Dynamical Systems II: Homework 4

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1 Questions from Silva

1.1 Section 2.8

Problem 1.

Suppose we have $n > 0$ disjoint subsets of $\mathbb{N}$, which we will denote as A_i for each $1 \leq i \leq n$. For now, suppose each A_i is a finite set. Let us take the union of these sets, $A = \sqcup_{i=0}^n A_i$. We have that,

$$\mu(A) = \sum_{k \in A} \frac{1}{2^k}$$

However, since the A_i are all disjoint, we have that each k is in one and only one A_i. Hence, we can re-index this summation as,

$$\mu(A) = \sum_{k \in A_1 \text{ or } k \in A_2 \text{ or ... or } k \in A_n} \frac{1}{2^k}$$

Since each of these possibilities are disjoint, let us split it up into separate summations,

$$\begin{aligned} \mu(A) &= \sum_{k \in A_1} \frac{1}{2^k} + \sum_{k \in A_2} \frac{1}{2^k} + \cdots + \sum_{k \in A_n} \frac{1}{2^k} \\ &= \mu(A_1) + \mu(A_2) + \cdots + \mu(A_n) \end{aligned}$$

So we have covered the case where each A_i is finite. Now suppose at least one is infinite. Then we must also have that A is infinite and so $\mu(A) = \infty$. Observe we also have,

$$\begin{aligned} \sum_{i=1}^n \mu(A_i) &= \mu(A_1) + \mu(A_2) + \cdots + \infty + \cdots + \mu(A_n) \\ &= \infty \end{aligned}$$

Hence, we have that $\mu(A) = \mu(A_1) + \mu(A_2) + \cdots + \mu(A_n)$ once again and thus, μ is finitely additive.

Now suppose we have a countable number of sets A_i, where $A_i = \{i\}$. Then we have,

$$\begin{aligned}\sum \mu(A_i) &= \mu(A_1) + \mu(A_2) + \cdots \\ &= \frac{1}{2} + \frac{1}{2^2} + \cdots \\ &= \sum_{i=1}^{\infty} \frac{1}{2^i} \\ &= 1\end{aligned}$$

However, if we consider $A = \sqcup A_i$, we have that $A = \mathbb{N}$ and hence is an infinite set. Thus, $\mu(A) = \infty$ and so,

$$\mu(A) \neq \sum \mu(A_i)$$

Thus, μ is not countably additive.

Problem 2.

Since $\mathcal{R}$ is a semi-ring, we have that $A \setminus \emptyset \in \mathcal{R}$ and,

$$\begin{aligned}A \setminus \emptyset &= A \\ &= \sqcup_{j=1}^{n} E_j\end{aligned}$$

for some disjoint sets $E_j \in \mathcal{R}$. Moreover, since $K = \cup K_i$ with $K_i \in \mathcal{R}$ for every i, we can apply Proposition 2.7.1 and get,

$$K = \sqcup_{k=1}^{\infty} C_k$$

where the sets $\{C_k\}$ are disjoint and in $\mathcal{R}$. Now consider $K \setminus A$ using these above definitions. We have,

$$\begin{aligned}K \setminus A &= (\sqcup_{k=1}^{\infty} C_k) \setminus \left(\sqcup_{j=1}^{n} E_j\right) \\ &= \left(\sqcup_{k=1}^{\infty} C_k \setminus \left(\sqcup_{j=1}^{n} E_j\right)\right)\end{aligned}$$

Observe that for each k, the set $C_k \setminus \left(\sqcup_{j=1}^{n} E_j\right)$ is in $\mathcal{R}$. Hence, we have that

$$C_k \setminus \left(\sqcup_{j=1}^{n} E_j\right) = \sqcup_{j=1}^{n} E_j^{(k)}$$

Note that for each k, we have that,

$$\begin{aligned}C_k \setminus \left(\sqcup_{j=1}^{n} E_j\right) \cap A &= \sqcup_{j=1}^{n} E_j^{(k)} \cap A \\ &= \emptyset\end{aligned}$$

Moreover, for $k_1 \neq k_2$, since $C_{k_1} \cap C_{k_2} = \emptyset$, then $\left(\sqcup_{j=1}^{n} E_j^{(k_1)}\right) \cap \left(\sqcup_{j=1}^{n} E_j^{(k_2)}\right)$ and so $E_j^{k_1} \cap E_i^{k-2} = \emptyset$ for any i, j.

Lastly, note that,

$$\begin{aligned} K \setminus A \sqcup A &= \sqcup_{k=1}^{\infty} \left(\sqcup_{j=1}^{n} E_j^{(k)}\right) \sqcup \left(\sqcup_{j=1}^{n} E_j\right) \\ &= K \end{aligned}$$

Hence,

$$\begin{aligned} \mu(K) &= \mu(K \setminus A) + \mu(A) \\ &= \mu(\sqcup_{k=1}^{\infty} \left(\sqcup_{j=1}^{n} E_j^{(k)}\right) + \mu(A) \\ &= \sum_{k=1}^{\infty} \mu(\sqcup_{j=1}^{n} E_j^{(k)}) + \mu(A) \\ &= \sum_{k=1}^{\infty} \sum_{j=1}^{n} \mu(E_j^{(k)}) + \mu(A) \end{aligned}$$

Since $\mu(E_j^{(k)}) \geq 0$ for all j, k, we must have from the above derivation that $\mu(A) \leq \mu(K)$. Moreover, we have that,

$$\begin{aligned} K &= \cup_i K_i \\ &= \sqcup_k C_k \end{aligned}$$

So,

$$\begin{aligned} \mu(K) &= \mu(\cup_i K_i) \\ &= \mu(\sqcup_k C_k) \\ &= \sum_k \mu(C_k) \end{aligned}$$

Problem 5.

Suppose μ is countably additive. Since finite collections are countable, we must have that for any disjoint sets A_i, $1 \leq i \leq n$ in $\mathcal{R}$, we have that,

$$\mu\left(\sqcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$$

Hence, μ is finitely additive. Moreover if we have a countably infinite collection disjoint of sets B_i where each $B_i \in \mathcal{R}$, then since μ is countably additive, we have,

$$\mu\left(\sqcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i)$$

which satisfies the definition of countable subadditivity.

Now assume μ is additive and countably subadditive.

1.2 Section 3.2

Problem 2.

Let $d \in \{x_1.x_2x_3x_4 \mid x_1 \in \{1, 2, \ldots, 9\} \text{ and } x_2, x_3, x_4 \in \{0, 1, 2, \ldots, 9\}\}$. Then d is the set of numbers between 1.000 and 9.999 (inclusive) that have terminating decimal expansion of length 4. Now suppose the decimal representation of 3^n starts with $d \cdot 10^3$. Then for some integer $k \geq 0$,

$$d \cdot 10^k \leq 3^n < (d + 0.001) \cdot 10^k$$

Thus,

$$\log_{10}(d \cdot 10^k) \leq \log_{10} 3^n < \log_{10}((d + 0.001) \cdot 10^k$$

which gives us,

$$\log_{10} d \leq n \log_{10} 3 - k < \log_{10}(d + 0.001)$$

and finally,

$$\log_{10} d \leq n \log_{10} 3 \pmod 1 < \log_{10}(d + 0.001)$$

But this is the same as saying that, letting $\alpha = \log_{10} 3$,

$$R_\alpha^n(0) \in [\log_{10} d, \log_{10}(d + 0.001))$$

Since $0 \leq \log_{10} d < 1$ based on our definition of d and α is irrational, we can apply Theorem 3.2.3. Thus, there are infinitely many integers n such that $R_\alpha^n(0) \in [\log_{10} d, \log_{10}(d + 0.001))$. Hence, there are infinitely many powers of 3 that start with 1984.

1.3 Section 3.4

Problem 1.

We have that the collection of left-closed, right-open dyadic intervals form a sufficient semi-ring for $(\mathbb{R}, \mathcal{L}, \lambda)$. Suppose $I \in \mathcal{C}$. Then we write $I = [k/2^i, (k+1)/2^i)$ for integers k, i with $i, k \in \mathbb{Z}$. We have $\mu(I) = \frac{1}{2^i}$. Moreover, we have $T^{-1}(x) = x \pm \sqrt{x^2 + 4}$ for $x \neq 0$ and $T^{-1}(0) = 0$. This gives us,

$$T^{-1}(I) = \left[k/2^i + \sqrt{k^2/2^{2i} + 4}, (k+1)/2^i + \sqrt{(k+1)^2/2^{2i} + 4}\right) \bigcup \\ \left[k/2^i - \sqrt{k^2/2^{2i} + 4}, (k+1)/2^i - \sqrt{(k+1)^2/2^{2i} + 4}\right)$$

$T^{-1}(I)$ is a finite union of intervals and is hence measurable.

Observe that $\sqrt{(k+1)^2/2^{2i}+4} > \sqrt{(k+1)^2/2^{2i}} = (k+1)/2^i$ and $\sqrt{k^2/2^{2i}+4} > \sqrt{k^2/2^{2i}} = k/2^i$. Hence, $T^{-1}(I)$ is a disjoint union and,

$$\begin{aligned}\mu\left(T^{-1}(I)\right) = \mu&\left(\left[k/2^i + \sqrt{k^2/2^{2i}+4}, (k+1)/2^i + \sqrt{(k+1)^2/2^{2i}+4}\right)\right) + \\ \mu&\left(\left[k/2^i - \sqrt{k^2/2^{2i}+4}, (k+1)/2^i - \sqrt{(k+1)^2/2^{2i}+4}\right)\right) \\ &= (k+1)/2^i + \sqrt{(k+1)^2/2^{2i}+4} - (k/2^i + \sqrt{k^2/2^{2i}+4}) + \\ &(k+1)/2^i - \sqrt{(k+1)^2/2^{2i}+4} - k/2^i + \sqrt{k^2/2^{2i}+4} \\ &= (k+1)/2^i - k/2^i + (k+1)/2^i - k/2^i \\ &= 2(k+1)/2^i - 2k/2^i \\ &= (k+1)/2^{i-1} - k/2^{i-1} \\ &= 1/2^{i-1}\end{aligned}$$

Problem 2.

Suppose $(X, \mathcal{S}, \mu)$ is a σ-finite measure-space and $T : X \to X$ is measure-preserving. Fix $X_0 \in \mathcal{S}(X)$ with $T^{-1}(X_0) = X_0$.We want to show that the system $(X_0, \mathcal{S}(X_0), \mu, T)$ is a measure-preserving dynamical system. That is $(X_0, \mathcal{S}(X_0), \mu)$ is a σ-finite measure space and $T : X_0 \to X_0$ is a measure preserving transformation.

By Proposition 2.5.1, since $X_0 \subset X$ is in $\mathcal{S}$, we have that $\mathcal{S}(X_0) = \{A : A \subset X_0 \text{ and } X_0 \in \mathcal{S}\}$ is a σ-algebra on X_0. Since the original measure space was σ-finite, there exist a sequence of measurable sets A_n of finite measure such that,

$$X = \bigcup_{n=1}^{\infty} A_n$$

Since X_0 is in the collection of measurable sets and is a subset of X, we can remove sets from the sequence B_n, creating a new sequence B_n such that,

$$X_0 = \bigcup_{n=1}^{\infty} B_n$$

Hence, the new measure space is σ-finite as well.

Problem 3.

Let $(X, \mathcal{S}, \mu)$ be a σ-finite measure space and let $X_0 \in \mathcal{S}(X)$ with $\mu(X \setminus X_0) = 0$. Suppose there exists a transformation T_0 so that $(X_0, \mathcal{S}(X_0), \mu, T_0)$ is a measure-preserving dynamical system. Since the original measure space is σ-finite

Problem 4.

Suppose $(X, \mathcal{S}, \mu, T)$ is a measure-preserving dynamical system.