Advanced Calculus II: Assignment 8

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Problem 1.

Let $F(x, y) = x^2 + y^2 - 1$. Consider the equation $F(x, y) = x^2 + y^2 - 1 = 0$.

In order to prove that the above equation can be solved for small values of x by a positive function $y = y(x)$, we need to show that F belongs to class $C^1(\mathbb{R}^2)$, that $F(a, b) = 0$, and that the linear map defined by $L_2(v) = DF(a, b)(0, v)$, $v \in \mathbb{R}$ is a bijection of $\mathbb{R}$ onto $\mathbb{R}$ (by Implicit Function Theorem).

First let $(a, b) = (0, 1)$. Observe that $F(a, b) = F(0, 1) = 0^2 + 1^2 - 1 = 0$ as required.

Now by Theorem 41.2, in order to show that F belongs to class $C^1(\mathbb{R}^2)$ (ie. the derivative exists and is continuous), it suffices to show that both partial derivatives are continuous on $\mathbb{R}^2$. Hence, we have,

$$\begin{aligned} D_x F &= 2x \\ D_y F &= 2y \end{aligned}$$

It is clear that both partial derivatives are continuous on all of $\mathbb{R}^2$ since both are comprised of a multiplication of two trivially continuous functions.

Finally, we need to show that the linear map defined by $L_2(v) = DF(0, 1)(0, v)$, $v \in \mathbb{R}$ is a bijection of $\mathbb{R}$ onto $\mathbb{R}$.

Note from Corollary 39.7 in the text that,

$$\begin{aligned} DF(0, 1)(0, v) &= (0)D_x F(0, 1) + (v)D_y F(0, 1) \\ &= v(2 \cdot 1) \\ &= 2v \end{aligned}$$

It is clear that this function is defined on all of $\mathbb{R}$.

Now suppose that $\exists v_1, v_2 \in \mathbb{R}$ such that $L_2(v_1) = L_2(v_2)$. Then we have that,

$$\begin{aligned} 2v_1 &= 2v_2 \\ \implies v_1 &= v_2 \end{aligned}$$

Hence L_2 is injective.

Now let $u \in \mathbb{R}$. Observe that, since $\mathbb{R}$ is closed under multiplication and $\frac{1}{2} \in \mathbb{R}$, we have that $\frac{u}{2} \in \mathbb{R}$. Hence, for any $u \in \mathbb{R}, \exists \frac{u}{2}$ such that,

$$L_2\left(\frac{u}{2}\right) = 2\left(\frac{u}{2}\right) = u$$

Thus, L_2 is surjective.

Since L_2 is both injective and surjective, it is a bijective function from $\mathbb{R} \to \mathbb{R}$.

Since $F(x, y)$ satisfies all of the above properites, we can apply the Implicit Function Theorem. Hence, there exists an open neighborhood W of $0 \in \mathbb{R}$ and a unique function $y(x) : W \to \mathbb{R}$ belonging to class $C^1(W)$ such that $y = y(x)$ and,

$$F(x, y(x)) = 0 \ \ \forall x \in W$$

Now we need to show that $y'(x) = \frac{-x}{y}$.

Applying Corollary 41.10 from the text yields,

$$\begin{aligned} Dy(x) &= -[D_{(2)}F(x, y(x))]^{-1} \circ [D_{(1)}F(x, y(x)] \\ &= -\frac{1}{2y} \circ 2x \\ &= -\frac{x}{y} \end{aligned}$$

as required.

Problem 2.

As above, we need to show we need that F belongs to class $C^1(\mathbb{R}^5)$, that $F(a, b) = 0$, and that the linear map defined by $L_2(v) = DF(a, b)(0, v)$, $v \in \mathbb{R}^2$ is a bijection of $\mathbb{R}^2$ onto $\mathbb{R}^2$ (by Implicit Function Theorem).

Observe that $(a, b) = (2, 1, 0, -1, 0)$ and we are given that $F(2, 1, 0, -1, 0) = (0, 0)$. Hence this condition is already satisfied.

Next, we need to show that F belongs to class C^1. First we will define the derivative of F,

$$DF(u, v, w, x, y) = \begin{bmatrix} y & v & 1 & v + 2x & u \\ vw & uw & uv & 1 & 1 \end{bmatrix}$$

Recall that each partial derivative of F is an entry in the above matrix and that the derivative of F is continuous iff every partial derivative is continuous.

From the above, it is clear that all of the partial derivatives are continuous on all of $\mathbb{R}^5$ as they are linear combinations of continuous functions. Hence, $DF(u, v, w, x, y)$ is continuous on all of $\mathbb{R}^5$ and F belongs to class $C^1(\mathbb{R}^5)$

Lastly, we need to show that the linear map defined by $L_2(v) = DF(a, b)(0, v)$, $v \in \mathbb{R}^2$ is a bijection of $\mathbb{R}^2$ onto $\mathbb{R}^2$ where $a = (2, 1, 0)$ and $b = (-1, 0)$.

Hence, we have,

$$\begin{aligned} L_2(v) &= \begin{bmatrix} 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} -v_1 + 2v_2 \\ v_1 + v_2 \end{bmatrix} \end{aligned}$$

Now suppose $L_2(v) = L_2(w)$ for $v, w \in \mathbb{R}^2$ with $v \neq w$. Then we have,

$$-v_1 + 2v_2 = -w_1 + 2w_2$$

and

$$v_1 + v_2 = w_1 + w_2$$

Subtracting 2 times the second equation from the first equation yields,

$$v_1 = w_1$$

Plugging this identity in yields,

$$v_2 = w_2$$

Hence $v = w$ and L_2 is injective.

Now suppose $w \in \mathbb{R}^2$. We will now consider the equation,

$$L_2(v) = w$$

for some $v \in \mathbb{R}^2$, which is equivalent to,

$$\begin{bmatrix} -v_1 + 2v_2 \\ v_1 + v_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

This system of equations yields $v_1 = \frac{1}{3}(2w_2 - w_1)$ and $v_2 = \frac{w_1+w_2}{3}$. Note that $w_1, w_2 \in \mathbb{R}$ and $\mathbb{R}$ is closed under addition and multiplication. Hence, $v_1, v_2 \in \mathbb{R}$ and, by extension, $v \in \mathbb{R}^2$.

Thus, for any $w \in \mathbb{R}^2$, $\exists v \in \mathbb{R}^2$ such that $L_2(v) = w$. As a result, L_2 is a bijection from $\mathbb{R}^2 \to \mathbb{R}^2$.

Thus, F satisfies all of the necessary properties. As a result, we can apply the Implicit Function Theorem. Hence, there exists an open neighborhood W of $(2, 1, 0) \in \mathbb{R}^3$ and a unique function $\varphi(x) : W \to \mathbb{R}^2$ belonging to class $C^1(W)$ such that $(x, y) = \varphi(u, v, w)$ and,

$$F(u, v, w, \varphi(u, v, w)) = 0 \ \ \forall (u, v, w) \in W$$

Now we need to compute $D\varphi(2, 1, 0)$.

We will start by using Corollary 41.10 from the text,

$$\begin{aligned} D\varphi(u, v, w) &= -[D_{(2)}F(u, v, w, \varphi(u, v, w)]^{-1} \circ [D_{(1)}F(u, v, w, \varphi(u, v, w)] \\ &= \frac{1}{v + 2x - u} \begin{bmatrix} 1 & -u \\ -1 & v + 2x \end{bmatrix} \circ \begin{bmatrix} y & v & 1 \\ vw & uw & uv \end{bmatrix} \\ &= \frac{1}{v + 2x - u} \begin{bmatrix} y - uvw & v - u^2w & 1 - u^2v \\ -y + v^2w + 2xvw & -v + vuw + 2xuw & -1 + uv^2 + 2xuv \end{bmatrix} \end{aligned}$$

Hence, we have,

$$\begin{aligned} D\varphi(2, 1, 0) &= \frac{1}{1 + 2x - 2} \begin{bmatrix} y - (2)(1)(0) & 1 - 2^2(0) & 1 - 2^2(1) \\ -y + 1^2(0) + 2x(1)(0) & -1 + (1)(2)(0) + 2x(2)(0) & -1 + 2(1^2) + 2x(2)(1) \end{bmatrix} \\ &= \frac{1}{2x - 1} \begin{bmatrix} y & 1 & -3 \\ -y & -1 & 4x + 1 \end{bmatrix} \end{aligned}$$

Problem 3.

Let $D \subset \mathbb{R}^2$ be open and $f \in C^1(D, \mathbb{R})$. Let $(x_0, y_0) \in D$. If $Df(x_0, y_0) = 0$ and if $\exists \delta > 0$ such that for every (x, y) with $||(x, y) - (x_0, y_0)|| < \delta$, we have $Df(x, y)(u, v) > 0 \ \ \forall (u, v) \in \mathbb{R}^2$, then f has a local maximum at (x_0, y_0)

First, observe that $Df(x_0, y_0) = 0$, and hence f has a critical point at (x_0, y_0). The set of points of relative extrema of f is a subset of the set of critical points of f (as stated on p. 398 in the text), so this is a necessary condition for f to have a relative maximum at (x_0, y_0).

Now observe that Df exists for all $x \in D$ and is a continuous mapping of D into $\mathscr{L}(\mathbb{R}^2, \mathbb{R})$ since $f \in C^1(D, \mathbb{R})$.

Problem 4.

In order to apply the Implicit Function theorem, we use the following equivalent formulation for this system of equations:

$$F(x, y, u, v) = (x^2 + y^2 - u^2 - v^2, x^2 + 2y^2 + 3u^2 + 4v^2 - 1) = (0, 0)$$

Note that $F(\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}) = (0,0)$. We will use this point as our (a,b) in the Implicit Function Theorem.

In addition, note that $\Omega = \mathbb{R}^4$ in this case. Hence, Ω is open as required.

We now need to show that F belongs to class $C^1(\mathbb{R}^4)$. Recall that F belongs to class $C^1(\mathbb{R}^4)$ iff all of the first partial derivatives of F are continuous on $\mathbb{R}^4$. We will verify that now,

$$DF(x,y,u,v) = \begin{bmatrix} 2x & 2y & -2u & -2v \\ 2x & 4y & 6u & 8v \end{bmatrix}$$

Again, clearly each partial derivative is continuous on $\mathbb{R}^4$ as they are just degree 1 polynomials. Hence, F belongs to class $C^{(}\mathbb{R}^4)$.

Now we need to show that the linear map defined by $L_2(v) = DF(\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}})(0,v) \quad v \in \mathbb{R}^2$ is a bijection of $\mathbb{R}^2$ onto $\mathbb{R}^2$.

We will start by defining this linear map,

$$\begin{aligned} L_2(v) &= DF\left(\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)(0,v) \\ &= \begin{bmatrix} \frac{2}{\sqrt{10}} & \frac{2}{\sqrt{10}} & \frac{-2}{\sqrt{10}} & \frac{-2}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} & \frac{4}{\sqrt{10}} & \frac{6}{\sqrt{10}} & \frac{8}{\sqrt{10}} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{-2v_1}{\sqrt{10}} - \frac{2v_2}{\sqrt{10}} \\ \frac{6v_1}{\sqrt{10}} + \frac{8v_2}{\sqrt{10}} \end{bmatrix} \end{aligned}$$

Now suppose there exists $v, w \in \mathbb{R}^2$ such that $L_2(v) = L_2(w)$. Then we have,

$$\begin{aligned} \begin{bmatrix} \frac{-2v_1}{\sqrt{10}} - \frac{2v_2}{\sqrt{10}} \\ \frac{6v_1}{\sqrt{10}} + \frac{8v_2}{\sqrt{10}} \end{bmatrix} &= \begin{bmatrix} \frac{-2w_1}{\sqrt{10}} - \frac{2w_2}{\sqrt{10}} \\ \frac{6w_1}{\sqrt{10}} + \frac{8w_2}{\sqrt{10}} \end{bmatrix} \\ \iff \begin{bmatrix} -2v_1 - 2v_2 \\ 6v_1 + 8v_2 \end{bmatrix} &= \begin{bmatrix} -2w_1 - 2w_2 \\ 6w_1 + 8w_2 \end{bmatrix} \end{aligned}$$

Solving the above system of equations yields $v_1 = w_1$ and $v_2 = w_2$. Hence $v = w$ and $L_2(v)$ is injective.

Now let $w \in \mathbb{R}^2$. Consider the equation,

$$L_2(v) = w$$

for some $v \in \mathbb{R}^2$. This is equivalent to,

$$\begin{bmatrix} \frac{-2v_1}{\sqrt{10}} - \frac{2v_2}{\sqrt{10}} \\ \frac{6v_1}{\sqrt{10}} + \frac{8v_2}{\sqrt{10}} \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

The first equation yields $v_1 = -\frac{\sqrt{10}}{2}w_1 - v_2$. Plugging into the second equation gives,

$$v_2 = \sqrt{\frac{5}{2}}(3w_1 + w_2)$$

Plugging this back into the previous equation yields,

$$\begin{aligned} v_1 &= -\frac{\sqrt{10}}{2}w_1 - \sqrt{\frac{5}{2}}(3w_1 + w_2) \\ &= \frac{1}{2}\sqrt{10}(-4w_1 - w_2) \end{aligned}$$

We have that $w_1, w_2 \in \mathbb{R}$ and $\mathbb{R}$ is closed under addition and multiplication. Hence, $v_1, v_2 \in \mathbb{R}$ and so $v \in \mathbb{R}^2$.

Thus, for every $w \in \mathbb{R}^2$, $\exists v \in \mathbb{R}^2$ such that $L_2(v) = w$. As a result, L_2 is a bijection from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Hence, we can apply the Implicit Function Theorem here. As a result, there exists a neighborhood W of $\left(\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right) \in \mathbb{R}^2$ and a unique function $\varphi : W \to \mathbb{R}^2$ belonging to class $C^1(W)$ such that $(u, v) = \varphi(x, y)$ and,

$$F(x, y, \varphi_1(x, y), \varphi_2(x, y)) = 0 \ \ \forall v \in \mathbb{R}^2$$

where $u(x, y) = \varphi_1(x, y)$ and $v(x, y) = \varphi_2(x, y)$.

We will now use Corollary 41.10 to calculate $Du(x, y) = D\varphi_1(x, y)$ and $Dv(x, y) = \varphi_2(x, y)$,

$$\begin{aligned} D\varphi(x, y) &= -[D_{(2)}F(x, y, \varphi_1(x, y), \varphi_2(x, y)]^{-1} \circ [D_{(1)}F(x, y, \varphi_1(x, y), \varphi_2(x, y)] \\ &= -\frac{1}{-4uv}\begin{bmatrix} 8v & 2v \\ -6u & -2u \end{bmatrix} \circ \begin{bmatrix} 2x & 2y \\ 2x & 4y \end{bmatrix} \\ &= -\frac{1}{-4uv}\begin{bmatrix} 16xv + 4xv & 16vy + 32vy \\ -12ux - 4ux & -12uy - 8uy \end{bmatrix} \\ &= -\frac{1}{-4uv}\begin{bmatrix} 20xv & 48vy \\ -16ux & -20uy \end{bmatrix} \\ &= \begin{bmatrix} \frac{-5x}{u} & \frac{-12y}{u} \\ \frac{4x}{v} & \frac{5y}{v} \end{bmatrix} \end{aligned}$$

Hence,

$$Du(x, y) = D\varphi_1 = \begin{bmatrix} \frac{-5x}{u} & \frac{-12y}{u} \end{bmatrix}$$

and,

$$Dv(x, y) = D\varphi_2 = \begin{bmatrix} \frac{4x}{v} & \frac{5y}{v} \end{bmatrix}$$

Problem 5.

We have $L \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$ and $r > 0$ such that $||L(x)|| \geq 2r||x|| \ \ \forall x \in \mathbb{R}^n$.

We need to show that $\exists \epsilon > 0$ such that if $L' \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$ with $|||L' - L||| < \epsilon$ (note that we are using the operator norm here), then

$$||L'(x)|| \geq r||x|| \ \ \forall x \in \mathbb{R}^n \tag{1}$$

Define $\epsilon = r$ and let $L' \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$ such that $|||L' - L||| < \epsilon$.

For the first case, we consider $x = 0$. We have that (1) clearly holds.

For the second case, consider $||x|| = 1$. Then we have,

$$\begin{aligned} ||L'(x)|| &= ||L(x) - L(x) + L'(x)|| \\ &\geq ||L(x)|| - ||L(x) - L'(x)|| \\ &\geq 2r||x|| - r \\ &= r||x|| \end{aligned}$$

Note that the last line only holds because we assumed $||x|| = 1$.

Now consider the case $x \in \mathbb{R}^n \setminus \{0\}$ with $||x|| \neq 1$.

Define $y = \frac{x}{||x||}$, and so $||y|| = 1$. Hence, from case 2, we have,

$$||L'(y)|| \geq r||y|| = \frac{1}{||x||} r||x|| \tag{2}$$

In addition, by properties of linear transformations, we have,

$$||L'(y)|| = \left|\left| \frac{1}{||x||} L'(x) \right|\right| = \frac{1}{||x||} ||L'(x)|| \tag{3}$$

Combining (2) and (3) yields,

$$\begin{aligned} &\frac{1}{||x||} ||L'(x)|| \geq \frac{1}{||x||} r||x|| \\ \iff &||L'(x)|| \geq r||x|| \end{aligned}$$

as required.

We have proven that the required property holds when $||x|| = 0, ||x|| = 1$, and $||x|| \neq 1, 0$. Hence, we have proven this statement for all possible cases of $x \in \mathbb{R}^n$.