Dynamical Systems: Homework 2

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September 29, 2020

Problem 1.

We have,

$$\Sigma = \Sigma_b = \{X = (x_k)_{k=-\infty}^{\infty} : x_k \in \{0, \cdots, b-1\}\}$$

and,

$$d(x,y) = d_\theta(x,y) = \begin{cases} 0 & \text{if} \quad \text{x = y} \\ \theta^{\min\{|k| : x_k \neq y_k\}} & \end{cases}$$

where $0 < \theta < 1$.

Need to show that (Σ, d) is compact.

Let us suppose that (Σ, d) is closed and bounded but not compact, and then prove that this gives us compactness.

Consider $C_{-k,k}(x) = \{y \in \Sigma : y_{-k} = x_{-k}, \cdots, y_k = x_k\}$. Then

$$\begin{aligned} C_{-k,k}(x) &= B(x, \theta^k) \\ &= \{y \in \Sigma : d_\theta(x,y) < \theta^k\} \end{aligned}$$

Hence, $C_{-k,k}(x)$ is open. $C_{-k,k}$ is also closed.

$C_{-1,1} = \{x_{-1}, x_0, x_1\}$, $x \in \Sigma$. There are b^3 possibilities.

For $C_{-1,1}$, we have

$$C_{-1,1}(x) = \cup_{y \in C_{-1,1}(x)} C_{-2,2}(y)$$

So there are b^2 cylinders.

Since Σ is assumed to be bounded, we have a sequence of cylinders C_ℓ in Σ such that $\Sigma \cap C_\ell$ does not have a finite subcover.

We claim $(\Sigma \cap C_\ell) \to 0$.

For every ℓ, there exists $x_\ell \in \Sigma \cap C_\ell$

This implies that $(x_\ell)_\ell$ is a Cauchy sequence. Since Σ is complete, the limit of this sequence exists and we write,

$$x = \lim_{\ell \to \infty} x_\ell$$

Since Σ is assumed to be closed, we have $x \in \Sigma$

Hence, $\exists \lambda_x$ such that $x \in U_{\lambda_x}$ with U_{λ_x} open.

Thus, $\exists r > 0$ such that $B(x, r) \subset U_{\lambda_x}$

Since claim $(C_\ell) \to 0$ and $x_\ell \in C_\ell$, we can conclude that,

$$C_\ell \subset B(x, r) \subset U_{\lambda_x}$$

Hence, $\Sigma \cap C_\ell \subset U_{\lambda_x}$ and thus $\{U_{\lambda_x}\}$ is a finite subcover of $\Sigma \cap C_\ell$, a contradiction. Thus, (Σ, d) is compact.

Problem 2.

Suppose Σ_A is not closed. That is, there exists a sequence $(y)_\ell \in \Sigma_A$ which converges to a limit not in Σ_A. Let us call this limit y.

Since $y \notin \Sigma_A$, there exists $j \in \mathbb{Z}$ such that $a_{y_j, y_{j+1}} = 0$. Let this be the index with smallest absolute value $|j|$ such that this occurs. Since every element of the sequence $(y_k)_\ell \in \Sigma_A$, we have that $a_{y_{m_j}, y_{m_{j+1}}} = 1$ for every $m \in \mathbb{N}$

Note then that $d(y, y_m) \geq \theta^j$ for every $m \in \mathbb{N}$. Hence, if we fix $\epsilon_j < \theta^j$, we have that,

$$d(y, y_m) \geq \theta^j > \epsilon_j$$

for every $m \in \mathbb{N}$. As a result, we have that $(y)_\ell$ does not converge, a contradiction. Hence, Σ_A must be closed.

Now consider the shift map σ (which shifts each entry in a sequence one space to the left) and the inverse shift map σ^{-1} (which shifts each entry in a sequence one space to the right).

Suppose $x \in \Sigma_A$. Then $a_{x_i, x_{i+1}} = 1$ for every $i \in \mathbb{Z}$.

Problem 3.

We have $f_\mu(x) = \mu x(1-x)$

(a) Suppose $\mu = 1$. Then we have,

$$f_\mu(x) = x(1-x)$$

Suppose $x \in (0,1]$. Then we have,

$$x(1-x) < x$$

since $0 \leq 1 - x < 1$. Hence, if we take $x_1 = f_\mu(x)$, then

$$f_\mu(x_1) = f_\mu^2(x) = x_1(1-x_1) < x_1 = f_\mu(x) < x$$

This extends for all $n \in \mathbb{N}$. Note also that,

$$f_\mu(0) = 0(1-0) = 0$$

and that $0 \leq f_\mu(x) \leq 0.25$ for every $x \in (0,1]$. That is, if our initial $x \in (0,1]$, then $f_\mu^n \in [0, 0.25]$ for any choice of n.

Thus, we have that $f_\mu^n(x)$ is a monotonically decreasing sequence of real numbers with lower bound 0 if the initial $x \in (0,1]$.

Now note that we have,

$$\begin{aligned} & x = f_\mu(x) \\ \iff & x = x(1-x) \\ \iff & 1 = 1 - x \\ \iff & x = 0 \end{aligned}$$

so 0 is the only fixed point of $f_\mu(x)$.

Hence, we cannot have $\lim_{n\to\infty} f_\mu^n > 0$ since there are no positive fixed points. In addition, we cannot have $\lim_{n\to\infty} f_\mu^n < 0$ since $f_\mu^n \geq 0$ for every n. Thus, we must have,

$$\lim_{n\to\infty} f_\mu^n = 0$$

for every $x \in (0,1]$.

(b) Let $1 < \mu < 3$ and consider $p_\mu = \frac{\mu-1}{\mu}$. We have,

$$\begin{aligned} f_\mu\left(\frac{\mu-1}{\mu}\right) &= \mu\left(\frac{\mu-1}{\mu}\right)\left(1-\frac{\mu-1}{\mu}\right) \\ &= (\mu-1)\left(1-\frac{\mu-1}{\mu}\right) \\ &= \mu-1-\frac{(\mu-1)(\mu-1)}{\mu} \\ &= \mu-1-\frac{\mu^2-2\mu+1}{\mu} \\ &= \frac{\mu^2-\mu-\mu^2+2\mu-1}{\mu} \\ &= \frac{\mu-1}{\mu} \end{aligned}$$

So we have shown that $p_\mu = \frac{\mu-1}{\mu}$ is a fixed point of $f_\mu(x)$.

Now consider,

$$\begin{aligned} f_\mu(x) &= \mu x(1-x) \\ \implies f'_\mu(x) &= \mu(1-2x) \end{aligned}$$

So,

$$\begin{aligned} f'_\mu\left(\frac{\mu-1}{\mu}\right) &= \mu\left(1-2\left(\frac{\mu-1}{\mu}\right)\right) \\ &= \mu-2\mu\left(\frac{\mu-1}{\mu}\right) \\ &= \mu-2\mu+2 \\ &= 2-\mu \end{aligned}$$

Since $1 < \mu < 3$, we have,

$$-1 < 2-\mu < 1$$

and,

$$\left|f'_\mu\left(\frac{\mu-1}{\mu}\right)\right| < 1$$

Thus, p_μ is an attracting fixed point.

Now we need to show that $W^s_{\text{loc}}(p_\mu) = (0,1)$. Note that,

$$\begin{aligned} W^s(a) &= \{x \in X : f^n(x) \to a \text{ as } n \to \infty\} \\ &= \{x \in X : \lim_{n\to\infty} |f^n(a) - f^n(x)| = 0\} \end{aligned}$$

and

$$\begin{aligned} W^s_{\text{loc}}(a) &= \text{the connected component of } W^s(a) \text{ containing } a \\ &= \cup_{n\in\mathbb{N}_0} f^{-n}\left(W^s_{\text{loc}}(a)\right) \end{aligned}$$

Problem 4.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{1}{2}(x^3 + x)$. We have,

$$\begin{aligned} x &= \frac{1}{2}(x^3 + x) \\ \iff & \frac{x^3}{2} + \frac{x}{2} - x = 0 \\ \iff & \frac{x^3}{2} - \frac{x}{2} = 0 \\ \iff & x^3 - x = 0 \\ \iff & x(x^2 - 1) = 0 \\ \iff & x = 0 \text{ or } x^2 = 1 \end{aligned}$$

So, we have that the fixed points of $f(x)$ are,

$$f(0) = 0, f(1) = 1, f(-1) = -1$$

Now observe that,

$$f'(x) = \frac{1}{2}(3x^2 + 1)$$

so we have,

$$\begin{aligned} f'(0) &= \frac{1}{2} \\ f'(1) &= 2 \\ f'(-1) &= 2 \end{aligned}$$

Hence, 0 is an attracting fixed point and $1, -1$ are repelling fixed points.

Observe that, if $x > 1$, then $x^3 - x > 2x$ and we get,

$$\frac{1}{2}(x^3 + x) > \frac{1}{2}(2x) = x$$

Hence, for every $x > 1$, we have,

$$f(x) > x$$

Now fix $x < -1$. Then we have $x^3 - x < 2x$ and we get

$$f(x)\frac{1}{2}(x^3 + x) < \frac{1}{2}(2x) = x$$

for every $x < -1$.

Now let $x \in (-1, 0)$. We have, $x^3 - x > 2x$, so

$$f(x) = \frac{1}{2}(x^3 + x) < \frac{1}{2}(2x) = x$$

for every $x \in (-1, 0)$. Since 0 is an attracting fixed point and $f^n(x)$ is a monotonically increasing sequence with supremum 0 when $x \in (-1, 0)$, we have that $\lim_{n\to\infty} f^n(x) = 0$ in this case.

Now assume $x \in (0, 1)$. Then we have $x^3 - x < 2x$ and we get

$$f(x)\frac{1}{2}(x^3 + x) < \frac{1}{2}(2x) = x$$

for every $x \in (0, 1)$. Since 0 is an attracting fixed point and $f^n(x)$ is a monotonically decreasing sequence with infimum 0 when $x \in (1, 0)$, we have that $\lim_{n\to\infty} f^n(x) = 0$ in this case.

Thus, we get the following limit,

$$\lim_{n\to\infty} f^n(x) = \begin{cases} -\infty & x < -1 \\ -1 & x = -1 \\ 0 & -1 < x < 1 \\ 1 & x = 1 \\ \infty & x > 1 \end{cases}$$