Advanced Mathematical Statistics: Assignment 2

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Problem 4.2.

a) $f(x_1, x_2) = \frac{1}{2\pi\sqrt{1.5}} \times \exp\{-\frac{2}{3}[\left(\frac{x_1}{\sqrt{2}}\right)^2 + (x_2-2)^2 - \left(\frac{x_1}{\sqrt{2}}\right)(x_2-2)]\}$

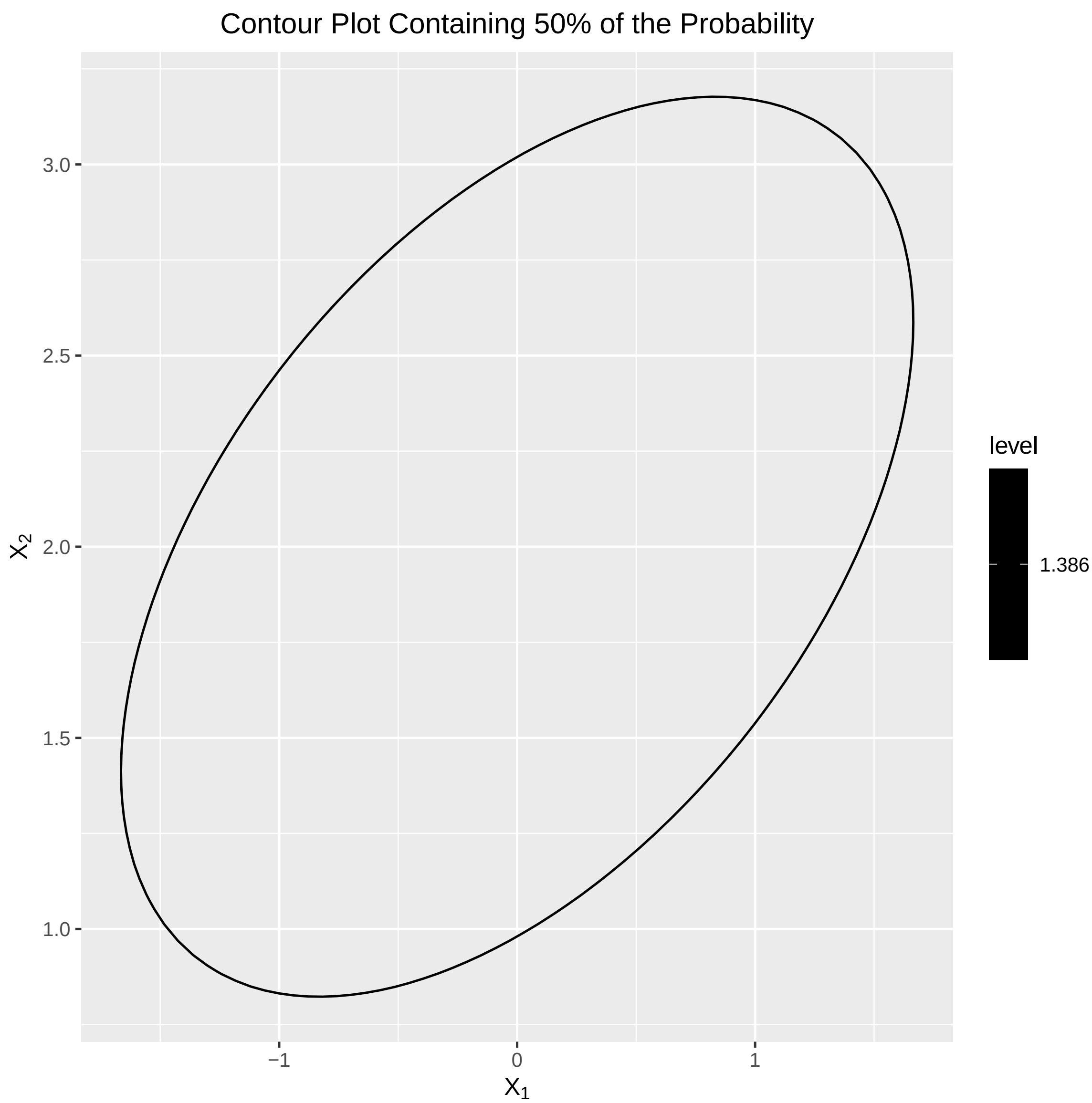
b) We have that $\rho_{12} = 0.5 \implies \sigma_{12} = 0.5(\sqrt{\sigma_{11}\sigma_{22}}) = 0.5(\sqrt{2})$. So,

$$
\begin{aligned}
&(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu) \\
&= \begin{bmatrix} x_1-\mu_1 & x_2-\mu_2 \end{bmatrix} \frac{1}{\sigma_{11}\sigma_{22}-\sigma_{12}^2} \begin{bmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{bmatrix} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix} \\
&= \begin{bmatrix} x_1 & x_2-2 \end{bmatrix} \frac{1}{2-(0.5\sqrt{2})^2} \begin{bmatrix} 1 & -0.5\sqrt{2} \\ -0.5\sqrt{2} & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2-2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{3}x_1 + (\frac{-\sqrt{2}}{3})(x_2-2) & (\frac{-\sqrt{2}}{3})(x_1) + \frac{4}{3}(x_2-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2-2 \end{bmatrix} \\
&= x_1\left(\frac{2}{3}x_1 + \left(\frac{-\sqrt{2}}{3}\right)(x_2-2)\right) + (x_2-2)\left(\left(\frac{-\sqrt{2}}{3}\right)x_1 + \frac{4}{3}(x_2-2)\right) \\
&= \frac{2}{3}x_1^2 + \left(\frac{-\sqrt{2}}{3}\right)(x_2-2)x_1 + (x_2-2)\left(\frac{-\sqrt{2}}{3}\right)x_1 + \frac{4}{3}(x_2-2)^2 \\
&= \frac{2}{3}x_1^2 + \frac{-2\sqrt{2}(x_2x_1-2x_1)}{3} + \frac{4}{3}x_2^2 - \frac{16}{3}x_2 + \frac{16}{3}
\end{aligned}
$$

c) $c^2 = \chi_2^2(0.5) \approx 1.386294$. So we take

$$\frac{2}{3}x_1^2 + \frac{-2\sqrt{2}(x_2x_1-2x_1)}{3} + \frac{4}{3}x_2^2 - \frac{16}{3}x_2 + \frac{16}{3} = 1.386294$$

to be the surface of the ellipsoid containing 50% of the probability. The graph for this can be seen below.



Problem 4.3.

a) X_1 and X_2 are not independent because $\sigma_{12} = \sigma_{21} = -2 \neq 0$.

b) X_2 and X_3 are independent because $\sigma_{23} = \sigma_{32} = 0$.

c) If we partition the covariance matrix into (X_1, X_2) and X_3 partitions, we get

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ -2 & 5 & 0 \\ \hline 0 & 0 & 2 \end{array}\right]$$

Thus, we can see that the two diagonal sections of the matrix have the forms $\mathbf{0}, \mathbf{0}'$. As a result, (X_1, X_2) and X_3 are independent.

d) Let $\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. From Result 4.3, we know $\mathbf{AX}$ is distributed as $N_q(\mathbf{A}\mu, \mathbf{A\Sigma A'})$ with $q = 2$ in this case.

So we have,

$$\begin{aligned}
\mathbf{A\Sigma A'} &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}
\end{aligned}$$

As can be clearly seen from the above matrix, $\mathbf{A\Sigma A'}$, the covariance between $\frac{X_1+X_2}{2}$ and X_3 is 0. As a result, $\frac{X_1+X_2}{2}$ and X_3 are independent.

e) Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{5}{2} & 1 & -1 \end{bmatrix}$. From Result 4.3, we know $\mathbf{AX}$ is distributed as $N_q(\mathbf{A}\mu, \mathbf{A\Sigma A'})$ with $q = 2$ in this case.

So we have,

$$\begin{aligned}
\mathbf{A\Sigma A'} &= \begin{bmatrix} 0 & 1 & 0 \\ -\frac{5}{2} & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -\frac{5}{2} \\ 1 & 1 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} -2 & 5 & 0 \\ -\frac{9}{2} & 10 & -2 \end{bmatrix} \begin{bmatrix} 0 & -\frac{5}{2} \\ 1 & 1 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 10 \\ 10 & \frac{93}{4} \end{bmatrix}
\end{aligned}$$

We can see from the above matrix that the covariance between the random variables is not 0. Thus, X_2 and $X_2 - \frac{5}{2}X_1 - X_3$ are not independent.

Problem 4.4.

a) Let $A = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}$. By Result 4.3, $3X_1 - 2X_2 + X_3$ is distributed as $N_1(\mathbf{A}\mu, \mathbf{A\Sigma A'})$ with mean vector and covariance matrix,

$$\begin{aligned}
\mathbf{A}\mu &= 6 + 6 + 1 = 13 \\
\mathbf{A\Sigma A'} &= \begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \mathbf{A'} = 6 + 2 + 1 = 9
\end{aligned}$$

b) Let $A = \begin{bmatrix} 0 & 1 & 0 \\ -a_1 & 1 & -a_2 \end{bmatrix}$.

Now find $\mathbf{A\Sigma A'}$:

$$\mathbf{A\Sigma A'} = \begin{bmatrix} 1 & 3 & 2 \\ -a_1 + 1 - a_2 & -a_1 + 3 - 2a_2 & a_1 + 2 - 2a_2 \end{bmatrix} \begin{bmatrix} 0 & -a_1 \\ 1 & 1 \\ 0 & -a_2 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & -a_1 + 3 - 2a_2 \\ -a_1 + 3 - 2a_2 & (-a_1)^2 - 2a_1 - 2a_1a_2 + 3 - 4a_2 + 2(-a_2)^2 \end{bmatrix}$$

In order for the covariance to be 0 (ie. X_2 and $-a_1X_1 + X_2 - a_3X_3$ independent), we need $-a_1 + 3 - 2a_2 = 0$. That is, $a_1 + 2a_2 = 3$. For instance, take $a_1 = 1$ and $a_2 = 1$. Then we have,

$$\mathbf{A\Sigma A'} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

Thus, if $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then we have that X_2 and $X_2 - \mathbf{a}' \begin{bmatrix} X_1 \\ X_3 \end{bmatrix}$ are independent.

Problem 4.5.

a) By Result 4.6, we have

$$\begin{aligned} \mu &= \mu_1 + \sigma_{12}(\sigma_{22})^{-1}(x_2 - \mu_2) \\ &= 0 + 0.5(\sqrt{2})(1)^{-1}(x_2 - 2) \\ &= 0.5(\sqrt{2})x_2 - \sqrt{2} \end{aligned}$$

In addition, we have

$$\begin{aligned} \sigma &= \sigma_{11} - \sigma_{12}(\sigma_{22})^{-1}\sigma_{21} \\ &= 2 - 0.5(\sqrt{2})(1)^{-1}0.5(\sqrt{2}) \\ &= 2 - 0.25(2) = 1.5 \end{aligned}$$

b) Let's rearrange the $\mathbf{\Sigma}$ matrix,

$$\mathbf{\Sigma} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 5 \end{bmatrix}$$
$$= \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 2 & 0 \\ \hline -2 & 0 & 5 \end{array}\right]$$

and the μ vector,

$$
\begin{aligned}
\mu &= \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} \\
&= \left[\begin{array}{c} -3 \\ 4 \\ \hline 1 \end{array}\right]
\end{aligned}
$$

Then we have,

$$
\begin{aligned}
\mu &= \mu_1 + \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}(x_2 - \mu_2) \\
&= \begin{bmatrix} -3 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}(\frac{1}{5})(x_2 - 1) \\
&= \begin{bmatrix} -3 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}(\frac{1}{5}x_2 - \frac{1}{5})
\end{aligned}
$$

and,

$$
\begin{aligned}
\mathbf{\Sigma} &= \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \end{bmatrix}\left(\frac{1}{5}\right)\begin{bmatrix} -2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \end{bmatrix}\begin{bmatrix} -\frac{2}{5} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} \frac{4}{5} & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & 2 \end{bmatrix}
\end{aligned}
$$

c) If we partition the covariance matrix into (X_1, X_2) and X_3 partitions, we get

$$
\begin{aligned}
\mathbf{\Sigma} &= \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 3 & 2 \\ \hline 1 & 2 & 2 \end{array}\right] \\
&= \left[\begin{array}{c|c} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \hline \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{array}\right]
\end{aligned}
$$

and,

$$
\begin{aligned}
\mu &= \left[\begin{array}{c} 2 \\ -3 \\ \hline 1 \end{array}\right] \\
&= \left[\begin{array}{c} \mu_1 \\ \hline \mu_2 \end{array}\right]
\end{aligned}
$$

So we have,

$$\begin{aligned}
\mu &= \mu_2 + \mathbf{\Sigma}_{21}(\mathbf{\Sigma}_{11})^{-1}(x_1 - \mu_1) \\
&= 1 + \begin{bmatrix} 1 & 2 \end{bmatrix} \left(\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \right) (x_1 - \begin{bmatrix} 2 \\ -3 \end{bmatrix}) \\
&= 1 + \left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right) (x_1 - \begin{bmatrix} 2 \\ -3 \end{bmatrix}) \\
&= 1 + \left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right) \left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} - \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right) \\
&= 1 + \left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right) \left(\begin{bmatrix} X_1 - 2 \\ X_2 + 3 \end{bmatrix} \right) \\
&= 1 + \frac{1}{2} (X_1 - 2 + X_2 + 3) \\
&= \frac{1}{2} X_1 + \frac{1}{2} X_2 + \frac{3}{2}
\end{aligned}$$

and,

$$\begin{aligned}
\mathbf{\Sigma} &= \mathbf{\Sigma}_{22} - \mathbf{\Sigma}_{21} \left(\mathbf{\Sigma}_{11} \right)^{-1} \mathbf{\Sigma}_{12} \\
&= 2 - \begin{bmatrix} 1 & 2 \end{bmatrix} \left(\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \right) \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= 2 - \left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right) \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= 2 - \frac{3}{2} = \frac{1}{2}
\end{aligned}$$

Problem 4.6.

a) We can see that $\sigma_{12} = \sigma_{21} = 0$. Thus, X_1 and X_2 are independent.

b) We have that $\sigma_{13} = \sigma_{31} = -1 \neq 0$. Thus, X_1 and X_3 are not independent.

c) We have that $\sigma_{23} = \sigma_{32} = 0$. Thus, X_2 and X_3 are independent.

d) Let's rearrange the covariance matrix,

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

We can partition it now as well,

$$\left[\begin{array}{cc|c} 4 & -1 & 0 \\ -1 & 2 & 0 \\ \hline 0 & 0 & 2 \end{array}\right]$$

So now we have the covariance atrix partitioned into (X_1, X_3) and X_2 blocks, along with their covariances. It is clear from this partitioning that (X_1, X_3) and X_2 have zero covariance. Thus, they are independent.

e) Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & -2 \end{bmatrix}$. From Result 4.3, we know $\mathbf{AX}$ is distributed as $N_q(\mathbf{A}\mu, \mathbf{A\Sigma A}')$ with $q = 2$ in this case.

So we have,

$$\begin{aligned}\mathbf{A\Sigma A}' &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 4 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 0 & -1 \\ 6 & 6 & -5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 6 \\ 6 & 34 \end{bmatrix}\end{aligned}$$

We can see from the above matrix that the covariance between the random variables is not 0. Thus, X_1 and $X_1 + 3X_2 - 2X_3$ are not independent.

Problem 4.7.

a) Let's define the covariance matrix,

$$\mathbf{\Sigma} = \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}$$

By Result 4.6, we have

$$\begin{aligned}\mu &= \mu_1 + \sigma_{12}(\sigma_{22})^{-1}(x_2 - \mu_2) \\ &= 1 + -1\left(\frac{1}{2}\right)(x_2 - 2) \\ &= 1 - \frac{1}{2}x_2 + 1 = \frac{1}{2} + 2\end{aligned}$$

and,

$$\begin{aligned}\sigma &= \sigma_{11} - \sigma_{12}(\sigma_{22})^{-1}\sigma_{21} \\ &= 4 - (-1)\left(\frac{1}{2}\right)(-1) \\ &= 4 - \frac{1}{2} = \frac{7}{2}\end{aligned}$$

b) Let's partition the $\boldsymbol{\Sigma}$ matrix,

$$\left[\begin{array}{cc|c} 4 & 0 & -1 \\ \hline 0 & 5 & 0 \\ -1 & 0 & 2 \end{array}\right] \tag{1}$$

and the μ vector,

$$\left[\begin{array}{c} 1 \\ \hline -1 \\ 2 \end{array}\right]$$

Then we have,

$$\begin{aligned} \mu &= \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{x}_2 - \mu_2) \\ &= 1 + -1 \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \left(\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right) \\ &= 1 + -1 \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \left(\begin{bmatrix} x_2 + 1 \\ x_3 - 2 \end{bmatrix} \right) \end{aligned}$$

and,

$$\begin{aligned} \boldsymbol{\Sigma} &= \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \\ &= \begin{bmatrix} 4 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 5 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -\frac{5}{2} \end{bmatrix} \\ &= \begin{bmatrix} 4 & \frac{5}{2} \end{bmatrix} \end{aligned}$$

Problem 4.10.

Problem 4.11.

Problem 4.12.

Problem 4.13.

Problem 4.14.

Problem 4.15.

Problem 4.16.

Problem 4.17.

Problem 4.18.

Problem 4.19.

Problem 4.20.

Problem 4.21.