Advanced Calculus II: Assignment 2

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Problem 1.

Let $x, y \in A$. Suppose $x \in C_y$ and $C_x \not\subset C_y$. That is, there are points in C_x which are not connected to y.

Let $C = C_x \cup C_y$. C is clearly disconnected by the above reasoning. Then there exist open sets B, D such that $C \cap B$, $C \cap D$ are disjoint, non-empty, and have union C.

Since $C \cap B$, $C \cap D$ are disjoint, $x \in B$ or $x \in D$. Assume $x \in B$ without loss of generality.

We know $x \in C_y$ by assumption. Since C_y is connected, it must all be contained in B, otherwise it would be split between $C_y \cap B$, $C_y \cap D$ where they are both non-empty, disjoint, and where the union is C_y, a contradiction.

In addition, we know $C \cap D$ is non-empty. Since $C_y \subset B$, then elements of C_x must be in D. However, $x \in B$.

Hence, if we take $C_x \cap B$ and $C_x \cap D$, we have two non-empty, disjoint sets with a union equal to C_x. However, C_x is connected by definition, so this is a contradiction.

Thus, we have that $C_x \subset C_y$.

If we swap x with y in the above proof, we get that $C_y \subset C_x$. Hence $C_x = C_y$ if $C_x \cap C_y \neq \emptyset$.

Problem 2.

Let $A = \mathbb{I}$, the set of irrational numbers.

Since $\mathbb{Q}$ is countable, $\mathbb{R}$ is uncountable, and $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$, we have that $\mathbb{I}$ must be uncountable. If it were not, then $\mathbb{R}$ would be the union of two countable sets and would hence be countable as well, a contradiction.

Now observe that $\mathbb{Q}$ is dense in $\mathbb{R}$. That is, for every $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$, $\exists y \in \mathbb{Q}$ such that $x_1 < y < x_2$. Since $\mathbb{I} \subset \mathbb{R}$, we see that $\mathbb{Q}$ divides $\mathbb{I}$ in such a manner.

Lastly, from Theorem 12.8 in the text, we know that a subset of $\mathbb{R}$ is connected if and only if it is an interval.

Now let $x \in \mathbb{I}$ and let $\epsilon > 0$ be arbitrarily small. Take the interval $[x-\epsilon, x+\epsilon]$. Since there exists a rational number between any two real numbers, and we have that $x, x+\epsilon \in \mathbb{R}$, we have that $\exists y \in \mathbb{Q}$ such that $y \in [x, x+\epsilon]$.

Hence, $[x-\epsilon, x+\epsilon] \not\subset \mathbb{I}$. This holds for every $\epsilon > 0$, so there are no interval subsets of $\mathbb{I}$. Since these are the only connected subsets of $\mathbb{R}$, $\mathbb{I}$ is totally disconnected.

Problem 3.

Let $C \subset \mathbb{R}^p$ be open and suppose C is connected.

Let $x \in C$. Let U be the set of points that can be connected to x by a path in C.

Let $V = C \setminus U$. That is, V is the set of points in C which cannot be connected to x via some path. So we have that $U \cap V = \emptyset$ and $U \cup V = C$.

We will show that U and V are both open.

Let $y \in U$ and let f be a path connecting x and y. Since C open, $\exists B_r(y) \subset C$ with $r > 0$.

For every $z \in B_r(y)$, there is a path f connecting y and z.

Thus, $B_r(y) \subset U$. Since this holds for each $y \in U$, U is open. A similar argument holds for V showing that it is open.

Since C is connected, either U or V must then be the empty set. Assume $V = \emptyset$ without loss of generality. Since $C = U \cup V$ and $V = \emptyset$, we have that $C = U$.

Since U is path connected, we then have that C is path connected as required.

Now suppose C is open and path connected. Also suppose that it is not connected for contradiction.

Then $C = U \cup V$ where $U \cap V = \emptyset$, U, V open, and $(U \cap C) \cup (V \cap C) = C$.

Let $x \in U \cap C$ and $y \in V \cap C$. Since C is path connected, $\exists$ a continuous function $f : [0,1] \to C$ with $f(0) = x$ and $f(1) = y$.

Then we have $[0,1] = f^{-1}(U) \cup f^{-1}(V)$, which would be a nontrivial splitting of $[0,1]$ by the continuity of f. This implies that $[0,1]$ is disconnected, a contradiction.

Hence C open and path connected implies that C is connected.

Problem 4.

Problem 5.

Let T be the topologist's sine curve. That is,

$$T = \left\{ \left(x, \sin \tfrac{1}{x} \right) : x \in (0,1] \right\} \cup \{(0,0)\}$$

T is not path connected because it includes the point $(0,0)$. However, the function $\left(x, \sin \frac{1}{x}\right)$ cannot be extended to include this point such that it forms a path.

Now observe that $A = \left\{ \left(x, \sin \frac{1}{x} \right) : x \in (0,1] \right\}$ is path connected and hence connected.

In addition, observe that the topologist's sine curve is the closure of A.

Thus, $A \subset T \subset \overline{A}$.

As a result, we have that T is connected.