Advanced Calculus II: Assignment 6

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Problem 1.

Let $D \subset \mathbb{R}^n$ be open and let v be a C^1-map from $D \to \mathbb{R}^n$.

Since the mapping $x \mapsto Dv(x)$ of D into $\mathcal{L}(D, \mathbb{R}^n)$ is continuous, we have that for, $\epsilon > 0$, $\exists \delta(\epsilon) > 0$ such that $x, y \in D$ with $||x - y|| < \delta(\epsilon)$ yields,

$$||Dv(x) - Dv(y)||_{nn} \leq \epsilon$$

By the triangle inequality we have that,

$$||Dv(x)||_{nn} \leq \epsilon + ||Dv(y)||_{nn} \tag{1}$$

If we fix y and allow x to vary such that $||x - y|| < \delta(\epsilon)$, we can see that (1) holds for every such x. That is, $||Dv(x)||_{nn}$ is bounded by $\epsilon + ||Dv(y)||_{nn}$ for every $x \in B(y, \delta(\epsilon))$. Let $M = \epsilon + ||Dv(y)||_{nn}$.

Now take the line segment S from x to y which joins these points in $B(y, \delta(\epsilon))$. We can do this because every open ball is a convex set. Since $S \subset B(y, \delta(\epsilon))$, it is clear that for every $c \in S$, we have that $||Dv(c)||_{nn} \leq M$.

Hence, by Corollary 40.6, we have,

$$||v(x) - v(y)|| \leq M||x - y||$$

Hence, v is Lipschitz continuous when restricted to the neighborhood $B(y, \delta(\epsilon))$ of y.

Since y was arbitrary, this holds for every $y \in D$. Thus, v is locally Lipschitz continuous.

Problem 2.

In order to show that $F_{\mu,\delta}$ is invertible, by the Inversion Theorem, we need to show that F belongs to class $C^1(\mathbb{R}^2)$. Moreover, for $c \in \mathbb{R}^2$, need to show that $DF(c)$ is a bijection.

We have that the Jacobian of $F_{\mu,\delta}$ is,

$$\begin{bmatrix} \mu - 2\mu x & \delta \\ \delta & 0 \end{bmatrix}$$

Clearly the derivative exists for all $(x, y) \in \mathbb{R}^2$. Now, as stated on p. 376 of the text, we just need to show that each of the partial derivatives are continuous on $\mathbb{R}^2$ in order to show that the derivative is continuous on $\mathbb{R}^2$.

Take the first entry: $\mu - 2\mu x$. Fix $a = (a_1, a_2) \in \mathbb{R}^2$. We have that, for every $x = (x_1, x_2) \in \mathbb{R}^2$,

$$\begin{aligned} ||D_1F_1(x) - D_1F_1(a)|| &= ||\mu - 2\mu x_1 - (\mu - 2\mu a_1)|| \\ &= ||2\mu(a_1 - x_1)|| \\ &= |2\mu| \cdot |a_1 - x_1| \\ &= 2\mu \cdot |a_1 - x_1| \end{aligned}$$

Since all norms are equivalent up to a constant, let us assume we are using the infinite norm here. Hence, let $\epsilon > 0$. If we choose $\delta(\epsilon) = \frac{\epsilon}{2\mu}$, then for every $||x - a||_\infty < \delta(\epsilon)$, we have

$$\begin{aligned} ||D_1F_1(x) - D_1F_1(a)|| &= 2\mu \cdot |a_1 - x_1| \\ &\leq 2\mu \cdot ||a - x||_\infty \\ &= 2\mu \cdot ||x - a||_\infty \\ &< 2\mu \cdot \delta \\ &= 2\mu \cdot \frac{\epsilon}{2\mu} \\ &= \epsilon \end{aligned}$$

We see that the choice for $\delta(\epsilon)$ does not depend on the choice of a. Hence, this holds for all $a \in \mathbb{R}^2$ and hence D_1F_1 is continuous on the domain.

Now take D_2F_1. We have that $||D_2F_1(x) - D_2F_1(a)|| = ||\delta - \delta|| = ||0|| = 0$ for any choice of $x, a \in \mathbb{R}^2$. Hence, we have that $||D_2F_1(x) - D_2F_1(a)|| < \epsilon$ for every $\epsilon > 0$, and thus any choice of δ works. Hence, D_2F_1 is continuous on $\mathbb{R}^2$.

By extension, D_1F_2 and D_2F_2 are continuous on $\mathbb{R}^2$ as well.

Hence, we have that $DF(x)$ is continuous on $\mathbb{R}^2$.

Now we need to show that for $c \in \mathbb{R}^2$, $DF(c)$ is a bijection. We know this is true if and only if the derivative has an inverse, which is true if and only if the Jacobian determinant is non-zero. Thus, we have,

$$\begin{aligned} J_F(c) &= \begin{vmatrix} \mu - 2\mu c_1 & \delta \\ \delta & 0 \end{vmatrix} \\ &= 0(\mu - 2\mu c_1) - \delta(\delta) \\ &= -\delta^2 \end{aligned}$$

Hence, $DF(c)$ is a bijection for any $c \in \mathbb{R}^2$.

As a result, $\forall c \in \mathbb{R}^2$, there exists an open neighborhood U of c such that $V = F(U)$ is an open neighborhood of $F(C)$ and the restriction of F to U is a bijection onto V with continuous inverse G.

Let $G(x, y) = (\frac{y}{\delta}, \frac{1}{\delta}[x - \mu\frac{y}{\delta}(1 - \frac{y}{\delta})])$.

Hence, we have that,

$$\begin{aligned} G(F_{\mu,\delta}(x, y)) &= (\frac{\delta x}{\delta}, \frac{1}{\delta}[\mu x(1 - x) + \delta y - \mu\frac{\delta x}{\delta}(1 - \frac{\delta x}{\delta})]) \\ &= (x, \frac{1}{\delta}[\mu x(1 - x) + \delta y - \mu x(1 - x)]) \\ &= (x, \frac{1}{\delta}[\delta y]) \\ &= (x, y) \end{aligned}$$

This holds for every $(x, y) \in \mathbb{R}^2$.

Now let us compute the fixed points of $F_{\mu,\delta}$. We know that any fixed point of G is a fixed point of $F_{\mu,\delta}$, so we'll attempt to find the fixed points of G,

$$G(\alpha) = (\frac{\alpha_2}{\delta}, \frac{1}{\delta}[\alpha_1 - \mu\frac{\alpha_2}{\delta}(1 - \frac{\alpha_2}{\delta})])$$

So, we must have that,

$$\begin{aligned} \alpha_1 &= \frac{\alpha_2}{\delta} \\ \alpha_2 &= \frac{1}{\delta}[\alpha_1 - \mu\frac{\alpha_2}{\delta}(1 - \frac{\alpha_2}{\delta})] \end{aligned}$$

Plugging the equation for α_1 into the α_2 equation yields,

$$\begin{aligned} \alpha_2 &= \frac{1}{\delta}[\frac{\alpha_2}{\delta} - \mu\frac{\alpha_2}{\delta}(1 - \frac{\alpha_2}{\delta})] \\ &= \frac{\alpha_2}{\delta^2} - \mu\frac{\alpha_2}{\delta^2} + \mu\frac{\alpha_2^2}{\delta^3} \end{aligned}$$

Hence, we have,

$$0 = \frac{\alpha_2}{\delta^2} - \mu\frac{\alpha_2}{\delta^2} + \mu\frac{\alpha_2^2}{\delta^3} - \alpha_2$$

The solutions to this equation are $\alpha_2 = 0$ and $\alpha_2 = \frac{\delta(\delta^2+\mu-1)}{\mu}$.

Thus, the two fixed points are the points $(0,0)$ and $(\frac{\delta^2+\mu-1}{\mu}, \frac{\delta(\delta^2+\mu-1)}{\mu})$.

Now to compute the eigenvalues at each fixed point. At $(0,0)$, we have

$$\begin{aligned} DF(0,0) &= \begin{bmatrix} \mu - 2\mu(0) & \delta \\ \delta & 0 \end{bmatrix} \\ &= \begin{bmatrix} \mu & \delta \\ \delta & 0 \end{bmatrix} \end{aligned}$$

The eignevalues at $(0,0)$ are given by,

$$\begin{vmatrix} \mu - \lambda & \delta \\ \delta & -\lambda \end{vmatrix} = \lambda^2 - \mu\lambda - \delta^2 = 0$$

Hence, we have,

$$\lambda_1 = \frac{1}{2}\left(\mu \pm \sqrt{\mu^2 + 4\delta^2}\right)$$

The eignevalues at $(\frac{\delta^2+\mu-1}{\mu}, \frac{\delta(\delta^2+\mu-1)}{\mu})$ are given by,

$$\begin{aligned} \begin{vmatrix} \mu - 2\delta^2 - 2\mu + 2 - \lambda & \delta \\ \delta & -\lambda \end{vmatrix} &= \lambda^2 - \mu\lambda + 2\delta^2\lambda + 2\mu\lambda - 2\lambda - \delta^2 \\ &= \lambda^2 + \mu\lambda + 2\delta^2\lambda - 2\lambda - \delta^2 = 0 \end{aligned}$$

Hence, we have,

$$\lambda_2 = \frac{1}{2}\left(\pm\sqrt{(\mu + 2\delta^2 - 2)^2 + 4\delta^2} - \mu - 2\delta^2 + 2\right)$$

Problem 3.

Let $D \subset \mathbb{R}^n$ be open and let $f \in C^1(D, \mathbb{R}^m)$. By 39.11 in the textbook, we know the derivative at a point $c \in D$ is the linear mapping of $\mathbb{R}^n$ into $\mathbb{R}^m$ determined by the $m \times n$ matrix whose elements are,

$$Df(c) = \begin{bmatrix} D_1 f_1(c) & D_2 f_1(c) & \cdots & D_n f_1(c) \\ D_1 f_2(c) & D_2 f_2(c) & \cdots & D_n f_2(c) \\ \cdots & \cdots & \cdots & \cdots \\ D_1 f_m(c) & D_2 f_m(c) & \cdots & D_n f_m(c) \end{bmatrix}$$

Now let $x, y \in D$. Then we have that,

$$Df(x) - Df(y) = \begin{bmatrix} D_1 f_1(x) - D_1 f_1(y) & D_2 f_1(x) - D_2 f_1(y) & \cdots & D_n f_1(x) - D_n f_1(y) \\ D_1 f_2(x) - D_1 f_2(y) & D_2 f_2(x) - D_2 f_2(y) & \cdots & D_n f_2(x) - D_n f_2(y) \\ \cdots & \cdots & \cdots & \cdots \\ D_1 f_m(x) - D_1 f_m(y) & D_2 f_m(x) - D_2 f_m(y) & \cdots & D_n f_m(x) - D_n f_m(y) \end{bmatrix}$$

In addition, from Exercise 21.P, we have that $|D_i f_j(x) - D_i f_j(y)| \leq ||Df(x) - Df(y)||_{nm}$ for all i, j.

Now let $\epsilon > 0$. Since Df is continuous on D by the fact that $f \in C^1(D, \mathbb{R}^m)$, $\exists \delta(\epsilon) > 0$ such that $\forall x, y \in D$ such that $||x - y|| < \delta(\epsilon)$, we have that $||Df(x) - Df(y)||_{nm} < \epsilon$.

Suppose that the x, y chosen above satisfy $||x - y|| < \delta(\epsilon)$. Then we have that,

$$|D_i f_j(x) - D_i f_j(y)| \leq ||Df(x) - Df(y)||_{nm} < \epsilon$$

Hence, all the partial derivatives $D_i f_j$ are continuous on D.

Problem 4.

We have that the functions $g(x, y) = x$, $h(x, y) = y$, and $j(x, y) = -3$ are all clearly continuous on all of $\mathbb{R}^2$. By Theorem 20.7, the following combination of these functions must also then be continuous,

$$\begin{aligned} f(x,y) \cdot f(x,y) \cdot f(x,y) + j(x,y) \cdot f(x,y) \cdot h(x,y) \cdot h(x,y) &= x^3 - 3xy^2 \\ &= f(x,y) \end{aligned}$$

Hence, we have that $f(x, y)$ is continuous on all of $\mathbb{R}^2$, and thus is continuous on a neighborhood of the origin.

The derivative of $f(x, y)$ is given by,

$$Df(x,y) = \begin{bmatrix} 3x^2 - 3y^2 & -6xy \end{bmatrix}$$

Again, it is clear from a similar argument as the above that both partial derivatives are continuous on all of $\mathbb{R}^2$. Hence, $Df(x, y)$ is continuous on all of $\mathbb{R}^2$ and thus $f \in C^1(\mathbb{R}^2)$.

Problem 5.

Let $h(x) = f(x) - g(x)$ for all $x \in D$. By Theorem 40.1(a), we have that,

$$Dh(c) = Df(c) - Dg(c) = 0$$

for all points c in the interior of D. Since D is open, all of its points are interior points.

Take two arbitrary points $x, y \in D$. Since D is open and connected, we can apply Theorem 12.7 from the text, which says that x and y can be joined by a polygonal curve lying entirely in D.

Let $z_1, \cdots, z_{n-1}$ denote the k endpoints of the lines between x and y. Note that each line $L_1 = \{t \in [0,1] : x + t(z_1 - x)\}, L_2 = \{t \in [0,1] : z_1 + t(z_2 - z_1)\}, \cdots, L_n = \{t \in [0,1] :$

$z_{n-1} + t(y - z_{n-1})\}$ is fully contained in D.

Thus, we can apply the Mean Value Theorem from $\mathbb{R}^n \to \mathbb{R}$ to each line.

Start with the first line, L_1. We have that,

$$\begin{aligned} h(z_1) - h(x) &= Dh(c_1)(z_{k-1} - x) \\ &= 0 \end{aligned}$$

This implies that $h(x) = h(z_1)$.

More generally, for every L_k, there exists a point c_k on L_k such that,

$$h(z_k) - h(z_{k-1}) = 0$$

and hence $h(z_k) = h(z_{k-1})$.

And lastly, we have L_n, which yields,

$$h(y) - h(z_{n-1}) = 0$$

which implies that $h(y) = h(z_{n-1})$.

Since equality is transitive, we have that $h(x) = h(y)$. Moreover, since x, y were arbitrary in D, this formulation applies to any pair of points in D. Hence, $h(x)$ is constant on D. That is, $h(x) = c$ for some $c \in \mathbb{R}$ and for all $x \in D$.

Now recall that $h(x) = f(x) - g(x)$ for all $x \in D$. We thus have,

$$\begin{aligned} &h(x) = f(x) - g(x) = c \\ \implies &f(x) = g(x) + c \end{aligned}$$

as required.

Problem 6.

Note: δA denotes the boundary of the set A. We say $x \in \mathbb{R}^n$ is a boundary point of A if for every $r > 0$, $\exists y_1, y_2 \in B(x, r)$ such that $y_1 \in A$ and $y_2 \in \mathbb{R}^n \setminus A = A^C$

Now take $A \cap B$. Let $x \in \delta(A \cap B)$. Since x is a boundary point of $A \cap B$, we have that for every $r > 0$, $\exists y_1, y_2 \in B(x, r)$ such that $y_1 \in A \cap B$ and $y_2 \in \mathbb{R}^n \setminus (A \cap B) = (A \cap B)^C = A^C \cup B^C$.

We know y_2 is in at most one of A or B. If y_2 is not in contained in either set, then we can see that $y_1 \in A \cap B \implies y_1 \in A$ and $y_2 \notin A \cup B \implies y_2 \in A^C$. This holds for every y_1 and y_2. Hence, $x \in \delta A$ and thus $x \in \delta A \cup \delta B$.

Now assume y_2 is in A without loss of generality. Then we must have that $y_2 \notin B$, otherwise $y_2 \notin (A \cap B)^C$. In addition, we have that $y_1 \in A \cap B \implies y_1 \in B$. This holds for every y_1 and y_2. Hence, $x \in \delta A$ and thus $x \in \delta B \cup \delta B$.

Since this holds for every boundary point x of $A \cap B$, we have that $\delta(A \cap B) \subset (\delta A \cup \delta B)$.

Now take $A \cup B$. Let $x \in \delta(A \cup B)$. Since x is a boundary point of $A \cup B$, we have that for every $r > 0$, $\exists y_1, y_2 \in B(x, r)$ such that $y_1 \in A \cup B$ and $y_2 \in \mathbb{R}^n \setminus (A \cup B) = (A \cup B)^C = A^C \cap B^C$.

Now let $r_n = \{\frac{1}{k} : k \in \mathbb{N}\}$ be a decreasing sequence of r. For each r_k, select a $y_{r_k} \in B(x, r_k)$. Thus, we now have an infinite sequence $\{y_{r_n}\}_{n \in \mathbb{N}} \in A \cup B$ with $\lim y_{r_n} = x$.

Since each $y_{r_k} \in A \cup B$, y_{r_k} must be in A, B, or both. Let $z_k = 1$ if $y_{r_k} \in A$ and $z_k = 0$ otherwise (ie. if $y_{r_k} \in B \setminus A$). Let $z_n = \{z_k : k \in \mathbb{N}\}$. Since the sequence $\{z_n\}$ is infinite, it is clear that we must have one of the following options: both infinite 1s and 0s, only infinite 1s, and only infinite 0s. Let us assume that we have infinite 1s without loss of generality.

Fix $k_1 \in \mathbb{N}$ and take r_{k_1}. Since there are infinite 1s, we must have an r_{k_2} with $k_2 > k_1$ such that the corresponding $z_{k_2} = 1$. If we did not, then that would mean there are at most k_1 1s in $\{z_n\}$, a contradiction.

Since $r_{k_2} < r_{k_1}$, we have that $B(x, r_{k_2}) \subset B(x, r_{k_1})$. Hence, $y_{r_{k_2}} \in B(x, r_{k_1})$. Note that $y_{r_{k_2}} \in A$ since $z_{k_2} = 1$.

This holds for any arbitrary $k_1 \in \mathbb{N}$. As a result, $\exists y_k \in A$ such that $y_k \in B(x, r_{k_\ell})$ for every $k_\ell \in \mathbb{N}$. Note by the Archimedean Property that if $r \in \mathbb{R}$, we can find an $r_k = \frac{1}{k} < r$. Hence we can use the above formulation for any $r \in \mathbb{R}/$

Thus, we have that $y_1 \in A$ and $y_2 \in A^C \cap B^C \implies y_2 \in A^C$ for any choice of r. As a result, $x \in \delta A$ and hence $x \in \delta A \cup \delta B$. Since x was an arbitrary boundary point, we have that $\delta(A \cup B) \subset (\delta A \cup \delta B)$. A similar argument applies if we assume $y_1 \notin A$.

Lastly, take $A \setminus B = A \cap B^C$. Let $x \in \delta(A \cap B^C)$. Since x is a boundary point of $A \cap B^C$, we have that for every $r > 0$, $\exists y_1, y_2 \in B(x, r)$ such that $y_1 \in A \cap B^C$ and $y_2 \in \mathbb{R}^n \setminus (A \cap B^C) = (A \cap B^C)^C = A^C \cup B$.

We have that $y_1 \in A$ and $y_1 \in B^C$ regardless of the choice of r. In addition, we have that $y_2 \in A^C$ or $y_2 \in B$. Using a similar sequence construction as done previously, we can show that there must be infinitely many $y_2 \in A^C$ or $y_2 \in B$. If we have infinitely many $y_2 \in A^C$, then we have that $y_1 \in A$ and $y_2 \in A^C$ $\forall r > 0$. Hence, $x \in \delta A \implies x \in \delta A \cup \delta B$.

If we have infinitely many $y_2 \in B$, then we have that $y_1 \in B^C$ and $y_2 \in B$ $\forall r > 0$. Hence, $x \in \delta B \implies x \in \delta A \cup \delta B$.

Thus, we have that $\delta(A \setminus B) \subset (\delta A \cup \delta B)$.