Core Mathematics,

Part of the MPH and MSc in Epidemiology Courses, Imperial College London

Session 2: Inequalities, Functions and Units

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Inequalities I

Inequalities are based on the following four symbols that we've already seen:

Symbol	Meaning	Examples
$<$	less than	$1 < 2$ (true); $2 < 2$ (false); $3 < 2$ (false)
$>$	greater than	$1 > 2$ (false); $2 > 2$ (false); $3 > 2$ (true)
$\leq, \leqslant$	less than or equal to	$1 \leq 2$ (true); $2 \leq 2$ (true); $3 \leq 2$ (false)
$\geq, \geqslant$	greater than or equal to	$1 \geq 2$ (false); $2 \geq 2$ (true); $3 \geq 2$ (true)

Inequalities II

Inequalities can be chained together to form composite statements. The arrows should all point in the same direction to avoid confusion.

$$1 < 2 < 10 \quad \text{true;}$$
$$1 < 3 < 2 < 5 \quad \text{false.}$$

Inequalities can be manipulated in a similar way to equalities, though sometimes more care is required. Adding to and subtracting from inequalities works just as with equalities:

$$a = b \implies a + c = b + c$$
$$a < b \implies a + c < b + c$$

Inequalities III

When it comes to multiplication, inequalities are trickier than equalities:

$$a = b \implies ac = bc$$

$$a < b \implies \begin{cases} ac < bc & \text{if } c > 0 \\ ac > bc & \text{if } c < 0 \end{cases}$$

In other words, you need to know whether the thing you're multiplying by is negative, and if so, you reverse the direction of the arrow. This is true because 'less than' does not mean 'closer to zero', it means 'less positive' which is equivalent to 'more negative', i.e. further to the left on the number line. An example is helpful: $2 < 3$ but $-2 > -3$.

The same is true for division: dividing by a negative number reverses the direction of the arrow.

Problem Session 1

1. x can be any number greater than or equal to -2, and less than 3: $-2 \leq x < 3$. What inequalities do
   1.1 $x - 3$,
   1.2 $x + 2$,
   1.3 $3x$, and
   1.4 $-2x$

   satisfy?
2. R_0 is the average number of new infections caused by a typical infected individual, in a totally susceptible population. If $R_0 > 1$, the disease will either spread or remain stably pandemic; if $R_0 < 1$, transmission will decrease and ultimately disappear. A public health intervention prevents a fraction σ of transmissions. Derive the inequality that σ must satisfy if the intervention is to achieve elimination.

Functions I

A function takes a number and gives you a number back.

We use functions to describe the dependence of one quantity on another.

The number you give it is called the function's *argument*.

A function can be described in words: for example "whatever number you give me, I give you it back plus two", or "whatever number you give me, I give you one back four times as big", but for brevity and clarity it's better to put a label on "whatever number you give me", i.e. the argument. Let's call it x. "whatever number you give me, I give you it back plus two" can be written more concisely as $x + 2$; the function for multiplication by 4 can be written as $4 \times x$ etc.

Functions II

If we call the number going into the function x, what do we call the function itself? Two notations are common:

- with x the independent variable – something we can choose freely – we say y is the dependent variable, depending on x.
- $f(x)$, read out loud as "f of x", meaning the function f is a function of x.

Sometimes the notations are mixed: you might see $y = y(x)$ or $y = f(x)$ written to mean y is a function of x.

You'll soon be seeing all sorts of variables, each with their own letter to represent their value; you will have to decide in each case what depends what on what, i.e. what is a function of what.

Simplifying Expressions

Preliminary nomenclature: an expression is a sum of terms; terms are things separated by $+$ signs in an expression. e.g. in the expression $\frac{a-b+c}{d}$, the terms are $\frac{a}{d}$, -$\frac{b}{d}$ and $\frac{c}{d}$.

The expression $2^3 - 2^2$ can be simplified: it equals $8 - 4 = 4$.
$x^3 - x^2$ cannot be simplified unless one picks one specific value of x, because the extent to which the second term cancels the first term depends on x.
$3x^2 - 2x^2$ can be simplified – it equals x^2 – because the two terms are the same function of x but with different *coefficients* (constant multipliers, 3 and -2 here).
By *collecting together like terms* one can simplify expressions and equations.

Polynomials

A *polynomial* of a variable, x say, is an expression consisting of non-negative integer powers of x.

Recall that $a(b+c) = ab + ac$. Define a to be equal to $d+e$, and consider this equation again:

$$(d+e)(b+c) = b(d+e) + c(d+e) = bd + be + cd + ce$$

In words: each term in one bracket gets multiplied by each term in the other, then everything is added together; this way of saying it remains true when the brackets contain more than two terms. This tells us how we can multiply together polynomials.

$$\begin{aligned}(2+x&+4x^2)(1+6x+3x^2)\\ &= 2(1+6x+3x^2) + x(1+6x+3x^2) + 4x^2(1+6x+3x^2)\\ &= 2+12x+6x^2 \quad + x + 6x^2 + 3x^3 \quad + 4x^2 + 24x^3 + 12x^4\\ &= 2+13x+16x^2+27x^3+12x^4\end{aligned}$$

Problem Session 2

Simplify

1. $x + 7 - 2\sin x + \frac{1}{x} - 7 + \frac{3x}{2} + \frac{2}{3x} + \sin x$
2. $(2 + x)^3 - (3 + 2x^2 - 5x^3)$

Unit Prefixes

Unit prefixes abbreviate multiplication by various powers of ten.
More are listed at http://en.wikipedia.org/wiki/Metric_prefix

Name	Prefix	Meaning
giga	G	10^9
mega	M	10^6
kilo	k	10^3
deci	d	10^{-1}
centi	c	10^{-2}
milli	m	10^{-3}
micro	μ	10^{-6}

Viewed as simply representing multiplication by their associated power of ten, these prefixes appear to break the operator precedence rule for powers: $1\text{km}^2 = 1(\text{km})^2 \neq 1\text{k}(\text{m})^2$. One can try to rationalise this as follows: these unit prefixes are so common that we think of the unit absorbing them to define a new unit: one 'km' is thought of as a unit in its own right, rather than $10^3 \times \text{m}$.

Understanding Units I

The *dimension* of a quantity X, $[X]$, is the kind of units X is measured in.
The quantities '3 hours', 'seven days' and 'one decade' all have dimension time: [3 hours] = [seven days] = [one decade] = [time].
Some quantities are *dimensionless* – they consistent of just a number (something on the number line e.g. $1, 2 \ldots$) and no units.
Examples include

- ratios of quantities that have the same dimension, e.g. the incidence in the UK / the incidence in France;
- relative changes in quantities: $(x_{\mathrm{new}} - x_{\mathrm{old}})/x_{\mathrm{old}}$;
- products of quantities that have inverse dimensions, e.g. $2\ \mathrm{day}^{-1} \times 1$ week.

Dimensionful quantities on the other hand require a unit to make sense, e.g. lengths, times.

Understanding Units II

Note that *per* literally means *divided by*; e.g. one per day $= 1/\text{day}$ $= 1\ \text{day}^{-1}$.

The dimension of the product of two quantities is equal to the product of the dimensions: $[XY] = [X][Y]$.
e.g. distance travelled $=$ speed $\times$ travelling time.

- [distance travelled] $=$ [length]
- [speed] $=$ [length][time]$^{-1}$
- [travelling time] $=$ [time]

Indeed, multiplying the dimensions of the latter two gives the dimension of the first, because the time dimension cancels.

Note that to see the cancellation of dimensions (and get the correct answer!) you might need to convert some units. e.g. sixty miles per hour $\times$ ten minutes is
$\frac{60\ \text{miles}}{1\ \text{hour}} \times 10\ \text{mins} = \frac{60\ \text{miles}}{60\ \text{mins}} \times 10\ \text{mins} = 10\ \text{miles}.$

Understanding Units III

The *units* on both sides of an equation do not have to be equal: ‘one week = seven days’ is fine. The *dimensions*, however, must be equal. Even stronger, every term in an equation must have the same dimension: if $a + b = c - d + e$, then a, b, c, d and e must all have the same dimension. This is because adding terms of different dimension is meaningless, e.g. ‘one mile + one day’.

Slightly Confusing Point I

It's common in this field for people to write equations that force a particular choice of units. e.g.

$$\begin{aligned}\text{number of pills taken} = {} & \text{number of pills taken in one day} \\ & \times \text{number of days during which pills are taken}\end{aligned}$$

Note that both quantities on the RHS are dimensionless.
It's better to say

$$\begin{aligned}\text{number of pills taken} = {} & \text{rate of taking pills} \\ & \times \text{time during which pills are taken}\end{aligned}$$

Here, the first quantity on the RHS has dimension $[\text{time}]^{-1}$ and the second has dimension [time].

Slightly Confusing Point II

Units are effectively optional for those variables that *count* things; truly, such variables are dimensionless, but a unit can be bolted on provided this is done consistently. e.g. if you want to vaccinate $100,000$ people at £5 each, you could say

$$\begin{aligned} \text{total cost} &= \text{number of people} \times \text{cost for one person} \\ &= 100,000 \times £5 \end{aligned}$$

Or alternatively,

$$\begin{aligned} \text{total cost} &= \text{population size} \times \text{per capita cost} \\ &= 100,000 \text{ persons} \times £5 \text{ per person} \\ &= 100,000 \times £5 \end{aligned}$$

The difference is subtle. In the second (more pedagogical) case we promote 'a person' to be a unit that measures population size, and the units 'person' and 'per person' cancel. In the first (more common) case we don't bother with that and jump more directly to the end result.

Problem Session 3, I

State the dimension, and give an example of an appropriate unit, for the following quantities:

1. the mass of a particular drug required,
2. the number of people infected with a given disease,
3. the length of time between being infected and becoming infectious,
4. the power to which we raise some quantity (e.g. a in x^a)
5. an amount of money,
6. a fractional increase in an amount of money,
7. a rate of fractional increase in an amount of money (e.g. the interest rate on a bank account),
8. the concentration of parasites in a patient's blood
9. the rate of change in the (population average) time from HIV infection to AIDS.

Problem Session 3, II

1. Calculate
    1.1 1 per week $\times$ 2 years
    1.2 5 people $\times$ 3 pills per person per day $\times$ 2 weeks $\times$ £1 per ten pills
2. A tiny parasite is found x times in one cm^2. Assuming constant number per unit area, how many would there be in an area that is a square, with each side 2 km long?
3. Two average men are drinking a wine that is 12.5% ABV. The density of alcohol (ethanol) is 789 kg/m^3. The UK limit for driving is 80mg of alcohol per 100ml of blood. Assuming their blood volume stays constant, and assuming the rate of entry into the bloodstream far exceeds the rate of removal, how many 75cl bottles can they work through before hitting the drink driving limit? Discuss.
   (An average man has 5 litres of blood. 1 litre = 1dm^3. ABV means alcohol by volume.)

Problem Session 3, III

1. Imagine that the following is a dimensionally correct (though admittedly bizarre) expression for R_0:

$$R_0 = x\, e^{ab^2 cy/d}$$

where a has dimensions of area, b of temperature, c of time, and d of length per unit time. (e is a dimensionless fundamental constant like π – more on this next lecture.) What are the dimensions of x and y?

2. Fundraising for a vaccination project resulted in N people steadily donating at an average rate r_d over a time T. It costs P to buy an amount D of the vaccine. A single vaccination requires an amount d of the vaccine, and one (disposable) needle which costs p. In the project, n nurses each administer the vaccine at a rate r_v, and are each paid at a rate r_p. A fraction f of all money raised must be used indirectly (for administration etc.). How many people can this project vaccinate? Hint: considering the dimensions of all these quantities helps you to see which ones should be multiplied together.

Problem Session 3, IV

Erythropoiesis – production of red blood cells – occurs at a rate of two million cells per second in humans. There are 270 million haemoglobin molecules in each red blood cell; each haemoglobin molecule contains four iron atoms. 1g of iron contains 1.1×10^{21} atoms. The iron content of grilled sirloin steak, by mass, is one part in 10^5; a typical serving is eight ounces. 100g is 3.5 ounces. Iron is lost from the body in various ways at a total rate of 1.5mg per day. Estimate:

1. the mass of iron used by the body each day for erythropoiesis[1]. Much of this is obtained from recycling old red blood cells.
2. the number of grilled sirloin steaks one should eat a week to get enough iron, pretending that such steaks are one's only source of iron and that the body consumes iron only for erythropoiesis. (Disclaimer: this crude, simplified question should *not* be used to inform a healthy balanced diet!)

[1] Your result should be significantly different from the value quoted at http://en.wikipedia.org/wiki/Human_iron_metabolism#Systemic_iron_regulation (in the figure caption). I don't know why; do you?

Extra Slide: Macdonald's Equation for R_0 for Malaria

$$R_0 = \frac{ma^2bc}{gr}\, e^{-gv},$$

- The mosquito death rate, g. $[g] = [\text{time}]^{-1}$
- The rate at which a mosquito feeds on humans, a. $[a] = [\text{time}]^{-1}$
- The probability a mosquito becomes infected after biting an infected human, c, which is dimensionless.
- The proportion of bites by infectious mosquitoes that infect a human, b, which is dimensionless.
- The rate each human recovers from infection, r. $[r] = [\text{time}]^{-1}$
- The time from infection to infectiousness in the mosquito, v. $[v] = [\text{time}]$
- The ratio of mosquitoes to humans, m, which is dimensionless.