Advanced Mathematical Statistics: Assignment 2

Chris Hayduk

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Problem 4.2.

a) $f(x_1, x_2) = \frac{1}{2\pi\sqrt{1.5}} \times \exp\{-\frac{2}{3}[\left(\frac{x_1}{\sqrt{2}}\right)^2 + (x_2-2)^2 - \left(\frac{x_1}{\sqrt{2}}\right)(x_2-2)]\}$

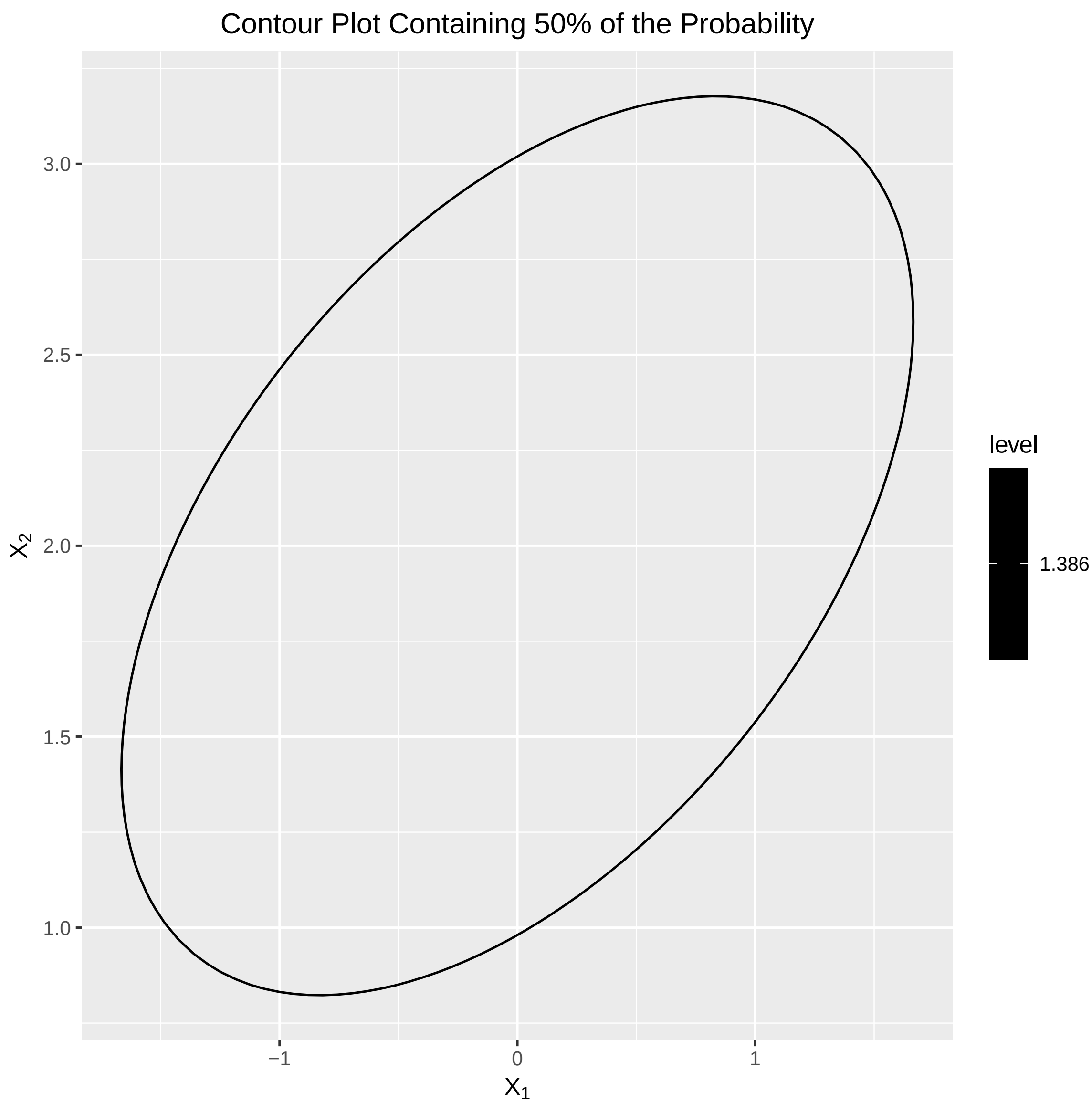
b) We have that $\rho_{12} = 0.5 \implies \sigma_{12} = 0.5(\sqrt{\sigma_{11}\sigma_{22}}) = 0.5(\sqrt{2})$. So,

$$
\begin{aligned}
&(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu) \\
&= \begin{bmatrix} x_1-\mu_1 & x_2-\mu_2 \end{bmatrix} \frac{1}{\sigma_{11}\sigma_{22}-\sigma_{12}^2} \begin{bmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{bmatrix} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix} \\
&= \begin{bmatrix} x_1 & x_2-2 \end{bmatrix} \frac{1}{2-(0.5\sqrt{2})^2} \begin{bmatrix} 1 & -0.5\sqrt{2} \\ -0.5\sqrt{2} & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2-2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{3}x_1 + (\frac{-\sqrt{2}}{3})(x_2-2) & (\frac{-\sqrt{2}}{3})(x_1) + \frac{4}{3}(x_2-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2-2 \end{bmatrix} \\
&= x_1\left(\frac{2}{3}x_1 + \left(\frac{-\sqrt{2}}{3}\right)(x_2-2)\right) + (x_2-2)\left(\left(\frac{-\sqrt{2}}{3}\right)x_1 + \frac{4}{3}(x_2-2)\right) \\
&= \frac{2}{3}x_1^2 + \left(\frac{-\sqrt{2}}{3}\right)(x_2-2)x_1 + (x_2-2)\left(\frac{-\sqrt{2}}{3}\right)x_1 + \frac{4}{3}(x_2-2)^2 \\
&= \frac{2}{3}x_1^2 + \frac{-2\sqrt{2}(x_2x_1-2x_1)}{3} + \frac{4}{3}x_2^2 - \frac{16}{3}x_2 + \frac{16}{3}
\end{aligned}
$$

c) $c^2 = \chi_2^2(0.5) \approx 1.386294$. So we take

$$\frac{2}{3}x_1^2 + \frac{-2\sqrt{2}(x_2x_1-2x_1)}{3} + \frac{4}{3}x_2^2 - \frac{16}{3}x_2 + \frac{16}{3} = 1.386294$$

to be the surface of the ellipsoid containing 50% of the probability. The graph for this can be seen below.



Problem 4.3.

a) X_1 and X_2 are not independent because $\sigma_{12} = \sigma_{21} = -2 \neq 0$.

b) X_2 and X_3 are independent because $\sigma_{23} = \sigma_{32} = 0$.

c) If we partition the covariance matrix into (X_1, X_2) and X_3 partitions, we get

$$\left(\begin{array}{cc|c} 1 & -2 & 0 \\ -2 & 5 & 0 \\ \hline 0 & 0 & 2 \end{array}\right)$$

Thus, we can see that the two diagonal sections of the matrix have the forms $\mathbf{0}, \mathbf{0}'$. As a result, (X_1, X_2) and X_3 are independent.

d) Let $\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. From Result 4.3, we know $\mathbf{AX}$ is distributed as $N_q(\mathbf{A}\mu, \mathbf{A\Sigma A}')$ with $q = 2$ in this case.

So we have,

$$\begin{aligned}
\mathbf{A\Sigma A}' &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}
\end{aligned}$$

As can be clearly seen from the above matrix, $\mathbf{A\Sigma A}'$, the covariance between $\frac{X_1+X_2}{2}$ and X_3 is 0. As a result, $\frac{X_1+X_2}{2}$ and X_3 are independent.

e) Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{5}{2} & 1 & -1 \end{bmatrix}$. From Result 4.3, we know $\mathbf{AX}$ is distributed as $N_q(\mathbf{A}\mu, \mathbf{A\Sigma A}')$ with $q = 2$ in this case.

So we have,

$$\begin{aligned}
\mathbf{A\Sigma A}' &= \begin{bmatrix} 0 & 1 & 0 \\ -\frac{5}{2} & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -\frac{5}{2} \\ 1 & 1 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} -2 & 5 & 0 \\ -\frac{9}{2} & 10 & -2 \end{bmatrix} \begin{bmatrix} 0 & -\frac{5}{2} \\ 1 & 1 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 10 \\ 10 & \frac{93}{4} \end{bmatrix}
\end{aligned}$$

We can see from the above matrix that the covariance between the random variables is not 0. Thus, X_2 and $X_2 - \frac{5}{2}X_1 - X_3$ are not independent.

Problem 4.4.

a) Let $A = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}$. By Result 4.3, $3X_1 - 2X_2 + X_3$ is distributed as $N_1(\mathbf{A}\mu, \mathbf{A\Sigma A}')$ with mean vector and covariance matrix,

$$\begin{aligned}
\mathbf{A}\mu &= 6 + 6 + 1 = 13 \\
\mathbf{A\Sigma A}' &= \begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \mathbf{A}' = 6 + 2 + 1 = 9
\end{aligned}$$

b) Let $A = \begin{bmatrix} 0 & 1 & 0 \\ -a_1 & 1 & -a_2 \end{bmatrix}$.

Now find $\mathbf{A\Sigma A'}$:

$$\mathbf{A\Sigma A'} = \begin{bmatrix} 1 & 3 & 2 \\ -a_1+1-a_2 & -a_1+3-2a_2 & a_1+2-2a_2 \end{bmatrix} \begin{bmatrix} 0 & -a_1 \\ 1 & 1 \\ 0 & -a_2 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & -a_1+3-2a_2 \\ -a_1+3-2a_2 & (-a_1)^2-2a_1-2a_1a_2+3-4a_2+2(-a_2)^2 \end{bmatrix}$$

In order for the covariance to be 0 (ie. X_2 and $-a_1X_1+X_2-a_3X_3$ independent), we need $-a_1+3-2a_2=0$. That is, $a_1+2a_2=3$. For instance, take $a_1=1$ and $a_2=1$. Then we have,

$$\mathbf{A\Sigma A'} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

Thus, if $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then we have that X_2 and $X_2 - \mathbf{a}' \begin{bmatrix} X_1 \\ X_3 \end{bmatrix}$ are independent.

Problem 4.5.

a) By Result 4.6, we have

$$\begin{aligned} \mu &= \mu_1 + \sigma_{12}(\sigma_{22})^{-1}(x_2-\mu_2) \\ &= 0 + 0.5(\sqrt{2})(1)^{-1}(x_2-2) \\ &= 0.5(\sqrt{2})x_2 - \sqrt{2} \end{aligned}$$

In addition, we have

$$\begin{aligned} \sigma &= \sigma_{11} - \sigma_{12}(\sigma_{22})^{-1}\sigma_{21} \\ &= 2 - 0.5(\sqrt{2})(1)^{-1}0.5(\sqrt{2}) \\ &= 2 - 0.25(2) = 1.5 \end{aligned}$$

b)

c) If we partition the covariance matrix into (X_1, X_2) and X_3 partitions, we get

$$\begin{aligned} \mathbf{\Sigma} &= \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 3 & 2 \\ \hline 1 & 2 & 2 \end{array}\right) \\ &= \left(\begin{array}{c|c} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \hline \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{array}\right) \end{aligned}$$

and,

$$\mu = \begin{pmatrix} 2 \\ -3 \\ \hline 1 \end{pmatrix}$$
$$= \begin{pmatrix} \mu_1 \\ \hline \mu_2 \end{pmatrix}$$

So we have,

$$\begin{aligned} \mu = &= \mu_1 + \mathbf{\Sigma}_{12}(\mathbf{\Sigma}_{22})^{-1}(x_2 - \mu_2) \\ &= \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \end{aligned}$$

Problem 4.6.

Problem 4.7.

Problem 4.10.

Problem 4.11.

Problem 4.12.

Problem 4.13.

Problem 4.14.

Problem 4.15.

Problem 4.16.

Problem 4.17.

Problem 4.18.

Problem 4.19.

Problem 4.20.

Problem 4.21.