Dynamical Systems II: Final

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Problem 1.

Recall that a rotation of the plane is a linear map of the form

$$R : \mathbb{R}^2 \to \mathbb{R}^2; \quad R\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

This map preserves Lebesgue measure, but we want to show that it is never ergodic.

We have that,

$$\begin{aligned} R\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix} \end{aligned}$$

$$\begin{aligned} R^{-1}\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{\cos^2\theta + \sin^2\theta}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{pmatrix} \end{aligned}$$

Let us now consider the unit disk in $\mathbb{R}^2$, denoted by $A = \{(x, y) \in \mathbb{R}^2 : -1 \leq x^2 + y^2 \leq 1\}$. We have that Lebesgue measure is a generalization of area in $\mathbb{R}^2$ and, since the unit disk has a well-defined area, we know its Lebesgue measure must be equal to 1.

Let us fix $(x, y) \in A$. Then $-1 \leq x^2 + y^2 \leq 1$. Applying R to this point, we get,

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \end{aligned}$$

And note that,

$$\begin{aligned}
(x')^2 + (y')^2 &= (x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2 \\
&= x^2\cos^2\theta - 2xy\cos\theta\sin\theta + y^2\sin^2\theta + x^2\sin^2\theta + 2xy\sin\theta\cos\theta + y^2\cos^2\theta \\
&= x^2\cos^2\theta + y^2\sin^2\theta + x^2\sin^2\theta + y^2\cos^2\theta \\
&= x^2(\cos^2\theta + \sin^2\theta) + y^2(\cos^2\theta + \sin^2\theta) \\
&= x^2 + y^2
\end{aligned}$$

And so we have $-1 \leq (x')^2 + (y')^2 \leq 1$ as well. Thus, $R\begin{pmatrix} x \\ y \end{pmatrix} \in A$ and, since $(x, y) \in A$ was arbitrary, we have that $(x, y) \in A \implies R\begin{pmatrix} x \\ y \end{pmatrix} \in A$.

Now applying R^{-1} to an arbitrary (x, y), we get,

$$\begin{aligned}
x' &= x\cos\theta + y\sin\theta \\
y' &= -x\sin\theta + y\cos\theta
\end{aligned}$$

And note that,

$$\begin{aligned}
(x')^2 + (y')^2 &= (x\cos\theta + y\sin\theta)^2 + (-x\sin\theta + y\cos\theta)^2 \\
&= x^2\cos^2\theta + 2xy\cos\theta\sin\theta + y^2\sin^2\theta + x^2\sin^2\theta - 2xy\sin\theta\cos\theta + y^2\cos^2\theta \\
&= x^2\cos^2\theta + y^2\sin^2\theta + x^2\sin^2\theta + y^2\cos^2\theta \\
&= x^2(\cos^2\theta + \sin^2\theta) + y^2(\cos^2\theta + \sin^2\theta) \\
&= x^2 + y^2
\end{aligned}$$

And so we have $-1 \leq (x')^2 + (y')^2 \leq 1$ as well. Thus, $R^{-1}\begin{pmatrix} x \\ y \end{pmatrix} \in A$ and, since $(x, y) \in A$ was arbitrary, we have that $(x, y) \in A \implies R^{-1}\begin{pmatrix} x \\ y \end{pmatrix} \in A$.

Thus, from the above derivations, we must have that A is strictly R-invariant. However, as we said above, note that A has measure 1. Moreover, $A^c = \{(x, y) \in \mathbb{R}^2 : -1 \leq x^2 + y^2 \leq 1\}$, which is a set of infinite measure (the area of the plane minus the unit disk). Hence, although A is strictly R-invariant, we do not have that $\mu(A) = 0$, nor that $\mu(A^c) = 0$. As a result, R is not ergodic for any value of θ.

Problem 2.

Problem 3.

a)

b)

Problem 4.

Problem 5.

a)

b)

Problem 6.

a)

b)