Dynamical Systems II: Homework 8

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1 Questions from Silva

1.1 Section 3.11

Problem 3. Revised problem: Let $(X, \mathcal{S}, \mu, T)$ be an invertible, recurrent, finite measure-preserving dynamical system. If A is a set of positive measure such that the transformation T_A is ergodic on A and $\mu(X \setminus \cup_{n\geq 0} T^{-n}(A)) = 0$, then T is ergodic.

Note: $X \setminus \cup_{n\geq 0} T^{-n}(A)$ is the set of points that never hit A.

1.2 Section 4.2

Problem 3.

Suppose A is measurable and consider $\mathbb{I}_A$. Observe that $\mathbb{I}_A(x) = 1$ if $x \in A$ and $\mathbb{I}_A(x) = 0$ if $x \notin A$. Then we must have that,

$$\{x \in X \ : \ \mathbb{I}_A(x) > 0\} = A$$

Hence, this set must be measurable and thus, by Proposition 4.2.1, we have that $\mathbb{I}_A$ is measurable.

Now suppose $\mathbb{I}_A$ is measurable. Then again by Proposition 4.2.1, we can say that $\{x \in X \ : \ \mathbb{I}_A(x) > 0\}$ is measurable. Since this set is equal to A by the definition of $\mathbb{I}_A$, we must have that A is measurable as well.

Problem 5.

Note that $f^{-1}(A \cup B) = \{x \in X : f(x) \in A \cup B\}$. But note that $f(x) \in A \cup B$ is the same as $f(x) \in A$ or $f(x) \in B$ (inclusive or). Hence we have that,

$$\begin{aligned}
f^{-1}(A \cup B) &= \{x \in X : f(x) \in A \cup B\} \\
&= \{x \in X : f(x) \in A \text{ or } f(x) \in B\} \\
&= \{x \in X : f(x) \in A\} \cup \{x \in X : f(x) \in B\} \\
&= f^{-1}(A) \cup f^{-1}(B)
\end{aligned}$$

Now consider $f^{-1}(A \cap B)$. We have that,

$$\begin{aligned} f^{-1}(A \cap B) &= \{x \in X : f(x) \in A \cap B\} \\ &= \{x \in X : f(x) \in A \text{ and } f(x) \in B\} \\ &= \{x \in X : f(x) \in A\} \cap \{x \in X : f(x) \in B\} \\ &= f^{-1}(A) \cap f^{-1}(B) \end{aligned}$$

Lastly, consider $f^{-1}(\mathbb{R} \setminus A)$. We have that,

$$\begin{aligned} f^{-1}(\mathbb{R} \setminus A) &= \{x \in X : f(x) \in \mathbb{R} \setminus A\} \\ &= \{x \in X : f(x) \in A^c\} \end{aligned}$$

Note that the final line in the aboe is the set of x in X that map to A^c in $\mathbb{R}$. In other words, it is the complement of the set of points that map to A in $\mathbb{R}$. Hence, we have,

$$\begin{aligned} \{x \in X : f(x) \in A^c\} &= \{x \in X : f(x) \in A\}^c \\ &= f^{-1}(A)^c \\ &= X \setminus f^{-1}(A) \end{aligned}$$

as required.

Now consider $f(A \cup B)$. Let $x \in f(A \cup B)$. Then $x \in \{y \in \mathbb{R} : f^{-1}(y) \in A \cup B)$. Thus, $x \in A$ or $x \in B$, or both. That is, $x \in \{y \in \mathbb{R} : f^{-1}(y) \in A)$ or $x \in \{y \in \mathbb{R} : f^{-1}(y) \in A)$ or both. Hence, we have that $f(A \cup B) \subset f(A) \cup f(B)$. Now fix $x \in f(A) \cup f(B)$. Then $f^{-1}(x) \in A$ or $f^{-1}(x) \in B$. But this is exactly the definition of $f(A \cup B)$ and so $x \in f(A \cup B)$. Thus, $f(A) \cup B \subset f(A \cup B)$ and hence, $f(A \cup B) = f(A) \cup f(B)$.

For $f(A \cap B)$, let us use a counterexample. Consider $f(x) = x^2$ and $A = [-1, 0]$, $B = [0, 1]$. Then $f(A \cap B) = f(\{0\}) = 0$, but $f(A) \cap f(B) = [0, 1] \cap [0, 1] = [0, 1]$. Hence, these are not equal and this does not hold.

Lastly, consider $f(X \setminus A)$. Again let us use $f(x) = x^2$ as a counterexample with $X = \mathbb{R}$ and $A = (0, 1]$. Then $f(X \setminus A) = \mathbb{R}$. However, $f(X) \setminus A = \mathbb{R} \setminus [0, 1]$. Hence, these are not equal and so this does not hold.

Problem 6.

Suppose that f is Lebesgue measurable. Then, by Proposition 4.2.1 and Lemma 4.2.2, the inverse image under f of any interval is a measurable set. Now fix $G \in \mathbb{R}$ such that G is an open set. Since every open subset of $\mathbb{R}$ is a countable union of disjoint open intervals, we have that $G = \sqcup_{k=1}^{\infty} I_k$ for some disjoint open intervals I_k. Now note that disjoint set in a function's image must have disjoint preimages. Otherwise, there would be an x such that $f(x)$ has two outputs, which is not possible for a validly defined function. Hence, we must have that,

$$\begin{aligned} f^{-1}(G) &= f^{-1}\left(\sqcup_{k=1}^{\infty} I_k\right) \\ &= \sqcup_{k=1}^{\infty} f^{-1}(I_k) \end{aligned}$$

Since each I_k is an interval, we have that $f^{-1}(I_k)$ is measurable. And since the countable union of measurable sets is measurable, we have that $f^{-1}(I_k) = f^{-1}(G)$ is measurable, as required.

Now suppose $f^{-1}(G)$ is measurable for every open set $G \subset \mathbb{R}$. Then, in particular, the preimage of every open interval in $\mathbb{R}$ is measurable. Hence,

$$\{x \in X \ : \ f(x) < a\}$$

is measurable for all $a \in \mathbb{R}$ and so f is measurable.

Problem 7.

Suppose $g : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable and $f : \mathbb{R} \to \mathbb{R}$ is continuous. We want to show that $f \circ g = f(g(x))$ is Lebesgue measurable.

Note that $\mathbb{R}$ is a metric space and f is continuous on $\mathbb{R}$. Hence, by Lemma 4.2.3, f is Lebesgue measurable as well. Now fix $B \in \mathcal{B}(\mathbb{R})$. We have that,

$$(f \circ g)^{-1}(B) = g^{-1}(f^{-1}(B))$$

We have that $f^{-1}(B) \in \mathcal{L}(\mathbb{R})$ since f is Lebesgue measurable. We need to use the continuity of f to show that $f^{-1}(B)$ is Borel.

A Borel set is any set that can be formed from open sets through the operations of countable union, countable intersection, and complement. The inverse images of open sets under a continuous function are open sets and inverse images of a countable union is the countable union of the inverse images. The same notions hold true for complements and countable intersections. Hence, we can write $f^{-1}(B)$ as an expression of open sets through countable unions, countable intersections, and complements.

Hence, $f^{-1}(B)$ is a Borel set and so $g^{-1}(f^{-1}(B))$ is Lebesgue measurable, and so we have that $f \circ g$ is measurable, as required.

Problem 8.

Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are Lebesgue measurable and g is such that for all null sets N, $g^{-1}(N)$ is measurable. We want to show that $f \circ g = f(g(x))$ is Lebesgue measurable.

Fix $B \in \mathcal{B}(\mathbb{R})$. We have that,

$$(f \circ g)^{-1}(B) = g^{-1}(f^{-1}(B))$$

We have that $f^{-1}(B) \in \mathcal{L}(\mathbb{R})$ since f is Lebesgue measurable. We need to use the property of g to show that $g^{-1}(f^{-1}(B))$ is measurable.

Since $f^{-1}(B)$ is measurable, then there exists a G_δ set G^* and a null set N such that $f^{-1}(B) = G^* \setminus N = G^* \cap N^C$. Note that G^* is a countable intersection of open sets, and hence is Borel. Hence, we have from Problem 5 that,

$$\begin{aligned} g^{-1}(f^{-1}(B)) &= g^{-1}(G^* \cap N^C) \\ &= g^{-1}(G^*) \cap g^{-1}(N)^C \end{aligned}$$

Since G^* is a Borel set, we have that $g^{-1}(G^*)$ is measurable and since N is a null set, we have that $g^{-1}(N)^C$ is measurable. Thus, we have that $g^{-1}(f^{-1}(B))$ is a finite intersection of measurable sets and hence is measurable. As a result, $f \circ g$ is Lebesgue measurable.

Problem 9.

Suppose that f is a Lebesgue measurable function. Then by Proposition 4.2.1, we have that

$$\{x \in X \ : \ f(x) \geq a\}$$

and

$$\{x \in X \ : \ f(x) \leq a\}$$

are both measurable sets for any $a \in \mathbb{R}$. Since the Lebesgue measurable sets form a sigma algebra, we can take the intersection of these sets and still have a measurable set. This gives us that,

$$\{x \in X \ : \ f(x) \geq a\} \cap \{x \in X \ : \ f(x) \leq a\} = \{x \in X \ : \ f(x) = a\}$$

is measurable for every $a \in \mathbb{R}$, as required.

Problem 10.

Let $x \in \{x \in X \ : \ \lim_{n\to\infty} f_n(x) > \alpha\}$ Then $\lim_{n\to\infty} f_n(x)$ converges to some number $f(x)$ such that $f(x) > \alpha$. Hence, if we fix $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, we have,

$$|f_n(x) - f(x)| < |f_n(x) - \alpha| < \epsilon$$