Advanced Mathematical Statistics: Assignment 2

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Problem 4.2.

a) $f(x_1, x_2) = \frac{1}{2\pi\sqrt{1.5}} \times \exp\{-\frac{2}{3}[\left(\frac{x_1}{\sqrt{2}}\right)^2 + (x_2-2)^2 - \left(\frac{x_1}{\sqrt{2}}\right)(x_2-2)]\}$

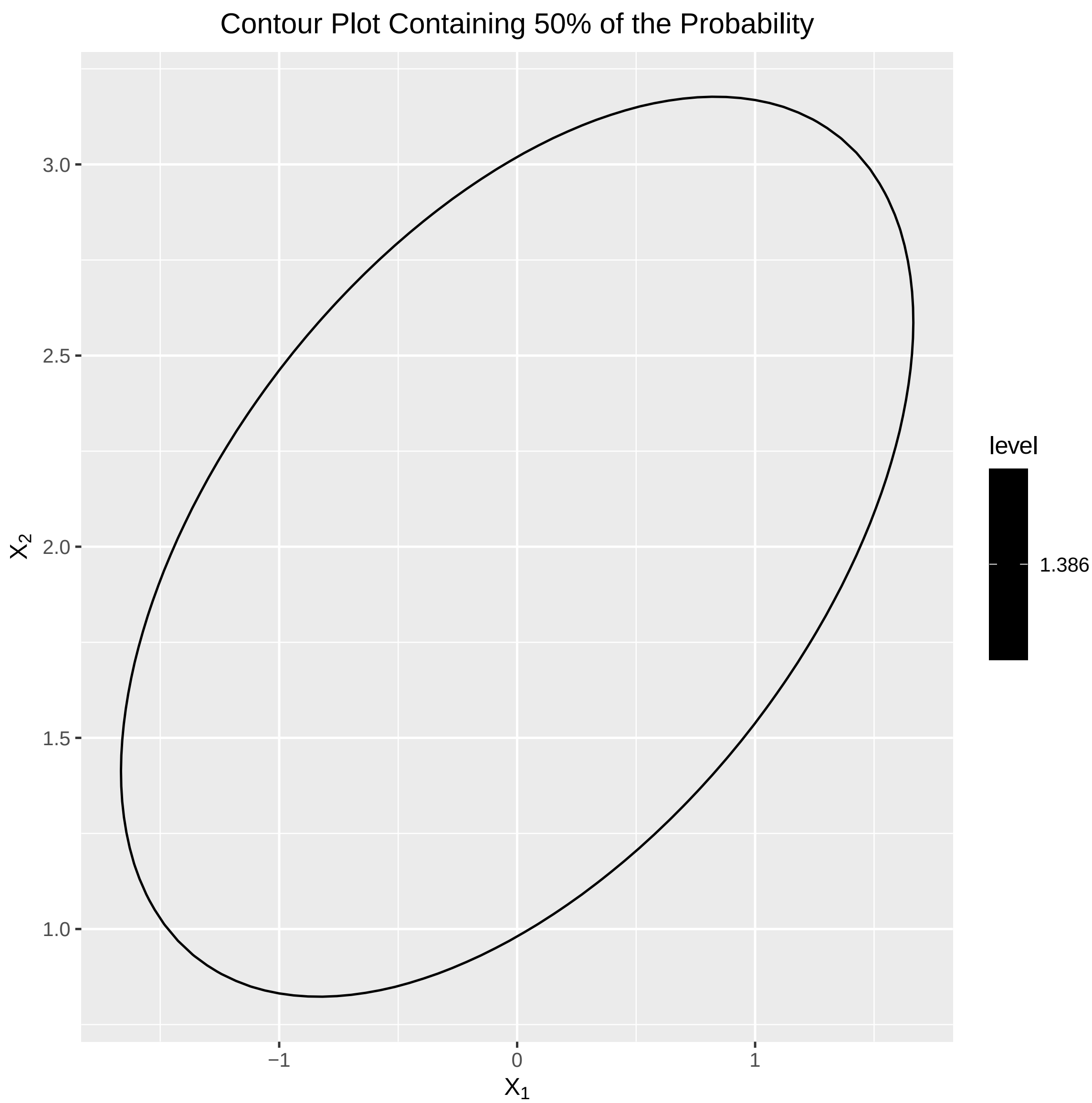
b) We have that $\rho_{12} = 0.5 \implies \sigma_{12} = 0.5(\sqrt{\sigma_{11}\sigma_{22}}) = 0.5(\sqrt{2})$. So,

$$
\begin{aligned}
&(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu) \\
&= \begin{bmatrix} x_1-\mu_1 & x_2-\mu_2 \end{bmatrix} \frac{1}{\sigma_{11}\sigma_{22}-\sigma_{12}^2} \begin{bmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{bmatrix} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix} \\
&= \begin{bmatrix} x_1 & x_2-2 \end{bmatrix} \frac{1}{2-(0.5\sqrt{2})^2} \begin{bmatrix} 1 & -0.5\sqrt{2} \\ -0.5\sqrt{2} & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2-2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{3}x_1 + (\frac{-\sqrt{2}}{3})(x_2-2) & (\frac{-\sqrt{2}}{3})(x_1) + \frac{4}{3}(x_2-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2-2 \end{bmatrix} \\
&= x_1\left(\frac{2}{3}x_1 + \left(\frac{-\sqrt{2}}{3}\right)(x_2-2)\right) + (x_2-2)\left(\left(\frac{-\sqrt{2}}{3}\right)x_1 + \frac{4}{3}(x_2-2)\right) \\
&= \frac{2}{3}x_1^2 + \left(\frac{-\sqrt{2}}{3}\right)(x_2-2)x_1 + (x_2-2)\left(\frac{-\sqrt{2}}{3}\right)x_1 + \frac{4}{3}(x_2-2)^2 \\
&= \frac{2}{3}x_1^2 + \frac{-2\sqrt{2}(x_2x_1-2x_1)}{3} + \frac{4}{3}x_2^2 - \frac{16}{3}x_2 + \frac{16}{3}
\end{aligned}
$$

c) $c^2 = \chi_2^2(0.5) \approx 1.386294$. So we take

$$\frac{2}{3}x_1^2 + \frac{-2\sqrt{2}(x_2x_1-2x_1)}{3} + \frac{4}{3}x_2^2 - \frac{16}{3}x_2 + \frac{16}{3} = 1.386294$$

to be the surface of the ellipsoid containing 50% of the probability. The graph for this can be seen below.



Problem 4.3.

a) X_1 and X_2 are not independent because $\sigma_{12} = \sigma_{21} = -2 \neq 0$.

b) X_2 and X_3 are independent because $\sigma_{23} = \sigma_{32} = 0$.

c) If we partition the covariance matrix into (X_1, X_2) and X_3 partitions, we get

$$\left(\begin{array}{cc|c} 1 & -2 & 0 \\ -2 & 5 & 0 \\ \hline 0 & 0 & 2 \end{array}\right)$$

Thus, we can see that the two diagonal sections of the matrix have the forms $\mathbf{0}, \mathbf{0}'$. As a result, (X_1, X_2) and X_3 are independent.

d) Let $\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. From Result 4.3, we know $\mathbf{AX}$ is distributed as $N_q(\mathbf{A}\mu, \mathbf{A\Sigma A'})$ with $q = 2$ in this case.

So we have,

$$\begin{aligned}
\mathbf{A\Sigma A'} &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}
\end{aligned}$$

As can be clearly seen from the above matrix, $\mathbf{A\Sigma A'}$, the covariance between $\frac{X_1+X_2}{2}$ and X_3 is 0. As a result, $\frac{X_1+X_2}{2}$ and X_3 are independent.

e) Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{5}{2} & 1 & -1 \end{bmatrix}$. From Result 4.3, we know $\mathbf{AX}$ is distributed as $N_q(\mathbf{A}\mu, \mathbf{A\Sigma A'})$ with $q = 2$ in this case.

So we have,

$$\begin{aligned}
\mathbf{A\Sigma A'} &= \begin{bmatrix} 0 & 1 & 0 \\ -\frac{5}{2} & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -\frac{5}{2} \\ 1 & 1 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} -2 & 5 & 0 \\ -\frac{9}{2} & 10 & -2 \end{bmatrix} \begin{bmatrix} 0 & -\frac{5}{2} \\ 1 & 1 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 10 \\ 10 & \frac{93}{4} \end{bmatrix}
\end{aligned}$$

We can see from the above matrix that the covariance between the random variables is not 0. Thus, X_2 and $X_2 - \frac{5}{2}X_1 - X_3$ are not independent.

Problem 4.4.

a) Let $A = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}$. By Result 4.3, $3X_1 - 2X_2 + X_3$ is distributed as $N_1(\mathbf{A}\mu, \mathbf{A\Sigma A'})$ with mean vector and covariance matrix,

$$\begin{aligned}
\mathbf{A}\mu &= 6 + 6 + 1 = 13 \\
\mathbf{A\Sigma A'} &= \begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \mathbf{A'} = 6 + 2 + 1 = 9
\end{aligned}$$

b) Let $A = \begin{bmatrix} 0 & 1 & 0 \\ -a_1 & 1 & -a_2 \end{bmatrix}$.

Now find $\mathbf{A\Sigma A}'$:

$$\mathbf{A\Sigma A}' = \begin{bmatrix} 1 & 3 & 2 \\ -a_1+1-a_2 & -a_1+3-2a_2 & a_1+2-2a_2 \end{bmatrix} \begin{bmatrix} 0 & -a_1 \\ 1 & 1 \\ 0 & -a_2 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & -a_1+3-2a_2 \\ -a_1+3-2a_2 & (-a_1)^2-2a_1-2a_1a_2+3-4a_2+2(-a_2)^2 \end{bmatrix}$$

In order for the covariance to be 0 (ie. X_2 and $-a_1X_1+X_2-a_3X_3$ independent), we need $-a_1+3-2a_2=0$. That is, $a_1+2a_2=3$. For instance, take $a_1=1$ and $a_2=1$. Then we have,

$$\mathbf{A\Sigma A}' = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

Thus, if $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then we have that X_2 and $X_2 - \mathbf{a}' \begin{bmatrix} X_1 \\ X_3 \end{bmatrix}$ are independent.

Problem 4.5.

a)

b)

c)

Problem 4.6.

Problem 4.7.

Problem 4.10.

Problem 4.11.

Problem 4.12.

Problem 4.13.

Problem 4.14.

Problem 4.15.

Problem 4.16.

Problem 4.17.

Problem 4.18.

Problem 4.19.

Problem 4.20.

Problem 4.21.