Dynamical Systems: Homework 4

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Problem 1.

Recall that the shift map $\sigma : \Sigma_A \to \Sigma_A$ is defined as follows:

$$\sigma((x_k)_{k=0}^{\infty}) = (x_{k+1})_{k=0}^{\infty}$$

That is, we are essentially cutting off the first element of the sequence.

In addition, recall that topological transitivity is defined as $\exists x \in \Sigma_A$ such that the orbit $O(x) = \{\sigma^n(x) : n \in \mathbb{N}_0\}$ of x is dense in Σ_A.

Lastly, we have that,

$$d(x,y) = d_\theta(x,y) = \begin{cases} 0 & \text{if } x = y \\ \theta^{\min\{k : x_k \neq y_k\}} & \text{if } x \neq y \end{cases}$$

where $0 < \theta < 1$.

Let us define A as follows,

$$A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \\ d-1 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & \cdots & d-1 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\ & & \vdots & & & \cdots & \vdots \\ 0 & 1 & 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \end{array}$$

That is, 0 cannot appear anywhere in a sequence $x \in \Sigma_A$ except for in the first position. Now let us fix some $x, y \in \Sigma_A$ with $x \neq y$. Let y be such that it begins with 0. That is,

$$y = (0 y_1 y_2 y_3 \cdots)$$

If x does not also start with 0, then it is clear that,

$$d(\sigma^n(x), y) = \theta^0 = 1$$

for every $n \in \mathbb{N}_0$. This is true because, by the definition of A, the element 0 can only appear as the first element in a sequence. Hence, if x does not start with a 0, then $O(x) \cap B(y, \epsilon) = \emptyset$ for every $y \in \Sigma_A$ that starts with 0 and every $\epsilon < 1$.

Now suppose x does indeed start with a 0 and fix $\epsilon = \theta^2$. Observe that following a 0, we have $d - 1$ choices for the next element in the sequence. Let us now fix $y \in \Sigma_A$ such that $y_0 = 0$ and $y_1 \neq x_1$. Then we have,

$$d(x, y) = \theta^1 = \theta \tag{1}$$

However, observe that since 0 can only appear at the beginning of a sequence, the first element of $\sigma^n(x)$ will never be 0 for any $n \geq 1$. Hence, we have,

$$d(\sigma^n(x), y) = \theta^0 = 1$$

for every $n > 1$. Combining this result with (1) yields

$$d(\sigma^n(x), y) \geq \theta > \epsilon = \theta^2$$

for all $n \in \mathbb{N}_0$. Thus, we have that $O(x) \cap B(y, \epsilon) = \emptyset$ in this case as well. Since we have covered all possible cases for $x \in \Sigma_A$, we have shown that there is no x such that $O(x)$ is dense in Σ_A.

Problem 2.

(a) Suppose $\lambda > 1/e$. Then,

$$\begin{aligned} E_\lambda(x) &= \lambda e^x \\ &> \frac{e^x}{e} \\ &= e^{x-1} \end{aligned}$$

Hence, if we define $f(x) = e^{x-1}$, it suffices to show that $\lim_{n\to\infty} f^n(x) = \infty$ in order to assert that $\lim_{n\to\infty} E_\lambda^n(x) = \infty$.

From Bernoulli's Inequality, we have that,

$$e^x \geq x + 1$$

for all $x \in \mathbb{R}$. Hence, applying this to our case, we have that,

$$f(x) = e^{x-1} \geq (x - 1) + 1 = x$$

Now let us define the function $y = x - 1$. Then,

$$f(y) = e^y \geq y + 1$$

Note that y ranges over all of $\mathbb{R}$. Define $g(y) = y + 1$. It is clear that,

$$\begin{aligned} g^n(y) &= y + 1 + 1 + \cdots + 1 \\ &= y + n \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{n\to\infty} g^n(y) &= y \lim_{n\to\infty} n \\ &= y \cdot \infty \\ &= \infty \end{aligned}$$

Since $E_\lambda(x) > f(x) = f(y) \geq g(y)$, we have that,

$$\lim_{n\to\infty} E_\lambda^n(x) > \lim_{n\to\infty} f^n(y) \geq \lim_{n\to\infty} g(y)$$

This yields,

$$\lim_{n\to\infty} E_\lambda^n(x) > \infty$$

and so,

$$\lim_{n\to\infty} E_\lambda^n(x) = \infty$$

(b) Now suppose $\lambda = 1/e$. Then we have,

$$\begin{aligned} E_\lambda(x) &= \lambda e^x \\ &= \frac{e^x}{e} \\ &= e^{x-1} \end{aligned}$$

Observe that $E_\lambda(1) = e^{1-1} = 1$. Hence, 1 is a fixed point of $E_\lambda(x)$. In addition, we have,

$$E_\lambda'(x) = e^{x-1}$$

So,

$$\begin{aligned} E_\lambda'(1) &= e^{1-1} \\ &= 1 \end{aligned}$$

Hence, 1 is a non-hyperbolic fixed point of $E_\lambda(x)$. Now suppose $x < 1$. Then,

$$\begin{aligned} E_\lambda(x) &= e^{x-1} \\ &< e^{1-1} = 1 \end{aligned} \tag{2}$$

In addition, from Bernoulli's Inequality, we have that,

$$e^x \geq x + 1$$

for all $x \in \mathbb{R}$. Hence, applying this to our case, we have that,

$$E_\lambda(x) = e^{x-1} \geq (x-1) + 1 = x$$

Note that since $x \neq 1$, we get the following inequality,

$$E_\lambda(x) = e^{x-1} > (x-1) + 1 = x \tag{3}$$

By combining (2) and (3) and fixing $x < 1$, we have that,

$$x < E_\lambda(x) < E_\lambda^2(x) < E_\lambda^3(x) < \cdots < 1$$

Hence, if we define the sequence $(x_n)_{n\in\mathbb{N}_0} = E_\lambda^n(x)$, we have that it is monotonically increasing and bounded above by 1. Thus, by the Monotone Convergence Theorem, (x_n) converges to its supremum.

Now suppose that $\sup(x_n) < 1$. Then there exists $m \in \mathbb{R}$ such that $x_n \leq m < 1$ for every $n \in \mathbb{N}_0$. But observe that, since m is a supremum of the set, for any $\epsilon > 0$, we have that $m < x_n + \epsilon$ for some x_n. If we choose ϵ small enough, we have,

$$\begin{aligned} x_{n+1} &= e^{x_n - 1} \\ &> x_n \end{aligned}$$

Hence, $x_{n+1} = x_n + \epsilon$ and so $x_{n+1} > m$, a contradiction. Thus the supremum of the set is 1. Hence, if $x < 1$, then,

$$\lim_{n\to\infty} E_\lambda^n(x) = 1$$

Now suppose $x > 1$. We have,

$$x < E_\lambda(x) < E_\lambda^2(x) < E_\lambda^3(x) < \cdots$$

as before. However, we also have that this set is unbounded. Suppose that it were bounded. Then there exists a number $m > 1$ such that,

$$x < E_\lambda(x) < E_\lambda^2(x) < E_\lambda^3(x) < \cdots < m$$

This implies that m is a fixed point of $E_\lambda^n(x)$. However, we know that there are no fixed point of this function greater than 1. Thus, we have a contradiction and hence there is no upper bound for this sequence. Thus, if $x > 1$, then,

$$\lim_{n\to\infty} E_\lambda^n(x) = \infty$$

(c) Suppose $0 < \lambda < 1/e$. Then

$$\begin{aligned} E_\lambda(x) &= \lambda e^x \\ &< \frac{e^x}{e} \\ &= e^{x-1} \end{aligned}$$

We have,

$$\begin{aligned} &\lambda e^x = x \\ \implies &e^x = x/\lambda \\ \implies &x = \ln(x/\lambda) \\ \implies &x = \ln(x) - \ln(\lambda) \end{aligned}$$

Problem 3.

Let $\alpha \in [0, 1) \cap \mathbb{Q}$. Hence, $\alpha = q/p$ for some $q, p \in \mathbb{Z}$ such that $0 \leq \alpha < 1$. Hence,

$$\begin{aligned} f(e^{2\pi i\phi}) &= e^{2\pi i(\alpha+\phi)} \\ &= e^{2\pi i(p/q+\phi)} \end{aligned}$$

Note that this map shifts the initial angle by $\alpha = p/q$. Now suppose we consider $f^q(e^{2\pi i\phi})$. That is, we iterate the function q times. We are thus have that f^q is equivalent to,

$$f^q(e^{2\pi i\phi}) = e^{2\pi i[(\alpha+\alpha+\cdots+\alpha)+\phi]}$$

That is, we are shifting the initial angle by $q \cdot \alpha = p$. Hence,

$$\begin{aligned} f^q(e^{2\pi i\phi}) &= e^{2\pi i(q\cdot\alpha+\phi)} \\ &= e^{2\pi i(p+\phi)} \\ &= e^{2\pi ip+2\pi i\phi} \end{aligned}$$

Since $p \in \mathbb{Z}$, we have that $2\pi ip$ is a multiple of 2π. Hence, $2\pi ip$ represents a full rotation around the circle, returning to the angle ϕ. Thus,

$$f^q(e^{2\pi i\phi}) = e^{2\pi i\phi}$$

for any angle ϕ. Hence, $\mathrm{Per}(f) = [0, 2\pi)$.

Now suppose $\alpha \in [0, 1) \cap \mathbb{I}$. I would guess that $\mathrm{Per}(f)$ would exclude any rational points, since shifting a rational number by an irrational number would result in an irrational number.

Problem 4. Bonus problem

Problem 5. Bonus problem

Let $4 < \mu \leq 2 + \sqrt{5}$ and let $\Lambda_\mu = \{x \in [0, 1] \mid f^n(x) \in [0, 1]\ \forall n \in \mathbb{N}\}$