# Advanced Calculus II: Assignment 1
# Chapter 2 - A Taste of Topology

Chris Hayduk

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**Problem 12 on p. 126.**

(a) The limit of a sequence is unaffected by rearrangement when $f$ is a bijective function. Since $f$ is bijective, each term from $(p_n)$ must be included one and only one time in the new sequence $(q_k)$.

We know that, if $(p_n) \to \ell$, this means that $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $n \geq N \implies \mathrm{d}(p_n, \ell) < \epsilon$.

Thus, for each choice of $\epsilon > 0$ there are only finitely many terms for which $\mathrm{d}(p_n, \ell) \geq \epsilon$, while there are infinitely many terms for which $\mathrm{d}(p_n, \ell) < \epsilon$.

As a result, the rearrangement $(q_k)$ will eventually exhaust all of the terms that have a distance from $\ell$ that is greater than or equal to $\epsilon$, and thus will have infinitely many terms left for which $\mathrm{d}(q_k, \ell) < \epsilon$.

Hence, $(q_k) \to \ell$ as well.

(b) A rearrangement of $(p_n)$ where $f$ is an injective function does not necessarily preserve the limit of $(p_n)$. For example, take $(p_n) = (-1)^n$. This sequence alternates between 1 and -1, and thus never converges.

Now let $f(n) = 2n$. This injective function maps the natural numbers to the evens. We can see that $\forall m$ where $m$ is even, we have that $p_m = 1$.

Thus, $q_k = p_{f(k)} = 1 \ \forall k \in \mathbb{N}$. Clearly, $(q_k) \to 1$ while $(p_n)$ does not converge.

(c) A rearrangement of $(p_n)$ where $f$ is a surjective function does not necessarily preserve the limit of $(p_n)$. For example, take $(p_n) = \frac{1}{n}$. This sequence converges to 0.

Now let $f(n) = \begin{cases} 1 & \text{n is odd} \\ 2 & \text{n} = 2 \\ f(n-2)+1 & \text{n is even and n} > 2 \end{cases}$

Call this new sequence $q_k = p_{f(k)}$. Then we have that $(q_k)$ contains all terms in the original sequence $(p_n)$ and, for every odd term $m$, $q_m = 1$. Thus, the even terms of $(q_k)$ converge to 0 while the odd terms converge to 1. Since we have two subsequences in $(q_k)$ that converge to different limits, $(q_k)$ does not converge.

**Problem 44 on p. 128.**

(a) Yes, prove from definitions.

(b) Graph is the image of $(I, f) : X \to X \text{ x } \mathbb{R}$ where $I$ is the identity function. Flesh out this proof.

(c) Graph is compact, so it has a limit point in the graph (Bolzano-Weierstrass). The limit point must be a limit point in each coordinate, so $f$ must approach this limit point. Thus, $f$ is continuous. FLESH THIS OUT MORE.

(d) Counterexample:
$f : [0, 1] \to \mathbb{R}$

$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$f$ is discontinuous because if you have a sequence $(x_n) \to 0$, $f(x_n) \not\to f(0)$ since $f(0)$ is undefined.

**Problem 76 on p. 131.**

(a) Give counterexample.

(b) Let $S_n = \{x : x \geq n\}$. Then each $S_n$ is connected and closed, and we have $S_1 \supset S_2 \supset \cdots$.

However, we have $\cap S_n = \emptyset$.

(c) Intersection is connected. Prove this.

(d) Connected but not path connected. Prove this.

**Problem 1 on p. 147.**

From assumptions, $f$ is continuous.

$x_n \to x_0$ have to prove $f(x_n) \to f(x_0)$

Suffice to prove $x_i$ are distinct and $x_m = \{x_m, x_{m+1}, \cdots] \cup x_0$

FLESH THIS OUT.