Dynamical Systems: Midterm

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Problem 1.

(a) Devaney's definition of a chaotic function is as follows,

Definition Let V be a set. $f : V \to V$ is said to be chaotic on V if

(a) f has *sensitive dependence on initial conditions.* That is, $\exists \delta > 0$ such that for any $x \in V$ and all $\epsilon > 0$, there exists $y \in B(x, \epsilon)$ and $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$

(b) f is *topologically transitive.* That is, $\exists x \in V$ such that $O(x)$ is dense in V

(c) periodic points of f (i.e. $\text{Per}(f)$) are dense in V

(b) Note that on HW3, Q4 we discussed the angle doubling map, $g : S^1 \to S^1$ given by $z \mapsto z^2$. This was equivalent to $\tilde{g} : S^1 \to S^1$, $\tilde{g}(\alpha) = e^{2\pi i(2\alpha)} = e^{4\pi i \alpha}$.

Let us now turn our attention to this function: $f : S^1 \to S^1$ given by $z \mapsto z^5$. This is equivalent to $\tilde{f} : S^1 \to S^1$, $\tilde{f}(\alpha) = e^{2\pi i(5\alpha)} = e^{10\pi i \alpha}$.

We proved in HW3, Q4 that $\tilde{g}$ (and hence g) was chaotic. Moreover, we proved in HW3, Q3 if two continuous maps defined on metric spaces are topologically conjugate, then one being chaotic implies that the other is chaotic. Since $\tilde{g}$ and $\tilde{f}$ are both continuous and defined on metric spaces, all we must show is that $\tilde{g}$ is topologically conjugate to $\tilde{f}$ in order to show that f is chaotic.

Note that $h(x) = e^{2/5}x$ is continuous since it is a constant multiplied by x, where x is in S^1. In addition, we have that $h^{-1}(x) = e^{5/2}x$ is continuous for the same reason. Both are also well-defined on S^1. Moreover, note that

$$\begin{aligned} h \circ \tilde{f} &= e^{2/5} \cdot e^{10\pi i \alpha} \\ &= e^{4\pi i \alpha} \end{aligned}$$

Problem 2.

(a) Let $\lambda = -e$. Then,

$$\begin{aligned} E_\lambda(x) &= \lambda e^x \\ &= -e \cdot e^x \\ &= -e^{x+1} \end{aligned}$$

Hence, we have,

$$\begin{aligned} E_\lambda(-1) &= -e^{-1+1} \\ &= -e^0 \\ &= -1 \end{aligned}$$

Now note that, by the chain rule, we have

$$\begin{aligned} E'_\lambda(x) &= -e^{x+1} \\ &= E_\lambda(x) \end{aligned}$$

Thus,

$$\begin{aligned} E'_\lambda(-1) &= E_\lambda(-1) \\ &= -1 \end{aligned}$$

(b) Let $\lambda > -e$. Then,

$$E_\lambda(x) > -e^{x+1}$$

(c) Let $\lambda < -e$. Then,

$$E_\lambda(x) < -e^{x+1}$$

Problem 3.

(a)

(b)

Problem 4.

(a) Idea: assign to each dynamical system a positive real number $h_{\text{top}}(f)$, where $f : X \to X$ is continuous and (X, d) is a compact metric space such that

i) if f and g are topologically conjugate then $h_{\text{top}}(f) = h_{\text{top}}(g)$

ii) if $0 < h_{\text{top}}(f) < h_{\text{top}}(g)$, then g is more chaotic than f

Now let $f : X \to X$. Entropy will be the exponential growth rate of the number of distinct orbits. We say $\epsilon > 0$ is the "margin of measurement".

Iterate $n-$times, x and y:

$$\begin{aligned} x &\to f(x) \to f^2(x) \to \cdots \to f^{n-1}(x) \\ y &\to f(y) \to f^2(y) \to \cdots \to f^{n-1}(y) \end{aligned}$$

Now define,

$$d_n(x,y) = \max_{k=0,\cdots,n-1} d(f^k(x), f^k(y))$$

That is, $d_n(x,y)$ denotes the maximum distance between x and y along the orbit segment of length n. d_n is also called the Bowen metric.

Now let $F \subset X$. Also let $\epsilon > 0$ and $n \in \mathbb{N}$ be fixed. F is called $(n,\epsilon)-$separated if $\forall x, y \in F$ with $x \neq y$, we have $d_n(x,y) \geq \epsilon$.

Assume $F_n(\epsilon)$ is a maximal $(n,\epsilon)-$separated set. That is

i) $F_n(\epsilon)$ is (n,ϵ) separated

ii) if $y \in X \setminus F_n(\epsilon)$, then $I_n(\epsilon) \cup \{y\}$ is not $(n,\epsilon)-$separated

Remark: Since X is compact, $F_n(\epsilon)$ is finite

Now define $h_{\text{top}}(f)$ as,

$$h_{\text{top}}(f) = \lim_{\epsilon \to 0} \overline{\lim_{n \to \infty}} \frac{1}{n} \log \left(\text{cardinality of } F_n \epsilon\right)$$

(b) Let $r > 0$. Let us define,

$$A = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{array}$$

Then we have that $\Sigma_A^+ = \Sigma_2^+$. Hence, let us define $f = \sigma_A$ as our continuous map from $\Sigma_2^+ \to \Sigma_2^+$. Thus, we have that,

$$\begin{aligned} h_{\text{top}}(f) &= h_{\text{top}}(\sigma_A) \\ &= \log \rho \end{aligned}$$

where ρ is the largest eigenvalue of A. Note that the two eigenvalues of A are 2 and 0. Hence, we have $\rho = 2$ and,

$$\begin{aligned} h_{\text{top}}(f) &= \log 2 \\ &\approx 0.693 \end{aligned}$$

Now let us use the fact that $2h_{\text{top}}(f) = h_{\text{top}}(f^2)$. Define $g = f^2$. Then we have,

$$2h_{\text{top}}(g) = h_{\text{top}}(g^2)$$

But this implies,

$$\begin{aligned} 2h_{\text{top}}(f^2) &= 2[2h_{\text{top}}(f)] \\ &= 4h_{\text{top}}(f) \end{aligned}$$

Proceeding by induction, we see that $2^k h_{\text{top}}(f) = h_{\text{top}}(f^{2k})$ for $k \in \mathbb{N}$. Thus, in our case,

$$h_{\text{top}}(f^{2k}) = 2^k \cdot \log 2$$

Now we need k such that,

$$r < 2^k \cdot \log 2$$

which implies,

$$\log_2\left(\frac{r}{\log 2}\right) < k$$

So if we choose $k \in \mathbb{N}$ such that $k > \log_2\left(\frac{r}{\log 2}\right)$, then we have that,

$$h_{\text{top}}(f^{2k}) > r$$

Now let $g = f^{2k} = \sigma_A^{2k}$. We have that $g : \Sigma_2^+ \to \Sigma_2^+$, g continuous, and

$$h_{\text{top}}(g) > r$$

(c) Yes, it is possible for a continuous $f : \mathbb{R} \to \mathbb{R}$ to have precisely one periodic orbit. For example, consider $f(x) = x/2$. Clearly this function is continuous on all of $\mathbb{R}$ and $f(0) = 0$, so 0 has period 1. But observe that, for any $x \neq 0$, we have,

$$\cdots < f^3(x) = x/8 < f^2(x) = x/4 < f(x) = x/2 < x$$

Thus, we see that the sequence is $(f^n(x))_{n=0}^{\infty}$ is actually equivalent to

$$(x/2^n)_{n=0}^{\infty}$$

This sequence is strictly decreasing. Hence, for any choice of x, there will never be any point $n_1 \in \mathbb{N}$ with $n_1 > 0$ such that $x_{n_1} = x$ since we must have that $x_{n_1} < x$. Since $x \neq 0$ was arbitrary in $\mathbb{R}$, we have shown that there are no periodic points in $\mathbb{R} \setminus \{0\}$. Thus, 0 is the only periodic point of f and has period 1.