Advanced Calculus II: Assignment 9

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Problem 1.

Note that $S^2 = \{x \in \mathbb{R}^3 : ||x|| = 1\}$ is the surface of the unit sphere in $\mathbb{R}^3$.

If $n \in \mathbb{N}$, define cubes such that

Problem 2.

Let $A \subset \mathbb{R}^n$ be a set with content and assume that f and g are integrable on A and $g(x) \geq 0$ for all x. Define $m = \inf f(A)$ and $M = \sup f(A)$.

Now take a closed cell $I \subset \mathbb{R}^n$ such that $A \subset I$.

Consider $\int_I f_I \cdot g_I$. Recall that $L = \int_I f_I \cdot g_I$ if, for every $\epsilon > 0$, there is a partition P_ϵ of I such that if P is any refinement of P_ϵ and $S(P; f_I \cdot g_I)$ is any Riemann sum according to P, then $|S(P; f_I \cdot g_I) - L| \leq \epsilon$.

For any partition $P = \{J_1, \cdots, J_n\}$, the Riemann sum of $g_I \cdot f_I$ is given by,

$$S(P, g_I \cdot f_I) = \sum_{k=1}^{n} f(x_k)g(x_k)c(J_k)$$

where x_k is any intermediate point in J_k.

Note that $m \leq f(x_k)$ and $M \geq f(x_k)$ for any choice of x_k. Hence, we have,

$$m \sum_{k=1}^{n} g(x_k)c(J_k) \leq \sum_{k=1}^{n} f(x_k)g(x_k)c(J_k) \leq M \sum_{k=1}^{n} g(x_k)c(J_k) \tag{1}$$

Now note that g is integrable on A. Let $L_g = \int_A g$. Moreover, $g(x) \geq 0$ for every $x \in A$. Hence, $L_g \geq 0$.

As a result, with small enough ϵ, (1) becomes,

$$mL_g \leq \sum_{k=1}^{n} f(x_k)g(x_k)c(J_k) \leq ML_g$$

If we denote the overall integral as L, then we have that $L \in [mL_g, ML_g]$

Now take the linear function $h(x) = x \cdot L_g$ defined on the interval $[m, M]$. Note that L is in the range of $h(x)$.

By Theorem 21.3 in the text, $h(x)$ is continuous on this interval since it is a linear function. Furthermore, $[m, M]$ is a connected set. Hence, we can apply Bolzano's Intermediate Value Theorem. Since we know $\inf\{h(x)\} = mL_g \leq L \leq ML_g = \sup\{h(x)\}$, we can assert that there is a point $\mu \in [m, M]$ where $h(\mu) = \mu L_g = L$.

Thus, since $L_g = \int_A g$ and $L = \int_A fg$, we have that,

$$\int_A fg = \mu L_g$$

as required.

Problem 3.

a) Suppose $F \subset \mathbb{R}^n$ is a bounded discrete set.

Now suppose F is uncountable. Then there does not exist a bijection between $\mathbb{N}$ and the elements of F.

However, from the fact that F is discrete, we have that for every $x \in F$, there exists a δ neighborhood of x $[B(x, \delta) \subset \mathbb{R}^n]$ such that $B(x, \delta) \cap F = x$. In addition, since F is bounded, $\exists M > 0$ such that $|x| \leq M$ for every $x \in F$.

Hence, construct

b) Suppose F is a bounded discrete set. Then by part a), F is countable. By the countable additivitiy of content and the fact $c(\{x\}) = 0$ for every $x \in \mathbb{R}$, we have that, for $f_k \in F$,

$$c(F) = \sum_{k=0}^{\infty} c(f_k) = 0$$

Hence, every bounded discrete set $F \subset \mathbb{R}$ has zero content.

Problem 4.

The four theorems that I enjoyed most were:

1. **Heine-Borel Theorem**: A subset of $\mathbb{R}^p$ is compact if and only if it is closed and bounded.

   I found that this theorem provided a much more intuitive notion for compactness (at least in $\mathbb{R}^n$) than the definition of compactness gave. In addition, I enjoyed getting a sampling of point set topology to begin the class, and the Heine-Borel theorem had one of the most difficult proofs from that section, so it was fun getting to learn about it.

2. **Theorem 12.7**: Let G be an open set in $\mathbb{R}^p$. Then G is connected if and only if any pair of points $x, y \in G$ can be joined by a polygonal curve lying entirely in G.

   I found this property of open connected sets to be extremely interesting. I am also currently enrolled in the Computational Geometry graduate course, so I found it very enjoyable to consider polygonal curves from both a topological and computational perspective.

3. **Theorem 39.9**: Let $A \subset \mathbb{R}^p$, let $f : A \to \mathbb{R}^q$, and let c be an interior point of A. If the partial derivatives of $D_i f_i$ $(i = 1, \cdots q, j = 1, \cdots p)$ exist in a neighborhood of c and are continuous at c, then f is differentiable at c. Moreover, $Df(c)$ is represented by the $q \times p$ matrix (39.11)

   This theorem and its proof really made the concept of the derivative "click" for me. I was finally able to wrap my head around how the derivative and partial derivatives related to one another on a theoretical level, as well as how the partial derivatives can be used to express the derivative.

4. **Integrability Theorem**: Let $I \subset \mathbb{R}^p$ be a closed cell and let $f : I \to \mathbb{R}$ be bounded. If there exists a subset $E \subset I$ with content zero such that f is continuous on $I \setminus E$, then f is integrable on I.

   I found the concept of removing a set of content zero from the set I while still preserving the value of the integral on I to be incredibly interesting.