Dynamical Systems II: Homework 1

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1 Notebook Question

Problem 1.

Let $K \subset \mathbb{R}^n$ be compact.

Problem 2.

Problem 3.

2 Questions from Silva

2.1 Section 2.1

Problem 3.

Problem 4.

Let $A \subset \mathbb{R}$ and let $t \in \mathbb{R}$. Let us define $A + t = \{a + t : a \in A\}$. Suppose that $C = \bigcup I_j$ is an open cover for A.

Now suppose that $C + t = \{c + t : c \in C\}$ is not an open cover for $A + t$. Then there exists some $a \in A$ such that $a \in C$ but $a + t \notin C + t$. Note that since C is a union of open intervals, $\exists j_a \in \mathbb{N}$ such that $a \in I_{j_a} = (x_{j_a}, y_{j_a})$. By our assumption that $a + t \notin C + t$, we must have that $a + t \notin (x_{j_a} + t, y_{j_a} + t)$. However, we know that

$$x_{j_a} < a < y_{j_a} \iff x_{j_a} + t < a + t < y_{j_a} + t$$

Thus, we have a contradiction and so $C + t$ is an open cover for $A + t$.

Now take some countable collection $D = \bigcup I_j$ of open intervals. Suppose $D + t$ is an open cover of $A + t$ but D is not an open cover of A. Then there exists some $a \in A$ such that $a + t \in D + t$ but $a \notin D$. Note that since D is a union of open intervals, $\exists k_a \in \mathbb{N}$ such that $a + t \in I_{k_a} + t = (c_{k_a} + t, d_{k_a} + t)$. By our assumption that $a \notin D$, we must have that $a \notin (c_{k_a}, d_{k_a})$. However, we know that

$$c_{k_a} + t < a + t < d_{k_a} + t \iff c_{j_a} < a < d_{j_a}$$

Thus, we have a contradiction and so D is an open cover for A.

From the above, we have shown that the unions of countable open intervals which cover A and cover $A + t$ are precisely the same covers, up to a shift by t. Now note that for $I = (b, a)$, $I + t = (a + t, b + t)$, we have

$$|I| = |b - a|$$

and

$$|I + t| = |b + t - (a + t)| = |b - a|$$

Hence, the intervals that make up each countable union retain their length when shifted by t. Thus,

$$\left\{ \sum_{j=1}^{\infty} |I_j| : \; A \subset \bigcup_{j=1}^{\infty} I_j, \text{ where } I_j \text{ are bounded open intervals} \right\} =$$
$$\left\{ \sum_{j=1}^{\infty} |I_j + t| : \; A + t \subset \bigcup_{j=1}^{\infty} I_j + t, \text{ where } I_j \text{ are open bounded intervals} \right\}$$

As a result, the above two sets must have the same infimum and so $\lambda^*(A) = \lambda^*(A + t)$.

Problem 5.

Suppose N is a null set. Then $\lambda^*(N) = 0$ by definition. Now fix some set $A \subset \mathbb{R}$. By countable subadditivity, we have that

$$\lambda^*(A \cup N) \leq \lambda^*(A) + \lambda^*(N) = \lambda^*(A)$$

Now observe that $A \subset A \cup N$. Hence, by Proposition 2.1.1 (3), we have that $\lambda^*(A) \leq \lambda^*(N \cup A)$. Thus, by both of the above statements, we have

$$\lambda^*(A \cup N) = \lambda^*(A)$$

Problem 7.

Suppose we have countably many null sets $N_1, N_2, \ldots$. By countable subadditivity, we have that,

$$\lambda^* \left(\bigcup_{k=1}^{\infty} N_k \right) \leq \sum_{k=1}^{\infty} \lambda^*(N_k) = 0$$

In addition, we know that $\lambda^* \left(\bigcup_{k=1}^{\infty} N_k \right)$ is bounded below by 0 because we are taking the infimum of a sum of interval lengths, where length is defined to be a non-negative real number. Hence, $\lambda^* \left(\bigcup_{k=1}^{\infty} N_k \right) = 0$ and thus is a null set, as required.

Problem 8.

Let $A \subset \mathbb{R}$ and $t \in \mathbb{R}$. Define $tA = \{ta : a \in A\}$. If $t = 0$, then $tA = \{0\}$ and so

$$\lambda^*(tA) = \lambda^*(\{0\}) = 0 = |t|\lambda^*(A)$$

as required.

Now suppose $t \neq 0$. Suppose that $C = \bigcup I_j$ is an open cover for A.

Now suppose that $tC = \{tc : \; c \in C\}$ is not an open cover for tA. Then there exists some $a \in A$ such that $a \in C$ but $ta \notin tC$. Note that since C is a union of open intervals, $\exists j_a \in \mathbb{N}$ such that $a \in I_{j_a} = (x_{j_a}, y_{j_a})$. By our assumption that $ta \notin tC$, we must have that $ta \notin (tx_{j_a}, ty_{j_a})$. However, we know that when t is positive

$$x_{j_a} < a < y_{j_a} \iff tx_{j_a} < ta < ty_{j_a}$$

and when t is negative

$$x_{j_a} < a < y_{j_a} \iff tx_{j_a} > ta > ty_{j_a}$$

Thus, we have a contradiction and so tC is an open cover for tA.

Now take some countable collection $D = \bigcup I_j$ of open intervals. Suppose tD is an open cover of tA but D is not an open cover of A. Then there exists some $a \in A$ such that $ta \in tD$ but $a \notin D$. Note that since D is a union of open intervals, $\exists k_a \in \mathbb{N}$ such that $ta \in tI_{k_a} = (tc_{k_a}, td_{k_a})$. By our assumption that $a \notin D$, we must have that $a \notin (c_{k_a}, d_{k_a})$. However, we know that when t is positive

$$tc_{k_a} < a < td_{k_a} \iff c_{j_a} < a < d_{j_a}$$

and when t is negative

$$tc_{k_a} < a < td_{k_a} \iff c_{j_a} > a > d_{j_a}$$

Thus, we have a contradiction and so D is an open cover for A.

From the above, we have shown that the unions of countable open intervals which cover A and cover tA are precisely the same covers, up to a scaling by t. Now note that for $I = (b, a)$, $tI = (ta, tb)$, we have

$$|I| = |b - a|$$

and

$$|tI| = |tb - ta| = |t| \cdot |b - a|$$

Hence, the intervals that make up each cover for A have their lengths scaled by $|t|$. Thus, $\sum |tI_j| = \sum |t||I_j| = |t| \sum |I_j|$. Thus, from the reasoning above about the length of each interval and the fact that the covers for tA are precisely the covers for A scaled by t, we have

$$\begin{aligned}
\lambda^*(tA) &= \inf\left\{\sum_{j=1}^{\infty}|tI_j| : \ tA \subset \bigcup_{j=1}^{\infty} tI_j, \text{ where } I_j \text{ are open bounded intervals}\right\} \\
&= \inf |t| \cdot \left\{\sum_{j=1}^{\infty}|I_j| : \ A \subset \bigcup_{j=1}^{\infty} I_j, \text{ where } I_j \text{ are open bounded intervals}\right\} \\
&= |t| \inf \left\{\sum_{j=1}^{\infty}|I_j| : \ A \subset \bigcup_{j=1}^{\infty} I_j, \text{ where } I_j \text{ are open bounded intervals}\right\} \\
&= |t| \cdot \lambda^*(A)
\end{aligned}$$

as required.

2.2 Section 2.2

Problem 5.

Problem 6.

Note that $K \subset [0,1]$, and for any $x, y \in [0,1]$, we have that $0 \leq x + y \leq 2$. Thus, $K + K \subset [0,2]$. Now we need to show that $[0,2] \subset K + K$