Advanced Calculus II: Assignment 7

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Problem 1.

- **Injective Mapping Theorem:** Suppose that $\Omega \subset \mathbb{R}^p$ is open, that $f : \Omega \to \mathbb{R}^q$ belongs to Class $C^1(\Omega)$, and that $L = Df(c)$ is an injection. Then there exists a number $\delta > 0$ such that the restriction of f to $B_\delta = \{x \in \mathbb{R}^p : ||x - c|| \leq \delta\}$ is an injection. Moreover, the inverse of the restriction $f|B_\delta$ is a continuous function on $f(B_\delta) \subset \mathbb{R}^q$ to $B_\delta \subset \mathbb{R}^p$.

  Sketch of Proof: By Corollary 22.8, since $L = Df(c)$ is injective, we can say that there is an $r > 0$ such that $r||u|| \leq ||Df(c)(u)||$ for $u \in \mathbb{R}^p$.

  We can then use this r in the approximation lemma as part of our ϵ expression. This yields,

$$||f(x_1) - f(x_2) - L(x_1 - x_2)|| \leq \frac{1}{2}r||x_1 - x_2||$$

  for every x_1, x_2 in some $\delta(\epsilon)$ ball centered at c.

  When we use the triangle inequality and combine the above statements with $u = x_1 - x_2$ and $L = Df(c)$, we get

$$\begin{aligned}
&||L(x_1 - x_2)|| - ||f(x_1) - f(x_2)|| \leq \frac{1}{2}r||x_1 - x_2|| \\
\Longrightarrow\ & r||x_1 - x_2|| - ||f(x_1) - f(x_2)|| \leq \frac{1}{2}r||x_1 - x_2|| \\
\Longrightarrow\ & -||f(x_1) - f(x_2)|| \leq -\frac{1}{2}r||x_1 - x_2|| \\
\Longrightarrow\ & \frac{1}{2}r||x_1 - x_2|| \leq ||f(x_1) - f(x_2)||
\end{aligned}$$

  We can again apply Corollary 22.8 to the above, which tells us that f is injective on this $\delta(\epsilon)$ ball centered at c.

  Now since this restriction of f is an injection, for each y in the image of the delta ball, there exists a unique x such that $f(x) = y$. Hence, if g denotes the inverse of

f restricted to the delta ball, then there exists a unique point $x = g(y)$ in the delta ball.

Now if we take the last line of the above equation and let $y_1 = f(x_1)$ and $y_2 = f(x_2)$, we get,

$$\begin{aligned} \frac{1}{2}r||x_1 - x_2|| \le ||f(x_1) - f(x_2)|| &\implies \frac{1}{2}r||g(y_1) - g(y_2)|| \le ||y_1 - y_1|| \\ &\implies ||g(y_1) - g(y_2)|| \le \frac{2}{r}||y_1 - y_1|| \end{aligned}$$

Now according to Definition 23.2, g satisfies a Lipschitz condition and hence g is uniformly continuous when restricted to the image of the delta ball under f.

- **Surjective Mapping Theorem:** Let $\Omega \subset \mathbb{R}^p$ be open and let $f : \Omega \to \mathbb{R}^q$ belong to class $C^1(\Omega)$. Suppose that for some $c \in \Omega$, the linear function $L = Df(c)$ is a surjection of $\mathbb{R}^p$ onto $\mathbb{R}^q$. Then there exist numbers $m > 0$ and $\alpha > 0$ such that if $y \in \mathbb{R}^q$ and $||y - f(c)|| \le \frac{\alpha}{2m}$, then there exists an $x \in \Omega$ such that $||x - c|| \le \alpha$ and $f(x) = y$

  Sketch of Proof: Let $L = Df(c)$ with c defined as above. Using the fact that L is surjective, we know that each standard basis vector in $\mathbb{R}^q$ is the image under L of some vector in $\mathbb{R}^p$.

  We then take the inverse linear function M from $\mathbb{R}^q$ to $\mathbb{R}^p$ which maps each basis vector to its pre-image under L. Hence, the composition of L with M is the identity mapping on $\mathbb{R}^q$.

  Using the triangle inequality and Schwarz inequality, we can show that the norm of M evaluated at any point $y \in \mathbb{R}^q$ is less than or equal to a constant multiple (called m) of the norm of that point y.

  We then use the Approximation Lemma to show that, on an α ball centered at a point c, the difference between the function f evaluated at two points, x_1 and x_2, and the derivative L evaluated at the difference $x_2 - x_1$ is bounded above by $\frac{1}{2m}||x_1 - x_2||$

  Now we choose $y \in \mathbb{R}^q$ such that the norm of $y - f(c)$ is bounded above by $\alpha/2m$.

  In addition, we construct x_k inductively in $\mathbb{R}^p$ such that $||x_k - x_{k-1}||$ is bounded above by $\alpha/2^k$ and $||x_k - c||$ is bounded above by $(1 - 1/2^k)\alpha$, where $c = x_0$.

  We can show that this sequence can be extended infinitely many times. In addition, the constructed sequence can be shown to be Cauchy.

Hence, this sequence (x_n) is Cauchy in $\mathbb{R}^p$ and thus converges to some eleemnt x.

We have that $||x-c|| \leq \alpha$, so x is in the ball of radius α centered at c.

Applying induction, we get that $L(x_{n+1}-x_n) - y - f(x_n)$., and it follows that $y = \lim f(x)_n) = f(x)$.

Thus, every y in the $\alpha/2m$ ball around $f(c)$ is the image under f of a some point $x \in \Omega$ with x in the α ball around c.

- **Inverse Mapping Theorem:** Let $\Omega \subset \mathbb{R}^p$ be open and suppose that $f : \Omega \to \mathbb{R}^p$ belongs to Class $C^1(\Omega)$. If $c \in \Omega$ is such that $Df(c)$ is a bijection, then there exists an open neighborhood U of c such that $V = f(U)$ is an open neighborhood of $f(c)$ and the restriction of f to U is a bijection onto V with continuous inverse g. Moreover, g belongs to Class $C^1(V)$ and $Dg(y) = [Df(g(y))]^{-1}$ for $y \in V$.

  Sketch of Proof: We can apply both the Injective Mapping Theorem and Surjective Mapping Theorem to quickly get that the restriction f to U is a bijection onto V with continuous inverse g.

  We take M_1 to be the inverse of the linear function $Df(x)$. If $x \in U$, then $x = g(y)$ for some $y = f(x) \in V$. Using elements of the proof of the Injective Mapping Theorem, we get that,

$$||y - y_1|| = ||f(x) - f(x_1)|| \geq \frac{1}{2}r||x - x_1||$$

Problem 2.

Let $r > 0$ and choose $(x, y) \in \mathbb{R}^2$ such that $|x| > r$ and $|x| \geq |y|$.

We have,

$$\begin{aligned}
F^2(x,y) &= [\mu(\mu x(1-x)+\delta y))(1-(\mu x(1-x)+\delta y))+\delta(\delta x), \delta(\mu x(1-x)+\delta y)] \\
&= [(\mu^2 x - \mu^2 x^2 + \mu\delta y)(1-\mu x+\mu x^2-\delta y)+\delta^2 x, \delta\mu x - \delta\mu x^2+\delta^2 y] \\
&= [\mu^2 x - \mu^3 x^2 + \mu^3 x^3 - \delta\mu^2 xy - \mu^3 x^2 + \mu^3 x^3 - \mu^3 x^4 - \mu\delta^2 y^2 + \mu\delta y - \mu^2\delta xy + \mu^2\delta x^2 y - \\
&\mu\delta^2 y^2 + \delta^2 x, \delta\mu x - \delta\mu x^2 + \delta^2 y] \\
&=
\end{aligned}$$

Problem 3.

We have that the derivative of f at a point (x, y) is given by,

$$Df(x,y) = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}$$

It is clear that each entry in the above matrix is continuous on all of $\mathbb{R}^2$ because each entry is a product of functions that are continuous on all of $\mathbb{R}^2$. Hence, we have that f is $C^1(\mathbb{R}^2)$

Now take the distinct points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$. Assume that $Df(x,y)(x_1,y_1) = Df(x,y)(x_2,y_2)$. Then we have the following system of equations,

$$\begin{bmatrix} e^x\cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} e^x\cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$
$$\implies \begin{bmatrix} x_1 e^x \cos y - y_1 e^x \sin y \\ x_1 e^x \sin y + y_1 e^x \cos y \end{bmatrix} = \begin{bmatrix} x_2 e^x \cos y - y_2 e^x \sin y \\ x_2 e^x \sin y + y_2 e^x \cos y \end{bmatrix}$$

Thus, we have,

$$\begin{aligned} x_1 e^x \cos y - y_1 e^x \sin y &= x_2 e^x \cos y - y_2 e^x \sin y \\ x_1 e^x \sin y + y_1 e^x \cos y &= x_2 e^x \sin y + y_2 e^x \cos y \end{aligned}$$

which yields, $x_1 = x_2$ and $y_1 = y_2$. Thus, $Df(x,y)$ is injective.

Now select $(x_0, y_0) \in \mathbb{R}^2$ and take an open neighborhood U of this point. By the Injective Mapping Theorem, we have that there exists a number $\delta > 0$ such that the restriction of f to B_δ is an injection. Moreover, the inverse of the restriction $f|B_\delta$ is a continuous function on $f(B_\delta) \subset \mathbb{R}^2$ to $B_\delta \subset \mathbb{R}^2$.

The local inverse of f is $f^{-1} = [\log(x) - \log(\cos(\tan^{-1}(y/x))), \tan^{-1}(y/x)]$

The derivative of the local inverse at (x_0, y_0) is,

$$Df^{-1}(x_0,y_0) = \begin{bmatrix} \frac{1}{x_0} - \log\left(\frac{1}{\sqrt{\frac{y_0^2}{x_0^2}+1}}\right) & -\log\left(\frac{1}{\sqrt{\frac{y_0^2}{x_0^2}+1}}\right) \\ -\frac{y_0}{x_0^2+y_0^2} & \frac{x_0}{x_0^2+y_0^2} \end{bmatrix}$$