Dynamical Systems II: Quiz 1

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March 11, 2021

Problem 1.

Let
$$T(x) = \frac{x - a_i}{a_{i+1} - a_i}$$
with $x \in [a_i, a_{i+1})$. $T : J \to J$ where $J = [0, 1)$. And $0 = a_0 < a_1 < \cdots < a_k = 1$. We want to show that T is a measure preserving transformation of $(J, \mathcal{L}(J), \lambda)$ regardless of choice of $\{a_i\}$.

We have that $T^{-1}(x) = x(a_{i+1} - a_i) + a_i$. Let $\mathcal{C}$ be the collection of left-closed, right-open dyadic intervals in $[0, 1)$. We saw in Section 2.7 that $\mathcal{C}$ is a sufficient semi-ring. For I we write $I = [k/2^i, (k+1)/2^i)$ for integers k, i with $i \geq 0$ and $k \in \{0, \dots, 2^i - 1\}$. Observe that $\lambda(I) = 1/2^i$ for all $I \in \mathcal{C}$. Assume for a fixed I that $I \subset [a_i, a_{i+1})$ for some $i \in \{0, \dots k\}$. Then,

$$T^{-1}(I) = \left[\frac{k}{2^i}(a_{i+1} - a_i) + a_i, \frac{k+1}{2^i}(a_{i+1} - a_i) + a_i\right)$$

$T^{-1}(I)$ is an interval for any I and is hence a measurable set. Moreover,

$$\begin{aligned}
\lambda(T^{-1}(I)) &= \frac{k+1}{2^i}(a_{i+1} - a_i) + a_i - (\frac{k}{2^i}(a_{i+1} - a_i) + a_i) \\
&= \frac{k+1}{2^i}a_{i+1} - \frac{k+1}{2^i}a_i - \frac{k}{2^i}a_{i+1} + \frac{k}{2^i}a_i \\
&= \frac{(k+1)(a_{i+1} - a_i) - k(a_{i+1} - a_i)}{2^i} \\
&= \frac{((k+1) - k)(a_{i+1} - a_i)}{2^i} \\
&= \frac{a_{i+1} - a_i}{2^i}
\end{aligned}$$

Observe that T maps some values onto I for each T defined on the intervals $[a_i, a_{i+1})$. Hence, there will be k such intervals resulting from I with the same length as the above when applying T^{-1}.

If we add this up over all intervals $[a_i, a_{i+1})$, we get,

$$\frac{a_k - a_0}{2^i} = \frac{1}{2^i} = \lambda(I)$$

as required. Hence, we can apply Theorem 3.4.1 in order to assert that T is then a measure-preserving transformation.