Advanced Calculus II: Assignment 1
Chapter 2 - A Taste of Topology

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Problem 12 on p. 126.

(a) The limit of a sequence is unaffected by rearrangement when f is a bijective function. Since f is bijective, each term from (p_n) must be included one and only one time in the new sequence (q_k).

We know that, if $(p_n) \to \ell$, this means that $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $n \geq N \implies \mathrm{d}(p_n, \ell) < \epsilon$.

Thus, for each choice of $\epsilon > 0$ there are only finitely many terms for which $\mathrm{d}(p_n, \ell) \geq \epsilon$, while there are infinitely many terms for which $\mathrm{d}(p_n, \ell) < \epsilon$.

As a result, the rearrangement (q_k) will eventually exhaust all of the terms that have a distance from ℓ that is greater than or equal to ϵ, and thus will have infinitely many terms left for which $\mathrm{d}(q_k, \ell) < \epsilon$.

Hence, $(q_k) \to \ell$ as well.

(b) A rearrangement of (p_n) where f is an injective function does not necessarily preserve the limit of (p_n). For example, take $(p_n) = (-1)^n$. This sequence alternates between 1 and -1, and thus never converges.

Now let $f(n) = 2n$. This injective function maps the natural numbers to the evens. We can see that $\forall m$ where m is even, we have that $p_m = 1$.

Thus, $q_k = p_{f(k)} = 1 \ \forall k \in \mathbb{N}$. Clearly, $(q_k) \to 1$ while (p_n) does not converge.

(c) A rearrangement of (p_n) where f is a surjective function does not necessarily preserve the limit of (p_n). For example, take $(p_n) = \frac{1}{n}$. This sequence converges to 0.

Now let $f(n) = \begin{cases} 1 & \text{n is odd} \\ 2 & \text{n} = 2 \\ f(n-2)+1 & \text{n is even and n} > 2 \end{cases}$

Call this new sequence $q_k = p_{f(k)}$. Then we have that (q_k) contains all terms in the original sequence (p_n) and, for every odd term m, $q_m = 1$. Thus, the even terms of (q_k) converge to 0 while the odd terms converge to 1. Since we have two subsequences in (q_k) that converge to different limits, (q_k) does not converge.

Problem 44 on p. 128.

(a) Since f is continuous, then $\forall (p_n) \in M$ with a limit $p \in M$, we have $(p_n) \to p \implies f(p_n) \to f(p)$.

Thus, for the graph of f, we have that $(p_n, f(p_n)) \to (p, f(p)) \in M \times f(M)$ for every sequence $(p_n, f(p_n)) \in M \times \mathbb{R}$.

Hence, the continuity of f implies that its graph is closed.

(b) Suppose M is compact and f is continuous. By Theorem 38, the continuous image of a compact set is compact. Thus, $f(M)$ is a compact subset of $\mathbb{R}$.

Hence, the graph of f is the Cartesian product of two compact sets, M and $f(M)$. By Corollary 29, the graph of f is compact as a result.

(c) Assume the graph of f is compact. Since the graph is compact, the graph must be bounded. By Bolzano-Weierstrass, any sequence in the graph must have a convergent subsequence. Since compact also implies closed, the limit of this subsequence must reside in the graph.

Let $(p_n, f(p_n))$ be one such sequence in the graph. Thus, $\exists (p_{n_k})$ such that $(p_{n_k}, f(p_{n_k}))$ converges to a limit point.

The limit point must be a limit point in each coordinate, so we have $(p_{n_k}, f(p_{n_k})) \to (p, f(p))$. Thus, f preserves sequential limits and, as a result, f is continuous.

(d) Counterexample:
$f : [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

f is discontinuous because if you have a sequence $(x_n) \to 0$, $f(x_n) \to \infty$ while $f(0) = 0$. Thus, $f(x_n) \not\to f(0)$. Hence, f does not preserve sequential limits.

Problem 76 on p. 131.

(a) Let $A = \{(x, y) : x^2 + y^2 = 1\}$, i.e. the unit circle.

Let $B = \{(x, 0) : x \in \mathbb{R}\}$, i.e. a straight line along the x-axis.

Both sets are connected. Their intersection is $A \cap B = \{(-1, 0), (1, 0)\}$. This set is disconnected in $\mathbb{R}^2$.

(b) Let $S_n = \{x : x \geq n\}$. Then each S_n is connected and closed, and we have $S_1 \supset S_2 \supset \cdots$.

However, we have $\cap S_n = \emptyset$.

(c) Intersection is connected. Prove this.

Assume that $S_1, S_2, ...$ are a sequence of connected, compact sets with $S_1 \supset S_2 \supset \cdots$.

By Theorem 34, their intersection $\cap S_n$ is compact and non-empty.

(d) Connected but not path connected. Prove this.

Problem 1 on p. 147.

We know that the intersection of a nested sequence of nonempty compact sets is a nonempty compact set. Thus, $\cap K_n$ is nonempty and compact. Hence, for every sequence $(a_n) \in \cap K_n$, there exists a subsequence (a_{n_k}) that converges to some limit a in $\cap K_n$.

By properties of intersections, every term in (a_{n_k}) is in K_n for every n. Furthermore, $a \in K_n$ for every n.

Now, by property (i), $f(K_n)$ is compact as well for every n. Thus, $\cap f(K_n)$ is compact.

We know that each term in (a_{n_k}) is in K_n for every n. As a result, each $f(a_{n_k})$ is in $f(K_n)$ for every n. Thus, the image of the sequence (a_{n_k}) is contained in $\cap f(K_n)$. In addition, $f(a) \in \cap f(K_n)$.

Since $\cap f(K_n)$ is compact,