Dynamical Systems II: Homework 9

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1 Questions from Silva

1.1 Section 4.3

Problem 1.

Suppose s_1 and s_2 are simple function with $s_1 = s_2$ a.e. Let E be the set of measure 0 where $s_1 \neq s_2$. Observe that,

$$\begin{aligned}
\int s_1 d\mu &= \sum_{i=1}^{n} a_i \mu(A_i) \\
&= \sum_{i=1}^{n} a_i \mu(A_i \setminus E) + a\mu(E) \\
&= \sum_{i=1}^{n} a_i \mu(A_i \setminus E) + a \cdot 0 \\
&= \sum_{i=1}^{n} a_i \mu(A_i \setminus E)
\end{aligned}$$

and,

$$\begin{aligned}
\int s_2 d\mu &= \sum_{i=1}^{m} b_i \mu(B_i) \\
&= \sum_{i=1}^{m} b_i \mu(B_i \setminus E) + b\mu(E) \\
&= \sum_{i=1}^{m} b_i \mu(B_i \setminus E) + a \cdot 0 \\
&= \sum_{i=1}^{m} b_i \mu(B_i \setminus E)
\end{aligned}$$

But observe that $\cup_{i=1}^{n} A_i \setminus E = \cup_{i=1}^{m} B_i \setminus E = X \setminus E$ and, by definition, $s_1 = s_2$ everywhere on this set. Since we are integrating the same function on the same set, we must have that $\int s_1 d\mu = \int s_2 d\mu$.

Problem 3.

Suppose s is a nonnegative simple function. Suppose $\int_X s d\mu = 0$. Then,

$$\begin{aligned}\int_x s d\mu &= \sum_{i=1}^{n} \alpha_i \mu(E_i) \\ &= 0\end{aligned}$$

Since $s \geq 0$, we have that $\alpha_i \geq 0$ for all i. Moreover, we have that $\mu(E_i) \geq 0$ for all i by properties of measure. Thus, in order for this sum to equal 0, we must have that $\alpha_i \mu(E_i) = 0$ for all i. Note that if $\mu(E_i) = 0$ for all i, then $\mu(X) = 0$. Hence, even if $s > 0$ on all of X, it is vacuously true that $s = 0$ a.e. since the whole space is measure 0. So let us assume that at least one E_i has positive measure. If $\alpha_i \mu(E_i) > 0$ for this i, then $\sum_{i=1}^{n} \alpha_i \mu(E_i) > 0$, a contradiction. Hence, for every set E_i of positive measure, we must have that $s = 0$. Thus, $s = 0$ a.e. in X.

Now let us assume $s = 0$ a.e. in X. Then we have that there is a set E with $\mu(E) = 0$ such that $s > 0$ on E and $s = 0$ on $X \setminus E$. Since $s = 0$ a.e. and 0 is a simple function (with every coefficient α_i set to 0), we can apply Exercise 1 and state that we must have,

$$\begin{aligned}\int_X s d\mu &= \int_X 0 d\mu \\ &= \sum_{i=1}^{n} 0 \cdot \mu(A_i) \\ &= 0\end{aligned}$$

as required.

1.2 Section 4.4

Problem 2.

Let f be a nonnegative measurable function and let A be a measurable set. Suppose $\int_A f \ d\mu = 0$. Then,

$$\int_A f \ d\mu = \sup\{\int_A s \ d\mu : s \text{ is simple and } 0 \leq s \leq f\} = 0$$

. Since the supremum of this set is 0 and every nonnegative simple function has a nonnegative integral, then for any s in the above set, we must have that,

$$\begin{aligned}\int_A s \ d\mu &= \sum_{i=1}^{n} a_i \mu(A_i) \\ &= 0\end{aligned}$$

By Section 4.3 Problem 3, we thus have that $s = 0$ a.e. in A. Since s was arbitrary, this holds for every s in the above set. Let us suppose that it is not the case that $f = 0$ a.e. That is, there is a set of positive measure where $f > 0$. Then there would be a simple function s with $0 \leq s \leq f$ such that $s > 0$ on this set as well and hence it would not be the case that $s = 0$ a.e., a contradiction. Hence, we must have that $f = 0$ a.e.

Now suppose that $f = 0$ a.e. Then there is a set E of measure 0 such that $f = 0$ everywhere on $A \setminus E$. Fix a simple function s on A such that $0 \leq s \leq f$. For this to hold, we must have that $s = 0$ on $A \setminus E$. We can have that $0 < s \leq f$ on E, but note that $\mu(E) = 0$. Thus,

$$\begin{aligned}\int_A s &= \sum_{i=1}^{n} a_i \ \mu(A_i) \\ &= \sum_{i=1}^{n} a_i \ \mu(A_i \setminus E) + \sum_{i=1}^{m} e_i \ \mu(E_i) \\ &= \sum_{i=1}^{n} a_i \ \mu(A_i \setminus E) \\ &= 0\end{aligned}$$

where $\cup E_i = E$ (note since $E_i \subset E$ we have $\mu(E_i) \leq \mu(E)$ and since measure is nonnegative and $\mu = 0$, we get $\mu(E_i) = 0$).

Since s was arbitrary on A with the property $0 \leq s \leq f$, this must hold for every element of the set $\{\int_A s \ d\mu : s \text{ is simple and } 0 \leq s \leq f\}$. Thus, $\int_A s = 0$ for every element s in the set and so,

$$\begin{aligned}\int_A f \ d\mu &= \sup\{\int_A s \ d\mu : s \text{ is simple and } 0 \leq s \leq f\} \\ &= 0\end{aligned}$$

as required.

Problem 3.

Suppose f is a nonnegative measurable function and let A, B be measurable sets with $A \subset B$. We have that

$$\int_A f = \sup\{\int_A s \ d\mu : s \text{ is simple and } 0 \leq s \leq f\}$$

and

$$\int_B f = \sup\{\int_B s \ d\mu : s \text{ is simple and } 0 \leq s \leq f\}$$

Observe that for every s in the set $\{\int_B s \ d\mu : s \text{ is simple and } 0 \leq s \leq f\}$, we can write,

$$\begin{aligned}\int_B s &= \sum_{i=1}^{n} b_n \mu(B_i) \\ &= \sum_{i=1}^{n} b_i \mu(B_i \setminus A) + \sum_{i=1}^{m} a_i \mu(A_i) \\ &= \int_{B \setminus A} s + \int_A s\end{aligned}$$

where $\cup A_i = A$. Since the integral of any nonnegative simple function is nonnegative, we must have from the above that $\int_B s \geq \int_A s$. Observe that every element of $\{\int_A s \ d\mu :$ s is simple and $0 \leq s \leq f\}$ can be extended to a simple function on B by taking $s = 0$ on $B \setminus A$, so we have $\{\int_A s \ d\mu : s \text{ is simple and } 0 \leq s \leq f\} \subset \{\int_B s \ d\mu : s \text{ is simple and } 0 \leq s \leq f\}$. Moreover, as we have shown above, if $\int_{B\setminus A} > 0$, then $\int_B s > \int_A s$. Thus, we must have that,

$$\sup\{\int_A s \ d\mu : s \text{ is simple and } 0 \leq s \leq f\} \leq \sup\{\int_B s \ d\mu : s \text{ is simple and } 0 \leq s \leq f\}$$

and so,

$$\int_A f \leq \int_B f$$

as required.

Problem 4.

Let f be a nonnegative measurable function and $\{A_j\}$ be a sequence of disjoint measurable sets. We have,

$$\int_{\sqcup A_j} f \ d\mu = \sup\{\int_{\sqcup A_j} s \ d\mu : s \text{ is simple and } 0 \leq s \leq f\}$$

Fix s in the above set. Then,

$$\int_{\sqcup A_j} s = \sum_{i=1}^{n} b_i \ \mu(B_i)$$

for some sets B_i such that $\cup B_i = \sqcup A_j$. However, we could alternatively divide up $\sqcup A_j$ so that each subset in our sum corresponds to only one A_j. That is, we can write,

$$\begin{aligned}
\int_{\sqcup A_j} s &= \sum_{j=1} \sum_{i=1}^{n} a_{ji} \ \mu(A_{ji}) \\
&= \sum_{i=1}^{n} a_{1i} \ \mu(A_{1i}) + \sum_{i=1}^{n} a_{2i} \ \mu(A_{2i}) + \cdots \\
&= \int_{A_1} s + \int_{A_2} s + \cdots
\end{aligned}$$

Now note that $\int_{A_i} s$ is contained in the above set for any i because we can set $\int_{A_j} s = 0$ for any $j \neq i$ and maintain its status as a simple function. Hence,

$$\int_{\sqcup A_i} f \ d\mu = \sum_j \int_{A_j} f \ d\mu$$

Problem 5.

Consider

$$f_n(x) = \begin{cases} \mathbf{1}_{[0,1/2]}(x) & n \text{ odd} \\ \mathbf{1}_{[1/2,1]}(x) & n \text{ even} \end{cases}$$

Observe that $\liminf_{n\to\infty} f_n(x) = 0$. Thus,

$$\begin{aligned} \int \liminf_{n\to\infty} f_n(x) d\mu &= \int 0 d\mu \\ &= 0 \end{aligned}$$

Now note that $\int f_n d\mu = \frac{1}{2}$ for all n. Thus,

$$\begin{aligned} \liminf_{n\to\infty} \int f_n d\mu &= \liminf_{n\to\infty} \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

Thus, we have that $\int \liminf_{n\to\infty} f_n(x) d\mu < \liminf_{n\to\infty} \int f_n d\mu$ as required.