# Dynamical Systems I: Important Definitions, Theorems, Lemmas, Propositions, and Corollaries

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## 1 Introduction

**Phase space:** space of all possible states: set $X, M, Y, \Omega$, etc. often with a structure:

- Topological space (metric space)
- Vector space: $\mathbb{R}^n, \mathbb{C}^n$
- Differentiable manifold: $S^1 = \{x \in \mathbb{R}^2 : ||x|| = 1\}$

**Metric space:** $(X, d)$ is a metric space if:

1. $X \neq \emptyset$
2. $d : X \times X \to \mathbb{R}_0^+$
3. $d(x, y) = 0 \iff x = y$
4. $d(x, y) = d(y, x)$ for all $x, y \in X$
5. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$

**Open set:** Let $(X, d)$ be a metric space. $U \subset X$ is open if for all $x \in U$, $\exists r > 0$ such that $B(x, r) = \{y \in X : d(x, y) < r\} \subset U$

**Compact set:** $X$ is a compact set if for every open cover of $X$ there exists a finite subcover.

**Heine-Borel Theorem:** A set $X \subset \mathbb{R}^n$ is compact if and only if $X$ is closed and bounded.

**Topology:** $(X, \tau)$ where $\tau = \{U \subset X : U \text{ open}\}$ is a topology if,

1. $\phi, X \in \tau$
2. if $U_1, \ldots, U_\ell \in \tau$, then $U_1 \cap \cdots \cap U_\ell \in \tau$

3. if $(U_\alpha)_{\alpha \in I}$ is a family of open sets, then $\bigcup_{\alpha \in I} U_\alpha$ is open, i.e., in $\tau$

**Continuity at a point:** The following are equivalent to $f : X \to Y$ being continuous at $x_0 \in X$:

1. if $\lim_{n\to\infty} = x_n = x_0$, then $\lim_{n\to\infty} f(x_n) = f(x_0)$
2. $\lim_{x\to x_0} f(x) = f(x_0)$
3. $\forall \epsilon > 0, \exists \delta > 0$ such that if $x \in X$ with $d(x, x_0) < \delta$, then $d'(f(x), f(x_0)) < \epsilon$

A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converges to $x_0$ if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n \geq N$, then $d(x_n, x_0) < \epsilon$. We write $\lim_{n\to\infty} x_n = x_0$

$f : X \to Y$ is continuous if $\forall U \subset Y$ open, $f^{-1}(U)$ is open in $X$

**Cauchy sequence:** $(X, d)$ metric space. We say $(x_n)_{n \in mathbbN} \subset X$ is a Cauchy sequence if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $d(x_k, x_n) < \epsilon$ whenever $k, n \geq N$.

**Complete metric space:** $(X, d)$ is a complete metric space if Cauchy sequences converge.

**Fact:** If $(X, d)$ is a complete metric space and $Y \subset X$ is closed, then $(Y, d|_Y)$ is a complete metric space.

**Disconnected set:** $Y \subset X$ is disconnected if there exist $U, V \subset X$ open such that,

1. $U \cap Y, V \cap Y \neq \emptyset$
2. $(U \cap Y) \cap (V \cap Y) = \emptyset$
3. $(U \cap Y) \cup (V \cap Y) = Y$

We say that $(U, V)$ is a disconnection of $Y$ in this case.

**Connected set:** $Y \subset X$ is connected if it is not disconnected.

**Theorem:** Let $Y \subset \mathbb{R}$. Then $Y$ is connected if and only if $Y$ is an interval.

**Connected component:** Let $Y \subset X$ and $a \in Y$. Then,

$$C_a = \bigcup_{A \subset Y \text{ connected}, a \in A} A$$

is called the connected component of $a$ relative to $Y$. It is "the largest connected set in $Y$ containing $a$".

**Cantor sets:** totally disconnected, compact, perfect metric space

## 1.1 Review from Advanced Calc I

Let $D \subset \mathbb{R}$, $x_0 \in D$ be an accumulation point of $D$, i.e., $\forall \epsilon > 0 \exists x \in D$ such that $|x - x_0| < \epsilon$ and $x \neq x_0$.

We say $f : D \to \mathbb{R}$ is differentiable at $x_0$ if,

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. This definition is equivalent to that $\exists L \in \mathbb{R}$ such that $\forall \epsilon > 0$, $\exists \delta > 0$ such that if $x \in D$ with $0 < |x - x_0| < \delta$, then

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - L \right| < \epsilon$$

$f : D \to \mathbb{R}$ is differentiable if $g$ is differentiable at all $x_0 \in D$

If $f'$ is continuous, then we say $f$ is one-times continuously differentiable, we write $f \in C^1(D, \mathbb{R})$

Suppose $x \mapsto f(x)$ is k-times differentiable and $x \mapsto f^{(k)}(x)$ is differentiable. We call,

$$f^{(k+1)}(x) = \left( f^{(k)}(x) \right)'$$

the (k+1)-th derivative of $f$.

If $f$ is k-times differentiable and $f \mapsto f^{(k)}(x)$ is continuous, we say,

$$f \in C^k(D, \mathbb{R})$$

If $f \in C^k(D, \mathbb{R}) \forall k \in \mathbb{N}$, then we write $f \in C^\infty(D, \mathbb{R})$

**Analytic functions:** $D \subset \mathbb{R}$ open, $f : D \to \mathbb{R}$. We say $f$ is analytic if $\forall x_0 \in D, \exists \epsilon > 0$ and $(a_k)_{k=0}^\infty \subset \mathbb{R}$ such that

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

If this is the case, then $a_k = \frac{f^{(k)}(x_0)}{k!}$.

We denote the set of all analytic functions on $D$ by $C^\omega(D, \mathbb{R})$.

**Fact:** $C^\infty(D, \mathbb{R}) \supset C^\omega(D, \mathbb{R})$ but not $C^\infty(D, \mathbb{R}) \subset C^\omega(D, \mathbb{R})$

**Min-Max Theorem:** Let $f : [a, b] \to \mathbb{R}$ be continuous. Define $m(f) = \inf\{f(x) : x \in [a, b]$ and $M(f) = \sup\{f(x) : x \in [a, b]\}$. Then $\exists \underline{x}, \overline{x} \in [a, b]$ such that $f(\underline{x}) = m(f)$,

$f(\overline{x}) = M(f)$

**Intermediate Value Theorem:** Let $f : [a, b] \to \mathbb{R}$ be continuous and let $m(f) < d < M(f)$. Then $\exists c \in [a, b]$ such that $f(c) = d$.

**Mean Value Theorem:** Let $f : [a, b] \to \mathbb{R}$ be continuous and $f$ differentiable in $(a, b)$. Then $\exists c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

**Cauchy Mean Value Theorem:** Let $f, g : [a, b] \to \mathbb{R}$ be continuous and differentiable on $(a, b)$. Then $\exists c \in (a, b)$ such that $f'(c)\,(g(b) - g(a)) = g'(c)\,(f(b) - f(a))$

**Rolle's Theorem:** Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable in $(a, b)$. Suppose $f(a) = f(b)$. Then $\exists c \in (a, b)$ such that $f'(c) = 0$.

**One-variable version of Implicit Function Theorem:** Let $U \subset \mathbb{R}^2$ be open and let $P : U \to \mathbb{R}$, $(u, x) \mapsto P(u, x)$ be $C^1$ and suppose that $(u_0, x_0) \in U$ such that $P(u_0, x_0) = 0$. Suppose $\frac{\partial P}{\partial x}(u_0, x_0) \neq 0$. Then $\exists \delta > 0$ and $\epsilon > 0$ such that the equation

$$P(u, x) = 0$$

has a unique solution $x = g(u)$ in $|u - u_0| < \delta$ and $|x - x_0| < \epsilon$. Moreover, $g$ is differentiable, and,

$$\frac{\partial g}{\partial u} = -\left[\frac{\partial P}{\partial x}\right]^{-1} \cdot \left[\frac{\partial P}{\partial u}\right]$$

**Banach Fixed Point Theorem:** Let $(X, d)$ be a complete metric space and let $f : X \to X$ be a contraction, i.e., $\exists 0 < L < 1$ such that $d(f(x), f(y)) \leq Ld(x, y)$ for all $x, y \in X$. Then $f$ has a unique fixed point.

# 2 One-Dimensional Dynamics

## 2.1 Intro

**Time evolution law:** discrete times $T = \{0, 1, 2, \cdots, \} = \mathbb{N} \cup \{0\}$. eg. $f : X \to X$ where $X$ is a phase space, then $x_0 \to f(x_0) \to f^2(x_0) = f(f(x_0)) \to f^3(x_0) \to \cdots$

**Forward orbit of $x_0$:** $O^+(x_0) = \{x_0, f(x_0), f^2(x_0), \ldots\}$

**Backward orbit of $x_0$:** If $f$ is invertible, then $f^{-(n+1)}(x_0) = f^{-1}(f^{-n}(x_0))$ and the backward orbit is given by $O^-(x_0) = \{x_0, f^{-1}(x_0), f^{-2}(x_0), \ldots\}$

**Fixed point:** A point $x$ is a fixed point of a function $f$ if $f(x) = x$

**Periodic Point:** $f : X \to X$ is a periodic point if $\exists p \in \mathbb{N}$ such that $f^p(x) = x$. $p$ is called the period of $x$ if it is the minimal natural number satisfying $f^p(x) = x$.

**Homeomorphism:** A function $h : X \to Y$ is a homeomorphism if $h$ is bijective, continuous, and $h^{-1}$ is continuous.

**S. Smale Conjecture:** A "typical" higher dimensional system has finitely many periodic points.

**Topologically conjugate:** $f : X \to X$ is topologically conjugate to $\sigma : Y \to Y$ if there exists a homeomorphism $h : X \to Y$ such that $f = h^{-1} \circ \sigma \circ h$. (Note this implies $f^n = h^{-1} \circ \sigma^n \circ h$)

**Chaos:** Let $V$ be a set. $f : V \to V$ is said to be chaotic on $V$ if

1. $f$ has sensitive dependence on initial conditions. That is, $\exists \delta > 0$ such that for any $x \in V$ and any neighborhood $N$ of $x$, there exists $y \in N$ and $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$.
2. $f$ is topologically transitive. That is, $\exists x \in V$ such that the orbit $O(x) = \{f^n(x) : n \in \mathbb{N}_0\}$ of $x$ is dense in $V$.
3. The periodic points of $f$ $(Per(f))$ are dense in $V$

## 2.2 Continuous Time Dynamics in $\mathbb{R}$

**Flow:** Let $(X, d)$ be a metric space. A map $\phi : \mathbb{R} \times X \to X$ is called a flow if:

1. $\phi(s + t, x) = \phi(s, \phi(t, x))$ for all $s, t \in \mathbb{R}$ and for all $x \in X$. (called "flow property")
2. for all $t \in \mathbb{R}$, $\phi(t, \cdot) = \phi_t : X \to X$ is a homeomorphism of $X$
3. $\phi(0, \cdot) = \mathrm{id}_X$

## 2.3 Entropy

**Idea:** assign to each dynamical system a positive real number $h_{top}(f)$ where $f : X \to X$ is continuous and $(X, d)$ is a compact metric space such that,

1. if $f$ and $g$ are topologically conjugate then $h_{top}(f) = h_{top}(g)$
2. if $h_{top}(f) > 0$, then $f$ is chaotic
3. if $0 < h_{top}(f) < h_{top}(g)$, then $g$ is more chaotic than $f$

Entropy will be the exponential growth rate of the number of distinc orbits.

**Bowen metric:** $d_n(x, y) = \max_{k=0,\dots,n-1} d(f^k(x), f^k(y))$

Let $\epsilon > 0$, $n \in \mathbb{N}$ be fixed. $I \subset X$ is called $(n, \epsilon)-$separated if $\forall x, y \in I$ with $x \neq y$, we have $d_n(x, y) \geq \epsilon$. Assume $I_n(\epsilon)$ is a *maximal* $(n, \epsilon)-$separated set, i.e.,

1. $I_n(\epsilon)$ is $(n, \epsilon)-$separated
2. if $y \in X \setminus I_n(\epsilon)$, then $I_n(\epsilon) \cup \{y\}$ is not $(n, \epsilon)-$separated.

**Remark:** Since $X$ is compact, $I_n(\epsilon)$ is finite.

**Entropy:** $h_{top}(f) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log(\text{card } I_n(\epsilon))$

Example: card $I_n(\epsilon) = 2^n \implies \frac{1}{n} \log 2^n = \log 2 \implies h_{top}(f) = \log 2$

## 2.4 Attractors and repellers

From now on, $f$ is a $C^2$-function. $f : X \to X$, $X \subset \mathbb{R}$ an interval ($X = [a, b]$). Suppose $f(a) = a$, $a \in X$. We say $a$ is **attracting** if $|f'(a)| < 1$ (stable fixed point) and if $f'(a) = 0$, we say $a$ is **superattracting**.

$a$ is **repelling** if $|f'(a)| > 1$. If $a$ is either attracting or repelling we say $a$ is **hyperbolic**.

Now let $a \in X$ be an attracting fixed point. Define the **stable set** of $a$ by,

$$W^s(a) = \{x \in X : f^n(x) \to a as n \to \infty\}$$

Fact: $W^s(a)$ is open set containing $a$.

**Lemma:** $\exists \epsilon > 0$ such that $B(a, \epsilon) \subset W^s(a)$

**Corollary:** $W^s(a) = \bigcup_{n \in \mathbb{N}} f^{-n}(B(a, \epsilon))$

**Definition:** $W^s_{loc}(a)$ is the connected component of $W^s(a)$ containing $a$. It is called the *immediate basin of attraction* of $a$.

**Higher period orbits:** $a \in Per_p(f) \iff f^p(a) = a$ and $f^k \neq a$ for all $k = 1, \ldots, p-1$. We say $a$ is attracting if it is an attracting fixed point of $f^p$ (that is, $|(f^p)'(a)| < 1$.

$W^s(a) = \{x : \lim_{n\to\infty} |f^n(a) - f^n(a)| = 0\}$

$W^s(a) = W^s(f^p, a)$

If $a$ is attracting withe period $p$, then $f(a), \ldots, f^{p-1}(a)$ are also attracting with period $p$.

$W^s(f(a)) = f(W^s(a))$

**Fact:** If $a$ is attracting fixed point, $f(W^s(a)) = f^{-1}(W^s(a)) = W^s(a)$, i.e. "stable set is $f$-invariant"

$f(W^s_{loc}(a)) = W^s_{loc}(a)$

**Definition:** We say $c$ is a critical point of $f$ if $f'(c) = 0$. We say the critical point $c$ is non-degenerate if $f''(x) \neq 0$, otherwise degenerate.

Two types of attracting fixed points:

1. Orientation preserving: $0 < f'(a) < 1$
2. Orientation reversing: $-1 < f'(a) < 0$

**Theorem:** $f : \mathbb{R} \to \mathbb{R}$ continuous, $\exists$ periodic point of prime period 3. Then $\exists$ periodic points of any other period.

**Sarkovskii's Theorem:** Order periods in the following manner:

$$3 \rhd 5 \rhd 7 \rhd \cdots 2 \cdot 3 \rhd 2 \cdot 5 \rhd 2 \cdot 7 \rhd \cdots \rhd 2^2 \cdot 3 \rhd 2^2 \cdot 5 \rhd 2^3 \cdot 7 \rhd \cdots \rhd 2^3 \cdot 3 \rhd 2^3 \cdot 5 \rhd 2^3 \cdot 7 \rhd \cdots$$

Now let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If $k \rhd \ell$ in the order above and if $f$ has a periodic point of prime period $k$, then $f$ has a periodic point of prime period $\ell$.

**Corollary:** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and suppose $f$ has finitely many periodic points. Then all these periods are powers of 2.

## 2.5 The Logistic Family

For every $\mu$ we consider the dynamical system $f_\mu : \mathbb{R} \to \mathbb{R}$, $f_\mu(x) = \mu x(1-x)$.

This is a "model in population dynamics". $0 \leq x \leq 1$, the function gives the percentage of max possible population.

For $0 < \mu < 1$, the only fixed point in $[0, 1]$ is $f(0) = 0$. For every $x \in [0, 1] : f^n(x) \to 0$ as $n \to \infty$.

0 is an attracting fixed point since $|f'(0) < 1$. 1 is eventually periodic.

$\frac{\mu-1}{\mu}$ is a fixed point in $(0, 1)$ whenever $\mu > 1$.

**Proposition:** If $\mu > 1$ and $x \in [0, 1]$, then $f_\mu^n(x) \to -\infty$ as $n \to \infty$.

**Proposition:** Let $1 < \mu < 3$:

(a) $f_\mu$ has an attracting fixed point $p_\mu = \frac{\mu-1}{\mu}$ and repelling fixed point 0

(b) if $0 < x < 1$, then $\lim_{n\to\infty} f_\mu^n(x) = p_\mu$. That is, $W^s_{loc}(p_\mu) = (0, 1)$

**For $\mu > 4$:**

**Theorem:** There exists Cantor set $C \subset [0, 1]$ such that if $x \in \mathbb{R} \setminus C$, then $|f^n(x)| \to \infty$ and $C$ is $f$-invariant, i.e., $f(C) = C = f^{-1}(C)$ where $f^{-1}(C) = \{x \in \mathbb{R} : f(x) \in C\}$. $f|_C$ is chaotic

## 2.6 Bifurcation Theory

"Change of the behavior in family of dynamical systems"
$f_\lambda : I \to I, \lambda$ in some interval

1. for $\lambda$ fixed, $f_\lambda$ is a $C^\infty$ map
2. $f_\lambda$ depends smoothly on $\lambda$ "$(\lambda, x) \mapsto f_\lambda(x)$ is at least $C^1$-map"

**Goal:** Hyperbolic periodic orbits are preserved

**Theorem:** Let $(f_\lambda)_\lambda$ be a smooth family of $C^\infty$ maps on an interval $I \subset \mathbb{R}$. Suppose $\lambda_0$ has the property that $f_{\lambda_0}$ has a hyperbolic periodic point $\alpha_{\lambda_0}$ of prime period $p$ (i.e. $|(f^p)'(\alpha_{\lambda_0})| \neq 1$. Then $\exists \epsilon > 0$ such that $\forall \lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, $f_\lambda$ has a periodic point $\alpha_\lambda$ with prime period $p$. Moreover, $\lambda \mapsto \alpha_\lambda$ is a $C^1$-map. Moreover, by making $\epsilon$ smaller if necessary, we can assure that $\alpha_\lambda$ is hyperbolic for all $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$

Now consider the logistic family, $f_\mu(x) = \mu x(1 - x)$. For $0 < \mu < 1$: one periodic point which is a fixed point. For $2 + \sqrt{5} < \mu$: $|Per_n(f)| = 2^n$ and $Per(f)$ is dense in $[0, 1]$.

**Saddle Node Bifurcation Theorem:** Suppose

1. $f_{\lambda_0}(0) = 0$
2. $f'_{\lambda_0}(0) = 1$
3. $f''_\lambda(0) \neq 0$

4. $\frac{\partial f_\lambda}{\partial \lambda}|_{\lambda=\lambda_0}(0) \neq 0$

Then $\exists$ interval $I$ about 0 and a smooth function $p : I \to \mathbb{R}$ with $p(0) = \lambda_0$ such that $f_{p(x)}(x) = x$ and $p'(0) = 0$, $p''(0) \neq 0$.

The signs in (3) and (4) determine the opening direction.

**Period Doubling Bifurcation:** Let $G(x, \lambda) = f_\lambda(x) - x$, $G(0, \lambda_0) = 0$. 0 is a fixed point of $f_{\lambda_0}$. $(x, \lambda) \mapsto f_\lambda(x) \in C^3$, $L(\lambda) = f'(0, \lambda)$. Assumption:

(a) $f_{\lambda_0}(0) = -1$

(b) $\frac{\partial L}{\partial \lambda}(\lambda_0) > 0$

(c) $2\frac{\partial^3 f_\lambda}{\partial x^3}(\lambda_0, 0) + 3\left(\frac{\partial^2 G}{\partial x^2}(\lambda_0, 0)\right) > 0$

Then $\exists$ non-empty intervals $(\lambda_1, \lambda_0)$ and $(\lambda_0, \lambda_2)$ such that,

1. if $\lambda \in (\lambda_0, \lambda_2)$ then $f$ has a repelling fixed point and an attracting 2-cycle in $(-\epsilon, \epsilon)$.
2. if $\lambda \in (\lambda_1, \lambda_0)$, then $f_\lambda$ has one attracting fixed point on $(-\epsilon, \epsilon)$.

# 3 Complex Dynamics

## 3.1 Introduction to Complex Analysis

$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}\}$ where $i^2 = -1$

Let $z = a_1 + ib_1, w = a_2 + ib_2$. Then addition and multiplication are defined as,

$$z + w = (a_1 + a_2) + i(b_1 + b_2)$$
$$z \cdot w = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)$$

$(\mathbb{C}, +, \cdot)$ is a field. $\mathbb{C}$ is a $\mathbb{C}$-vector space

$|z| = \sqrt{a^2 + b^2}$

$z = |z| \cdot e^{2\pi i\rho}$ for some $\rho \in [0, 1)$.

We can compactify $\mathbb{C}$ by using the one-point compactification:

$$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

We say $U$ is a neighborhood of $\infty$ if it is $\{\infty\}$ union a complement of a compact set in $\mathbb{C}$

$\overline{\mathbb{C}}$ with this topology is called the Riemann sphere

**Fact:** $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is compact

**Definition:** Let $D \subset \mathbb{C}$ be open, $f : D \to \mathbb{C}$ be a function. Let $z_0 \in D$. We say $f$ is complex differentiable at $z_0$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists and we call this limit $f'(z_0)$

**Definition:** We say $f$ is *holomorphic* on $D$ if $f$ is complex differentiable at all $z_0 \in D$.

$f(a + ib) = u(a, b) + iv(a, b)$. $u(a, b) = Re(f(z)), iv(a, b) = Im(f(z))$

**Fact:** $f$ is holomorphic in $D$ if and only if the partial derivatives of $u$ and $v$ exist and are continuous and satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1}$$

$f : D \to \mathbb{C}$ is analytic if for all $\forall z_0 \in D \exists r > 0$ such that,

$$f|_{B(z_0,r)} = \Sigma_{k=0}^{\infty} a_k(z - z_0)^k$$

where $a_k = \frac{f^{(k)}(z_0)}{k!}$. In particular, $f$ analytic implies $f$ is $C^\infty$.

**Fact:** The following are equivalent:

(a) $f$ holomorphic on $D$

(b) $f$ analytic on $D$

**Results:**

(a) Cauchy's Integral Formula: $f$ holomorphic, $\gamma$ rectifiable, curve simply connected $\implies \int_\gamma f(z)dz = 0$

(b) $f'(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^2} dz$

(c) Identity Theorem: Suppose $D \subset \mathbb{C}$ is open and connected, let $g, f : D \to \mathbb{C}$ be holomorphic. Suppose $\exists z_0 \in D$ and $r > 0$ such that $f|_{B(z_0,r)} = g|_{B(z_0,r)}$. Then $f = g$.

## 3.2 Introduction to Complex Dynamics

**Normal families:** $D \subset \mathbb{C}$ open: $\mathcal{F} \subset \{f : D \to \mathbb{C} : f \text{ hol } \} = Hol(D)$

$\mathcal{F}$ is normal if $\forall$ sequences $(f_n)_n \subset \mathcal{F}$ $\exists$ subsequence $(f_{n_k})_{k\in\mathbb{N}}$ which converges uniformly on compact subsets of $D$.

$f_{n_k} \to f : D \to \mathbb{C}$ for all $K \subset D$ compact $\forall \epsilon > 0$ $\exists N \in \mathbb{N}$ such that if $n_k \geq N$ we have $|f_{n_k}(z) - f(z)| < \epsilon$ $\forall z \in K \implies f$ is also holomorphic.

Convergence to $\infty$ is also possible.

**Montel's theorem:** Let $\mathcal{F} \subset Hol(D)$. Then,

(a) $\mathcal{F}$ is normal if $\{f(D) : f \in \mathcal{F}$ is bounded

(b) $\mathcal{F}$ is normal if $|\mathbb{C} \setminus \{f(D) : f \in \mathcal{F}\}| \geq 2$ (This the Big Theorem of Montel)

## 3.3 Julia and Fatou Sets

**Main idea:** Split up $\mathbb{C}$ into two sets $J$ ad $F$ such that $f|_J$ is chaotic and $f|_F$ is stable. We call $J$ the Julia set of $f$ and $F$ the Fatou set of $f$.

**Definition:** Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial with degree $d \geq 2$. Then we call $F_f = \{z \in \mathbb{C} : \{f^n : n \in \mathbb{N}\}$ is normal in a neighborhood of z$\}$ the Fatou set of $f$.

We call $\mathbb{C} \setminus F_f = J_f$ the Julia set $f$.

**Lemma:** $F_f$ is open and $J_f$ is closed.

$f$ is holomorphic if one of the following holds:

(a) $f$ is complex differentiable at all $z \in D$

(b) $f$ has continuous partial derivatives and satisfies equation (1) from Section 3.1 above

(c) $f$ is analytic

Often $D$ domain means $D$ open and connected

**Big Theorem of Montel:** If $\exists \alpha, \beta \in \mathbb{C}$ such that $\bigcup_{f \in \mathcal{F}} f(D) \subset \mathbb{C} \setminus \{\alpha, \beta\}$, then $\mathcal{F}$ is normal.

## 3.4 Polynomials

**Proposition:** Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d \geq 2$. Then $\exists r > 0$ such that if $z \in \mathbb{C}$ with $|z| > r$ then $f^n(z) \to \infty$ as $n \to \infty$.

**Claim:** $W^s(\infty) \neq \mathbb{C}$, i.e. all periodic points cannot converge to $\infty$ and $f(z)$ has $d$ fixed points counted by multiplicity.

**Definition:** Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d \geq 2$. We call $K = K_f = \{z : \{f^n(z) : n \in \mathbb{N}\}$ is bounded$\}$ the filled-in Julia set of $f$

**Theorem:** Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d \geq 2$. Then $J_f = \partial W^s(\infty) = \partial K_f$

**Corollary:** If $f : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree $d \geq 2$, then $J_f \neq \emptyset$

**Definition:** Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d \geq 2$. If $f(z) = z$, then we say,

- $z$ is attracting if $|f'(z)| < 1$
- $z$ is repelling if $|f'(z)| > 1$
- $z$ is parabolic if $f'(z) = e^{2\pi i q}$ for some $q \in \mathbb{Q}$
- elliptic if $f'(z) = e^{2\pi i \rho}$ for some $\rho \in [0, 1) \setminus \mathbb{Q}$

**Proposition:** Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d \geq 2$. Let $z \in Per_n(f)$. Then,

- if $z$ is attracting, then $z \in F_f$
- if $z$ is repelling, then $z \in J_f$

**Theorem:** $J_f$ and $F_f$ are completely invariant sets (that is, $J_f = f(J_f) = f^{-1}(J_f)$ and $F_f = f(F_f) = f^{-1}(F_f)$

**Fatou component:** We define $C_z(F_f)$ as the connected component of $F_f$ which contains $z$ and call it a Fatou component. The exact defintion for $C_z(F_f)$ is:

$$C_z(F_f) = \bigcup_{U \subset F_f \text{ connected}, \ z \in U} U$$

$C_z(F_f)$ is the largest connected subset of $F_f$ which contains $z$.

**Lemma:** If $C$ is a Fatou component, then $f(c)$ is also a Fatou component.

**Possibilities for Fatou Set:**

(a) Periodic Fatou Component: $\exists n \in \mathbb{N}$ such that $f^n(C) = C$

(b) Preperiodic Fatou Component: $\exists n \geq 1$ such that $f^n(C)$ is periodic and $C, \ldots, f^{n-1}(C)$ is not periodic

(c) Wandering Fatou Component: $f^i(C) \cap f^j(C) = \emptyset$ for all $i, j \in \mathbb{N}_0$ with $i \neq j$

**Conjecture:** There are no wandering Fatou components.

**Theorem (Sullivan, 1983):** Absence of wandering Fatou components for rational maps

**Theorem:** If $C$ is a periodic Fatou component, then $C$ is one and only of the following types:

1. $C$ is immediate basin of attraction of an attracting fixed point
2. $C$ is a parabolic domain
3. $C$ is Siegel disk
4. $C$ is a Herman ring

**Proposition:** Let $z$ be a periodic point of period $p \in \mathbb{N}$ and suppose $z$ is attracting (i.e. $|(f^p)'(z)| < 1$. Then $\exists r > 0$ such that if $w \in B(z, r)$ then $(f^p)^n(w) \to z$ as $n \to \infty$

**Basin of attraction:** The basion of attraction of the attracting fixed point $z$ is given by $W^s(z) = \{w \in \mathbb{C} : f^n(w) \to z \text{ as } n \to \infty\} = \bigcup_{n \in \mathbb{N}} f^{-n}(B(z, r))$

We define $W^s_\epsilon(z)$ to be the connected component of $W^s(z)$ which contains $z$. We call $W^s_\epsilon(z)$ the immediate basin of attraction.

**Claim:** $f(W^s_\epsilon(z)) = W^s_\epsilon(z)$

We can also define the basin of attraction of $z$ by

$$W^s(z) = \{w \in \mathbb{C} : |f^n(w) - f^n(z)| \to 0 \text{ as } n \to \infty\}$$

**Theorem:** $f$ is hyperbolic if and only if all critical points of $f$ are contained in the basins of attraction of attracting periodic cycles including $\infty$. Moreover, every basin of attraction of an attracting periodic orbit contains at least one critical point.

**Corollary:** Let $f(z) = z^2 + c$, $c \in \mathbb{C}$. Then 0 is the only critical point of $f$. Therefore, if $f$ has an attracting periodic orbit, then $f$ is hyperbolic.

**Fact:** $J_f \neq 0$, uncountable, perfect (without isolated points), transitive, $J_f = \overline{\text{Rep Per}(f)}$

**Lemma:** Let $z \in \mathbb{C}$ be a repelling periodic point with period $p$. Then $z \in J_f$.

$c \in \mathbb{C}$ is a critical point of $f$ if $f'(c) = 0$. In that case, we call $f(c)$ a critical value.

**Theorem:** Critical points

(a) $J_f$ is connected if and only if all critical points have bounded orbits

(b) $f$ is hyperbolic if and only if all critical points are contained in immediate basin of attraction (including infinity)

**Corollary:** If $f(z) = z^2 + c, c \in \mathbb{C}$ then $f$ is hyperbolic if and only if either $0 \in W^s(\infty)$ or $f$ has an attracting periodic orbit

## 3.5 Hausdorff Dimension

Let $K \subset \mathbb{C}$ be a set, $\delta > 0$. We say $\{D_i : i \in \mathbb{N}\}$ is a $\delta$-cover of $K$ if

1. $D_i$ are balls of diameter less than or equal to $\delta$
2. $K \subset \cup_i D_i$

Let $s \in [0, 2]$. We define,

$$H^s_\delta(K) = \inf\{\Sigma_{i\in\mathbb{N}}(\text{diam}(D_i))^s : (D_i)_i is \delta - \text{cover of } K\}$$

Define $H^s(K) = \lim_{\delta\to 0} H^s_\delta(K) \in [0, \infty]$. $H^s(K)$ is called the s-dimensional outer Hausdorff measure of $K$.

If $s > \alpha$, then $H^s(K) = 0$. If $s < \alpha$, then $H^s(K) = \infty$

We call $\alpha$ the Hausdorff dimension of $K$ $(HD(K))$.

We can define $HD(K)$ in the following ways:

$$HD(K) = \inf\{s \geq 0 : H^s(K) = 0\}$$
$$HD(K) = \sup\{s \geq 0 : H^s(K) = \infty\}$$

**Basic facts of $H^s(K)$:**

1. If $H^s(K) < \infty$, then $H^{s'}(K) = 0$ for all $s' > s$
2. If $H^s(K) = \infty$, then $H^{s'}(K) = \infty$ for all $s' < s$

3. $H^0(K)$ is the cardinality of $K$

4. $H^s(a + K) = H^s(K)$ for all $a \in \mathbb{R}^n$ and $H^s(tA) = t^s H^s(A)$ for all $t \geq 0$ where $a + K = \{a + x : x \in K\}$ and $tK = \{tx : x \in K\}$

5. If $K$ has non-empty interior, then $H^n(K) = C(n) \cdot \mathrm{vol}(K)$ where $C(n) = \frac{2^n}{\mathrm{Vol}(B(0,1))}$

6. If $s > n$, then $H^s(K) = 0$

# 4 Symbolic Dynamics

## 4.1 Intro and Full Shift

**Full shift:** $\Sigma_{\mathcal{A}} = \{x = (x_k)_{k\in\mathbb{Z}} : x_k \in \mathcal{A}\}$. $\mathcal{A}$ is called the alphabet of the full shift.

A special case of the full shift is $\Sigma_d$, where the alphabet is given by $\{0, 1, \ldots, d-1\}$

$0 < \theta < 1$,

$$d(x, y) = d_\theta(x, y) = \begin{cases} 0 & x = y \\ \theta^{\min\{|k| : x_k \neq y\}} & x \neq y \end{cases}$$

**Fact:** $(\Sigma_d, d_\theta)$ compact metric space

**Word:** For all $n \in \mathbb{N}_0$ we call the $n-$tuple $w = w_1 \cdot w_n$, $w_i \in \mathcal{A}$ a word in $\Sigma_{\mathcal{A}}$ of length $n$. The empty word $\epsilon$ has length 0.

If $i \leq j \in \mathbb{Z}$, we write $x_{[i,j]} = x_i x_{i+1} \cdots x_j$. If $i > j$, then $x_{[i,j]}$ is the empty word $\epsilon$. Note that $|x_{[i,j]}| = j - i + 1$.

**Subword:** Let $w = w_1 \cdots w_n$ be a word of length $n$ and let $1 \leq i \leq j \leq n$. Then $v = w_i w_{i+1} \cdots w_j$ is a subword of $w$.

If $w = w_1 \cdots w_n$ and $v = v_1 \cdots v_m$ are words of length $n$ and $m$ respectively, we define the concatenation of $w$ and $v$ as $wv = w_1 \cdots w_n v_1 \cdots v_m$ and hence $|wv| = n + m$.

For all $k \in \mathbb{N}$ we write $w^k = ww \cdots w$, $w^\infty = www \cdots$, ${}^\infty w = \cdots www$

**Shift map:** We define the (left) shift map on $\Sigma_{\mathcal{A}}$ denoted by $\sigma$ as $(\sigma(x))_k = x_{k+1}$ for all $k \in \mathbb{Z}$.

**Entropy of shift map:** If $\sigma : \Sigma_b \to \Sigma_b$ is the shift map on the alphabet with $b$ symbols, then $h_{top}(\sigma) = \log b$.

**Entropy from transition matrix:** If $A$ is the transition matrix of a subshift of finite type, then $h_{top}(\sigma_A) = \log \zeta$ where $\zeta$ is the largest eigenvalue of $A$.

**Proof of entropy of shift map:** Now let $\Sigma_b = \{x = (x_k)_{k=-\infty}^{+\infty} : x_k \in \{0, \ldots, b-1\}\}$ and define $\sigma(x)_k = x_{k+1}$. Fix $\theta \in (0, 1)$ and define $d_\theta$ as above. Define $\epsilon_k = \theta^k$. Then,

$$\begin{aligned} d(x, y) \geq \epsilon_k &\iff x_i \neq y_i \text{ for some } i \in \{-k, \ldots, k\} \\ d(\sigma(x), \sigma(y)) \geq \epsilon_k &\iff x_i \neq y_i \text{ for some } i \in \{-k+1, \ldots, k+1\} \\ &\vdots \\ d(\sigma^{n-1}(x), \sigma^{n-1}(y)) \geq \epsilon_k &\iff x_i \neq y_i \text{ for some } i \in \{-k+n-1, \ldots, k+n+1\} \end{aligned}$$

Thus, we have that,

$$d_n(x,y) \geq \epsilon_k \text{ if } x_i \neq y_i \text{ for some } i \in \{-k, \dots, k+n-1\}$$

The length of this segment ($\{-k, \dots, k+n-1\}$) is $2k+n$, so $|I_n(\epsilon_k) = b^{2k+n}$. Thus,

$$\begin{aligned}\limsup_{n\to\infty} \log b^{2k+n} &= \limsup_{n\to\infty}\left(\frac{2k}{n}\log b\right) + \limsup_{n\to\infty}\left(\frac{n}{n}\log b\right)\\ &= \log b\end{aligned}$$

Thus,

$$h_{top}(\sigma) = \lim_{\epsilon_k\to 0} \log b = \log b$$

**Cylinder set** ($\epsilon_k = \theta^{k+1}, k \in \mathbb{N}, \epsilon > 0$):

$$\begin{aligned}B(x,\epsilon_k) &= \{y \in \Sigma : d(x,y) < \epsilon_k\}\\ &= \{y \in \Sigma : y_{-k} = x_{-k}, y_{-k+1} = x_{-k+1}, \cdots, y_k = x_k\}\\ &= [x]_{-k}^k\\ &= C_{-k,k}(x)\end{aligned}$$

**Fact:** $C_{-k,k}$ is open and closed (clopen).

**Corollary:** $\Sigma_{\mathcal{A}}$ is totally disconnected.

**Shift space:** We say $X \subset \Sigma_{\mathcal{A}}$, $X \subset \Sigma_{\mathcal{A}}$ is a shift space if $X$ is closed and $\sigma$ invariant, in the case of the one-sided shift this means $\sigma(X) = X$. In the case of the two-sided shift space, this means $\sigma(X) = X = \sigma^{-1}(X)$.

We call $(X, \sigma|_X)$ a subshift.

## 4.2 One-Sided Shift Map

$b \in \mathbb{N} \setminus \{1\}$

$$\Sigma^+ = \Sigma_b^+ = \{x = (x_k)_{k=0}^\infty : x_k \in \{0, \dots, b-1\}$$

**Shift map:** $\sigma((x_k)_{k=0}^\infty) = (x_{k+1})_{k=0}^\infty$. We are "cutting off the first element in the sequence"

$\sigma$ is b-to-one. That is, $|\sigma^{-1}(x)| = b$.

Let $0 < \theta < 1$. Then the distance metric (similarly to the full shift) is given by,

$$d(x,y) = d_\theta(x,y) = \begin{cases} 0 & x = y \\ \theta^{\min\{k : x_k \neq y\}} & x \neq y \end{cases}$$

**Fact:** $\sigma$ is chaotic.

**Definition:** $X \subset \Sigma_b^+$ is a subshift if

(a) $\sigma(x) \in X$ for all $x \in X$

(b) $X$ is closed

$\mathcal{L}_n(X) = \{$words of length n that occur in at least one $x \in X\}$

$\mathcal{L}(X) = \cup_{n\in\mathbb{N}}\mathcal{L}_n(X)$

"$\mathcal{L}(X)$ is called the language of $X$"

**Definition:** $\mathcal{L} \subset \mathcal{L}(\Sigma_b^+)$ is a language if

(a) if $w \in d$ and $v$ is a subword of $w$ then $v \in \mathcal{L}$

(b) if $w \in \mathcal{L}$ then $\exists a \in \{0, \ldots, b-1\}$ such that $wa \in \mathcal{L}$

**Theorem:** If $\mathcal{L}$ is a language, then $\exists$ unique subshift $X$ such that $\mathcal{L}(X) = \mathcal{L}$

**Theorem:** $h_{top}(\sigma|_X) = \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{L}_n(X)|$

## 4.3 Languages

Let $X \subset \Sigma_\mathcal{A}$ be a shit space. We define $\mathcal{L}_n(X) = \{w : w$ is a word of length $n$ that occurs in some $x \in X\}$. Note that $\mathcal{L}_0(X) = \{\epsilon\}$.

We call,

$$\mathcal{L}(X) = \bigcup_{n\in\mathbb{N}_0} \mathcal{L}_n(X)$$

the **language** of the shift space $X$.

On the other hand, we say $\mathcal{L} \subset \mathcal{L}(\Sigma_d) = \{w_1 \cdots w_n : n \in \mathbb{N}_0, w_i \in \{1, \ldots, d-1\}\}$ is a language if the following hold,

1. If $w \in \mathcal{L}$ and $v$ is a subword of $w$, then $v \in \mathcal{L}$
2. For all $w \in \mathcal{L}$, $\exists i \in \{0, \ldots, d-1\}$ such that $wi \in \mathcal{L}$
3. For all $w \in \mathcal{L}$, $\exists j \in \{0, \ldots, d-1\}$ such that $jw \in \mathcal{L}$

The same notions hold for languages of one-sided shift spaces with the exception that (3) is not needed.

**Proposition:** Let $\mathcal{L} \subset \mathcal{L}(\Sigma_{\mathcal{A}})$ be a language. Then,

$$X_{\mathcal{L}} = \{x \in \Sigma_{\mathcal{A}} : x_i \cdots x_j \in \mathcal{L} \text{ for all } i, j \in \mathbb{Z}\}$$

is a shift space whose language is $\mathcal{L}$.

Now fix $d \geq 2$. We define,

$$\Sigma_{cs} = \{X \subset \Sigma_{\mathcal{A}} : X \text{ shift space }\}$$

Here "cs" stands for closed and shift invariant.

We define the **lexicographic order** on $L = \mathcal{L}(\Sigma_{\mathcal{A}})$. Let $v, w \in L$:

- If $v = \epsilon$ and $w \neq \epsilon$, then $v < w$
- If $|v| < |w|$, then $v < w$
- If $|v| = |w|$, define $k = \min\{k : v_k \neq w_k\}$. We define $v < w$ if $v_k < w_k$ and $w < v$ if $w_k < v_k$.

This define the lexicographic order on $L$

Let $X$ be a shift space. We call $\mathcal{F} = \mathcal{F}_X = \{w \in \mathcal{L}(\Sigma_{\mathcal{A}} : w \notin \mathcal{L}(X)\}$ **the set of forbidden words** of $X$. In other words, $\mathcal{F}_X = \mathcal{L}(\Sigma_{\mathcal{A}}) \setminus \mathcal{L}(X)$.

Let $\mathcal{F} \subset \mathcal{L}(\Sigma_{\mathcal{A}})$, we denote by $\overline{\mathcal{F}}$ the set of all words $w \in \mathcal{L}(\Sigma_{\mathcal{A}})$ such that $w$ has a subword that belongs to $\mathcal{F}$. We say that $\mathcal{F} \subset \mathcal{L}(\Sigma_{\mathcal{A}})$ is a forbidden set if $L = \mathcal{L}(\Sigma_{\mathcal{A}}) \setminus \overline{\mathcal{F}}$ is a language. In this case, we write $X_{\mathcal{F}} = X_L$.

**Definition:** We say a shift space $X$ is irreducible if for all $u, v \in \mathcal{L}(X)$ there exists $w \in \mathcal{L}(X)$ such that $uvw \in \mathcal{L}(X)$.

**Definition:** Suppose $X$ is a shift space. We say $X$ is a subshift of finite type if there exists a finite forbidden set $\mathcal{L}$ such that $X = X_{\mathcal{L}}$

## 4.4 Higher Block Order

$d \geq 2$ fixed, $\Sigma_d = \mathcal{A}^{\mathbb{Z}}$, $\mathcal{A} = \{0, \ldots, d-1\}$

$X$ shiftspace in $\Sigma_d$. Fix $N \geq 2$ and let $B$ be the set of words of length $N$ in $X$, i.e., $B = \mathcal{L}_N(X)$

$B = \{0, \ldots, d' - 1\}$ where $d' = |\mathcal{L}_N(X)|$. Define $B_N : X \to B^{\mathbb{Z}}$.

$B_N(x)_k = x_k x_{k+1} \cdots x_{k+N-1}$

We define $X^{[N]} = B_N(X) \implies B_N$ is surjective.

**Proposition:** $X^{[N]}$ is a shift space.

**Remark:** We shall see that $(X, \sigma)$ and $(X^{[N]}, \sigma)$ are topologically conjugate.

We call $X^{[N]}$ a block code of length $N$.

**Remark:** If $u = u_1 \cdots u_N$ and $v = v_1 \cdots v_N$. We say $u$ and $v$ progressively overlap if $v_2 \cdots v_N = u_1 \cdots u_{N-2}$

**Proposition:** Let $X$ be shift space and let $N \in \mathbb{N} \setminus \{1\}$. Then $X^{[N]}$ is a shift space.

**Definition:** A subshift of finite type is $M-$step if it can be described y a forbidden set of words all of which are of length $M+1$.

**Proposition:** If $X$ is a subshift of finite type, then $\exists M \in \mathbb{N}_0$ such that $X$ is $M$-step

**Theorem:** A subshift $X$ is an $M$-step subshift of finite type if and only if whenever $uv, vw \in \mathcal{L}(X)$ and $|v| \geq M$ then $uvw \in \mathcal{L}(X)$

**Theorem:** A shift space $X$ which is conjugate to a subshift of finite type is a subshift of finite type.

**Corollary:** Let $X$ be an $M-$step subshift of finite type. Then $X$ is conjugate to a 1-step subshift of finite type.

## 4.5 Shift Spaces via Graphs

**Definition:** A graph $G$ is a finite set $V = V(G)$ of vertices (also called states) together with a finite set of edge $\epsilon = \epsilon(G)$. Each edge $e \in \epsilon(G)$ starts at a vertex $i(e) \in V$ and ends at vertex $t(e) \in V$. There may be multiple edges that have the same starting vertex and terminal vertex. The set of edges is called a multiple set.

For a vertex $I \in V$ we denote by $\epsilon_I$ the set of all edges going out from $I$ and by $\epsilon^I$ the set of all edges coming into $I$.

An edge $e$ with $i(e) = t(e)$ is called a self loop.

**Definition:** Let $G$ be a graph with vertex set $V$ and edge set $\epsilon$. For vertices $I$ and $J$ in $V$, let $A_{I,J}$ be the number of edges with initial state $I$ and terminal state $J$. Then the adjacency matrix of $G$ is defined by $A = [A_{I,J}] \implies A = A(G)$

Let $A$ be $r \times r$ matrix with non-negative integral values. Then the graph $G = G(A) = G_A$ is defined by the vertex set $V(G) = \{1, \ldots r\}$ and $A_{I,J}$ distinct edges starting at $I$ and terminated at $J$. We clearly have $A = A(G_A)$

**Definition:** Let $G = (V, \epsilon)$ be a graph and $A$ its adjacency matrix. The edge shift $X_G$ or $X_A$ is the shift space of the alphabet $\mathcal{A} = \epsilon$ defined by

$$X_G = X_A = \{\xi = (\xi_i)_i \in \epsilon^{\mathbb{Z}} : t(\xi_i) = i(\xi_{i+1})\}$$

The shift map on $X_G$ or $X_A$ is called the edge shift map and is denoted by $\sigma_A$ or $\sigma_G$.

$X_G$ is given by the bi-infinite paths on the graph $G$.

**Proposition:** Let $G$ be a graph with adjacency matrix $A$. Then the associated edge shift $X_G = X_A$ is a one-step subshift of finite type.

## 4.6 Sofic Shifts

**Definition:** A labeled graph $\mathcal{G}$ is a pair $(G, L)$ where $G$ is a graph with edge set $\epsilon$ and $L : \epsilon \to \mathcal{A}$ is a function that assigns to each edge $e$ a label $L(e)$ in a finite alphabet $\mathcal{A}$

Suppose $\mathcal{G} = (V, G, L)$ is a labeled graph. If $\xi = \cdots e_{-2}e_{-1}e_0e_1e_1 \cdots$ is a bi-infinite path in $G \iff \xi \in X_G$

We define $L_\infty(\xi) = \cdots L(e_{-2})L(e_{-1})L(e_0)L(e_1)L(e_2) \cdots$

We define $X_{\mathcal{G}} = \{x \in \mathcal{A}^{\mathbb{Z}} : x = L_\infty(\xi) \text{ for some } \xi \in X_G\}$

A subshift $\mathcal{A}^{\mathbb{Z}}$ is called a **sofic shift** if $X = X_{\mathcal{G}}$ for some labeled graph $\mathcal{G} = (G, L)$. $\mathcal{G}$ is called a representation of $X$.

**Corollary:** Every subshift of finite type is sofic.

**Theorem:** A shift space $Y \subset \mathcal{A}^{\mathbb{Z}}$ is sofic if and only if it is a factor of some subshift of finite type $X$

**Definition:** We say a non-zero transition matrix $A$ is irreducible if for all $i, j \in \{0, \dots, d-1\}$, there exists $n \in \mathbb{N}$ such that $(A^n)_{i,j} > 0$

**Fact:** $A$ is irreducible if and only if $X_A$ is topologically transitive.

**Theorem:** Let $A \neq 0$ be a transition matrix and further assume that $A$ is irreducible. Then $A$ has a positive eigenvector $v_A$ with positive eigenvalue $\lambda_A$. Further, $\lambda_A$ is algebraically and geometrically simple. If $\mu$ is any other eigenvalue of $A$ then $|\mu| \leq \lambda_A$. Any positive eigenvector is a positive multiple of $v_A$.

**Corollary:** Let $A$ be a transition matrix of a topologically transitive subshift of finite type $X_A$. Then $h(X_A) = \log \lambda_A$, where $\lambda_A$ is the eigenvalue of $A$ described above.