Dynamical Systems II: Final

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May 11, 2021

Problem 1.

Recall that a rotation of the plane is a linear map of the form

$$R : \mathbb{R}^2 \to \mathbb{R}^2; \quad R\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

This map preserves Lebesgue measure, but we want to show that it is never ergodic.

We have that,

$$\begin{aligned} R\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix} \end{aligned}$$

$$\begin{aligned} R^{-1}\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{\cos^2\theta + \sin^2\theta}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{pmatrix} \end{aligned}$$

Let us now consider the unit disk in $\mathbb{R}^2$, denoted by $A = \{(x, y) \in \mathbb{R}^2 : \ -1 \leq x^2 + y^2 \leq 1\}$. We have that Lebesgue measure is a generalization of area in $\mathbb{R}^2$ and, since the unit disk has a well-defined area, we know its Lebesgue measure must be equal to 1.

Let us fix $(x, y) \in A$. Then $-1 \leq x^2 + y^2 \leq 1$. Applying R to this point, we get,

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \end{aligned}$$

And note that,

$$
\begin{aligned}
(x')^2 + (y')^2 &= (x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2 \\
&= x^2\cos^2\theta - 2xy\cos\theta\sin\theta + y^2\sin^2\theta + x^2\sin^2\theta + 2xy\sin\theta\cos\theta + y^2\cos^2\theta \\
&= x^2\cos^2\theta + y^2\sin^2\theta + x^2\sin^2\theta + y^2\cos^2\theta \\
&= x^2(\cos^2\theta + \sin^2\theta) + y^2(\cos^2\theta + \sin^2\theta) \\
&= x^2 + y^2
\end{aligned}
$$

And so we have $-1 \leq (x')^2 + (y')^2 \leq 1$ as well. Thus, $R\begin{pmatrix} x \\ y \end{pmatrix} \in A$ and, since $(x, y) \in A$ was arbitrary, we have that $(x, y) \in A \implies R\begin{pmatrix} x \\ y \end{pmatrix} \in A$.

Now applying R^{-1} to an arbitrary (x, y), we get,

$$
\begin{aligned}
x' &= x\cos\theta + y\sin\theta \\
y' &= -x\sin\theta + y\cos\theta
\end{aligned}
$$

And note that,

$$
\begin{aligned}
(x')^2 + (y')^2 &= (x\cos\theta + y\sin\theta)^2 + (-x\sin\theta + y\cos\theta)^2 \\
&= x^2\cos^2\theta + 2xy\cos\theta\sin\theta + y^2\sin^2\theta + x^2\sin^2\theta - 2xy\sin\theta\cos\theta + y^2\cos^2\theta \\
&= x^2\cos^2\theta + y^2\sin^2\theta + x^2\sin^2\theta + y^2\cos^2\theta \\
&= x^2(\cos^2\theta + \sin^2\theta) + y^2(\cos^2\theta + \sin^2\theta) \\
&= x^2 + y^2
\end{aligned}
$$

And so we have $-1 \leq (x')^2 + (y')^2 \leq 1$ as well. Thus, $R^{-1}\begin{pmatrix} x \\ y \end{pmatrix} \in A$ and, since $(x, y) \in A$ was arbitrary, we have that $(x, y) \in A \implies R^{-1}\begin{pmatrix} x \\ y \end{pmatrix} \in A$.

Thus, from the above derivations, we must have that A is strictly R-invariant. However, as we said above, note that A has measure 1. Moreover, $A^c = \{(x, y) \in \mathbb{R}^2 : -1 \leq x^2 + y^2 \leq 1\}$, which is a set of infinite measure (the area of the plane minus the unit disk). Hence, although A is strictly R-invariant, we do not have that $\mu(A) = 0$, nor that $\mu(A^c) = 0$. As a result, R is not ergodic for any value of θ.

Problem 2.

Let $(X, \mathcal{S}, \mu)$ be a measure space and suppose that $S : X \to X$ and $T : X \to X$ are measure-preserving transformations. We want to show that their composition, $S \circ T$, preserves measure as well.

Since S and T are measure-preserving transformations, for any $A \in \mathcal{S}(X)$ we have that $\mu(S^{-1}(A)) = \mu(A)$ and $\mu(T^{-1}(A)) = \mu(A)$. Let us consider $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$. We have,

$$(T^{-1} \circ S^{-1})(A) = T^{-1}(S^{-1}(A))$$

We have that, since S is measure-preserving, then $\mu(S^{-1}(A)) = \mu(A)$. Moreover, since $S^{-1}(A)$ is measurable, we have that $S^{-1}(A) \in \mathcal{S}(X)$ and hence. Let us call this set B. Then we have,

$$T^{-1}(S^{-1}(A)) = T^{-1}(B)$$

and, since T is measure-preserving, we have

$$\begin{aligned} \mu(T^{-1})(B) &= \mu(B) \\ &= \mu(S^{-1}(A)) \\ &= \mu(A) \end{aligned}$$

Sine $A \in \mathcal{S}(X)$, this holds for every set in $\mathcal{S}(X)$ and so $S \circ T$ is measure-preserving.

Problem 3.

a) Let us begin by showing that D is measurable using the Dyadic squares. Fix a set A in the Dyadic squares and consider $D^{-1}(A)$. Observe that for $x, y \in [0, 1/2)$, we have that $2x \in [0, 1)$ and $y \in [0, 1)$. Hence, we can consider $D(x, y)$ on $[0, 1/2) \times [0, 1/2)$ to be just $D(x, y) = (2x, 2y)$. Now, for $x, y \in (1/2, 1)$, we have that $D(x, y) \in (0, 1)$. Note that for x, y in this interval, we have $2x \mod 1 = 2x - 1$ and $2y \mod 1 = 2y - 1$. Thus, for $x, y \in (1/2, 1)$ we have $D(x, y) = (2x - 1, 2y - 1)$. Hence, rephrasing D, we have,

$$D(x, y) = \begin{cases} (2x, 2y) & x, y \in [0, 1/2) \\ (2x, 2y - 1) & x \in [0, 1/2), y \in (1/2, 1) \\ (2x - 1, 2y) & x \in (1/2, 1), y \in [0, 1/2) \\ (2x - 1, 2y - 1) & x, y \in (1/2, 1) \end{cases}$$

Hence, we can think of $D(x, y)$ as a cross product of the tent map on $[0, 1)$ with $\mu = 2$. We can now formulate the inverse of $D(x, y)$ as,

$$D^{-1}(x, y) = \begin{cases} (x/2, y/2) & x, y \in [0, 1/2) \\ (x/2, y/2 + 1/2) & x \in [0, 1/2), y \in (1/2, 1) \\ (x/2 + 1/2, y/2) & x \in (1/2, 1), y \in [0, 1/2) \\ (x/2 + 1/2, y/2 + 1/2) & x, y \in (1/2, 1) \end{cases}$$

Denote each of the four cases of $D^{-1}(x,y)$ as $D_i^{-1}(x,y)$ for $i \in \{1,\dots,4\}$ and observe that $\lambda(A) = 1/2^k \cdot 1/2^k = 1/4^k$. Now let us check $D^{-1}(A)$. We have,

$$\lambda(D^{-1}(A)) = \lambda(D_1^{-1}(A)) + \lambda(D_2^{-1}(A)) + \lambda(D_3^{-1}(A)) + \lambda(D_4^{-1}(A))$$

Let us now split up each D_i into D_{i_x} and D_{i_y} for the x and y components. In addition, write A_x for the x component of A and A_y for the y component. Then,

$$\lambda(D_i^{-1}(A)) = \lambda(D_{i_x}^{-1}(A_x)) \cdot \lambda(D_{i_y}^{-1}(A_y))$$

Further, observe that $\lambda(D_{i_x}^{-1}(A_x)) = \frac{1}{2}\lambda(A_x)$ and $\lambda(D_{i_y}^{-1}(A_y)) = \frac{1}{2}\lambda(A_y)$ by the proof of Theorem 3.3.1 in the text. Note that $\lambda(A_x) = 1/2^k$ and $\lambda(A_y) = 1/2^k$, so

$$\begin{aligned}\lambda(D_i^{-1}(A)) &= \frac{1}{2}\lambda(A_x) \cdot \frac{1}{2}\lambda(A_y)\\ &= \frac{1}{4} \cdot \frac{1}{4^k}\end{aligned}$$

This is true for every i and thus,

$$\begin{aligned}\lambda(D^{-1}(A)) &= \lambda(D_1^{-1}(A)) + \lambda(D_2^{-1}(A)) + \lambda(D_3^{-1}(A)) + \lambda(D_4^{-1}(A))\\ &= \frac{1}{4}\cdot\frac{1}{4^k} + \frac{1}{4}\cdot\frac{1}{4^k} + \frac{1}{4}\cdot\frac{1}{4^k} + \frac{1}{4}\cdot\frac{1}{4^k}\\ &= \frac{1}{4^k}\\ &= \lambda(A)\end{aligned}$$

Since A was an arbitrary Dyadic square, we thus have that D is measure-preserving.

b) We want to show that D is ergodic. Hence, we want to show that for any D-invariant measurable set A (that is, $D^{-1}(A) = A$), we have either $\lambda(A) = 0$ or $\lambda(A^c) = 0$. Again we can consider a Dyadic square A since the Dyadic squares form a sufficient semiring for Lebesgue measure. Hence, $A = [\frac{p}{2^k}, \frac{p+1}{2^k}) \times [\frac{q}{2^k}, \frac{q+1}{2^k})$ for k, p, q integers with $k \geq 0$. Observe that $D^{-1}(A) = A$ implies that

$$\begin{aligned}D^{-1}(A) &= D_1^{-1}(A) \sqcup D_2^{-1}(A) \sqcup D_3^{-1}(A) \sqcup D_4^{-1}(A)\\ &= [\frac{p}{2^{k+1}}, \frac{p+1}{2^{k+1}}) \times [\frac{q}{2^{k+1}}, \frac{q+1}{2^{k+1}}) \bigsqcup\\ &[\frac{p}{2^{k+1}}, \frac{p+1}{2^{k+1}}) \times [\frac{q+1}{2^{k+1}}, \frac{q+2}{2^{k+1}}) \bigsqcup\\ &[\frac{p+1}{2^{k+1}}, \frac{p+2}{2^{k+1}}) \times [\frac{q}{2^{k+1}}, \frac{q+1}{2^{k+1}}) \bigsqcup\\ &[\frac{p+1}{2^{k+1}}, \frac{p+2}{2^{k+1}}) \times [\frac{q+1}{2^{k+1}}, \frac{q+2}{2^{k+1}})\\ &= [\frac{p}{2^{k+1}}, \frac{p+2}{2^{k+1}}) \times [\frac{q}{2^{k+1}}, \frac{q+2}{2^{k+1}})\end{aligned}$$

Observe that $\frac{p+2}{2^{k+1}} < \frac{p+1}{2^k}$ and so $D^{-1}(A) \neq A$ for all Dyadic squares.

Problem 4.

Recall that $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra containing the open sets. Since $X \in \mathcal{B}(\mathbb{R})$, we have that $\mathcal{B}(X) = \{A \cap X : \ A \in \mathcal{B}(\mathbb{R})\}$.

We are given that $T : X \to X$ is a Borel-measurable transformation for a set $X \in \mathcal{B}(\mathbb{R})$ (that is, X is a Borel set). Since T is Borel measurable, we have $T^{-1}(B) \in \mathcal{B}(X)$ for all $B \in \mathcal{B}(X)$.

Define $p : X \to [-\infty, +\infty]$ by,

$$x \mapsto \begin{cases} k & \text{if } x \text{ is periodic and has least period } k \\ +\infty & \text{if } x \text{ is not periodic} \end{cases}$$

where *least period* k means that $k \geq 1$ is the minimal integer such that $T^k(x) = x$.

We want to prove that p is Borel measurable. Recall that $p : X \to \mathbb{R}^*$ is $\mathcal{B}(X)$ measurable iff $p^{-1}([-\infty, a)) \in \mathcal{B}(X)$ for every $a \in \mathbb{R}$. Equivalently, $p : X \to \mathbb{R}^*$ is $\mathcal{B}(X)$-measurable iff $p^{-1}(C) \in \mathcal{B}(X)$ for every $C \in \mathcal{B}(\mathbb{R}^*) = \mathcal{B}(\mathbb{R}) \sqcup P(\{\pm\infty\})$.

Note that $x \in X$ is period k if and only if $T^k(x) - x = 0$. So consider the set $[-\infty, a)$ for some $a \in \mathbb{R}$. We have $p^{-1}([-\infty, a))$ is the set of $x \in X$ with $T^k(x) - x = 0$ for some $k \in \mathbb{N}$ with $k < a$. Since we know $k \geq 1$ and $p(x) \mapsto +\infty$ when x is not periodic, we have that

$$p^{-1}([-\infty, a)) = p^{-1}([1, a))$$

for $a \geq 1$ and

$$p^{-1}([-\infty, a)) = \emptyset$$

for $a < 1$. Since the empty set is trivially measurable, let us only consider $a \geq 1$.

For $a \geq 1$, we have that,

$$\begin{aligned} p^{-1}([-\infty, a)) &= p^{-1}([1, a)) \\ &= \bigcup_{i=1}^{a} \{x \in X \ : \ T^i(x) - x = 0\} \\ &= \bigcup_{i=1}^{a} \{x \in X \ : \ T^i(x) = 0\} \end{aligned}$$

Now, note that by Lemma 4.2.6, since $T : X \to X$ is a measurable transformation and $X \subset \mathbb{R}$, we then have that $T \circ T = T^2$ is measurable. Proceeding inductively, we have that T^j is measurable for any $j \geq 1$. Moreover, by Exercise 4.4.9, we have that since T^j is Borel measurable (and hence Lebesgue measruable), then $\{x \in X \ : \ T^j(x) = a\}$ is measurable for any a. In particular, $\{x \in X \ : \ T^j(x) = x\}$ is measurable. Hence, for $a \geq 1$, we have that

$p^{-1}([-\infty, a)) = p^{-1}([1, a))$ is just a countable union of elements in $\mathcal{B}(X)$ and hence, since $\mathcal{B}(X)$ is a σ-algebra, we have,

$$\begin{aligned} p^{-1}([-\infty, a)) &= p^{-1}([1, a)) \\ &= \left(\bigcup_{i=1}^{a} \{x \in X \ : \ T^i(x) = 0\} \right) \in \mathcal{B}(X) \end{aligned}$$

Thus, $p^{-1}([-\infty, a))$ is measurable for any $a \geq 1$. We have now covered every case of $a \in \mathbb{R}$ and so, we have that $p^{-1}([-\infty, a)) \in \mathcal{B}(X)$ for every $a \in \mathbb{R}$. Thus, p is measurable.

Problem 5.

a) Recall that a point x is periodic if $T^k(x)$ for some $k \geq 1$. Since almost every point of x is periodic under T, there is a set N with $\lambda(N) = 0$ which contains all of the point which are *not* periodic under T. Now fix a measurable set $A \subset X$. Consider the set $A \setminus N$. By the definitions of T and N, we have that $A \setminus N$ must consist only of points which are recurrent. Thus, for every $x \in A \setminus N$, there exists $k = k(x) > 0$ such that $T^k(x) = x \in A \setminus N$. Since $\lambda(N) = 0$ and A was arbitrary, we have that T is recurrent.

b) Fix $A \subset X$ and a point $x \in A \setminus N$. Hence, n is periodic with period $k \geq 1$. Clearly we have that $\{x, T(x), T^2(x), \ldots, T^{k-1}(x)\}$ is a T-invariant set. Applying T^{-1} gives us,

$$\begin{aligned} \{T^{-1}(x), x, T(x), \ldots, T^{k-2}(x)\} &= \{T^{k-1}(x), x, T(x), \ldots, T^{k-2}(x)\} \\ &= \{x, T(x), T^2(x), \ldots, T^{k-1}(x)\} \end{aligned}$$

Observe that, since there are a finite number of elements in this orbit, we have that $\lambda(A) = 0$ and $\lambda(X \setminus (A \cup N)) = \lambda(X)$. Since $\lambda(X) > 0$, there must be "more" of these orbits outside of A in some sense. Moreover, observe that the orbits of each point are either disjoint or equal to each other. To show this, suppose that the orbits of two points x and y have a non-empty intersection. Then $T^j(x) = T^i(y)$ for some $i, j \geq 1$. But note that, if x has period k, then $T^{k-j}(T^j(x)) = T^k(x) = x$ and thus $T^{k-j}(T^i(y)) = T^{k-j+i}(y) = x$. Thus we must have,

$$\begin{aligned} T^{k-j+1}(T^j(x)) = T^{k+1}(x) &= T^2(x) \\ &= T^{k-j+1}(T^i(y)) \\ &= T^{k-j+1+i}(y) \end{aligned}$$

Continuing by induction, we see that the orbit of y equals the orbit of x for every point after $T^i(y)$. But note that y is also periodic with period ℓ. Then there is some point in the orbit of x such that $T^m(x) = y$. Hence, for all n such that $1 \leq n < i$, we must have that $T^n(y)$ is equal to an element of the orbit of x as well. Thus, these are the same orbit.

In the case where the intersection of the orbits is empty, it is clear that the two orbits are disjoint.

Let us now divide X into two sets, A containing some of the orbits and B containing orbits which are disjoint from all of the orbits in A. Observe that it is possible to construct two such sets of positive measure. If it were not, it would mean that every element, up to a set of 0 measure, would be contained within a single orbit. But note that each orbit has finite elements and thus has measure 0. Hence, this is a contradiction and there must be more than 1 orbit.

Thus, we have $\lambda(A) > 0$ and $\lambda(B) > 0$, with $A \cap B = \emptyset$. Observe further that $A^c = B \cup N$ because B contains all the orbits which are not in A and N contains all the points which are not part of any orbit (we know the points of N are not part of any orbit because they are not periodic and every point of an orbit is periodic).

Thus, we have that $\lambda(A) > 0$ and $\lambda(A^c) = \lambda(B \sqcup N) = \lambda(B) > 0$ as well. In addition, by our discussion above, we have that every orbit in A is T-invariant and thus $T^{-1}(A) = A$. Moreover, all of the orbits in B are T-invariant. Lastly, as previously stated, no element of N can be an element of an orbit. Since the orbits are exactly the set N^c, we have that N is T-invariant and so $T^{-1}(N) = N$ as well. Thus,

$$T^{-1}(A) = A$$

and

$$T^{-1}(A^c) = T^{-1}(B \sqcup N) = B \sqcup N$$

but $\lambda(A) > 0$ and $\lambda(A^c) > 0$. As a result, we have violated the definition of ergodicity and T is not ergodic.

Problem 6.

a) We have that the set of intervals with rational endpoints forms a sufficent semi-ring for $\mathbb{R}$. Hence, in order to show that S preserves Lebesgue measure, we must show that $S^{-1}(I)$ is measurable and that $\lambda(S^{-1}(I)) = \lambda(I)$ for any interval I with rational endpoints.

Observe that,

$$S^{-1}(y_1) = \{y \in \mathbb{R} : \; y = x + n \text{ for some } x \in [0, 1), \; n \in \mathbb{Z} \text{ with } T(x) + n + \phi(x) = x_1 + n_1 = y_1\}$$

Now let us fix an interval with rational endpoints $(p_1/q_1, p_2/q_2)$ with $p_1, q_1, p_2, q_2 \in \mathbb{Z}$.

b)