Advanced Calculus II: Assignment 1

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February 9, 2020

Problem 1.

a)

b) Let $D_f \subset \mathbb{R}^n$ and let $f : D_f \to \mathbb{R}^m$. Also let $a \in D_f$.

Suppose f is continuous at a and suppose $||\cdot||_{n1}, ||\cdot||_{m1}$ are norms on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

Thus, by the definition of continuity, for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in D_f$ with $||x-a||_{n1} < \delta$, then $||f(x) - f(a)||_{m1} < \epsilon$.

By part (a), for any other arbitrary norms $||\cdot||_{n2}, ||\cdot||_{m2}$ on $\mathbb{R}^n$ and $\mathbb{R}^m$, we have that

$$C_{n1}||x-a||_{n1} \leq ||x-a||_{n2} \leq C_{n2}||x-a||_{n1}$$
$$C_{m1}||f(x)-f(a)||_{m1} \leq ||f(x)-f(a)||_{m2} \leq C_{m2}||f(x)-f(a)||_{m1}$$

for $C_{n1}, C_{n2}, C_{m1}, C_{m2} > 0$ and for every $x \in \mathbb{R}^n, f(x) \in \mathbb{R}^m$.

Hence, $||x-a||_{n1} < \delta \implies C_{n2}||x-a||_{n1} < C_{n2}\delta \implies ||x-a||_{n2} < C_{n2}\delta$ for every $x \in \mathbb{R}^n$.

In addition, the reverse is true. We can see that if $||x-a||_{n2} < C_{n2}\delta$, then we have that

$$\frac{1}{C_{n2}}||x-a||_{n2} < \delta$$
$$\implies$$