Dynamical Systems II: Homework 8

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1 Questions from Silva

1.1 Section 3.11

Problem 3.

1.2 Section 4.2

Problem 3.

Suppose A is measurable and consider $\mathbb{I}_A$. Observe that $\mathbb{I}_A(x) = 1$ if $x \in A$ and $\mathbb{I}_A(x) = 0$ if $x \notin A$. Then we must have that,

$$\{x \in X \ : \ \mathbb{I}_A(x) > 0\} = A$$

Hence, this set must be measurable and thus, by Proposition 4.2.1, we have that $\mathbb{I}_A$ is measurable.

Now suppose $\mathbb{I}_A$ is measurable. Then again by Proposition 4.2.1, we can say that $\{x \in X \ : \ \mathbb{I}_A(x) > 0\}$ is measurable. Since this set is equal to A by the definition of $\mathbb{I}_A$, we must have that A is measurable as well.

Problem 5.

Problem 6.

Suppose that f is Lebesgue measurable. Then, by Proposition 4.2.1 and Lemma 4.2.2, the inverse image under f of any interval is a measurable set. Now fix $G \in \mathbb{R}$ such that G is an open set. Since every open subset of $\mathbb{R}$ is a countable union of disjoint open intervals, we have that $G = \sqcup_{k=1}^{\infty} I_k$ for some disjoint open intervals I_k. Now note that disjoint set in a function's image must have disjoint preimages. Otherwise, there would be an x such that $f(x)$ has two outputs, which is not possible for a validly defined function. Hence, we must have that,

$$\begin{aligned} f^{-1}(G) &= f^{-1}\left(\sqcup_{k=1}^{\infty} I_k\right) \\ &= \sqcup_{k=1}^{\infty} f^{-1}(I_k) \end{aligned}$$

Since each I_k is an interval, we have that $f^{-1}(I_k)$ is measurable. And since the countable union of measurable sets is measurable, we have that $f^{-1}(I_k) = f^{-1}(G)$ is measurable, as required.

Now suppose $f^{-1}(G)$ is measurable for every open set $G \subset \mathbb{R}$. Then, in particular, the preimage of every open interval in $\mathbb{R}$ is measurable. Hence,

$$\{x \in X \ : \ f(x) < a\}$$

is measurable for all $a \in \mathbb{R}$ and so f is measurable.

Problem 7.

Problem 8.

Problem 9.

Suppose that f is a Lebesgue measurable function. Then by Proposition 4.2.1, we have that

$$\{x \in X \ : \ f(x) \geq a\}$$

and

$$\{x \in X \ : \ f(x) \leq a\}$$

are both measurable sets for any $a \in \mathbb{R}$. Since the Lebesgue measurable sets form a sigma algebra, we can take the intersection of these sets and still have a measurable set. This gives us that,

$$\{x \in X \ : \ f(x) \geq a\} \cap \{x \in X \ : \ f(x) \leq a\} = \{x \in X \ : \ f(x) = a\}$$

is measurable for every $a \in \mathbb{R}$, as required.

Problem 10.