Advanced Mathematical Statistics: Assignment 2

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Problem 4.2.

a) $f(x_1, x_2) = \frac{1}{2\pi\sqrt{1.5}} \times \exp\{-\frac{2}{3}[\left(\frac{x_1}{\sqrt{2}}\right)^2 + (x_2-2)^2 - \left(\frac{x_1}{\sqrt{2}}\right)(x_2-2)]\}$

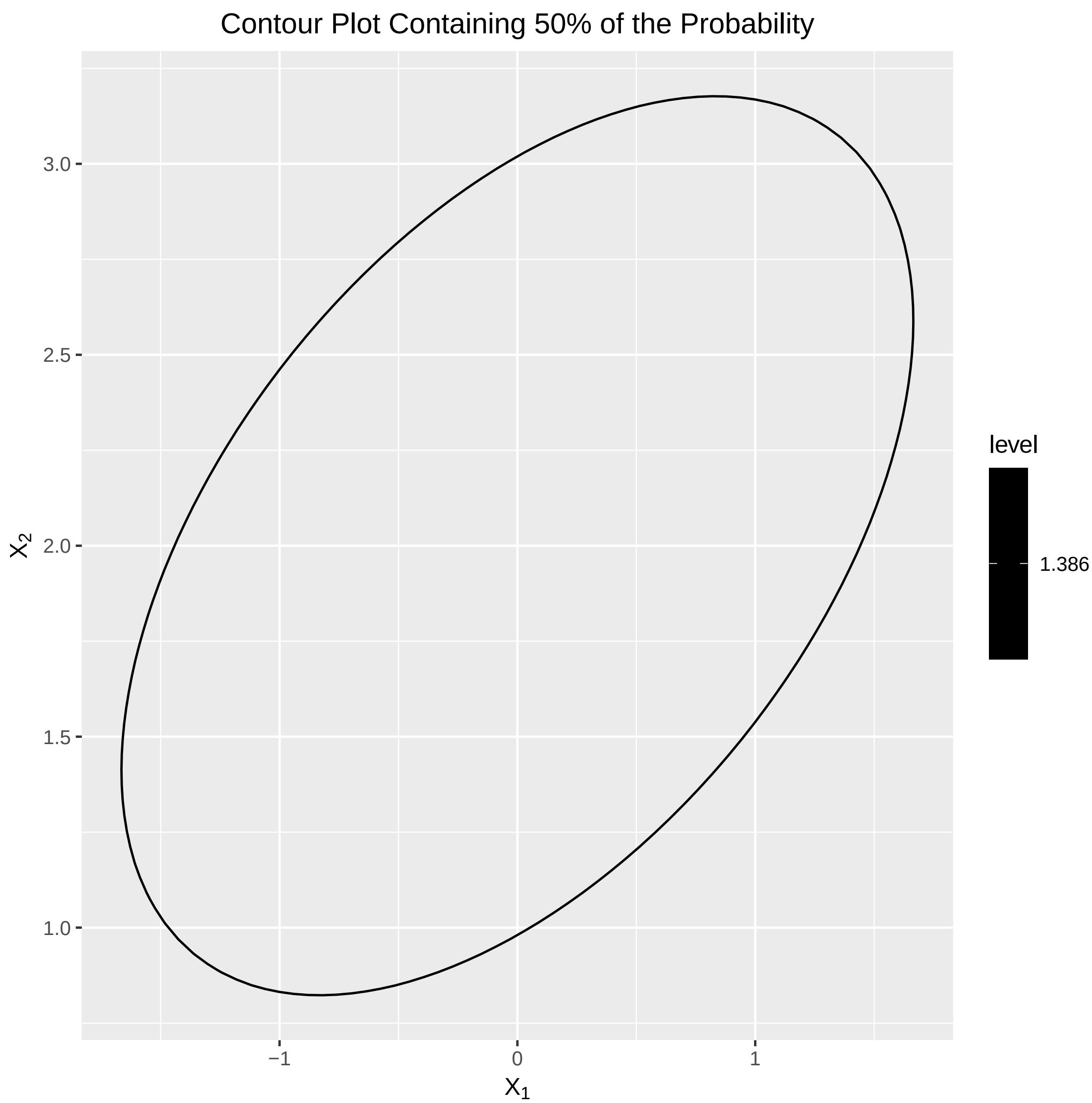
b) We have that $\rho_{12} = 0.5 \implies \sigma_{12} = 0.5(\sqrt{\sigma_{11}\sigma_{22}}) = 0.5(\sqrt{2})$. So,

$$\begin{aligned}
&(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu) \\
&= \begin{bmatrix} x_1-\mu_1 & x_2-\mu_2 \end{bmatrix} \frac{1}{\sigma_{11}\sigma_{22}-\sigma_{12}^2} \begin{bmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{bmatrix} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix} \\
&= \begin{bmatrix} x_1 & x_2-2 \end{bmatrix} \frac{1}{2-(0.5\sqrt{2})^2} \begin{bmatrix} 1 & -0.5\sqrt{2} \\ -0.5\sqrt{2} & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2-2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{3}x_1 + (\frac{-\sqrt{2}}{3})(x_2-2) & (\frac{-\sqrt{2}}{3})(x_1) + \frac{4}{3}(x_2-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2-2 \end{bmatrix} \\
&= x_1\left(\frac{2}{3}x_1 + \left(\frac{-\sqrt{2}}{3}\right)(x_2-2)\right) + (x_2-2)\left(\left(\frac{-\sqrt{2}}{3}\right)x_1 + \frac{4}{3}(x_2-2)\right) \\
&= \frac{2}{3}x_1^2 + \left(\frac{-\sqrt{2}}{3}\right)(x_2-2)x_1 + (x_2-2)\left(\frac{-\sqrt{2}}{3}\right)x_1 + \frac{4}{3}(x_2-2)^2 \\
&= \frac{2}{3}x_1^2 + \frac{-2\sqrt{2}(x_2x_1-2x_1)}{3} + \frac{4}{3}x_2^2 - \frac{16}{3}x_2 + \frac{16}{3}
\end{aligned}$$

c) $c^2 = \chi_2^2(0.5) \approx 1.386294$. So we take

$$\frac{2}{3}x_1^2 + \frac{-2\sqrt{2}(x_2x_1-2x_1)}{3} + \frac{4}{3}x_2^2 - \frac{16}{3}x_2 + \frac{16}{3} = 1.386294$$

to be the surface of the ellipsoid containing 50% of the probability. The graph for this can be seen below.



Problem 4.3.

a) X_1 and X_2 are not independent because $\sigma_{12} = \sigma_{21} = -2 \neq 0$.

b) X_2 and X_3 are independent because $\sigma_{23} = \sigma_{32} = 0$.

c) If we partition the covariance matrix into (X_1, X_2) and X_3 partitions, we get

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ -2 & 5 & 0 \\ \hline 0 & 0 & 2 \end{array}\right]$$

Thus, we can see that the two diagonal sections of the matrix have the forms $\mathbf{0}, \mathbf{0}'$. As a result, (X_1, X_2) and X_3 are independent.

d) Let $\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. From Result 4.3, we know $\mathbf{AX}$ is distributed as $N_q(\mathbf{A}\mu, \mathbf{A\Sigma A}')$ with $q = 2$ in this case.

So we have,

$$\begin{aligned}
\mathbf{A\Sigma A}' &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}
\end{aligned}$$

As can be clearly seen from the above matrix, $\mathbf{A\Sigma A}'$, the covariance between $\frac{X_1+X_2}{2}$ and X_3 is 0. As a result, $\frac{X_1+X_2}{2}$ and X_3 are independent.

e) Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{5}{2} & 1 & -1 \end{bmatrix}$. From Result 4.3, we know $\mathbf{AX}$ is distributed as $N_q(\mathbf{A}\mu, \mathbf{A\Sigma A}')$ with $q = 2$ in this case.

So we have,

$$\begin{aligned}
\mathbf{A\Sigma A}' &= \begin{bmatrix} 0 & 1 & 0 \\ -\frac{5}{2} & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -\frac{5}{2} \\ 1 & 1 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} -2 & 5 & 0 \\ -\frac{9}{2} & 10 & -2 \end{bmatrix} \begin{bmatrix} 0 & -\frac{5}{2} \\ 1 & 1 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 10 \\ 10 & \frac{93}{4} \end{bmatrix}
\end{aligned}$$

We can see from the above matrix that the covariance between the random variables is not 0. Thus, X_2 and $X_2 - \frac{5}{2}X_1 - X_3$ are not independent.

Problem 4.4.

a) Let $A = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}$. By Result 4.3, $3X_1 - 2X_2 + X_3$ is distributed as $N_1(\mathbf{A}\mu, \mathbf{A\Sigma A}')$ with mean vector and covariance matrix,

$$\begin{aligned}
\mathbf{A}\mu &= 6 + 6 + 1 = 13 \\
\mathbf{A\Sigma A}' &= \begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \mathbf{A}' = 6 + 2 + 1 = 9
\end{aligned}$$

b) Let $A = \begin{bmatrix} 0 & 1 & 0 \\ -a_1 & 1 & -a_2 \end{bmatrix}$.

Now find $\mathbf{A\Sigma A'}$:

$$\mathbf{A\Sigma A'} = \begin{bmatrix} 1 & 3 & 2 \\ -a_1+1-a_2 & -a_1+3-2a_2 & a_1+2-2a_2 \end{bmatrix} \begin{bmatrix} 0 & -a_1 \\ 1 & 1 \\ 0 & -a_2 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & -a_1+3-2a_2 \\ -a_1+3-2a_2 & (-a_1)^2-2a_1-2a_1a_2+3-4a_2+2(-a_2)^2 \end{bmatrix}$$

In order for the covariance to be 0 (ie. X_2 and $-a_1X_1+X_2-a_3X_3$ independent), we need $-a_1+3-2a_2=0$. That is, $a_1+2a_2=3$. For instance, take $a_1=1$ and $a_2=1$. Then we have,

$$\mathbf{A\Sigma A'} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

Thus, if $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then we have that X_2 and $X_2 - \mathbf{a}' \begin{bmatrix} X_1 \\ X_3 \end{bmatrix}$ are independent.

Problem 4.5.

a) By Result 4.6, we have

$$\begin{aligned} \mu &= \mu_1 + \sigma_{12}(\sigma_{22})^{-1}(x_2-\mu_2) \\ &= 0 + 0.5(\sqrt{2})(1)^{-1}(x_2-2) \\ &= 0.5(\sqrt{2})x_2 - \sqrt{2} \end{aligned}$$

In addition, we have

$$\begin{aligned} \sigma &= \sigma_{11} - \sigma_{12}(\sigma_{22})^{-1}\sigma_{21} \\ &= 2 - 0.5(\sqrt{2})(1)^{-1}0.5(\sqrt{2}) \\ &= 2 - 0.25(2) = 1.5 \end{aligned}$$

b) Let's rearrange the $\mathbf{\Sigma}$ matrix,

$$\mathbf{\Sigma} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 5 \end{bmatrix}$$
$$= \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 2 & 0 \\ \hline -2 & 0 & 5 \end{array}\right]$$

and the μ vector,

$$\begin{aligned}\mu &= \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} \\ &= \left[\begin{array}{c} -3 \\ 4 \\ \hline 1 \end{array}\right]\end{aligned}$$

Then we have,

$$\begin{aligned}\mu &= \mu_1 + \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}(x_2 - \mu_2) \\ &= \begin{bmatrix} -3 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}(\frac{1}{5})(x_2 - 1) \\ &= \begin{bmatrix} -3 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}(\frac{1}{5}x_2 - \frac{1}{5})\end{aligned}$$

and,

$$\begin{aligned}\mathbf{\Sigma} &= \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \end{bmatrix}\left(\frac{1}{5}\right)\begin{bmatrix} -2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \end{bmatrix}\begin{bmatrix} -\frac{2}{5} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} \frac{4}{5} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & 2 \end{bmatrix}\end{aligned}$$

c) If we partition the covariance matrix into (X_1, X_2) and X_3 partitions, we get

$$\begin{aligned}\mathbf{\Sigma} &= \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 3 & 2 \\ \hline 1 & 2 & 2 \end{array}\right] \\ &= \left[\begin{array}{c|c} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \hline \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{array}\right]\end{aligned}$$

and,

$$\begin{aligned}\mu &= \left[\begin{array}{c} 2 \\ -3 \\ \hline 1 \end{array}\right] \\ &= \left[\begin{array}{c} \mu_1 \\ \hline \mu_2 \end{array}\right]\end{aligned}$$

So we have,

$$\begin{aligned}
\mu &= \mu_2 + \mathbf{\Sigma}_{21}(\mathbf{\Sigma}_{11})^{-1}(x_1 - \mu_1) \\
&= 1 + \begin{bmatrix} 1 & 2 \end{bmatrix} \left(\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \right) (x_1 - \begin{bmatrix} 2 \\ -3 \end{bmatrix}) \\
&= 1 + \left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right) (x_1 - \begin{bmatrix} 2 \\ -3 \end{bmatrix}) \\
&= 1 + \left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right) \left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} - \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right) \\
&= 1 + \left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right) \left(\begin{bmatrix} X_1 - 2 \\ X_2 + 3 \end{bmatrix} \right) \\
&= 1 + \frac{1}{2} (X_1 - 2 + X_2 + 3) \\
&= \frac{1}{2} X_1 + \frac{1}{2} X_2 + \frac{3}{2}
\end{aligned}$$

and,

$$\begin{aligned}
\mathbf{\Sigma} &= \mathbf{\Sigma}_{22} - \mathbf{\Sigma}_{21} \left(\mathbf{\Sigma}_{11} \right)^{-1} \mathbf{\Sigma}_{12} \\
&= 2 - \begin{bmatrix} 1 & 2 \end{bmatrix} \left(\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \right) \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= 2 - \left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right) \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= 2 - \frac{3}{2} = \frac{1}{2}
\end{aligned}$$

Problem 4.6.

a) We can see that $\sigma_{12} = \sigma_{21} = 0$. Thus, X_1 and X_2 are independent.

b) We have that $\sigma_{13} = \sigma_{31} = -1 \neq 0$. Thus, X_1 and X_3 are not independent.

c) We have that $\sigma_{23} = \sigma_{32} = 0$. Thus, X_2 and X_3 are independent.

d) Let's rearrange the covariance matrix,

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

We can partition it now as well,

$$\left[\begin{array}{cc|c} 4 & -1 & 0 \\ -1 & 2 & 0 \\ \hline 0 & 0 & 2 \end{array}\right]$$

So now we have the covariance atrix partitioned into (X_1, X_3) and X_2 blocks, along with their covariances. It is clear from this partitioning that (X_1, X_3) and X_2 have zero covariance. Thus, they are independent.

e) Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & -2 \end{bmatrix}$. From Result 4.3, we know $\mathbf{AX}$ is distributed as $N_q(\mathbf{A}\mu, \mathbf{A\Sigma A}')$ with $q = 2$ in this case.

So we have,

$$\begin{aligned}
\mathbf{A\Sigma A}' &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 4 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 0 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 4 & 0 & -1 \\ 6 & 6 & -5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 0 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 4 & 6 \\ 6 & 34 \end{bmatrix}
\end{aligned}$$

We can see from the above matrix that the covariance between the random variables is not 0. Thus, X_1 and $X_1 + 3X_2 - 2X_3$ are not independent.

Problem 4.7.

a) Let's define the covariance matrix,

$$\mathbf{\Sigma} = \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}$$

By Result 4.6, we have

$$\begin{aligned}
\mu &= \mu_1 + \sigma_{12}(\sigma_{22})^{-1}(x_2 - \mu_2) \\
&= 1 + -1\left(\frac{1}{2}\right)(x_2 - 2) \\
&= 1 - \frac{1}{2}x_2 + 1 = \frac{1}{2} + 2
\end{aligned}$$

and,

$$\begin{aligned}
\sigma &= \sigma_{11} - \sigma_{12}(\sigma_{22})^{-1}\sigma_{21} \\
&= 4 - (-1)\left(\frac{1}{2}\right)(-1) \\
&= 4 - \frac{1}{2} = \frac{7}{2}
\end{aligned}$$

b) Let's partition the $\mathbf{\Sigma}$ matrix,

$$\left[\begin{array}{cc|c} 4 & 0 & -1 \\ \hline 0 & 5 & 0 \\ -1 & 0 & 2 \end{array}\right] \tag{1}$$

and the μ vector,

$$\left[\begin{array}{c} 1 \\ \hline -1 \\ 2 \end{array}\right]$$

Then we have,

$$\begin{aligned} \mu &= \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{x}_2 - \mu_2) \\ &= 1 + -1 \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \left(\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right) \\ &= 1 + -1 \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \left(\begin{bmatrix} x_2 + 1 \\ x_3 - 2 \end{bmatrix} \right) \end{aligned}$$

and,

$$\begin{aligned} \mathbf{\Sigma} &= \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \\ &= \begin{bmatrix} 4 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 5 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -\frac{5}{2} \end{bmatrix} \\ &= \begin{bmatrix} 4 & \frac{5}{2} \end{bmatrix} \end{aligned}$$

Problem 4.10.

a) We know that

$$\begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} = \begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix} \begin{vmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix}$$

If we expand the determinant of the first matrix in the above expression, which we will denote as $\mathbf{C}$ and assume it is $k \times k$, by the last row, we get

$$\begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix} = |\mathbf{C}| = \sum_{j=1}^{k} c_{kj}|\mathbf{C}_{kj}|(-1)^{1+j}$$

Then we have 1 times a determinant of the same form, with the order of $\mathbf{I}$ reduced by one. If we continue this procedure, we get $1 \times |\mathbf{A}|$.

Similarly, if we expand the determinant of the second matrix in the above expression, which we will denote as $\mathbf{D}$ and assume it is $k \times k$, by the first row, we get

$$\begin{vmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} = |\mathbf{D}| = \sum_{j=1}^{k} c_{1j} |\mathbf{C}_{1j}| (-1)^{1+j}$$

Then we have 1 times a determinant of the same form, with the order of $\mathbf{I}$ reduced by one. If we continue this procedure, we get $1 \times |\mathbf{B}|$.

Thus, we have,

$$\begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} = \begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix} \begin{vmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} = |\mathbf{A}||\mathbf{B}|$$

b) We know that

$$\begin{vmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} = \begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} \begin{vmatrix} \mathbf{I} & \mathbf{A}^{-1}\mathbf{C} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix}$$

However, if we expand $\begin{vmatrix} \mathbf{I} & \mathbf{A}^{-1}\mathbf{C} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix}$ by the last row, we get 1.

Thus, from part (a), we have,

$$\begin{vmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} = \begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}' & \mathbf{B} \end{vmatrix} \begin{vmatrix} \mathbf{I} & \mathbf{A}^{-1}\mathbf{C} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix} = |\mathbf{A}||\mathbf{B}|1 = |\mathbf{A}||\mathbf{B}|$$

Problem 4.11.

First, partition $\mathbf{A}$ such that,

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

Then we have that,

$$\begin{bmatrix} \mathbf{I} & -\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{21} & \mathbf{A}_{12} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

Moreover, we have,

Thus, taking determinants on both sides and using the result from Exercise 4.10, we have,

$$\begin{aligned} \begin{vmatrix} \mathbf{I} & -\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ \mathbf{0}' & \mathbf{I} \end{vmatrix} \begin{vmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{vmatrix} \begin{vmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I} \end{vmatrix} &= 1|\mathbf{A}|1 \\ &= |\mathbf{A}| \\ &= \begin{vmatrix} \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{0} \\ \mathbf{0}' & \mathbf{A}_{22} \end{vmatrix} \\ &= |\mathbf{A}_{22}||\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}| \end{aligned}$$

Problem 4.12.

Problem 4.13.

a) First, partition $\mathbf{\Sigma}$ such that,

$$\mathbf{\Sigma} = \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{bmatrix}$$

Then we can apply the answer to problem 4.11, which yields,

$$|\mathbf{\Sigma}| = |\mathbf{\Sigma}_{22}||\mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}|$$

b)

c)

Problem 4.14.

Problem 4.15.

Problem 4.16.

a) By Result 4.8, we have

$$\begin{aligned} \mu_{V_1} &= \left(\frac{1}{4} - \frac{1}{4} + \frac{1}{4} - \frac{1}{4}\right)\mu = \mathbf{0} \\ \mu_{V_2} &= \left(\frac{1}{4} + \frac{1}{4} - \frac{1}{4} - \frac{1}{4}\right)\mu = \mathbf{0} \end{aligned}$$

and

$$\begin{aligned} \mathbf{\Sigma}_{V_1} &= \left(\frac{1}{4}^2 + \left(-\frac{1}{4}\right)^2 + \frac{1}{4}^2 + \left(-\frac{1}{4}\right)^2\right)\mathbf{\Sigma} = \frac{1}{4}\Sigma \\ \mathbf{\Sigma}_{V_2} &= \left(\frac{1}{4} + \frac{1}{4} + \left(-\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right)^2\right)\mathbf{\Sigma} = \frac{1}{4}\Sigma \end{aligned}$$

b) By Result 4.8, the joint density is given by,

$$\left(\frac{1}{4}\left(\frac{1}{4}\right) + \left(-\frac{1}{4}\right)\frac{1}{4} + \frac{1}{4}\left(-\frac{1}{4}\right) + \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)\right)\Sigma = 0\Sigma = \mathbf{0}$$

Thus, the two linear combinations are independent and have a joint $2p$-variate normal distribution.

Problem 4.17.

The means of the linear combination are given by,

$$\mu_1 = \left(\frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5}\right)\mu = 1\mu = \mu$$
$$\mu_2 = (1 - 1 + 1 - 1 + 1)\,\mu = 1\mu = \mu$$

Similarly, the covariance matrices are given by,

$$\mathbf{\Sigma}_1 = \left(\frac{1}{5}^2 + \frac{1}{5}^2 + \frac{1}{5}^2 + \frac{1}{5}^2 + \frac{1}{5}^2\right)\mathbf{\Sigma} = \frac{1}{5}\mathbf{\Sigma}$$
$$\mathbf{\Sigma}_1 = \left(1^2 + (-1)^2 + 1^2 + (-1)^2 + 1^2\right)\mathbf{\Sigma} = 5\mathbf{\Sigma}$$

The covariance between the two linear combinations is given by,

$$\left(\frac{1}{5}(1) + \frac{1}{5}(-1) + \frac{1}{5}(1) + \frac{1}{5}(-1) + \frac{1}{5}(1)\right)\mathbf{\Sigma} = \frac{1}{5}\mathbf{\Sigma}$$

Problem 4.18.

The maximum likelihood estimate for μ is $\overline{\mathbf{X}}$, given by,

$$\overline{\mathbf{X}} = \begin{bmatrix} (3+4+5+4)/4 \\ (6+4+7+7)/4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

The maximum likelihood estimate for $\mathbf{\Sigma}$, $\hat{\mathbf{\Sigma}}$, is given by,

$$\hat{\mathbf{\Sigma}} = \frac{1}{n}\sum_{j=1}^{n}\left(\mathbf{X}_j - \overline{\mathbf{X}})(\mathbf{X}_j - \overline{\mathbf{X}}\right)'$$

which yields,

```
##           [,1]      [,2]
## [1,] 0.6666667 0.3333333
## [2,] 0.3333333 2.0000000
```

Problem 4.19.

a)

b) $\overline{\mathbf{X}}$ is distributed as $N_6(\mu, (1/20)\Sigma)$.
$\sqrt{n}(\overline{\mathbf{X}} - \mu) = \sqrt{20}(\overline{\mathbf{X}} - \mu)$ is distributed as $N_6(\mathbf{0}, \mathbf{\Sigma})$.

c) $(n-1)\mathbf{S} = 19\mathbf{S}$ is distributed as a Wishart random matrix with 19 degrees of freedom, ie. $W_{19}(19\mathbf{S}|\mathbf{\Sigma})$.

Problem 4.20.

By properties of the Wishart distribution, we have that $\mathbf{B}(19\mathbf{S})\mathbf{B}'$ is distributed as $W_{19}(\mathbf{B}(19\mathbf{S})\mathbf{B}'|\mathbf{B\Sigma B}')$

Problem 4.21.

a) The distribution of $\overline{\mathbf{X}}$ is given by $N_4(\mu, (1/60)\Sigma)$

b)

c) $n(\overline{\mathbf{X}} - \mu)'\mathbf{\Sigma}^{-1}(\overline{\mathbf{X}} - \mu)$ has a χ_4^2 distribution

d) $n(\overline{\mathbf{X}} - \mu)'\mathbf{S}^{-1}(\overline{\mathbf{X}} - \mu)$ has an approximately χ_4^2 distribution