Advanced Calculus II: Assignment 4

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March 10, 2020

Problem 1.

Suppose $g : \mathbb{R} \to \mathbb{R}$ is differentiable with $g'(x) \neq 0$ for all $x \in \mathbb{R}$.

Now let $x_1, x_2 \in \mathbb{R}$ such that $x_1 \neq x_2$ and $g(x_1) = g(x_2)$. Assume without loss of generality that $x_1 < x_2$.

Take $h(x) = g(x) - g(x_1)$. We have that $h(x)$ is clearly defined and differentiable for all $x \in \mathbb{R}$ because it is a linear combination of a constant and a differentiable function.

In addition, note that $h(x_1) = 0 = h(x_2)$. Hence, we can apply Rolle's Theorem on the interval $[x_1, x_2]$. As a result, we have that $\exists c \in [x_1, x_2]$ such that $h'(c) = 0$.

Now observe the defintion of the dervative of h at c:

$$\begin{aligned}
h'(c) &= \lim_{x \to c} \frac{h(x) - h(c)}{x - c} \\
&= \lim_{x \to c} \frac{[g(x) - g(x_1)] - [g(c) - g(x_1)]}{x - c} \\
&= \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\
&= g'(c)
\end{aligned}$$

Hence, we have that $h'(c) = 0 = g'(c)$. However, we know that $g'(x) \neq 0$ for all $x \in \mathbb{R}$. Thus, we have a contradiction and we must have that $x_1 = x_2$. That is, $g(x)$ is injective from $\mathbb{R} \to g(\mathbb{R})$.

By the definition of $g(\mathbb{R})$ (ie. the range of g), we have that g is surjective from $\mathbb{R} \to g(\mathbb{R})$.

As a result, g is a bijection of $\mathbb{R}$ onto $g(\mathbb{R})$.

Problem 2.

Let $g(x) = f(x) - C(x - a)$. We have that $g(x)$ is a linear combination of continuous functions on $[a, b]$, so it is continuous on $[a, b]$ as well.

Thus, since g is continuous on the compact set $[a, b]$, it attains its maximum value on the interval $[a, b]$ by the extreme value theorem.

We have $g'(x) = f'(x) - C$. Note that $g'(a) = f'(a) - C$. Since $f'(a) < C < f'(b)$, we have $g'(a) = f'(a) - C < 0$. By the Interior Maximum Theorem, then $g(a)$ cannot be the maximum value of g.

Similarly, we have that $g'(b) = f'(b) - C(b - a) > 0$, hence $g(b)$ cannot be the maximum value of g.

Thus, g must attain its maximum value at some point in $c \in (a, b)$. At this point, we have that $g'(c) = 0$, again by Interior Maximum Theorem, which yields,

$$\begin{aligned} & g'(c) = 0 \\ \implies & f'(c) - C = 0 \\ \implies & f'(c) = C \end{aligned}$$

Problem 3.

Show that f is partial differentiable at $(0, 0)$ with respect to any $(a, b) \in \mathbb{R}^2$:

Let $u = (a, b)$ be arbitrary in $\mathbb{R}^2$ with $(a, b) \neq (0, 0)$. In addition, let $c = (0, 0)$. Then, by Definition 39.1 in the text, we have

$$\begin{aligned} L_u &= \lim_{t \to 0} \frac{1}{t} [f(c + tu) - f(c)] \\ &= \lim_{t \to 0} \frac{1}{t} [f(ta, tb) - f(0, 0)] \\ &= \lim_{t \to 0} \frac{1}{t} \left[\frac{(ta)(tb)^2}{(ta)^2 + (tb)^4} \right] \\ &= \lim_{t \to 0} \frac{1}{t} \left[\frac{t^3 ab^2}{t^2 a^2 + t^4 b^4} \right] \\ &= \lim_{t \to 0} \frac{t^3 ab^2}{t^3 a^2 + t^5 b^4} \\ &= \lim_{t \to 0} \frac{ab^2}{a^2 + t^2 b^4} \\ &= \frac{ab^2}{a^2} \\ &= \frac{b^2}{a} \end{aligned}$$

If $u = (0,0)$, then clearly we have,

$$\begin{aligned} L_u &= \lim_{t\to 0} \frac{1}{t}[f(c+tu) - f(c)] \\ &= \lim_{t\to 0} \frac{1}{t}[f(0,0) - f(0,0)] \\ &= \lim_{t\to 0} \frac{1}{t}(0) \\ &= 0 \end{aligned}$$

Hence, L_u is defined at $(0,0)$ for every $u \in \mathbb{R}^2$.

Show that f is not continuous at $(0,0)$:

Observe that f is defined for every point in $\mathbb{R}^2$ except for $(0,0)$. Hence, we can approach $(0,0)$ on any line. Thus, let $x = y^2$. Then, we have,

$$\begin{aligned} f(y^2, y) &= \frac{y^2 y^2}{(y^2)^2 + y^4} \\ &= \frac{y^4}{y^4 + y^4} \\ &= \frac{y^4}{y^4(1+1)} \\ &= \frac{1}{2} \end{aligned}$$

Thus, $\lim_{(y^2,y)\to(0,0)} f(x,y) = \frac{1}{2}$.

Now approach on the line $x = 0$. This yields,

$$\begin{aligned} f(0, y) &= \frac{(0)y^2}{(0)^2 + y^4} \\ &= \frac{0}{y^4} \\ &= 0 \end{aligned}$$

Hence, $\lim_{(0,y)\to(0,0)} f(x,y) = 0 \neq \lim_{(y^2,y)\to(0,0)} f(x,y)$.

We can see that the limit as we approach $(0,0)$ is different depending upon the line we approach it from. Hence f is not continuous at $(0,0)$.

Problem 4.

By Corollary 39.7 in the text, since f is differentiable at c, we have that $Df(c)(u) = u_1D_1f(c) + \cdots + u_nD_nf(c)$.

Thus, let $v_c = \begin{bmatrix} D_1f(c) \\ \vdots \\ D_nf(c) \end{bmatrix}$. Then clearly we have,

$$Df(c)(u) = v_c \cdot u$$

where $u = \begin{bmatrix} u_1, & \cdots, & u_n \end{bmatrix} \in \mathbb{R}^n$. Now need to show that v_c is unique.

Suppose there exists $v_c' \neq v_c$ such that

$$Df(c)(u) = v_c' \cdot u$$

Then we have that,

$$v_c \cdot u = v_c' \cdot u$$

That is,

$$D_1f(c)u_1 = v_1'u_1$$
$$\vdots$$
$$D_nf(c)u_n = v_n'u_n$$

Each equation above is an equation of real numbers, so by using the field properties we can divide both sides by u_i for each $i \in \{1, \cdots, n\}$, which yields,

$$D_1f(c) = v_1'$$
$$\vdots$$
$$D_nf(c) = v_n'$$

Hence, we have that $v'c$ has the same components as v_c. As a result, $v_c = v_c'$ and thus v_c (ie. the gradient of f at c) is unique.

Now suppose $||u|| = 1$ and u is a positive multiple of grad $f(c)$. That is, $u = av_c$ for some real number $a > 0$. Hence, we have that,

$$\begin{aligned} D_uf(c) &= Df(c)(u) \\ &= v_c \cdot u \\ &\leq ||v_c|| \cdot ||u|| \\ &= ||v_c|| \cdot ||av_c|| \\ &= |a| \cdot ||v_c||^2 \end{aligned}$$

Problem 5.

Let Df_1 and Df_2 exist at c and let Df_1 be continuous in a neighborhood around c.