Advanced Calculus II: Important Theorems, Lemmas, and Corollaries

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Note that all of the theorem, lemma, and corollary numbers are taken from *The Elements of Real Analysis* (2nd edition) by Robert G. Bartle. The page numbers are also taken from the same text.

1 Topology

Section 8 - Vector and Cartesian Spaces

Triangle Inequality. Let V be a vector space with a norm defined on V to $\mathbb{R}$ denoted by $x \mapsto ||x||$. Then we have, for all $x, y \in V$,

$$||x + y|| \leq ||x|| + ||y||$$

Schwarz Inequality. Let V be an inner product space and define $||x||$ by $||x|| = \sqrt{x \cdot x}$ for $x \in V$.

Then $x \mapsto ||x||$ is a norm on V and satisfies the property that

$$|x \cdot y| \leq ||x|| \; ||y||$$

Moreover, if x and y are non-zero, then the equality holds iff there is some strictly positive real numbers c such that $x = cy$.

Section 9 - Open and Closed Sets

Open Set Properties. The following are properties of open sets:

a) The empty set $\emptyset$ and the entire space $\mathbb{R}^p$ are open in $\mathbb{R}^p$

b) The intersection of any two open sets is open in $\mathbb{R}^p$.

c) The union of any collection of open sets is open in $\mathbb{R}^p$

Closed Set Properties. The following are properties of closed sets:

a) The empty set $\emptyset$ and the entire space $\mathbb{R}^p$ are closed in $\mathbb{R}^p$

b) The union of any two closed sets is closed in $\mathbb{R}^p$.

c) The intersection of any collection of closed sets is closed in $\mathbb{R}^p$

Theorem 9.11. A subset of $\mathbb{R}$ is open if and only if it is the union of a countable collection of open intervals.

Section 10 - The Nested Cells and Bolzano-Weierstrass Theorem

Nested Cells Theorem. Let (I_k) be a sequence of non-empty closed cells in $\mathbb{R}^p$ which is nested in the sense that $I_1 \supset I_2 \supset \cdots \supset I_k \supset \cdots$. Then there exists a point in $\mathbb{R}^p$ which belongs to all of the cells.

Theorem 10.5. A set $F \subset \mathbb{R}^n$ is closed if and only if it contains all of its cluster points.

Bolzano-Weierstrass Theorem. Every bounded infinite subset of $\mathbb{R}^p$ has a cluster point.

Section 11 - The Heine-Borel Theorem

Heine-Borel Theorem. A subset of $\mathbb{R}^p$ is compact if and only if it is closed and bounded.

Cantor Intersection Theorem. Let F_1 be a non-empty closed, bounded subset of $\mathbb{R}^p$ and let

$$F_1 \supset F_2 \supset \cdots \supset F_n \supset \cdots$$

be a sequence of non-empty closed sets. Then there exists a point belonging to all of the sets $\{F_k : k \in \mathbb{N}\}$

Lebesgue Covering Theorem. Suppose that $\mathscr{G} = \{G_\alpha\}$ is a covering of a compact subset K of $\mathbb{R}^p$. There exists a strictly positive number λ such that if x, y belong to K and $||x - y|| < \lambda$, then there is a set in $\mathscr{G}$ containing both x and y.

Nearest Point Theorem. Let F be a non-void closed subset of $\mathbb{R}^p$ and let x be a point outside of F. Then there exists at least one point y belonging to F such that $||z-x|| \geq ||y-x||$ for all $z \in F$.

Circumscribing Theorem. Let F be a closed and bounded set in $\mathbb{R}^2$ and let G be an open set which contains F. Then there exists a closed curve C, lying entirely in G and made up of arcs of a finite number of circles, such that F is surrounded by C.

Section 12 - Connected Sets

Lemma 12.6. An open subset of $\mathbb{R}^p$ is connected if and only if it cannot be expressed as the union of two disjoint non-empty open sets.

Theorem 12.7. Let G be an open set in $\mathbb{R}^p$. Then G is connected if and only if any pair of points x, y in G can be joined by a polygonal curve lying entirely in G.

Theorem 12.8. A subset of $\mathbb{R}$ is connected if and only if it is an interval.

2 Sequences

Section 14 - Introduction to Sequences

Lemma 14.6. A convergent sequence in $\mathbb{R}^p$ is bounded.

Theorem 14.7. A sequence (x_n) in $\mathbb{R}^p$ with $x = (x_{1n}, x_{2n}, \cdots, x_{pn}), n \in \mathbb{N}$ converges to an element $y = (y_1, y_2, \cdots, y_p)$ if and only if the corresponding p sequences of real numbers $(x_{1n}), (x_{2n}), \cdots (x_{pn})$ converge to $y_1, y_2, \cdots, y_p$ respectively.

Theorem 14.9. Let $X = (x_n)$ be a sequence in $\mathbb{R}^p$ and let $x \in \mathbb{R}^p$. Let $A = (a_n)$ be a sequence in $\mathbb{R}$ which is such that

i) $\lim(a_n) = 0$

ii) $||x_n - x|| \leq C|a_n|$ for some $C > 0$ and all $n \in \mathbb{N}$

Then $\lim(x_n) = x$

Section 15 - Subsequences and Combinations

Lemma 15.2. If a sequence X in $\mathbb{R}^p$ converges to an element x, then any subsequence of X also converges to x.

Theorem 15.4. If $X = (x_n)$ is a sequence in $\mathbb{R}^p$, then the following statements are equivalent:

a) X does not converge to x

b) There exists a neighborhood V of x such that if n is any natural number, then there is a natural number $m = m(n) \geq n$ such that x_m does not belong to V.

c) There exists a neighborhood V of x and a subsequence X' of X such that none of the elements of X' belongs to V.

Theorem 15.6. The following statements apply to combinations of sequences:

a) Let X and Y be sequences in $\mathbb{R}^p$ which converge to x and y respectively. Then the sequences $X + Y$, $X - Y$, and $X \cdot Y$ converge to $x + y$, $x - y$, and $x \cdot y$ respectively.

b) Let $X = (x_n)$ be a sequence in $\mathbb{R}^p$ which converges to x and let $A = (a_n)$ be a sequence in $\mathbb{R}$ which converges to a. Then the sequence $(a_n x_n)$ in $\mathbb{R}^p$ converges ax.

c) Let $X = (x_n)$ be a sequence in $\mathbb{R}^p$ which converges to x and let $B = (b_n)$ be a sequence of non-zero real numbers which converges to a non-zero number b. Then the sequence $(b_n^{-1} x_n)$ in $\mathbb{R}^p$ converges to $b^{-1}x$

Section 16 - Two Criteria for Convergence

Monotone Convergence Theorem. Let $X = (x_n)$ be a sequence of real numbers which is monotone increasing in the sense that

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots$$

Then the sequence X converges if and only if it is bounded, in which case $\lim(x_n) = \sup\{x_n\}$

Bolzano-Weierstrass Theorem. A bounded sequence in $\mathbb{R}^p$ has a convergent subsequence.

Lemma 16.7. If $X = (x_n)$ is a convergent sequence in $\mathbb{R}^p$, then X is a Cauchy sequence.

Lemma 16.8. A Cauchy sequence in $\mathbb{R}^p$ is bounded.

Lemma 16.9. If a subsequence X' of a Cauchy sequence X in $\mathbb{R}^p$ converges to an element x, then the entire sequence X converges to x.

Cauchy Convergence Criterion. A sequence in $\mathbb{R}^p$ is convergent if and only if it is a Cauchy sequence.

3 Continuity

Section 20 - Local Properties of Continuous Functions

Theorem 20.2. Let a be a point in the domain $D(f)$ of the function f. The following statements are equivalent:

a) f is continuous at a

b) If $\epsilon > 0$, there exists a number $\delta(\epsilon) > 0$ such that if $x \in D(f)$ is any element such that $||x - a|| < \delta(\epsilon)$, then $||f(x) - f(a)|| < \epsilon$

c) If (x_n) is any sequence of elements of $D(f)$ which converges to a, then the sequence $(f(x_n))$ converges to $f(a)$

Discontinuity Criterion. The function f is not continuous at a point a in $D(f)$ if and only if there is a sequence (x_n) of elements in $D(f)$ which converges to a but such that the sequence $(f(x_n))$ of images does not converge to $f(a)$

Theorem 20.4. The function f is continuous at a point a in $D(f)$ if and only if for every neighborhood V of $f(a)$ there is a neighborhood V_1 of a such that

$$V_1 \cap D(f) = f^{-1}(V)$$

Theorem 20.6. Let $f : D(f) \subset \mathbb{R}^p \to \mathbb{R}^q$, $g : D(g) \subset \mathbb{R}^p \to \mathbb{R}^q$, $\varphi : D(\varphi) \subset \mathbb{R}^p \to \mathbb{R}$, and $c \in \mathbb{R}$. If the functions f, g, φ are continuous at a point, then the algebraic combinations,

$$f + g, \quad f - g, \quad f \cdot g, \quad cf, \quad \varphi f, \quad f/\varphi$$

are also continuous at this point

Theorem 20.7. If f is continuous at a point, then $|f|$ is also continuous there.

Theorem 20.8. If f is continuous at a and g is continuous at $b = f(a)$, then the composition $g \circ f$ is continuous at a.

Section 21 - Linear Functions

Theorem 21.2. If f is a linear function with domain $\mathbb{R}^p$ and range in $\mathbb{R}^q$, then there are pq real numbers (c_{ij}), $1 \le i \le q, 1 \le j \le p$, such that if $x = (x_1, x_2, \cdots, x_p)$ is any point in $\mathbb{R}^p$, and if $y = (y_1, y_2, \cdots, y_q) = f(x)$ is its image under f, then

$$\begin{aligned}
y_1 &= c_{11}x_1 + c_{12}x_2 + \cdots + c_{1p}x_p \\
y_2 &= c_{21}x_1 + c_{22}x_2 + \cdots + c_{2p}x_p \\
&\vdots \\
y_q &= c_{q1}x_1 + c_{q2}x_2 + \cdots + c_{qp}x_p
\end{aligned}$$

Conversely, if (c_{ij}) is a collection of pq real numbers, then the function which assigns x in $\mathbb{R}^p$ the element y in $\mathbb{R}^q$ according to the equations above is a linear function with domain $\mathbb{R}^p$ and range in $\mathbb{R}^q$,

Theorem 21.3. If f is a linear function with $\mathbb{R}^p$ and range in $\mathbb{R}^q$, then there exists a positive constant A such that if u, v are any two vectors in $\mathbb{R}^p$, then

$$||f(u) - f(v)|| \leq A||u - v||$$

Therefore, a linear function on $\mathbb{R}^p$ to $\mathbb{R}^q$ is continuous at every point.

Section 22 - Global Properties of Continuous Functions

Global Continuity Theorem. The following statements are equivalent:

a) f is continuous on its domain $D(f)$

b) If G is any open set in $\mathbb{R}^q$, then there exists an open set in G_1 in $\mathbb{R}^p$ such that $G_1 \cap D(f) = f^{-1}(G)$

c) If H is any closed set in $\mathbb{R}^q$, then there exists a closed set H_1 in $\mathbb{R}^p$ such that $H_1 \cap D(f) = f^{-1}(H)$

Corollary 22.2. Let f be defined on all of $\mathbb{R}^p$ and with range in $\mathbb{R}^q$. Then the following statements are equivalent:

a) f is continuous on $\mathbb{R}^p$

b) If G is open in $\mathbb{R}^q$, then $f^{-1}(G)$ is open in $\mathbb{R}^p$

c) If H is closed in $\mathbb{R}^q$, then $f^{-1}(H)$ is closed in $\mathbb{R}^p$

Preservation of Connectedness. If $H \subset D(f)$ is connected in $\mathbb{R}^p$ and f is continuous on H, then $f(H)$ is connected in $\mathbb{R}^q$

Bolzano's Intermediate Value Theorem. Let $H \subset D(f)$ be a connected subset of $\mathbb{R}^p$ and let f be continuous on H and with values in $\mathbb{R}$. If k is any real number satisfying

$$\inf\{f(x) : x \in H\} < k < \sup\{f(x) : x \in H\}$$

then there is at least one point of H where f takes the value k

Preservation of Compactness. If $K \subset D(f)$ is compact and f is continuous on K, then $f(K)$ is compact.

Maximum and Minimum Value Theorem. Let $K \subset D(f)$ be compact in $\mathbb{R}^p$ and let f be a continuous real valued function. Then there are points x^* and x_* in K such that,

$$f(x^*) = \sup\{f(x) : x \in K\}, \quad f(x_*) = \inf\{f(x) : x \in K\}$$

Corollary 22.7. Let f be a function on $D(f) \subset \mathbb{R}^p$ to $\mathbb{R}^q$ and let $K \subset D(f)$ be compact. If f is continuous on K, then there are points x^* and x_* in K such that

$$||f(x^*)|| = \sup\{||f(x)|| : x \in K\}, \quad ||f(x_*)|| = \inf\{||f(x)|| : x \in K\}$$

Corollary 22.8. Let $f : \mathbb{R}^p \to \mathbb{R}^q$ be a linear function. Then f is injective if and only if there exists $m > 0$ such that $||f(x)|| \geq m||x||$ for all $x \in \mathbb{R}^p$

Continuity of the Inverse Function. Let K be a compact subset of $\mathbb{R}^p$ and let f be a continuous injective function with domain K and range $f(K)$ in $\mathbb{R}^q$. Then the inverse function is continuous with domain $f(K)$ and range K.

Section 23 - Uniform Continuity and Fixed Points

Lemma 23.2. A necessary and sufficient condition that the function f is not uniformly continuous on $A \subset D(f)$ is that there exists $\epsilon_0 > 0$ and two sequences $X = (x_n)$, $Y = (y_n)$ in A such that if $n \in \mathbb{N}$, then $||x_n - y_n|| \leq 1/n$ and $||f(x_n) - f(y_n)|| > \epsilon_0$.

Uniform Continuity Theorem. Let f be a continuous function with domain $D(f)$ in $\mathbb{R}^p$ and range in $\mathbb{R}^q$. If $K \subset D(f)$ is compact, then f is uniformly continuous on K.

Fixed Point Theorem. Let f be a contraction with domain $\mathbb{R}^p$ and range contained in $\mathbb{R}^p$. Then f has a unique fixed point.

4 The Derivative in $\mathbb{R}$

Section 27 - The Mean Value Theorem

Lemma 27.2. If f has a derivative at c, then f is continuous there.

Lemma 27.3. a) If f has a derivative at c and $f'(c) > 0$, there exists a number $\delta > 0$ such that if $x \in D$ and $c < x < c + \delta$, then $f(c) < f(x)$

b) If $f'(c) < 0$, there exists a number $\delta > 0$ such that if $x \in D$ and $c - \delta < x < c$, then $f(c) < f(x)$

Interior Maximum Theorem. Let c be an interior point of D at which f has a relative maximum. If the derivative of f at c exists, then it must be equal to zero.

Rolle's Theorem. Suppose that f is continuous on a closed interval $J = [a, b]$, that the derivative f' exists in the open interval (a, b) and that $f(a) = f(b) = 0$. Then there exists a point c in (a, b) such that $f'(c) = 0$.

Mean Value Theorem. Suppose that f is continuous on a closed interval $J = [a, b]$ and has a derivative in the open interval (a, b). Then there exists a point c in (a, b) such that

$$f(b) - f(a) = f'(c)(b - a)$$

Cauchy Mean Value Theorem. Let f, g be continuous on $J = [a, b]$ and have derivatives inside (a, b). Then there exists a point c in (a, b) such that

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)]$$

Theorem 27.9. Suppose that f is continuous on $J = [a, b]$ and that its derivative exists in (a, b)

i) If $f'(x) = 0$ for $a < x < b$, then f is constant on J

ii) If $f'(x) = g'(x)$ for $a < x < b$, then f and g differ on J by a constant

iii) If $f'(x) \geq 0$ for $a < x < b$ and if $x_1 \leq x_2$ belongs to J, then $f(x_1) \leq f(x_2)$

iv) If $f'(x) > 0$ for $a < x < b$ and if $x_1 < x_2$ belongs to J, then $f(x_1) < f(x_2)$

v) If $f'(x) \geq 0$ for $a < x < a + \delta$, then a is a relative minimum point of f

vi) If $f'(x) \geq 0$ for $b - \delta < x < b$, then b is a relative maximum point of f

vii) If $|f'(x)| \leq M$ for $a < x < b$, then f satisfies the Lipschitz condition:

$$|f(x_1) - f(x_2)| \leq M|x_1 - x_2| \quad \text{for } x_1, x_2 \in J$$

Section 28 - Further Applications of the Mean Value Theorem

Taylor's Theorem. Suppose that n is a natural number, that f and its derivatives $f', f'', \cdots, f^{(n-1)}$ are defined and continuous on $J = [a, b]$, and that $f^{(n)}$ exists in (a, b). If α, β belong to J, then there exists a number γ between α and β such that

$$f(\beta) = f(\alpha) + \frac{f'(\alpha)}{1!}(\beta - \alpha) + \frac{f''(\alpha)}{2!}(\beta - \alpha)^2 + \cdots + \frac{f^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^{n-1} + \frac{f^{(n)}(\gamma)}{n!}(\beta - \alpha)^n$$

The last term $R_n = \frac{f^{(n)}(\gamma)}{n!}(\beta - \alpha)^n$ is known as the Lagrange form of the remainder.

5 The Derivative in $\mathbb{R}^p$

Section 39 - The Derivative in $\mathbb{R}^p$

Lemma 39.5. If $f : A \to \mathbb{R}^q$ is differentiable at $c \in A$, then there exist strictly positive numbers δ, K such that if $||x - c|| < \delta$, then

$$||f(x) - f(c)|| \leq K||x - c||$$

In particular, it follows that f is continuous at $x = c$.

Theorem 39.6. If $A \subset \mathbb{R}^p$, if $f : A \to \mathbb{R}^q$ is differentiable at a point $c \in A$, and if u is any element of $\mathbb{R}^p$, then the partial derivative $D_u f(c)$ of f at c with respect to u exists. Moreover,

$$D_u f(c) = Df(c)(u)$$

Corollary 39.7. Let $A \subset \mathbb{R}^p$, let $f : A \to \mathbb{R}$ and let c be an interior point of A. If the derivative $Df(c)$ exists, then each of the partial derivatives $D_1 f(c), \cdots, D_p f(c)$ exist in $\mathbb{R}$ and if $u = (u_1, \cdots, u_p) \in \mathbb{R}^p$, then

$$Df(c)(u) = u_1 D_1 f(c) + \cdots + u_p D_p f(c)$$

Theorem 39.9. Let $A \subset \mathbb{R}^p$, let $f : A \to \mathbb{R}^q$, and let c be an interior point of A. If the partial derivatives $D_j f_i$ $(i = 1, \cdots, q, j = 1, \cdots p)$ exist in a neighborhood of c and are continuous at c, then f is differentiable at c. Moreover, $Df(c)$ is represented by a $q \times p$ matrix:

$$Df(c) = \begin{bmatrix} D_1 f_1(c) & D_2 f_1(c) & \cdots & D_p f_1(c) \\ D_1 f_2(c) & D_2 f_2(c) & \cdots & D_p f_2(c) \\ \cdots & \cdots & \cdots & \cdots \\ D_1 f_q(c) & D_2 f_q(c) & \cdots & D_p f_q(c) \end{bmatrix}$$

This is called the Jacobian matrix of the system at point c. When $p = q$, the determinant of the matrix is called the Jacobian determinant, or simply the Jacobian of the system at the point c. Frequently, the Jacobian determinant is denoted by,

$$\left. \frac{\partial(f_1, f_2, \cdots, f_p)}{\partial(x_1, x_2, \cdots, x_p)} \right|_{x=c} \quad \text{or} \quad J_f(c)$$

Section 40 - The Chain Rule and Mean Value Theorems

Theorem 40.1. Let $A \subset \mathbb{R}^p$ and let c be an interior point of A.

a) If f and g are defined on A to $\mathbb{R}^q$ and are differentiable at c, and if $\alpha, \beta \in \mathbb{R}$, then the function $h = \alpha f + \beta g$ is differentiable at c and

$$Dh(c) = \alpha Df(c) + \beta Dg(c)$$

b) If $\varphi : A \to \mathbb{R}$ and $f : A \to \mathbb{R}^q$ are differentiable at c, then the product function $k = \varphi f : A \to \mathbb{R}^q$ is differentiable at c and

$$Dk(c)(u) = \{D\varphi(c)(u)\}f(c) + \varphi(c)\{Df(c)(u)\} \quad \text{for } u \in \mathbb{R}^p$$

Chain Rule. Let f have domain $A \subset \mathbb{R}^p$ and range in $\mathbb{R}^q$, and let g have domain $B \subset \mathbb{R}^q$ and range in $\mathbb{R}^r$. Suppose that f is differentiable at c and that g is differentiable at $b = f(c)$. Then the composition $h = g \circ f$ is differentiable at c and

$$Dh(c) = Dg(b) \circ Df(c)$$

Alternatively, we write,

$$D(g \circ f)(c) = Dg(f(c)) \circ Df(c)$$

Mean Value Theorem 40.4. Let f be defined on an open subset Ω of $\mathbb{R}^p$ and have values in $\mathbb{R}$. Suppose that the set Ω contains the points a, b and the line segment S joining them, and that f is differentiable at every point of this line segment. Then there exists a point c on S such that

$$f(b) - f(a) = Df(c)(b - a)$$

Mean Value Theorem 40.5. Let $\Omega \subset \mathbb{R}^p$ be an open set and let $f : \Omega \to \mathbb{R}^q$. Suppose that Ω contains the points a, b and the line segment S joining these points, and that f is differentiable at every point of S. Then there exists a point c on S such that

$$||f(b) - f(a)|| \leq ||Df(c)(b - a)||$$

Corollary 40.6. Suppose the hypotheses of Theorem 40.5 are satisfied and that there exists $M > 0$ such that $||Df(x)||_{pq} \leq M$ for all $x \in S$. Then we have

$$||f(b) - f(a)|| \leq M||b - a||$$

Theorem 40.8. Suppose that f is defined on a neighborhood U of a point $(x, y) \in \mathbb{R}^2$ with values in $\mathbb{R}$. Suppose that the partial derivative $D_x f$, $D_y f$, and $D_{yx} f$ exist in U and that $D_{yx} f$ is continuous at (x, y). Then the partial derivative $D_{xy} f$ exists at (x, y) and $D_{xy} f(x, y) = D_{yx} f(x, y)$

Taylor's Theorem 40.9. Suppose that f is a function with open domain Ω in $\mathbb{R}^p$ and range in $\mathbb{R}$, and suppose that f has continuous partial derivatives of of order n in a neighborhood of every point on a line segment S joining two points $a, b = a + u$ in Ω. Then there exists a point c on S such that

$$f(a+u) = f(a) + \frac{1}{1!}Df(a)(u) + \frac{1}{2!}D^2f(a)(u)^2 + \cdots + \frac{1}{(n-1)!}D^{n-1}f(a)(u)^{n-1} + \frac{1}{n!}D^n f(c)(u)^n$$

Note that:

$$\begin{aligned} D^2f(a)(u)^2 &= D^2f(a)(u,u) \\ D^3f(a)(u)^3 &= D^3f(a)(u,u,u) \\ &\vdots \\ D^nf(a)(u)^n &= D^nf(a)(u,u,\cdots,u) \end{aligned}$$

Also note that:

$$D^2f(a)(u,u) = \sum_{i,j=1}^{p} D_{ji}f(c)u_iu_j$$

Section 41 - Mapping Theorems and Implicit Functions

Lemma 41.3. Let $\Omega \subset \mathbb{R}^p$ be an open set and let $f : \Omega \to \mathbb{R}^q$ be differentiable on Ω. Suppose that Ω contains the points a, b and the line segment S joining these points, and let $x_0 \in \Omega$. Then we have

$$||f(b) - f(a) - Df(x_0)(b-a)|| \leq ||b-a|| \sup_{x \in S}\{||Df(x) - Df(x_0)||_{pq}\}$$

Approximation Lemma. Let $\Omega \subset \mathbb{R}^p$ be open and let $f : \Omega \to \mathbb{R}^q$ belong to Class $C^1(\Omega)$. If $x_0 \in \Omega$ and $\epsilon > 0$, then there exists $\delta(\epsilon) > 0$ such that if $||x_k - x_0|| \leq \delta(\epsilon)$, $k = 1, 2$, then $x_k \in \Omega$ and

$$||f(x_1) - f(x_2) - Df(x_0)(x_1 - x_2)|| \leq \epsilon ||x_1 - x_2||$$

Injective Mapping Theorem. Suppose that $\Omega \subset \mathbb{R}^p$ is open, that $f : \Omega \to \mathbb{R}^q$ belongs to Class $C^1(\Omega)$, and that $L = Df(c)$ is an injection. Then there exists a number $\delta > 0$ such that the restriction of f to $B_\delta = \{x \in \mathbb{R}^p : ||x - c|| \leq \delta\}$ is an injection. Moreover, the inverse of the restriction $f|B_\delta$ is a continuous function on $f(B_\delta) \subset \mathbb{R}^q$ to $B_\delta \subset \mathbb{R}^p$.

Surjective Mapping Theorem. Let $\Omega \subset \mathbb{R}^p$ be open and let $f : \Omega \to \mathbb{R}^q$ belong to Class $C^1(\Omega)$. Suppose that for some $c \in \Omega$, the linear function $L = Df(c)$ is a surjection of $\mathbb{R}^p$ onto $\mathbb{R}^q$. Then there exist numbers $m > 0$ and $\alpha > 0$ such that if $y \in \mathbb{R}^q$ and $||y - f(c)|| \leq \alpha/2m$, then there exists an $x \in \Omega$ such that $||x - c|| \leq \alpha$ and $f(x) = y$.

Open Mapping Theorem. Let $\Omega \subset \mathbb{R}^p$ be open and let $f : \Omega \to \mathbb{R}^q$ belong to class $C^1(\Omega)$. If for each $x \in \Omega$, the derivative $Df(x)$ is a surjection, and if $G \subset \Omega$ is open, then $f(G)$ is open in $\mathbb{R}^q$.

Inversion Mapping Theorem. Let $\Omega \subset \mathbb{R}^p$ be open and suppose that $f : \Omega \to \mathbb{R}^p$ belongs to Class $C^1(\Omega)$. If $c \in \Omega$ is such that $Df(c)$ is a bijection, then there exists an open neighborhood U of c such that $V = f(U)$ is an open neighborhood of $f(c)$ and the restriction of f to U is a bijection onto V with continuous inverse g. Moreover, g belongs to Class $C^1(V)$ and

$$Dg(y) = [Df(g(y))]^{-1} \quad \text{for } y \in V$$

Implicit Function Theorem. Let $\Omega \subset \mathbb{R}^p \times \mathbb{R}^q$ be open and let $(a, b) \in \Omega$. Suppose that $F : \Omega \to \mathbb{R}^q$ belongs to Class $C^1(\Omega)$, that $F(a, b) = 0$, and that the linear map defined by

$$L_2(v) = DF(a, b)(0, v), \quad v \in \mathbb{R}^q,$$

is a bijection of $\mathbb{R}^q$ onto $\mathbb{R}^q$

a) Then there exists an open neighborhood W of $a \in \mathbb{R}^p$ and a unique function $\varphi : W \to \mathbb{R}^q$ belonging to Class $C^1(W)$ such that $b\varphi(a)$ and

$$F(x, \varphi(x)) = 0 \quad \text{for all } x \in W$$

b) There exists an open neighborhood U of (a, b) in $\mathbb{R}^p \times \mathbb{R}^q$ such that the pair $(x, y) \in U$ satisfies $F(x, y) = 0$ if and only if $y = \varphi(x)$ for $x \in W$.

Note that $\mathbb{R}^p \times \mathbb{R}^q$ is equivalent to $\mathbb{R}^{p+q}$

Corollary 41.10. With the hypotheses of the theorem, there exists a $\gamma > 0$ such that if $||x - a|| < \gamma$, then the derivative of φ at x is the element of $\mathscr{L}(\mathbb{R}^p, \mathbb{R}^q)$ given by

$$D\varphi(x) = -[D_{(2)}F(x, \varphi(x))]^{-1} \circ [D_{(1)}F(x, \varphi(x))]$$

Section 42 - Extremum Problems

Theorem 42.1. Let $\Omega \subset \mathbb{R}^p$, and let $f : \Omega \to \mathbb{R}$. If an interior point c of Ω is a point of relative extremum of f, and if the partial derivative $D_u f(c)$ of f with respect to a vector $u \in \mathbb{R}^p$ exists, then $D_u f(c) = 0$

Corollary 42.2. Let $\Omega \subset \mathbb{R}^p$, and let $f : \Omega \to \mathbb{R}$. If an interior point c of Ω is a point of relative extremum of f, and if the derivative $Df(c)$ exists, then $Df(c) = 0$.

Theorem 42.4. Let $\Omega \subset \mathbb{R}^P$ be open and let $f : \Omega \to \mathbb{R}$ have continuous second partial derivatives on Ω. If $c \in \Omega$ is a point of relative minimum [respectively, maximum] of f, then

$$D^2 f(c)(w)^2 = \sum_{i,j=1}^{p} D_{ij} f(c) w_i w_j \geq 0$$

[respectively, $D^2 f(c)(w)^2 \leq 0$] for all $w \in \mathbb{R}^p$.

Theorem 42.5. Let $\Omega \subset \mathbb{R}^p$ be open, let $f : \Omega \to \mathbb{R}$ have continuous second partial derivatives on Ω, and let $c \in \Omega$ be a critical point of f

a) If $D^2 f(c)(w)^2 > 0$ for all $w \in \mathbb{R}^p$, $w \neq 0$, then f has a relative strict minimum at c

b) If $D^2 f(c)(w)^2 < 0$ for all $w \in \mathbb{R}^p$, $w \neq 0$, then f has a relative strict maximum at c

c) If $D^2 f(c)(w)^2$ takes on both strictly positive and strictly negative values for $w \in \mathbb{R}^p$, then f has a saddle point at c

6 The Integral in $\mathbb{R}^p$

Section 43 - The Integral in $\mathbb{R}^p$

Cauchy Criterion. A bounded function $f : I \to \mathbb{R}$ is integrable on I if and only if for every $\epsilon > 0$ there exists a partition Q_ϵ of I such that if P and Q are partitions of I which are refinements of Q, and $S(P; f)$ and $S(Q; f)$ are any corresponding Riemann sums, then

$$|S(P; f) - S(Q; f)| \leq \epsilon$$

Theorem 43.5. Let f and g be functions on A to $\mathbb{R}$ which are integrable on A and let $\alpha, \beta \in \mathbb{R}$. Then the function $\alpha f + \beta g$ is integrable on A and

$$\int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g$$

Theorem 43.6. If $f : A \to \mathbb{R}$ is integrable on A and if $f(x) \geq 0$ for $x \in A$, then $\int_A f \geq 0$

Theorem 43.7. Let $f : A \to \mathbb{R}$ be a bounded function and suppose that A has content zero. Then f is integrable on A and $\int_A f = 0$

Theorem 43.8. Let $f, g : A \to \mathbb{R}$ be bounded functions and suppose that f is integrable on A. Let $E \subset A$ have content zero and suppose that $f(x) = g(x)$ for all $x \in A \setminus E$. Then g is integrable on A and

$$\int_A f = \int_A G$$

Integrability Theorem. Let $I \subset \mathbb{R}^p$ be a closed cell and let $f : I \to \mathbb{R}$ be bounded. If there exists a subset $E \subset I$ with content zero such that f is continuous on $I \setminus E$, then f is integrable on I.

Section 44 - Content and the Integral

Lemma 44.3. A set $A \subset \mathbb{R}^p$ has content zero if and only if it has content and $c(A) = 0$.

Theorem 44.4. Let A, B belong to $\mathscr{D}(\mathbb{R}^p)$ and let $x \in \mathbb{R}^p$. Note that $\mathscr{D}(R^p)$ is the collection of all subsets of $\mathbb{R}^p$ which have content.

a) The sets $A \cap B$ and $A \cup B$ belong to $\mathscr{D}(\mathbb{R}^p)$ and

$$c(A) + c(B) = c(A \cap B) + c(A \cup B)$$

b) The sets $A \setminus B$ and $B \setminus A$ belong to $\mathscr{D}(\mathbb{R}^p)$ and

$$c(A \cup B) = c(A \setminus B) + c(A \cap B) + c(B \setminus A)$$

c) If $x + A = \{x + a : a \in A\}$, then $x + A$ belongs to $\mathscr{D}(\mathbb{R}^p)$ and

$$c(x + A) = c(A)$$

Corollary 44.5. Let A and B belong to $\mathscr{D}(\mathbb{R}^p)$.

a) If $A \cap B = \emptyset$, then $c(A \cup B) = c(A) + c(B)$

b) If $A \subset B$, then $c(B \setminus A) = c(B) - c(A)$

Theorem 44.6. Let $\gamma : \mathscr{D}(\mathbb{R}^p) \to \mathbb{R}$ be a function with the following properties:

(i) $\gamma(A) \geq 0$ for all $A \in \mathscr{D}(\mathbb{R}^p)$

(ii) if $A, B \in \mathscr{D}(\mathbb{R}^p)$ and $A \cap B = \emptyset$, then $\gamma(A \cup B) = \gamma(A) + \gamma(B)$

(iii) if $A \in \mathscr{D}(\mathbb{R}^p)$ and $x \in \mathbb{R}^p$, then $\gamma(A) = \gamma(x + A)$

(iv) $\gamma(K_0) = 1$

Then we have $\gamma(A) = c(A)$ for all $A \in \mathscr{D}(\mathbb{R}^p)$

Corollary 44.7. Let $\mu : \mathscr{D}(\mathbb{R}^p) \to \mathbb{R}$ be a function satisfying properties (i), (ii), (iii) from above. Then there exists a constant $m \geq 0$ such that $\mu(A) = mc(A)$ for all $A \in \mathscr{D}(\mathbb{R}^p)$

Theorem 44.8. Let $A \in \mathscr{D}(\mathbb{R}^p)$ and let $f : A \to \mathbb{R}$ be bounded and continuous on A. Then f is integrable on A

Theorem 44.9. a) Let A_1 and A_2 belong to $\mathscr{D}(\mathbb{R}^p)$ and suppose that $A_1 \cap A_2$ has content zero. If $A = A_1 \cup A_2$ and if $f : A \to \mathbb{R}$ is integrable on A_1 and A_2, then f is integrable on A and

$$\int_A f = \int_{A_1} f + \int_{A_2} f$$

b) Let A belong to $\mathscr{D}(\mathbb{R}^p)$ and let $A_1, A_2 \in \mathscr{D}(\mathbb{R}^p)$ be such that $A = A_1 \cup A_2$ and such that $A_1 \cap A_2$ has content zero. If $f : A \to \mathbb{R}$ is integrable on A, and if the restrictions of f to A_1 and A_2 are integrable, then Definition 44.1 holds.

Theorem 44.10. Let $A \in \mathscr{D}(\mathbb{R}^p)$ and let $f : A \to \mathbb{R}$ be integrable on A and such that $|f(x)| \leq M$ for all $x \in A$. Then

$$\left| \int_A f \right| \leq Mc(A)$$

More generally if f is real valued and $m \leq f(x) \leq M$ for all $x \in A$, then

$$mc(A) \leq \int_A f \leq Mc(A)$$

Mean Value Theorem 44.11. Let $A \in \mathscr{D}(\mathbb{R}^p)$ be a connected set and let $f : A \to \mathbb{R}$ be bounded and continuous on A. Then there exists a point $p \in A$ such that

$$\int_A f = f(p)c(A)$$

Theorem 44.12. If f is continuous on the closed cell $J = [a,b] \times [c,d]$ to $\mathbb{R}$, then

$$\begin{aligned}\int_J f &= \int_c^d \left\{ \int_a^b f(x,y)dx \right\} dy \\ &= \int_a^b \left\{ \int_c^d f(x,y)dy \right\} dx\end{aligned}$$

Theorem 44.13. Let f be integrable on the rectangle $J = [a,b] \times [c,d]$ to $\mathbb{R}$ and suppose that, for each $y \in [c,d]$, the function $x \mapsto f(x,y)$ of $[a,b]$ into $\mathbb{R}$ is continuous except possibly for a finite number of points, at which it has one-sided limits. Then

$$\int_J f = \int_c^d \left\{ \int_a^b f(x,y)dx \right\} dy$$

Theorem 44.14. Let $A \subset \mathbb{R}^2$ be given by,

$$A = \{(x,y) : \alpha(y) \leq x \leq \beta(y), c \leq y \leq d\}$$

where α and β are continuous functions on $[c,d]$ with values in $[a,b]$. If f is continuous on $A \to \mathbb{R}$, then f is integrable on A and

$$\int_A f = \int_c^d \left\{ \int_{\alpha(y)}^{\beta(y)} f(x,y)dx \right\} dy$$

Section 45 - Transformation of Sets and Integrals

Change of Variables Theorem. Let $\Omega \subset \mathbb{R}^p$ be open and suppose that $\varphi : \Omega \to \mathbb{R}^p$ belongs to Class $C^1(\Omega)$, is injective on Ω, and $J_\varphi(x) \neq 0$ for $x \in \Omega$. Suppose that A has content, $A^- \subset \Omega$, and $f : \varphi(A) \to \mathbb{R}$ is bounded and continuous. Then,

$$\int_{\varphi(A)} f = \int_A (f \circ \varphi)|J_\varphi|$$