Advanced Calculus II: Assignment 1
Chapter 2 - A Taste of Topology

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Problem 12 on p. 126.

(a) The limit of a sequence is unaffected by rearrangement when f is a bijective function. Since f is bijective, each term from (p_n) must be included one and only one time in the new sequence (q_k).

We know that, if $(p_n) \to \ell$, this means that $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $n \geq N \implies \mathrm{d}(p_n, \ell) < \epsilon$.

Thus, for each choice of $\epsilon > 0$ there are only finitely many terms for which $\mathrm{d}(p_n, \ell) \geq \epsilon$, while there are infinitely many terms for which $\mathrm{d}(p_n, \ell) < \epsilon$.

As a result, the rearrangement (q_k) will eventually exhaust all of the terms that have a distance from ℓ that is greater than or equal to ϵ, and thus will have infinitely many terms left for which $\mathrm{d}(q_k, \ell) < \epsilon$.

Hence, $(q_k) \to \ell$ as well.

(b) A rearrangement of (p_n) where f is an injective function does not necessarily preserve the limit of (p_n). For example, take $(p_n) = (-1)^n$. This sequence alternates between 1 and -1, and thus never converges.

Now let $f(n) = 2n$. This injective function maps the natural numbers to the evens. We can see that $\forall m$ where m is even, we have that $p_m = 1$.

Thus, $q_k = p_{f(k)} = 1 \; \forall k \in \mathbb{N}$. Clearly, $(q_k) \to 1$ while (p_n) does not converge.

(c) A rearrangement of (p_n) where f is a surjective function does not necessarily preserve the limit of (p_n). For example, take $(p_n) = \frac{1}{n}$. This sequence converges to 0.

Now let $f(n) = \begin{cases} 1 & \text{n is odd} \\ 2 & \text{n} = 2 \\ f(n-2)+1 & \text{n is even and n} > 2 \end{cases}$

Call this new sequence $q_k = p_{f(k)}$. Then we have that (q_k) contains all terms in the original sequence (p_n) and, for every odd term m, $q_m = 1$. Thus, the even terms of (q_k) converge to 0 while the odd terms converge to 1. Since we have two subsequences in (q_k) that converge to different limits, (q_k) does not converge.

Problem 44 on p. 128.

Problem 76 on p. 131.

Problem 1 on p. 147.