Dynamical Systems: Homework 3

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Problem 1.

Recall that Σ_b^+ is a one-sided shift map. That is,

$$\Sigma_b^+ = \{x = (x_k)_{k=0}^{\infty} : x_k \in \{0, \cdots b-1\}\}$$

The shift map $\sigma : \Sigma_b^+ \to \Sigma_b^+$ is defined as follows:

$$\sigma((x_k)_{k=0}^{\infty}) = (x_{k+1})_{k=0}^{\infty}$$

That is, we are essentially cutting off the first element of the sequence.

Lastly, we have that,

$$d(x,y) = d_\theta(x,y) = \begin{cases} 0 & \text{if } x = y \\ \theta^{\min\{k : x_k \neq y_k\}} & \text{if } x \neq y \end{cases}$$

where $0 < \theta < 1$.

We will start now by proving that σ has sensitive dependence.

Now fix $x = (x_0 x_1 x_2 \cdots) \in \Sigma_b^+$ and let $\delta > 0$. Fix $k \in \mathbb{N}$ such that it is the smallest natural number with $\theta^k < \delta$. Now define y such that $y_i = x_i$ for $0 \leq i \leq k-1$ and such that, for every index $i > k$, we have $y_i \in \{0, \cdots, b-1\}$ and $y_i \neq x_i$. Clearly $y \in \Sigma_b^+$ and we have,

$$y = (x_0 x_1 x_2 \cdots x_{k-1} y_k y_{k+1} \cdots)$$
$$x = (x_0 x_1 x_2 \cdots x_{k-1} x_k x_{k+1} \cdots)$$

This yields,

$$d(x,y) = \theta^k < \delta$$

Now apply the shift map k times:

$$\sigma^k(x) = (x_k x_{k+1} x_{k+2} \cdots) = (\sigma^k(x)_0 \sigma^k(x)_1 \cdots)$$
$$\sigma^k(y) = (x_k y_{k+1} y_{k+2} \cdots) = (\sigma^k(y)_0 \sigma^k(y)_1 \cdots)$$

Then we have,

$$d(\sigma^k(x), \sigma^k(y)) = \theta^1$$

If we fix $\epsilon = \theta^2$, then we have,

$$d(\sigma^k(x), \sigma^k(y)) = \theta^1 > \epsilon$$

Since x and δ were arbitrary, this holds for every $x \in \Sigma_b^+$ and every $\delta > 0$. Hence, σ has sensitive dependence.

Now we will prove that $\text{Per}(\sigma)$ is dense in Σ_b^+.

For a point $x \in \Sigma_b^+$ to be periodic, it must be the repetition for some block of symbols in $\{1, \cdots, b-1\}$. That is, there is some $n \in \mathbb{N}$ such that $x_{[0,n-1]} = x_{[n,2n-1]} = x_{[2n,3n-1]} = \cdots$.

Now fix $y \in \Sigma_b^+$ and fix $\epsilon > 0$. Then consider the set,

$$B_\epsilon(y) = \{x \in \Sigma_b^+ : d(x, y)\} < \epsilon$$

So for every $x \neq y \in B_\epsilon(y)$, we have $d(x, y) = \theta^{\min\{k: x_k \neq y_k\}} < \epsilon$

Now let $k \in \mathbb{N}$ be the smallest natural number such that $\theta^k < \epsilon$. Then define the block $y_{[0,k-1]}$ using elements of y. Next, define $x_k = y_k + 1 \mod b$. Let $x_y = (y_{[0,k-1]} x_k y_{[0,k-1]} x_k y_{[0,k-1]} x_k \cdots) = (y_0 y_1 \cdots y_{k-1} x_k y_0 y_1 \cdots y_{k-1} x_k \cdots)$. Clearly we have,

$$d(x_y, y) = \theta^k < \epsilon$$

Moreover, we have that $x_{[0,k]} = x_{[k+1,2k+1]} = x_{[2k+2,3k+2]} = \cdots$. So we have that,

$$\sigma^{k+1}(x) = x$$

Hence, x is a periodic point with period $k + 1$.

Since $\epsilon > 0$ and $y \in \Sigma_b^+$ were arbitrary, this holds for any $y \in \Sigma_b^+$ with any choice of $\epsilon > 0$. Hence, we have $\text{Per}(\sigma)$ is dense in Σ_b^+.

Lastly we need to show that σ is topologically transitive. That is, $\exists x \in \Sigma_b^+$ such that the orbit $O(x) = \{\sigma^n(x) : n \in \mathbb{N}_0\}$ of x is dense in Σ_b^+.

Define x to be the sequence of all valid n blocks in Σ_b^+. That is, for every valid block of symbols in Σ_b^+, such as $z_1 z_2 z_3 \cdots z_n$, this block is contained somewhere in x. The number of valid blocks is countable because for each block size n, there are a finite number of combinations. Hence, x is essentially a countable union of finite sets and thus is countable. As a result, x is a valid element of Σ_b^+.

Now fix $y \in \Sigma_b^+$. Let $\epsilon > 0$ and define $k \in \mathbb{N}$ such that it is the smallest natural number such that $\theta^k < \epsilon$.

Now consider the first k elements of y. That is,

$$y_{[0,k-1]} = y_0 y_1 \cdots y_{k-1}$$

We have that $y_{[0,k-1]}$ is a k-block in Σ_b^+ and that x contains every possible block at some point in the sequence. So we know we can apply σ some number m times which yields,

$$\sigma^m(x)_{[0,k-1]} = y_{[0,k-1]}$$

Then we have that,

$$d(\sigma^m(x), y) = \theta^k < \epsilon$$

So we have $\sigma^m(x) \in B_\epsilon(y)$. Since $y \in \Sigma_b^+$ and $\epsilon > 0$ were arbitrary, this holds for any $y \in \Sigma_b^+$ with any choice $\epsilon > 0$. Hence, we have that $O(x)$ is dense in Σ_b^+.

Hence, we have that σ satisfies all three properties and hence σ is chaotic.

Problem 2.

Recall that f and g being topologically conjugate means there exists a homeomorphism $h : A \to B$ such that,

$$h \circ f \circ h^{-1} = g$$

Or equivalently,

$$h \circ f = g \circ h$$

In addition, h as a homeomorphism means that it is bijective, continuous, and its inverse h^{-1} is continuous.

Now suppose f is chaotic. We know that f is topologically transitive. That is, $\exists x \in A$ such that $O(x)$ is dense in A.

Fix $y \in B$. Let $\epsilon > 0$ and consider the open set $B_\epsilon(y)$. We have that h^{-1} is continuous. Take $h^{-1}(y_1) = x$

Problem 3.

Problem 4.