Advanced Mathematical Statistics: Assignment 4

Chris Hayduk

October 17, 2019

Problem 5.1.

a) We have,

$$\begin{aligned}\overline{\mathbf{x}} &= \begin{bmatrix}\overline{x_1}\\ \overline{x_2}\end{bmatrix}\\ &= \begin{bmatrix}\frac{2+8+6+8}{4}\\ \frac{12+9+9+10}{4}\end{bmatrix}\\ &= \begin{bmatrix}6\\ 10\end{bmatrix}\end{aligned}$$

and,

$$\begin{aligned}s_{11} &= \frac{(2-6)^2+(8-6)^2+(6-6)^2+(8-6)^2}{3} = 8\\ s_{12} &= \frac{(2-6)(12-10)+(8-6)(9-10)+(6-6)(9-10)+(8-6)(10-10)}{3} = -\frac{10}{3} = s_{21}\\ s_{22} &= \frac{(12-10)^2+(9-10)^2+(9-10)^2+(10-10)^2}{3} = 2\end{aligned}$$

Thus,

$$\mathbf{S} = \begin{bmatrix}8 & -\frac{10}{3}\\ -\frac{10}{3} & 2\end{bmatrix}$$

and,

$$\begin{aligned}\mathbf{S}^{-1} &= \frac{1}{(8)(2)-(-\frac{10}{3})(-\frac{10}{3})}\begin{bmatrix}2 & \frac{10}{3}\\ \frac{10}{3} & 8\end{bmatrix}\\ &= \begin{bmatrix}\frac{9}{22} & \frac{15}{22}\\ \frac{15}{22} & \frac{18}{11}\end{bmatrix}\end{aligned}$$

Finally, by (5-4), we have,

$$\begin{aligned}
T^2 &= 4\begin{bmatrix}6-7 & 10-11\end{bmatrix}\begin{bmatrix}\frac{9}{22} & \frac{15}{22}\\ \frac{15}{22} & \frac{18}{11}\end{bmatrix}\begin{bmatrix}6-7\\ 10-11\end{bmatrix}\\
&= 4\begin{bmatrix}-1 & -1\end{bmatrix}\begin{bmatrix}\frac{9}{22} & \frac{15}{22}\\ \frac{15}{22} & \frac{18}{11}\end{bmatrix}\begin{bmatrix}-1\\ -1\end{bmatrix}\\
&\approx 13.63636
\end{aligned}$$

b) T^2 has the distribution of a

$$\frac{(4-1)2}{(4-2)}F_{2,4-2} = 3F_{2,2}$$

random variable.

c) Let $\alpha = 0.05$. Thus,

$$3F_{2,2}(0.05) = 3(19) = 57$$

We see that

$$T^2 \approx 13.63636 < 57$$

Thus, we cannot reject H_0. That is, we cannot reject the hypothesis that $\mu = \begin{bmatrix}7\\ 11\end{bmatrix}$.

Problem 5.5.

We have,

$$\begin{aligned}
T^2 &= n(\overline{\mathbf{X}} - \mu_0)'\mathbf{S}^{-1}(\overline{\mathbf{X}} - \mu_0)\\
&= 42\left(\begin{bmatrix}0.564-0.55 & 0.603-0.6\end{bmatrix}\right)\begin{bmatrix}203.018 & -163.391\\ & -163.391 & 200.228\end{bmatrix}\left(\begin{bmatrix}0.564-0.55\\ 0.603-0.6\end{bmatrix}\right)\\
&\approx 1.170487
\end{aligned}$$

In addition, we have that T^2 is distributed as $\frac{82}{40}F_{2,40}$. Thus, with $\alpha = 0.05$, we have,

$$\frac{82}{40}F_{2,40}(0.05) \approx \frac{82}{40}(3.231727) = 6.62504$$

We have that $T^2 \approx 1.170487 < 6.62504$, so we cannot reject H_0. That is, we cannot reject the hypothesis that $\mu = \begin{bmatrix}0.55\\ 0.60\end{bmatrix}$.

This result is consistent with the 95% confidence ellipse for μ pcitured in Figure 5.1 because the point $(\mu_1, \mu_2) = (0.55, 0.60)$ lies within the confidence ellipse.

Problem 5.10(c).

$$\mathbf{X} = \begin{bmatrix} 157-141 & 183-168 \\ 168-140 & 170-174 \\ 162-145 & 177-172 \\ 159-146 & 171-176 \\ 158-150 & 175-168 \\ 140-142 & 189-178 \\ 171-139 & 175-176 \end{bmatrix}$$
$$= \begin{bmatrix} 16 & 15 \\ 28 & -4 \\ 17 & 5 \\ 13 & -5 \\ 8 & 7 \\ -2 & 11 \\ 32 & -1 \end{bmatrix}$$

We need to find a $100(1-0.05)\%$ confidence region for μ. Thus, $\alpha = 0.05$.
Now, we have,

$$\overline{\mathbf{x}} = \begin{bmatrix} 16 \\ 4 \end{bmatrix}$$

and,

$$\mathbf{S} = \begin{bmatrix} 133 & -49.66667 \\ -49.66667 & 58.33333 \end{bmatrix}$$

Thus,

$$\mathbf{S}^{-1} = \begin{bmatrix} 0.011023854 & 0.009386024 \\ 0.009386024 & 0.025134386 \end{bmatrix}$$

As a result, we have,

$$\begin{aligned} T^2 &= n(\overline{\mathbf{x}}-\mu)'S^{-1}(\overline{\mathbf{x}}-\mu) \\ &= 7\begin{bmatrix} 16-\mu_1 & 4-\mu_2 \end{bmatrix} \begin{bmatrix} 0.011023854 & 0.009386024 \\ 0.009386024 & 0.025134386 \end{bmatrix} \begin{bmatrix} 16-\mu_1 \\ 4-\mu_2 \end{bmatrix} \\ &= 0.0771673(\mu_1^2 + 1.70285\mu_1\mu_2 - 38.8114\mu_1 + 2.27999(\mu_2)^2 - 45.4855\mu_2 + 401.462) \end{aligned}$$

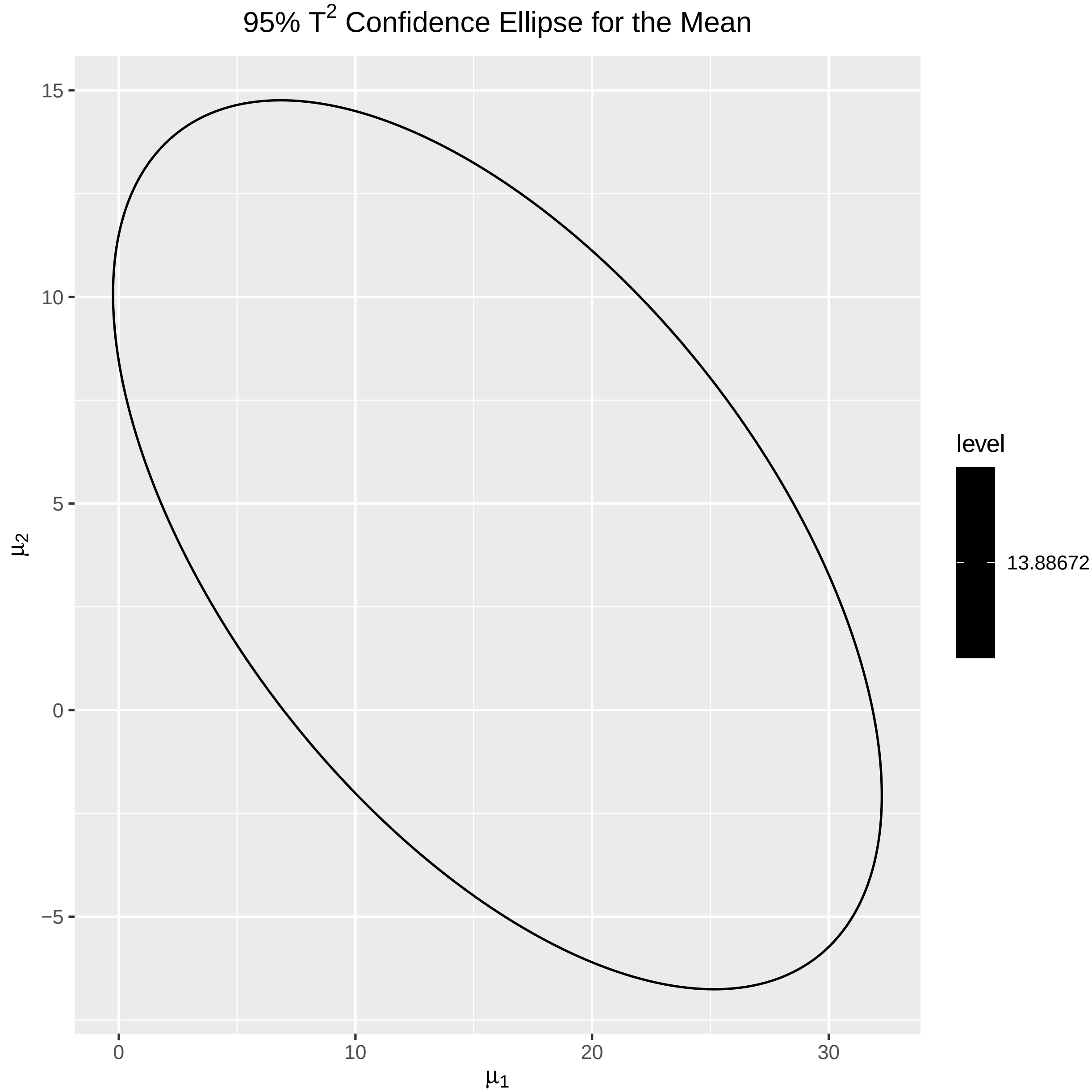
In addition, we have the test statistic equal to,

$$\begin{aligned} \frac{p(n-1)}{(n-p)}F_{p,n-p}(\alpha) &= \frac{2(7-1)}{(7-2)}F_{2,7-2}(0.05) \\ &= \frac{12}{5}F_{2,5}(0.05) \approx 13.88672 \end{aligned}$$

Thus, the equation for the ellipse is given by,

$$0.0771673(\mu_1^2 + 1.70285\mu_1\mu_2 - 38.8114\mu_1 + 2.27999(\mu_2)^2 - 45.4855\mu_2 + 401.462) \leq 13.88672$$

which yields the following graph,



Problem 5.11(a).

We have,

$$\overline{\mathbf{x}} = \begin{bmatrix} 5.185556 \\ 16.070000 \end{bmatrix}$$

and,

$$\mathbf{S} = \begin{bmatrix} 176.0042 & 287.2412 \\ 287.2412 & 527.8493 \end{bmatrix}$$

which implies,

$$\mathbf{S}^{-1} = \begin{bmatrix} 0.05077341 & -0.02762951 \\ -0.02762951 & 0.01692971 \end{bmatrix}$$

Thus,

$$\begin{aligned} T^2 &= n(\overline{\mathbf{x}} - \mu)' S^{-1} (\overline{\mathbf{x}} - \mu) \\ &= 9 \begin{bmatrix} 5.185556 - \mu_1 & 16.070000 - \mu_2 \end{bmatrix} \begin{bmatrix} 0.05077341 & -0.02762951 \\ -0.02762951 & 0.01692971 \end{bmatrix} \begin{bmatrix} 5.185556 - \mu_1 \\ 16.070000 - \mu_2 \end{bmatrix} \\ &= 0.456961(\mu_1^2 - 1.08835(\mu_1)(\mu_2) + 7.11859\mu_1 + 0.333436(\mu_2)^2 - 5.07297\mu_2 + 22.3043) \end{aligned}$$

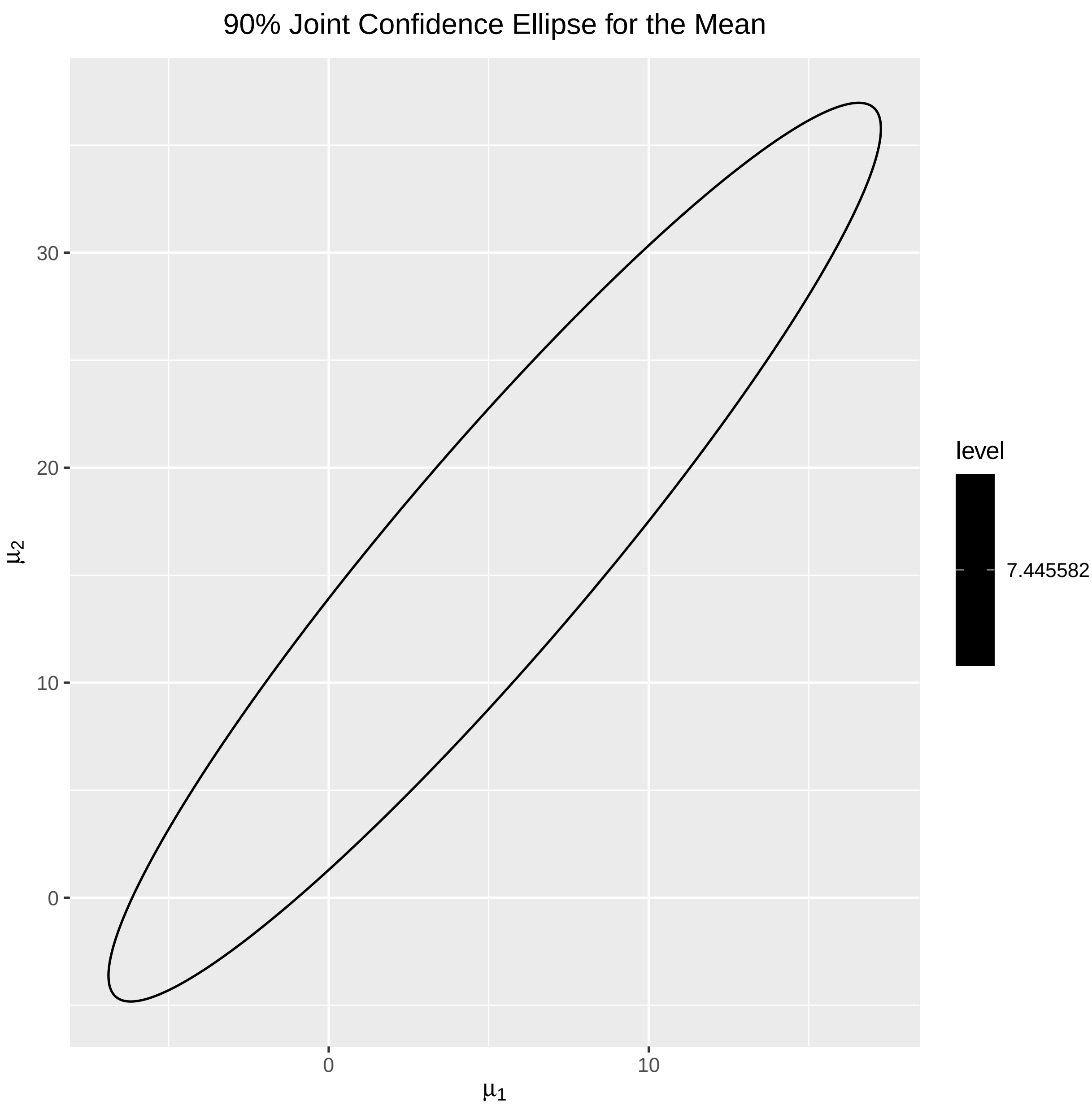
In addition, the test statistic (with $\alpha = 0.1$) is,

$$\begin{aligned} \frac{p(n-1)}{(n-p)} F_{p,n-p}(\alpha) &= \frac{2(9-1)}{(9-2)} F_{2,9-2}(0.1) \\ &= \frac{16}{7} F_{2,7}(0.1) \approx 7.445582 \end{aligned}$$

Thus, by Example 5.5 in the textbook, the equation for the joint confidence ellipse is,

$$0.456961(\mu_1^2 - 1.08835(\mu_1)(\mu_2) + 7.11859\mu_1 + 0.333436(\mu_2)^2 - 5.07297\mu_2 + 22.3043) \leq 7.445582$$

which yields the following graph,


90% Joint Confidence Ellipse for the Mean
30
20
10
0
0
10
μ_1
μ_2
level
7.445582

Problem 5.13.

We know that $\Lambda^{(2/n)} = \left(\frac{|\hat{\mathbf{\Sigma}}|}{|\hat{\mathbf{\Sigma}}_0|}\right)$ with,

$$\Lambda = \frac{\max_{\mathbf{\Sigma}} L(\mu_0, \mathbf{\Sigma})}{\max_{\mathbf{\Sigma},\mu} L(\mu, \mathbf{\Sigma})}$$

Thus, we have,

$$\begin{aligned} -n \ln \left(|\hat{\mathbf{\Sigma}}|/|\hat{\mathbf{\Sigma}}_0|\right) &= -n \ln \left(\Lambda^{(2/n)}\right) \\ &= -n(2/n) \ln \left(\frac{\max_{\mathbf{\Sigma}} L(\mu_0, \mathbf{\Sigma})}{\max_{\mathbf{\Sigma},\mu} L(\mu, \mathbf{\Sigma})}\right) \\ &= -2 \ln \left(\frac{\max_{\mathbf{\Sigma}} L(\mu_0, \mathbf{\Sigma})}{\max_{\mathbf{\Sigma},\mu} L(\mu, \mathbf{\Sigma})}\right) \end{aligned} \tag{1}$$

We know from the discussion on p. 219 in the textbook that we have $v = p + p(p+1)/2$ and $v_0 = p(p+1)/2$ in this case. Thus, using the data from Table 1, we have $v = 3 + 3(4)/2 = 9$ and $v_0 = 6$. As a result, $v - v_0 = 3$.

Hence, from result 5.2 and derivation (1), we have that $n \ln \left(|\hat{\mathbf{\Sigma}}|/|\hat{\mathbf{\Sigma}}_0|\right)$ is distributed approximately as χ_3^2.