Advanced Calculus II: Assignment 8

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April 26, 2020

Problem 1.

Let $F(x,y) = x^2 + y^2 - 1$. Consider the equation $F(x,y) = x^2 + y^2 - 1 = 0$.

In order to prove that the above equation can be solved for small values of x by a positive function $y = y(x)$, we need to show that F belongs to class $C^1(\mathbb{R}^2)$, that $F(a,b) = 0$, and that the linear map defined by $L_2(v) = DF(a,b)(0,v)$, $v \in \mathbb{R}$ is a bijection of $\mathbb{R}$ onto $\mathbb{R}$ (by Implicit Function Theorem).

First let $(a,b) = (0,1)$. Observe that $F(a,b) = F(0,1) = 0^2 + 1^2 - 1 = 0$ as required.

Now by Theorem 41.2, in order to show that F belongs to class $C^1(\mathbb{R}^2)$ (ie. the derivative exists and is continuous), it suffices to show that both partial derivatives are continuous on $\mathbb{R}^2$. Hence, we have,

$$D_x F = 2x$$
$$D_y F = 2y$$

It is clear that both partial derivatives are continuous on all of $\mathbb{R}^2$ since both are comprised of a multiplication of two trivially continuous functions.

Finally, we need to show that the linear map defined by $L_2(v) = DF(0,1)(0,v)$, $v \in \mathbb{R}$ is a bijection of $\mathbb{R}$ onto $\mathbb{R}$.

Note from Corollary 39.7 in the text that,

$$DF(0,1)(0,v) = (0)D_x F(0,1) + (v)D_y F(0,1)$$
$$= v(2 \cdot 1)$$
$$= 2v$$

It is clear that this function is defined on all of $\mathbb{R}$.

Now suppose that $\exists v_1, v_2 \in \mathbb{R}$ such that $L_2(v_1) = L_2(v_2)$. Then we have that,

$$2v_1 = 2v_2$$
$$\implies v_1 = v_2$$

Hence L_2 is injective.

Now let $u \in \mathbb{R}$. Observe that, since $\mathbb{R}$ is closed under multiplication and $\frac{1}{2} \in \mathbb{R}$, we have that $\frac{u}{2} \in \mathbb{R}$. Hence, for any $u \in \mathbb{R}, \exists \frac{u}{2}$ such that,

$$L_2\left(\frac{u}{2}\right) = 2\left(\frac{u}{2}\right) = u$$

Thus, L_2 is surjective.

Since L_2 is both injective and surjective, it is a bijective function from $\mathbb{R} \to \mathbb{R}$.

Since $F(x, y)$ satisfies all of the above properites, we can apply the Implicit Function Theorem. Hence, there exists an open neighborhood W of $0 \in \mathbb{R}$ and a unique function $y(x) : W \to \mathbb{R}$ belonging to class $C^1(W)$ such that $y = y(x)$ and,

$$F(x, y(x)) = 0 \quad \forall x \in W$$

Problem 2.

As above, we need to show we need that F belongs to class $C^1(\mathbb{R}^5)$, that $F(a, b) = 0$, and that the linear map defined by $L_2(v) = DF(a, b)(0, v)$, $v \in \mathbb{R}^2$ is a bijection of $\mathbb{R}^2$ onto $\mathbb{R}^2$ (by Implicit Function Theorem).

Observe that $(a, b) = (2, 1, 0, -1, 0)$ and we are given that $F(2, 1, 0, -1, 0) = 0$. Hence this condition is already satisfied.

Next, we nee to show that F belongs to class C^1. First we will define the derivative of F,

$$DF(u, v, w, x, y) = \begin{bmatrix} y & v & 1 & v + 2x & u \\ vw & uw & uv & 1 & 1 \end{bmatrix}$$

Recall that each partial derivative of F is an entry in the above matrix and that the derivative of F is continuous iff every partial derivative is continuous.

From the above, it is clear that all of the partial derivatives are continuous on all of $\mathbb{R}^5$ as they are linear combinations of continuous functions. Hence, $DF(u, v, w, x, y)$ is continuous on all of $\mathbb{R}^5$ and F belongs to class $C^1(\mathbb{R}^5)$

Lastly, we need to show that the linear map defined by $L_2(v) = DF(a, b)(0, v)$, $v \in \mathbb{R}^2$ is a bijection of $\mathbb{R}^2$ onto $\mathbb{R}^2$ where $a = (2, 1, 0)$ and $b = (-1, 0)$.

Hence, we have,

$$L_2(v) = \begin{bmatrix} 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ v_1 \\ v_2 \end{bmatrix}$$

$$= \begin{bmatrix} -v_1 + 2v_2 \\ v_1 + v_2 \end{bmatrix}$$

Now suppose $L_2(v) = L_2(w)$ for $v, w \in \mathbb{R}^2$ with $v \neq w$. Then we have,

$$-v_1 + 2v_2 = -w_1 + 2w_2$$

and

$$v_1 + v_2 = w_1 + w_2$$

Subtracting 2 times the second equation from the first equation yields,

$$v_1 = w_1$$

Plugging this identity in yields,

$$v_2 = w_2$$

Hence $v = w$ and L_2 is injective.

Now suppose $w \in \mathbb{R}^2$

Problem 3.

Problem 4.

Problem 5.