Dynamical Systems: Homework 1

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Problem 1.

The following are definitions:

(a) (X, d) is a metric space if

  (i) $X \neq \emptyset$
  (ii) $d : X \times X \to \mathbb{R}_0^+$
  (iii) $d(x, y) = 0 \iff x = y$
  (iv) $d(x, y) = d(y, x) \; \forall \; x, y \in X$
  (v) $d(x, y) \leq d(x, z) + d(z, y) \; \forall \; x, y, z \in X$

(b) A metric space (X, d) is a complete metric space if every Cauchy sequence in X converges to a limit in X.

(c) The topology $\tau = \tau(d)$ induced on X by d is the set $\tau = \{U \subset X : U \text{ open}\}$. It has the following properties:

  (i) $\emptyset, X \in \tau$
  (ii) if $U_i, \cdots, U_\ell \in \tau$, then $U_1 \cap \cdots \cap U_\ell \in \tau$
  (iii) if $(U_\alpha)_{\alpha \in I}$ is a family of open sets, then $\cup_{\alpha \in I} U_\alpha$ is open, ie., is in τ

(d) A metric space (X, d) is compact if for every open cover of X there exists a finite subcover.

Problem 2.

We have that,

$$\Sigma = \Sigma_b = \{X = (x_k)_{k=-\infty}^{\infty} : x_k \in \{0, \cdots, b-1\}\}$$

and,

$$d(x, y) = d_\theta(x, y) = \begin{cases} 0 & \text{if} \quad \text{x = y} \\ \theta^{\min\{|k| : x_k \neq y_k\}} \end{cases}$$

where $0 < \theta < 1$

We use these definitions in the following problems,

(a) We need to show that (Σ, d) satisfies the properties of a metric space as described in 1a.

We know that $\Sigma \neq \emptyset$ because $(x_k) = \{0, 0, 0, 0, \cdots\} \in \Sigma$. From the definition of d above, we also have that $d : \Sigma \times \Sigma \to \mathbb{R}_0^+$.

Now suppose $x \neq y$ and suppose $d(x, y) = \theta^{\min\{|k| : x_k \neq y_k\}} = 0$. Let $k_0 = \min\{|k| : x_k \neq y_k\}$. Then,

$$\theta^{k_0} = 0$$
$$\implies \theta = 0$$

However, we know $0 < \theta < 1$. Hence, we have a contradiction and thus $d(x, y) > 0$ if $x \neq y$. Thus, $d(x, y) = 0 \iff x = y$.

Now let $x, y \in \Sigma$. Suppose $x = y$. Then clearly $d(x, y) = 0 = d(y, x)$. Now suppose $x \neq y$. Then,

$$d(x, y) = \theta^{\min\{|k| : x_k \neq y_k\}}$$

and

$$d(y, x) = \theta^{\min\{|k| : y_k \neq x_k\}}$$

Note that $x_k = y_k \iff y_k = x_k$ and $x_k \neq y_k \iff y_k \neq x_k$.

Hence, we can rewrite $d(y, x)$ as

$$d(y, x) = \theta^{\min\{|k| : x_k \neq y_k\}} = d(x, y)$$

as required.

Now let $x, y \in \Sigma$. Suppose $x = y$. Then $d(x, y) = 0$ and we have trivially that $d(x, y) \leq d(x, z) + d(z, y)$ for any $z \in \Sigma$ because $d(x, z), d(z, y) \geq 0$ since $\theta > 0$.

Now suppose $x \neq y$. Then,

$$d(x, y) = \theta^{\min\{|k| : x_k \neq y_k\}}$$

Let $k_0 = \min\{|k| : x_k \neq y_k\}$

Fix $z \in \Sigma$. Then,

$$d(x, z) = \theta^{\min\{|k| : x_k \neq z_k\}}$$

Let $k_1 = \min\{|k| : x_k \neq z_k\}$

We also have

$$d(z, y) = \theta^{\min\{|k|: z_k \neq y_k\}}$$

Let $k_2 = \min\{|k| : z_k \neq y_k\}$.

Suppose $k_1 \geq k_2$. So we have,

$$\begin{aligned} d(x, z) + d(z, y) &= \theta^{k_1} + \theta^{k_2} \\ &= \theta^{k_2}(\theta^{k_1 - k_2} + 1) \\ &\geq \theta^{k_2} \end{aligned}$$

(b) Let (y_k) be a Cauchy sequence of elements in Σ. Then for every $\epsilon > 0$, there exists an integer $N > 0$ such that for all integers $m, n > N$, we have,

$$d(y_m, y_n) < \epsilon$$

That is,

$$d(y_m, y_n) = \theta^{\min\{|k|: y_m \neq y_n\}} < \epsilon$$

(c) Let the shift map $\sigma : \Sigma \to \Sigma$ be defined by $\sigma(x) = (x_{k-1})_{k=-\infty}^{\infty}$

Problem 3.

(a) Let $0 < \mu < 1$ and $x \in (0, 1]$.

Then,

$$\begin{aligned} f_\mu(x) &= \mu x(1 - x) \\ &< x(1 - x) \\ &< x \end{aligned}$$

since $0 \leq 1 - x < 1$ and $0 < \mu < 1$.

Let $x_1 = f_\mu(x)$. Then,

$$\begin{aligned} & f_\mu(x_1) < x_1 \\ \iff & f_\mu(f_\mu(x)) < f_\mu(x) \\ \iff & f_\mu^2(x) < f_\mu(x) < x \end{aligned}$$

In general, for $i \in \{1, 2, \cdots\}$, we have that,

$$f_\mu^i(x) < f_\mu^{i-1}(x) < \cdots < f_\mu(x) < x$$

In addition, we have that $0 \leq f_\mu(x) < 1 \; \forall \; x \in (0, 1]$, so we get,

$$0 \leq f_\mu^i(x), \; \forall \; i$$

Hence,

$$0 \leq \cdots < f_\mu^i(x) < f_\mu^{i-1}(x) < \cdots < f_\mu(x) < x$$

is a montonically decreasing sequence of real numbers with lower bound of 0.

Hence, by the Monotone Convergence Theorem, we get

$$\lim_{n \to \infty} f_\mu^n(x) = 0$$

for all $x \in (0, 1]$

(b) Now let $\mu > 1$ and $x \in \mathbb{R} \setminus [0, 1]$.

Note that if $x > 1$ or $x < 0$, we have that $x(1 - x) < 0$.

If $x > 1$, we have,

$$\begin{aligned} f_\mu(x) &= \mu x(1 - x) \\ &< 0 \end{aligned}$$

and thus,

$$f_\mu(x) = \mu x(1 - x) < x$$

In addition, if $x < 0$, we have

$$\begin{aligned} f_\mu(x) &= \mu x(1 - x) \\ &< x(1 - x) \\ &< x \end{aligned}$$

since $(1 - x) > 1$ when $x < 0$.

Now we just need to show that there is no lower bound for the sequence $\lim_{n \to \infty} f_\mu^n(x)$

Fix x and suppose we have a lower bound m for the above sequence.

Then $m \leq f_\mu^i(x)$ for all i.

Note that as x becomes more negative, the rate of changes gets larger. That is,

$$|f_\mu^2(x) - f_\mu(x)| < |f_\mu^3(x) - f_\mu^2(x)|$$

because $|x(1-x)|$ gets larger as x becomes more negative.

However, if m is a lower bound, then this rate of change must decrease at some point.

We know that it doesn't so there cannot be a lower bound for the sequence. Hence,

$$\lim_{n\to\infty} f_\mu^n(x) = -\infty$$

when $x \in \mathbb{R} \setminus [0,1]$ and $\mu > 1$