Advanced Calculus II: Assignment 1

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February 11, 2020

Problem 1.

a)

b) Let $D_f \subset \mathbb{R}^n$ and let $f : D_f \to \mathbb{R}^m$. Also let $a \in D_f$.

Suppose f is continuous at a and suppose $||\cdot||_{n1}, ||\cdot||_{m1}$ are norms on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

Thus, by the definition of continuity, for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in D_f$ with $||x-a||_{n1} < \delta$, then $||f(x) - f(a)||_{m1} < \epsilon$.

By part (a), for any other arbitrary norms $||\cdot||_{n2}, ||\cdot||_{m2}$ on $\mathbb{R}^n$ and $\mathbb{R}^m$, we have that

$$C_{n1}||x-a||_{n1} \leq ||x-a||_{n2} \leq C_{n2}||x-a||_{n1}$$
$$C_{m1}||f(x)-f(a)||_{m1} \leq ||f(x)-f(a)||_{m2} \leq C_{m2}||f(x)-f(a)||_{m1}$$

for $C_{n1}, C_{n2}, C_{m1}, C_{m2} > 0$ and for every $x \in \mathbb{R}^n, f(x) \in \mathbb{R}^m$.

Hence, $||x-a||_{n1} < \delta \iff C_{n2}||x-a||_{n1} < C_{n2}\delta \implies ||x-a||_{n2} < C_{n2}\delta$ for every $x \in \mathbb{R}^n$.

Now let $\delta' = \min\{\delta, C_{n2}\delta\}$.

Then clearly we have that $||x-a||_{n1} < \delta' \implies ||f(x)-f(a)||_{m1} < \epsilon$ and $||x-a||_{n2} < \delta' \implies ||f(x)-f(a)||_{m1} < \epsilon$.

Problem 2.

Suppose A is open. Then for every $x \in A$, $\exists r > 0$ such that $B(x,r) \subset A$.

We know that a point $x' \in \mathbb{R}^n$ is a boundary point of A if $\forall r > 0$, $\exists y_1, y_2 \in B(x', r)$ such that $y_1 \in A$ and $y_2 \in \mathbb{R}^n \setminus A$.

Now suppose there exists a point $z \in \partial A$ such that $z \in A$. Then, by definition of the openness of A, $\exists r_z > 0$ such that $B(z, r_z) \subset A$. However, since $z \in \partial A$, we also have that $\exists y_z \in \mathbb{R}^n \setminus A$ such that $y_z \in B(z, r_z)$. Hence, $B(z, r_z) \not\subset A$, a contradiction.

Thus we must have that, if A is open, then $A \cap \partial A = \emptyset$.

Now suppose $A \cap \partial A = \emptyset$. Then A does not contain any of its boundary points. That is, there is no $x \in \mathbb{R}^n$ such that $\forall r > 0, \exists y_1, y_2 \in B(x, r)$ such that $y_1 \in A$ and $y_2 \in \mathbb{R}^n \setminus A$. Note that there is always a $y_1 \in B(x, r)$ with $y_1 \in A$ since $x \in B(x, r)$ and $x \in A$.

Hence, for each $x \in A$, there must exist an $r > 0$ such that $B(x, r) \cap (\mathbb{R}^n \setminus A) = \emptyset$.

Thus, $B(x, r)$ must be contained in the complement of $\mathbb{R}^n \setminus A$, which is A. Hence, A is open.

Now suppose A is closed. Then we have that $A^\mathsf{c} = \mathbb{R}^n \setminus A$ is open.

By the above proof, we have that $A^\mathsf{c} \cap \partial A^\mathsf{c} = \emptyset$. In addition, observe that the definition for ∂A^c is the same as for ∂A:

We know that a point $x \in \mathbb{R}^n$ is a boundary point of A^c if $\forall r > 0$, $\exists y_1, y_2 \in B(x, r)$ such that $y_1 \in A^\mathsf{c}$ and $y_2 \in \mathbb{R}^n \setminus A^\mathsf{c} = A$.

This precisely the same definition given for a boundary point of A. Hence, we can rewrite the statement from above as: $A^\mathsf{c} \cap \partial A^\mathsf{c} = A^\mathsf{c} \cap \partial A = \emptyset$.

Hence $\partial A \subset (A^\mathsf{c})^\mathsf{c} = A$, as required.

Now suppose $\partial A \subset A$. Take A^c.

As we showed above, $\partial A = \partial A^\mathsf{c}$. Thus, we have $A^\mathsf{c} \cap \partial A = A^\mathsf{c} \cap \partial A^\mathsf{c} = \emptyset$.

By the earlier proof, we thus have that A^c is open. Then, by definition, A is closed.

Problem 3.

Problem 4.

Problem 5.

Let $f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q}, p, q \in \mathbb{Z} \\ 0 & x \text{ is irrational} \end{cases}$