Advanced Calculus II: Assignment 5

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Problem 1.

Suppose each f is differentiable at $c \in D$. We know from Theorem 8.10 that for every $x \in \mathbb{R}^n$, we have that $|x_i| \leq ||x|| \leq \sqrt{n} \sup\{|x_1|, \cdots, |x_n|\}$.

Now let $\epsilon > 0$. By the differentiability of f at c, $\exists \delta(\epsilon) > 0$ such that for every $u \in \mathbb{R}^n$ with $||u|| \leq \delta(\epsilon)$, we have,

$$\lim_{||u|| \to 0} \frac{||f(c+u) - f(c) - Df(c)(u)||}{||u||} = 0$$

Note that $f(c+u)$, $f(c)$, and $Df(c)$ are all vectors in $\mathbb{R}^n$. Hence, Theorem 8.10 applies here and we have,

$$\lim_{||u|| \to 0} \frac{|f_i(c+u) - f_i(c) - Df_i(c)(u)|}{||u||} \leq \lim_{||u|| \to 0} \frac{||f(c+u) - f(c) - Df(c)(u)||}{||u||} = 0 \qquad (1)$$

for every $i \in \{1, \cdots, n\}$.

Since $|f_i(c+u) - f_i(c) - Df_i(c)(u)|,\ ||u|| \geq 0$, we have from (1) that

$$\lim_{||u|| \to 0} \frac{|f_i(c+u) - f_i(c) - Df_i(c)(u)|}{||u||} = 0$$

Thus, each f_i is differentiable at c.

Now assume that each f_i is differentiable at c. We need to show that f is differentiable at c.

By the second inequality in Theorem 8.10, we have that $||x|| \leq \sqrt{n} \sup\{|x_1|, \cdots, |x_n|\}$ for every $x \in \mathbb{R}^n$.

Let $|f_j(c+u) - f_i(c) - Df_i(c)(u)| = \sup\{|f_1(c+u) - f_1(c) - Df_1(c)(u)|, \cdots, |f_n(c+u) - f_n(c) - Df_n(c)(u)|\}$. Then,

$$\begin{aligned}
\lim_{||u|| \to 0} \frac{||f(c+u) - f(c) - Df(c)(u)||}{||u||} &\leq \lim_{||u|| \to 0} \frac{\sqrt{n} \cdot |f_j(c+u) - f_j(c) - Df_j(c)(u)|}{||u||} \\
&= \sqrt{n} \lim_{||u|| \to 0} \frac{|f_j(c+u) - f_j(c) - Df_j(c)(u)|}{||u||} \\
&= \sqrt{n}(0) = 0
\end{aligned}$$

The last line is obtained from the differentiablity of each f_i at c.

Since $||f(c+u) - f(c) - Df(c)(u)||, \ ||u|| \geq 0$, we have from the above that,

$$\lim_{||u|| \to 0} \frac{||f(c+u) - f(c) - Df(c)(u)||}{||u||} = 0$$

Hence, f is differentiable at c.

Problem 2.

Take the function $f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

f is a composition of differentiable functions and hence is differentiable when $(x,y) \neq (0,0)$. When $(x,y) = (0,0)$, we have,

$$\begin{aligned}
\lim_{||u|| \to 0} \frac{||f(0+u_1, 0+u_2) - f(0,0) - L(u)||}{||u||} &= \lim_{||u|| \to 0} \frac{||(u_1^2+u_2^2)\sin\left(\frac{1}{\sqrt{u_1^2+u_2^2}}\right) - L(u)||}{||u||} \\
&\leq \lim_{||u|| \to 0} \frac{||(u_1^2+u_2^2)||}{||u||} = 0
\end{aligned}$$

Hence f is also differentiable at $(0,0)$.

Now consider the partial derivatives of f,

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y) &= 2x\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{x\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)}{\sqrt{x^2+y^2}} \\
\frac{\partial f}{\partial y}(x,y) &= 2y\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{y\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)}{\sqrt{x^2+y^2}}
\end{aligned}$$

Both of these functions oscillate wildly near the origin and are thus discontinuous.

Problem 3.

Suppose $h(x) = f(x) \cdot g(x)$ with $f, g : D \to \mathbb{R}^m$ differentiable at $c \in D$. Then we have,

$$\begin{aligned}
&h(x) - h(c) - [Df(c)(x-c)g(c) + f(c)Dg(c)(x-c)] \\
&= [f(x) - f(c) - Df(c)(x-c)]g(x) + Df(c)(x-c)[g(x) - g(c)] + \\
&f(c)[g(x) - g(c) - Dg(c)(x-c)]
\end{aligned}$$

Since g is continuous at c, we have that $\exists M$ such that $||g(x)|| < M$ for $||x - c|| < \delta$. Thus, if we choose $||x - c||$ small enough, we have that

$$h(x) - h(c) - [Df(c)(x-c)g(c) + f(c)Dg(c)(x-c)] = 0$$

and thus,

$$\lim_{x \to c} \frac{||h(x) - h(c) - [Df(c)(x-c)g(c) + f(c)Dg(c)(x-c)]||}{||x-c||} = 0$$

Hence, h is differentiable at c with derivative

$$Dh(c)(x-c) = (Df(c)(x-c)) \cdot g(c) + f(c) \cdot (Dg(c)(x-c))$$

Problem 4.

Suppose f differentiable on an open cell $J \subset \mathbb{R}^n$ and $f : J \to \mathbb{R}$. Also suppose that $D_1 f(x) = 0$ for all $x \in J$. By Corollary 39.7, we have that, for every $c \in J$,

$$\begin{aligned}
Df(c)(u) &= u_1 D_1 f(c) + u_2 D_2 f(c) + \cdots + u_n D_n f(c) \\
&= 0 + u_2 D_2 f(c) + \cdots + u_n D_n f(c) \\
&= u_2 D_2 f(c) + \cdots + u_n D_n f(c)
\end{aligned}$$

Now let $y, z \in J$ with $y_2 = z_2, \cdots, y_n = z_n$. Also let $\epsilon > 0$. Then $\exists \delta_1(\epsilon), \delta_2(\epsilon) > 0$ such that,

$$||f(x) - f(y) - Df(y)(x-y)|| \leq \epsilon ||x - y|| \tag{2}$$

and

$$||f(x) - f(z) - Df(z)(x-z)|| \leq \epsilon ||x - z|| \tag{3}$$

Now, using the facts that $D_1 f(x) = 0$ for all $x \in J$ and that $y_2 = z_2, \cdots, y_n = z_n$ along with Corollary 39.7, we have that,

$$\begin{aligned}
Df(y)(x-y) &= (x_1 - y_1) D_1 f(y) + (x_2 - y_2) D_2 f(y) + \cdots + (x_n - y_n) D_n f(y) \\
&= (x_2 - y_2) D_2 f(y) + \cdots + (x_n - y_n) D_n f(y) \\
&= (x_2 - z_2) D_2 f(y) + \cdots + (x_n - z_n) D_n f(y)
\end{aligned}$$

Using (2) and (3) along with the above equations yields,

$$\begin{aligned}
&||f(x) - f(y) - Df(y)(x-y) - (f(x) - f(z) - Df(z)(x-z))|| \\
&= ||f(z) - f(y) + Df(z)(x-y) - Df(y)(x-y)|| \\
&= ||f(z) - f(y)|| \\
&\leq ||f(x) - f(y) - Df(y)(x-y)|| + ||f(x) - f(z) - Df(z)(x-z)|| \\
&\leq \epsilon(||x-y|| + ||x-z||)
\end{aligned}$$

Since $\epsilon(||x-y|| + ||x-z||)$ can be made arbitrarily small, we have $f(y) = f(z)$.