Advanced Calculus II: Assignment 3

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Problem 1.

Let $L_1, L_2, L_3 \in L(\mathbb{R}^n, \mathbb{R}^m)$. Then all of these functions are linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. We will now use these properties to show that $L(\mathbb{R}^n, \mathbb{R}^m)$ is a vector space:

1. Associativity of addition

$$\begin{aligned}(L_1 + L_2)(x) + L_3(x) &= L_1(x) + L_2(x) + L_3(x) \\ &= L_1(x) + (L_2 + L_3)(x)\end{aligned}$$

2. Commutativity of addition

$$(L_1 + L_2)(x) = L_1(x) + L_2(x)$$

We have that $L_1(x), L_2(x) \in \mathbb{R}^m$ for every $x \in \mathbb{R}^n$. Since $\mathbb{R}^m$ is a vector space, we must have that,

$$\begin{aligned}(L_1 + L_2)(x) &= L_1(x) + L_2(x) \\ &= L_2(x) + L_1(x) \\ &= (L_2 + L_1)(x)\end{aligned}$$

as required.

3. Identity element of addition

Let L_0 be the function that assigns the 0 vector in $\mathbb{R}^m$ to every vector in $\mathbb{R}^n$. We must first show that this is a linear transformation.

Let $u, v \in \mathbb{R}^n$ and let $c \in \mathbb{R}$. Then,

$$L_0(u + v) = 0 = L_0(u) + L_0(v)$$

and,

$$L_0(cu) = 0 = c0 = cL(u)$$

Thus $L_0 \in L(\mathbb{R}^n, \mathbb{R}^m)$. Now to show that it is the identity element of addition in that set:

$$\begin{aligned}(L_0 + L_1)(x) &= L_0(x) + L_1(x) \\ &= 0 + L_1(x) \\ &= L_1(x)\end{aligned}$$

4. Inverse elements of addition

   First we will show that $-L_1(x)$ is a linear transformation.

   Let $u, v \in \mathbb{R}^n$ and let $c \in \mathbb{R}$. Then,

$$\begin{aligned}-L_1(u + v) &= -(L_1(u) + L_1(v)) \\ &= -L_1(u) + -L_1(v)\end{aligned}$$

   and,

$$c(-L_1(u)) = (-c)L_1(u) = -L_1(cu)$$

   Thus $-L_1 \in L(\mathbb{R}^n, \mathbb{R}^m)$. Now to show that it is the inverse element of addition in that set:

$$\begin{aligned}(L_1 + -L_1)(x) &= L_1(x) + -L_1(x) \\ &= 0\end{aligned}$$

5. Compatibility of scalar multiplication with field multiplication

   Let $a, b \in \mathbb{R}$. Then, using properties of linear transformations, we have

$$\begin{aligned}a(bL_1(x)) &= a(L_1(bx)) \\ &= L_1(a(bx)) \\ &= L_1((ab)x) \\ &= (ab)L_1(x)\end{aligned}$$

6. Identity element of scalar multiplication

   Let $1 \in \mathbb{R}$. Since $\mathbb{R}^n$ is a vector space with 1 as the scalar multiplication identity, we have that $1x = x$ for every $x \in \mathbb{R}^n$. Moreover, by properties of linear transformations, we have,

$$\begin{aligned}(1)L_1(x) &= L_1(1x) \\ &= L_1(x)\end{aligned}$$

7. Distributivity of scalar multiplication with respect to vector addition

   Let $c \in \mathbb{R}$. Since $L_1(u), L_2(v) \in \mathbb{R}^m$ and $\mathbb{R}^m$ is an $\mathbb{R}$ vector space, we must necessarily have that

$$c(L_1(x) + L_2(x)) = cL_1(x) + cL_2(x)$$

8. Distributivity of scalar multiplication with respect to field addition

   Let $a, b \in \mathbb{R}$. Moreover, let $d = a + b$. Then,

$$\begin{aligned}(a+b)L_1(x) &= dL_1(x)\\ &= L_1(dx)\\ &= L_1((a+b)x)\\ &= L_1(ax+bx)\\ &= L_1(ax) + L_1(bx)\\ &= aL_1(x) + bL_1(x)\end{aligned}$$

Hence, $L(\mathbb{R}^n, \mathbb{R}^m)$ together with the standard addition and scalar multiplication is an $\mathbb{R}$ vector space.

Problem 2.

We will show that $||T|| = \sup\{||T(x)|| : ||x|| \leq 1\}$ defines a norm on $L(\mathbb{R}^n, \mathbb{R}^m)$.

Let $T_1, T_2 \in L(\mathbb{R}^n, \mathbb{R}^m)$. We must check the three norm properties:

1. We have that,

$$\begin{aligned}||T_1 + T_2|| &= \sup\{||(T_1 + T_2)(x)|| : ||x|| \leq 1\}\\ &= \sup\{||T_1(x) + T_2(x)|| : ||x|| \leq 1\}\end{aligned}$$

   Since $T_1(x), T_2(x) \in \mathbb{R}^m$ for every $x \in \mathbb{R}^n$, we have that $||T_1(x) + T_2(x)|| \leq ||T_1(x)|| + ||T_2(x)||$ for any valid norm on $\mathbb{R}^m$. Hence, we have,

$$\begin{aligned}||T_1 + T_2|| &= \sup\{||T_1(x) + T_2(x)|| : ||x|| \leq 1\}\\ &\leq \sup\{||T_1(x)|| + ||T_2(x)|| : ||x|| \leq 1\}\\ &= ||T_1|| + ||T_2||\end{aligned}$$

2. Let $a \in \mathbb{R}$. By properties of norms and supremum, we have

$$\begin{aligned}||aT_1|| &= \sup\{||aT_1(x)|| : ||x|| \leq 1\}\\ &= \sup\{|a| \cdot ||T_1(x)|| : ||x|| \leq 1\}\\ &= |a| \sup\{||T_1(x)|| : ||x|| \leq 1\}\\ &= |a| \cdot ||T_1||\end{aligned}$$

3. Suppose $||T|| = 0$. Then need to show that $T = \mathbf{0}$ where $\mathbf{0}$ is the 0 vector in $\mathbb{R}^m$. We have,

$$||T|| = \sup\{||T(x)|| : ||x|| \leq 1\}$$

Since any valid norm on $\mathbb{R}^m$ is non-negative and the supremum of $||T(x)||$ with $||x|| \leq 1$ is 0, we must have that $||T(x)|| = 0$ for all $||x|| \leq 1$ and hence $T(x) = 0$ for all $||x|| \leq 1$.

Now suppose $\exists x_1$ with $||x_1|| > 1$ and $||T(x_1)|| \neq 0$ (that is, $T(x_1) = 0$).

Write x_1 as a linear combination of basis vectors. Hence, $x_1 = c_1e_1 + c_2e_2 + \cdots + c_ne_n$. Then we have,

$$\begin{aligned} ||T(x_1)|| &= ||T(c_1e_1 + c_2e_2 + \cdots + c_ne_n)|| \\ &= ||T(c_1e_1) + \cdots + T(c_ne_n)|| \qquad = ||c_1T(e1) + \cdots + c_nT(e_n)|| = ||c_1(0) + \cdots c_n(0)|| \\ &= ||0|| = 0 \end{aligned}$$

However, we supposed that $||T(x_1)|| \neq 0$, a contradiction. Thus, $T(x) = \mathbf{0}$.

Thus, the operator norm defines a valid norm on $L(\mathbb{R}^n, \mathbb{R}^m)$.

Problem 3.

Problem 4.

Suppose f is even and differentiable. Then,

$$\begin{aligned} f'(-x) &= \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} \\ &= \lim_{h\to 0} \frac{f(x-h) - f(x)}{h} \\ &= -\lim_{h\to 0} \frac{f(x) - f(x-h)}{h} \\ &= -f'(x) \end{aligned}$$

Hence, f' is odd.

Problem 5.

We have,

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{h^2\sin(1/h) - 0}{h} \\ &= \lim_{h\to 0} h\sin(1/h) \end{aligned}$$

We have that $|\sin(1/h)| \leq 1$ for every value of h. Hence,

$$|h\sin(1/h)| \leq |h|$$

Thus,

$$\begin{aligned}
-\lim_{h\to 0}|h| &\leq \lim_{h\to 0} h\sin(1/h) \leq \lim_{h\to 0}|h| \\
\implies 0 &\leq \lim_{h\to 0} h\sin(1/h) \leq 0 \\
\implies \lim_{h\to 0} h &= 0
\end{aligned}$$

Hence, $f'(0)$ exists and equals 0.

Now need to show that f' is not continuous at 0. We have that,

$$f'(x) = \begin{cases} 2x\sin(1/x) - \cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Observe that,

$$\begin{aligned}
\lim_{x\to 0} f'(x) &= \lim_{x\to 0} 2x\sin(1/x) - \cos(1/x) \\
&= \lim_{x\to 0} 2x\sin(1/x) - \lim_{x\to 0}\cos(1/x) \\
&= -\lim_{x\to 0}\cos(1/x)
\end{aligned}$$

This limit is undefined, and so $f'(0) \neq \lim_{x\to 0} f'(x)$. Hence, $f'(x)$ is not continuous at 0.