Dynamical Systems: Homework 3

Chris Hayduk

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Problem 1.

Recall that Σ_b^+ is a one-sided shift map. That is,

$$\Sigma_b^+ = \{x = (x_k)_{k=0}^{\infty} : x_k \in \{0, \cdots b-1\}\}$$

The shift map $\sigma : \Sigma_b^+ \to \Sigma_b^+$ is defined as follows:

$$\sigma((x_k)_{k=0}^{\infty}) = (x_{k+1})_{k=0}^{\infty}$$

That is, we are essentially cutting off the first element of the sequence.

Lastly, we have that,

$$d(x,y) = d_\theta(x,y) = \begin{cases} 0 & \text{if } x = y \\ \theta^{\min\{k : x_k \neq y_k\}} & \text{if } x \neq y \end{cases}$$

where $0 < \theta < 1$.

We will start now by proving that σ has sensitive dependence.

Now fix $x = (x_0 x_1 x_2 \cdots) \in \Sigma_b^+$ and let $\delta > 0$. Fix $k \in \mathbb{N}$ such that it is the smallest natural number with $\theta^k < \delta$. Now define y such that $y_i = x_i$ for $0 \leq i \leq k-1$ and such that, for every index $i > k$, we have $y_i \in \{0, \cdots, b-1\}$ and $y_i \neq x_i$. Clearly $y \in \Sigma_b^+$ and we have,

$$y = (x_0 x_1 x_2 \cdots x_{k-1} y_k y_{k+1} \cdots)$$
$$x = (x_0 x_1 x_2 \cdots x_{k-1} x_k x_{k+1} \cdots)$$

This yields,

$$d(x,y) = \theta^k < \delta$$

Now apply the shift map k times:

$$\sigma^k(x) = (x_k x_{k+1} x_{k+2} \cdots) = (\sigma^k(x)_0 \sigma^k(x)_1 \cdots)$$
$$\sigma^k(y) = (x_k y_{k+1} y_{k+2} \cdots) = (\sigma^k(y)_0 \sigma^k(y)_1 \cdots)$$

Then we have,

$$d(\sigma^k(x), \sigma^k(y)) = \theta^1$$

If we fix $\epsilon = \theta^2$, then we have,

$$d(\sigma^k(x), \sigma^k(y)) = \theta^1 > \epsilon$$

Since x and δ were arbitrary, this holds for every $x \in \Sigma_b^+$ and every $\delta > 0$. Hence, σ has sensitive dependence.

Now we will prove that $\text{Per}(\sigma)$ is dense in Σ_b^+.

For a point $x \in \Sigma_b^+$ to be periodic, it must be the repetition for some block of symbols in $\{1, \cdots, b-1\}$. That is, there is some $n \in \mathbb{N}$ such that $x_{[0,n-1]} = x_{[n,2n-1]} = x_{[2n,3n-1]} = \cdots$.

Now fix $y \in \Sigma_b^+$ and fix $\epsilon > 0$. Then consider the set,

$$B_\epsilon(y) = \{x \in \Sigma_b^+ : d(x, y)\} < \epsilon$$

So for every $x \neq y \in B_\epsilon(y)$, we have $d(x, y) = \theta^{\min\{k: x_k \neq y_k\}} < \epsilon$

Now let $k \in \mathbb{N}$ be the smallest natural number such that $\theta^k < \epsilon$. Then define the block $y_{[0,k-1]}$ using elements of y. Next, define $x_k = y_k + 1 \mod b$. Let $x_y = (y_{[0,k-1]} x_k y_{[0,k-1]} x_k y_{[0,k-1]} x_k \cdots) = (y_0 y_1 \cdots y_{k-1} x_k y_0 y_1 \cdots y_{k-1} x_k \cdots)$. Clearly we have,

$$d(x_y, y) = \theta^k < \epsilon$$

Moreover, we have that $x_{[0,k]} = x_{[k+1,2k+1]} = x_{[2k+2,3k+2]} = \cdots$. So we have that,

$$\sigma^{k+1}(x) = x$$

Hence, x is a periodic point with period $k + 1$.

Since $\epsilon > 0$ and $y \in \Sigma_b^+$ were arbitrary, this holds for any $y \in \Sigma_b^+$ with any choice of $\epsilon > 0$. Hence, we have $\text{Per}(\sigma)$ is dense in Σ_b^+.

Lastly we need to show that σ is topologically transitive. That is, $\exists x \in \Sigma_b^+$ such that the orbit $O(x) = \{\sigma^n(x) : n \in \mathbb{N}_0\}$ of x is dense in Σ_b^+.

Define x to be the sequence of all valid n blocks in Σ_b^+. That is, for every valid block of symbols in Σ_b^+, such as $z_1 z_2 z_3 \cdots z_n$, this block is contained somewhere in x. The number of valid blocks is countable because for each block size n, there are a finite number of combinations. Hence, x is essentially a countable union of finite sets and thus is countable. As a result, x is a valid element of Σ_b^+.

Now fix $y \in \Sigma_b^+$. Let $\epsilon > 0$ and define $k \in \mathbb{N}$ such that it is the smallest natural number such that $\theta^k < \epsilon$.

Now consider the first k elements of y. That is,

$$y_{[0,k-1]} = y_0 y_1 \cdots y_{k-1}$$

We have that $y_{[0,k-1]}$ is a k-block in Σ_b^+ and that x contains every possible block at some point in the sequence. So we know we can apply σ some number m times which yields,

$$\sigma^m(x)_{[0,k-1]} = y_{[0,k-1]}$$

Then we have that,

$$d(\sigma^m(x), y) = \theta^k < \epsilon$$

So we have $\sigma^m(x) \in B_\epsilon(y)$. Since $y \in \Sigma_b^+$ and $\epsilon > 0$ were arbitrary, this holds for any $y \in \Sigma_b^+$ with any choice $\epsilon > 0$. Hence, we have that $O(x)$ is dense in Σ_b^+.

Hence, we have that σ satisfies all three properties and hence σ is chaotic.

Problem 2.

We have that $f : A \to A$ and $g : B \to B$ are continuous maps on compact metric spaces A and B. Recall that f and g being topologically conjugate means there exists a homeomorphism $h : A \to B$ such that,

$$h \circ f \circ h^{-1} = g$$

Or equivalently,

$$h \circ f = g \circ h$$

In addition, h as a homeomorphism means that it is bijective, continuous, and its inverse h^{-1} is continuous.

Now suppose f is chaotic. We know that $\mathrm{Per}(f)$ is dense in X. Let $I \subset B$ be a non-empty open set. Then $h^{-1}(I)$ is a union of open sets by the continuity of h. Let K be one of these open sets in A, and fix a point $y \in \mathrm{Per}(f) \cap K$, which exists since $\mathrm{Per}(f)$ is dense in A. But then, $h(y) \in h(\mathrm{Per}(f)) \cap I$. Thus, $h(\mathrm{Per}(f))$ is dense in B.

Now note that, since y is periodic under f, we have for some $n \in \mathbb{N}$,

$$\begin{aligned} &f^n(p) = p \\ \implies &g^n(h(p)) = h(f^n(p)) = h(p) \end{aligned}$$

Hence, $h(\mathrm{Per}(f)) \subset \mathrm{Per}(g)$. Since $h(\mathrm{Per}(f))$ is dense in B, we must then have that $\mathrm{Per}(g)$, as required.

We also know that f is topologically transitive. That is, $\exists x \in A$ such that $O(x)$ is dense in A.

Fix $y \in B$. Let $\epsilon > 0$ and consider the open set $B_\epsilon(y)$. Now fix J as an open set in B such that $J \cap B_\epsilon(y) = \emptyset$. We want to show that $g^n(y) \in I$ for some $n \in \mathbb{N}$.

Since h is continuous, then $h^{-1}(J)$ maps to a union of open sets in A. In addition, since h is surjective, we have that $h^{-1}(J)$ is non-empty. Let L be one of the open sets in $h^{-1}(J)$. We also have $h^{-1}(B_\epsilon(y))$ is a union of open sets in A. Let K be one of these open sets.

By the transitivity of f, we can find an n and a $x \in K$ with $f^n(x) \in L$. Let $y = h(x)$. Then $x \in I$ and $g^n(y) = g^n(h(x)) = h(f^n(x)) \in h(L) \subset J$.

Hence, for any open set in B, we can find an n such that $g^n(y)$ is in that open set. Thus, $O(y)$ is dense in B and g is topologically transitive.

Lastly, we need to show that the sensitive dependence on initial conditions of f implies the same for g. Let $x \in A$ and let $\epsilon > 0$. Then for all $\delta > 0$, there exists $y \in A$ such that $d(x, y) < \delta$ and there exists $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) \geq \epsilon$.

Since h is continuous, we have that $h(y) \in B_{\delta_1}(h(x))$ for some $\delta_1 > 0$ since $y \in B_\delta(x)$. Now consider $g^n(h(x)) = h(f^n(x))$ and $g^n(h(y)) = h(f^n(y))$. Consider ϵ as fixed in the above paragraph. Since h^{-1} is continuous, we have that if $g^n(h(y)) \in B_{\epsilon_1}(g^n(h(x)))$ for some $\epsilon_1 > 0$, then $h^{-1}(g^n(h(y))) \in B_\epsilon(h^{-1}(g^n(h(y))))$. By the identities shown above, this is equivalent to,

$$f^n(x) \in B_{\epsilon_1}(f^n(y))$$

However, we assumed that $d(f^n(x), f^n(y)) \geq \epsilon$. Hence, we have a contradiction and so we must have that $d(g^n(h(x)), g^n(h(y))) \geq \epsilon_1$. Thus, we have that $d(h(x), h(y)) < \delta_1$, but $d(g^n(h(x)), g^n(h(y))) \geq \epsilon_1$. Since x and ϵ_1 were arbitrary, we have that this holds for every $x \in A$ and every $\epsilon_1 > 0$. Thus, g also has the sensitive dependence property.

From the above 3 parts, we have proved that if f is chaotic and topologically conjugate to g, then g is chaotic.

The other direction of the proof is completed analogously to the first direction. We must simply swap the roles of f and g as well as h and h^{-1},

Problem 3.

Let $f_\mu = \mu x(1-x)$ where $\mu > 2+\sqrt{5}$. Recall that sensitive dependence on initial conditions means there exists $\epsilon > 0$ such that for all $x \in X$ and all $\delta > 0$, there exists $y \in X$ with $d(x,y) < \delta$ and there exists $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) \geq \epsilon$.

Let us now see if there are multiple fixed points for f_μ. We clearly have that,

$$0 = \mu \cdot 0(1-0) = f_\mu(0)$$

So then $x = 0$ is a fixed point of f_μ. In addition, we have,

$$\begin{aligned} f_\mu(1-\frac{1}{\mu}) &= \mu \cdot (1-\frac{1}{\mu}) \cdot (1-(1-\frac{1}{\mu}) \\ &= (\mu-1)\cdot(\frac{1}{\mu}) \\ &= 1-\frac{1}{\mu} \end{aligned}$$

Hence, we have that $1-\frac{1}{\mu}$ is a fixed point of f_μ as well. Note that,

$$\begin{aligned} 1-\frac{1}{\mu} &> 1-\frac{1}{2+\sqrt{5}} \\ &= 3-\sqrt{5} \end{aligned}$$

So $3-\sqrt{5} \approx 0.7639 < 1-\frac{1}{\mu} < 1$ for every $\mu > 2+\sqrt{5}$. Hence, both 0 and $1-\frac{1}{\mu}$ are in the interval of interest $[0,1]$ for every choice of $\mu > 2+\sqrt{5}$.

As a result, f_μ has at least two distinct periodic orbits.

Now fix two distinct periodic points, r and s so that,

$$d(f_\mu^k(r), f_\mu^\ell(s)) > 0$$

for all k and ℓ. Choose c such that $2c < \min d(f_\mu^k(r), f_\mu^\ell(s))$. Then for all k and ℓ, and for any $x \in [0,1]$, we have,

$$\begin{aligned} 2c &< d(f_\mu^k(r), f_\mu^\ell(s)) \\ &\leq d(f_\mu^k(r), x) + d(x, f_\mu^\ell(s)) \end{aligned}$$

If $d(f_\mu^k(r), x) \leq c$, then by the above derivation, $d(x, f_\mu^\ell(s)) \geq c$. Similarly, if $d(x, f_\mu^\ell(s)) \leq c$, then $d(f_\mu^k(r), x) \geq c$.

Since f_μ maps the unit interval into itself, for any $f_\mu^k(x_1)$, we can find an $x = f^n(x_1)$ with $x \in [0,1]$ and $d(x,p) < \delta$ for any $\delta > 0$ and some periodic point p. Hence, by what we have proved above, we have that,

$$d(f_\mu^k(p), x = f_\mu^k(x_1)) \geq c > 0$$

If we let $\epsilon = c$, then we have proved sensitive dependence for f_μ with $\mu > 2+\sqrt{5}$.

Problem 4. Bonus problem

Under the angle doubling map, we have that,

$$\theta \mapsto 2\theta \pmod{2\pi}$$