Advanced Calculus II: Assignment 8

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Problem 1.

Let $F(x, y) = x^2 + y^2 - 1$. Consider the equation $F(x, y) = x^2 + y^2 - 1 = 0$.

In order to prove that the above equation can be solved for small values of x by a positive function $y = y(x)$, we need to show that F belongs to class $C^1(\mathbb{R}^2)$, that $F(a, b) = 0$, and that the linear map defined by $L_2(v) = DF(a, b)(0, v)$, $v \in \mathbb{R}$ is a bijection of $\mathbb{R}$ onto $\mathbb{R}$ (by Implicit Function Theorem).

First let $(a, b) = (0, 1)$. Observe that $F(a, b) = F(0, 1) = 0^2 + 1^2 - 1 = 0$ as required.

Now by Theorem 41.2, in order to show that F belongs to class $C^1(\mathbb{R}^2)$ (ie. the derivative exists and is continuous), it suffices to show that both partial derivatives are continuous on $\mathbb{R}^2$. Hence, we have,

$$\begin{aligned} D_x F &= 2x \\ D_y F &= 2y \end{aligned}$$

It is clear that both partial derivatives are continuous on all of $\mathbb{R}^2$ since both are comprised of a multiplication of two trivially continuous functions.

Finally, we need to show that the linear map defined by $L_2(v) = DF(0, 1)(0, v)$, $v \in \mathbb{R}$ is a bijection of $\mathbb{R}$ onto $\mathbb{R}$.

Note from Corollary 39.7 in the text that,

$$\begin{aligned} DF(0, 1)(0, v) &= (0)D_x F(0, 1) + (v)D_y F(0, 1) \\ &= v(2 \cdot 1) \\ &= 2v \end{aligned}$$

It is clear that this function is defined on all of $\mathbb{R}$.

Now suppose that $\exists v_1, v_2 \in \mathbb{R}$ such that $L_2(v_1) = L_2(v_2)$. Then we have that,

$$\begin{aligned} 2v_1 &= 2v_2 \\ \implies v_1 &= v_2 \end{aligned}$$

Hence L_2 is injective.

Now let $u \in \mathbb{R}$. Observe that, since $\mathbb{R}$ is closed under multiplication and $\frac{1}{2} \in \mathbb{R}$, we have that $\frac{u}{2} \in \mathbb{R}$. Hence, for any $u \in \mathbb{R}, \exists \frac{u}{2}$ such that,

$$L_2\left(\frac{u}{2}\right) = 2\left(\frac{u}{2}\right) = u$$

Thus, L_2 is surjective.

Since L_2 is both injective and surjective, it is a bijective function from $\mathbb{R} \to \mathbb{R}$.

Since $F(x, y)$ satisfies all of the above properites, we can apply the Implicit Function Theorem. Hence, there exists an open neighborhood W of $0 \in \mathbb{R}$ and a unique function $y(x) : W \to \mathbb{R}$ belonging to class $C^1(W)$ such that $y = y(x)$ and,

$$F(x, y(x)) = 0 \;\; \forall x \in W$$

Now we need to show that $y'(x) = \frac{-x}{y}$.

Problem 2.

As above, we need to show we need that F belongs to class $C^1(\mathbb{R}^5)$, that $F(a, b) = 0$, and that the linear map defined by $L_2(v) = DF(a, b)(0, v)$, $v \in \mathbb{R}^2$ is a bijection of $\mathbb{R}^2$ onto $\mathbb{R}^2$ (by Implicit Function Theorem).

Observe that $(a, b) = (2, 1, 0, -1, 0)$ and we are given that $F(2, 1, 0, -1, 0) = (0, 0)$. Hence this condition is already satisfied.

Next, we need to show that F belongs to class C^1. First we will define the derivative of F,

$$DF(u, v, w, x, y) = \begin{bmatrix} y & v & 1 & v + 2x & u \\ vw & uw & uv & 1 & 1 \end{bmatrix}$$

Recall that each partial derivative of F is an entry in the above matrix and that the derivative of F is continuous iff every partial derivative is continuous.

From the above, it is clear that all of the partial derivatives are continuous on all of $\mathbb{R}^5$ as they are linear combinations of continuous functions. Hence, $DF(u, v, w, x, y)$ is continuous on all of $\mathbb{R}^5$ and F belongs to class $C^1(\mathbb{R}^5)$

Lastly, we need to show that the linear map defined by $L_2(v) = DF(a, b)(0, v)$, $v \in \mathbb{R}^2$ is a bijection of $\mathbb{R}^2$ onto $\mathbb{R}^2$ where $a = (2, 1, 0)$ and $b = (-1, 0)$.

Hence, we have,

$$L_2(v) = \begin{bmatrix} 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 2 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ v_1 \\ v_2 \end{bmatrix}$$
$$= \begin{bmatrix} -v_1 + 2v_2 \\ v_1 + v_2 \end{bmatrix}$$

Now suppose $L_2(v) = L_2(w)$ for $v, w \in \mathbb{R}^2$ with $v \neq w$. Then we have,

$$-v_1 + 2v_2 = -w_1 + 2w_2$$

and

$$v_1 + v_2 = w_1 + w_2$$

Subtracting 2 times the second equation from the first equation yields,

$$v_1 = w_1$$

Plugging this identity in yields,

$$v_2 = w_2$$

Hence $v = w$ and L_2 is injective.

Now suppose $w \in \mathbb{R}^2$. We will now consider the equation,

$$L_2(v) = w$$

for some $v \in \mathbb{R}^2$, which is equivalent to,

$$\begin{bmatrix} -v_1 + 2v_2 \\ v_1 + v_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

This system of equations yields $v_1 = \frac{1}{3}(2w_2 - w_1)$ and $v_2 = \frac{w_1 + w_2}{3}$. Note that $w_1, w_2 \in \mathbb{R}$ and $\mathbb{R}$ is closed under addition and multiplication. Hence, $v_1, v_2 \in \mathbb{R}$ and, by extension, $v \in \mathbb{R}^2$.

Thus, for any $w \in \mathbb{R}^2$, $\exists v \in \mathbb{R}^2$ such that $L_2(v) = w$. As a result, L_2 is a bijection from $\mathbb{R}^2 \to \mathbb{R}^2$.

Thus, F satisfies all of the necessary properties. As a result, we can apply the Implicit Function Theorem. Hence, there exists an open neighborhood W of $(2, 1, 0) \in \mathbb{R}^3$ and a unique function $\varphi(x) : W \to \mathbb{R}^2$ belonging to class $C^1(W)$ such that $(x, y) = \varphi(u, v, w)$ and,

$$F(u, v, w, \varphi(u, v, w)) = 0 \;\; \forall (u, v, w) \in W$$

Now we need to compute $D\varphi(2, 1, 0)$.

Problem 3.

Problem 4.

Problem 5.