the mnemonic 'sohcahtoa'.
   (c) Two years
   (d) i. $0 < e^{-1} < 1$
      ii. $e^0 = 1$
      iii. $e^1 > 1$
      iv. $\ln(-1)$ is undefined. (It can be defined using *imaginary* numbers, but that is beyond the scope of this course: for our purposes it is undefined.)
      v. $\ln(0)$ is undefined. If x is positive, then as it approaches zero, $\ln x$ diverges to $-\infty$. (c.f. $\frac{1}{0}$ is undefined, and for positive x, $\frac{1}{x}$ diverges to ∞ as x approaches 0.)
      vi. $-1 < \ln(0.5) < 0$
      vii. $\ln(1) = 0$
   (e) i. 2.9 per day (2 s.f.)
      ii. $1\frac{1}{2}$ months
      iii. $\frac{\ln 2}{2 \ln 10}$ months $= 0.15$ months (2 s.f.) $\approx 4\frac{1}{2}$ days
      iv. $2 \times \ln(10)$ per month $= 4.6$ per month $(2s.f.)$.

Differentiation, Nov 4th

4. (a) The derivative of a function is the rate of change of the value of the function with respect to its argument. In other words, it is the amount by which the function changes when its argument changes by a small amount, relative to that change in the argument. It is the gradient/slope of the function, at a given point.
   (b) $\frac{dS}{dt} < 0$
   (c) FN/C
   (d) i. 12
      ii. $2\ \text{day}^{-1} e^{2\ \text{day}^{-1} t}$
      iii. $6\pi°\text{C}\,\text{week}^{-1} \cos\left(\frac{2\pi t}{1\ \text{week}}\right)$

What Functions Look Like, Nov 11th

5. (a) See Figures 1-5.
   (b) $Af(x-c)$ looks like $f(x)$ but shifted to the right by c and stretched vertically / away from the x axis by a factor A.

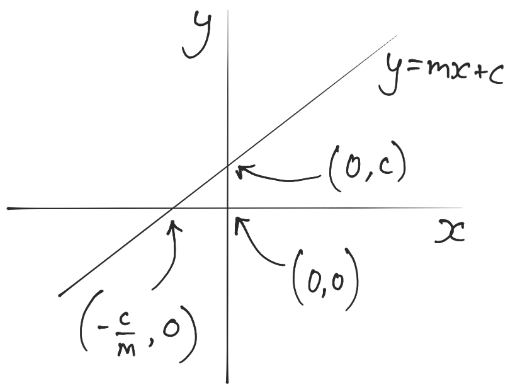

Figure 1: $y = mx + c$.
$y \to \pm\infty$ as $x \to \pm\infty$.

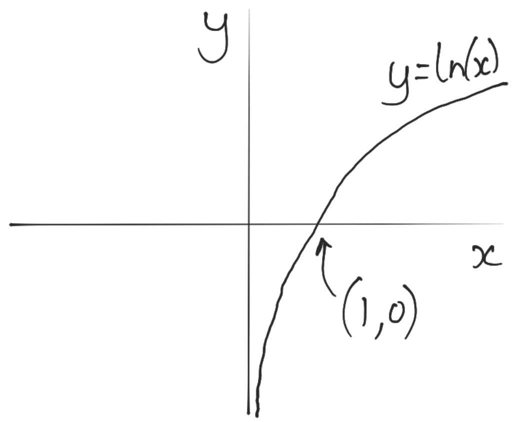

Figure 2: $y = \ln(x)$.
$y \to \infty$ as $x \to \infty$;
$y \to -\infty$ as $x \to 0$;
y is undefined for $x \leq 0$.

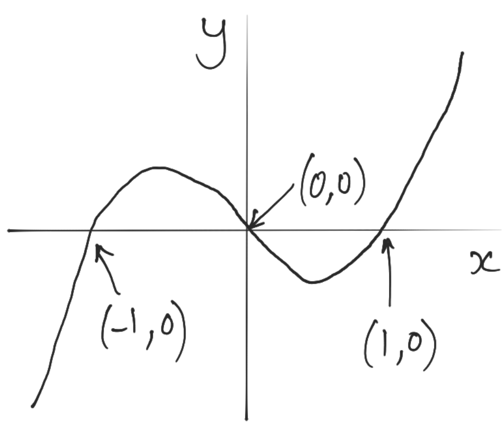

Figure 3: $y = x^3 - x$.
$y \to \pm\infty$ as $x \to \pm\infty$.

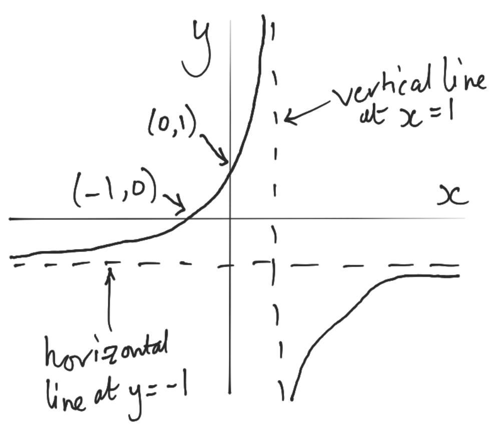

Figure 4: $y = \frac{1+x}{1-x}$.
$y \to -1$ as $x \to \pm\infty$;
y is undefined at $x = 1$.

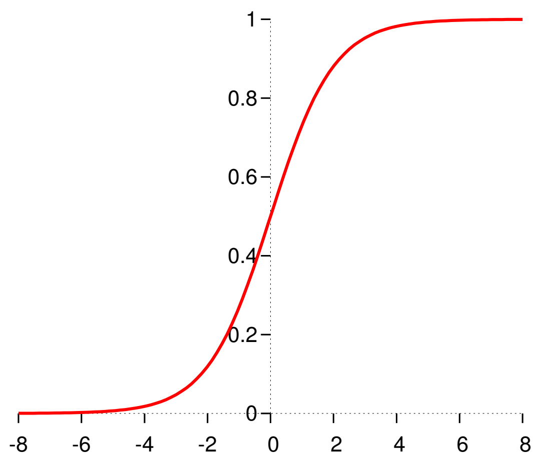

Figure 5: $\frac{e^x}{1+e^x}$.
$y \to 0$ as $x \to -\infty$, $y(0) = 0.5$, and $y \to 1$ as $x \to \infty$.

(c)

$$\text{for } |x| \ll 1, \qquad \frac{e^{ax} - e^{bx}}{e^{cx} + e^{dx}} \approx \tfrac{1}{2}(a-b)x$$

Integration, Nov 18th

6. (a) Integration, basically, is the process of calculating the area under a curve. Though naively a simple geometrical problem, this is highly relevant in real-world problems. As a simple example, any quantity which is a constant rate – *something* occurring per unit time – when multiplied by a time duration will give the total amount of that *something* occurring in that time. However if the rate itself is changing, the rate must be integrated between two times to get the total, and not simply multiplied by the time duration.

   (b) $\int_{t=0}^{T} I(t)dt$ is the number of new infections that occur in the population between times 0 and T.

   (c) i. The indefinite integral of x is $\frac{1}{2}x^2 + C$, for arbitrary constant C. $\int_a^b x\,dx = [\frac{1}{2}x^2]_a^b$.

      ii. The indefinite integral of x^2 is $\frac{1}{3}x^3 + C$, for arbitrary constant C. $\int_a^b x^2\,dx = [\frac{1}{3}x^3]_a^b$.

      iii. The indefinite integral of $e^{x/2}$ is $2e^{x/2} + C$, for arbitrary constant C. $\int_a^b e^{x/2}\,dx = [2e^{x/2}]_a^b$.

      iv. The indefinite integral of $\frac{1}{x}$ is $\ln(x) + C$, for arbitrary constant C. $\int_a^b \frac{1}{x}\,dx = [\ln(x)]_a^b$.

   (d) i. 1

      ii. 1

Probability I, Nov 25th

7. (a) i. $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) = 1 - P(\text{neither } A \text{ nor } B)$

      ii. If A and B are independent, $P(A \text{ and } B) = P(A)P(B)$. If A and B tend to occur together, $P(A \text{ and } B) > P(A)P(B)$; if they tend to anticorrelate (i.e. one of A or B being true makes it less likely for the other to be true) then $P(A \text{ and } B) < P(A)P(B)$.

   (b) i. 0.03 or 3%

ii. 0.36 or 36%

(c)

$$\sum_{k=2n}^{\infty} \frac{n^k e^{-n}}{n!}$$

(d)

$$\sum_{k=2}^{N} \frac{N!}{k!(N-k)!} p^k (1-p)^{N-k} \quad \text{or, simpler,} \quad 1 - \left((1-p)^N + Np(1-p)^{N-1}\right)$$

Probability II, Dec 2nd

8. (a) μ is the mean, σ is the standard deviation (from the mean). The probability that x is somewhere in the range from $\mu - \sigma$ to $\mu + \sigma$ is

$$\int_{x=\mu-\sigma}^{\mu+\sigma} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

The value of this integral (which many people in the field know by heart) is 0.68 or 68% (2 s.f.), i.e. about two-thirds. Note that this is true regardless of the value of μ or σ.

(b) i. $e^{-1} = 0.37$ (2 d.p.)

ii. $\lambda \int_0^\infty x^2 e^{-\lambda x}\, dx$

(c)

$$P(A|B) = \frac{P(B|A)P(A)}{p(B)}$$

or

$$P(\text{explanation}|\text{observation}) = \frac{P(\text{observation}|\text{explanation})P(\text{explanation})}{P(\text{observation})}$$

or

$$P(\text{model}|\text{data}) = \frac{P(\text{data}|\text{model})P(\text{model})}{P(\text{data})}$$

Bayes' Theorem is very useful when attempting to model phenomena because, while 'the probability of observing the data we have observed given that this model is true' is a simpler quantity to calculate, it can only be used to deduce the (subjective) probability that a model is true by using Bayes' Theorem. Comparing the probabilities of different models being true in light of the data we observe is often at the heart of trying to describe the world around us with mathematical models.

Matrices, Dec 9th

9. (a) While $y = mx + c$ describes the linear dependence of one number (y) on another number (x), matrices can describe the linear dependence of a *collection* of numbers (a vector) on another collection of numbers. They are therefore invaluable in problems naturally formulated in terms of vectors. (An example relevant for epidemiology is the splitting up of a population into compartments, and keeping track of the number of people in each compartment.)

The eigenvectors of a matrix are those vectors which, upon multiplication by the matrix, continue to point in the same direction, in other words are merely multiplied by constant. That constant is the eigenvalue associated with the eigenvector. When a vector is multiplied by the same matrix a very large number of times, the direction in space it ends up pointing in is the direction of the eigenvector with the eigenvalue of largest magnitude.

(b)

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \cdot \begin{pmatrix} j \\ k \\ l \end{pmatrix} = \begin{pmatrix} aj + bk + cl \\ dj + ek + fl \\ gj + hk + il \end{pmatrix}$$

(c)

$$\text{i) } \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2, \qquad \text{ii) } \begin{vmatrix} 6 & 2 \\ 7 & 4 \end{vmatrix} = 10, \qquad \text{iii) } \begin{vmatrix} 2 & 8 \\ 1 & 4 \end{vmatrix} = 0$$

(d)

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \\ 5 \end{pmatrix}$$