Dynamical Systems II: Midterm

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Problem 1.

Let

$$T(x) = \begin{cases} \sqrt{x} & x < \frac{1}{4} \\ \frac{1}{2} - \sqrt{\frac{1}{2} - x} & x \geq 4 \end{cases}$$

Note then that,

$$T^{-1}(x) = \begin{cases} x^2 & x < \frac{1}{4} \\ -x^2 + x + \frac{1}{4} & x \geq \frac{1}{4} \end{cases}$$

In addition, note that for $x \in [0, \frac{1}{4})$, we have $T(x) \in [0, \frac{1}{2})$ and for $x \in [\frac{1}{4}, \frac{1}{2})$, we have $T(x) \in [0, \frac{1}{2})$. Thus, for any y in the image of T, we will need to consider two pre-images under the inverse function $T^{-1}(x)$.

Now let $\mathcal{C}$ be the set of intervals in $[0, \frac{1}{2})$. This is a sufficient semi-ring for the domain. Let us fix some interval I with endpoints $a, b \in \mathcal{C}$ with $b \geq a$ (whether it is open or closed will not change its Lebesgue measure). We will assume the interval is open for now. Then,

$$\lambda(I) = b - a$$

Now let us consider $T^{-1}(I)$. We have two inverse images to consider,

$$T_1^{-1}(I) = (a^2, b^2)$$

and,

$$T_2^{-1}(I) = (-a^2 + a + \frac{1}{4}, -b^2 + b + \frac{1}{4})$$

Note that the inverse of T preserves the interval structure, so $T^{-1}(I)$ is a measurable set.

Now let us check $\lambda(T^{-1}(I))$. We have,

$$\begin{aligned} \lambda(T^{-1}(I)) &= \lambda(T_1^{-1}(I)) + \lambda(T_2^{-1}(I)) \\ &= b^2 - a^2 + (-b^2 + b + \frac{1}{4} - (-a^2 + a + \frac{1}{4})) \\ &= b^2 - a^2 - b^2 + b + \frac{1}{4} + a^2 - a - \frac{1}{4} \\ &= b - a \\ &= \lambda(I) \end{aligned}$$

Hence, by Theorem 3.4.1, T is measure preserving.

Problem 2.

We have that T is continuous and measure preserving and that f is continuous and $f(T(x)) \geq f(x)$.

Now suppose T is recurrent. Then for every measurable set A of positive measure, there is a null set $N \subset A$ such that for all $x \in A \backslash N$ there is an integer $n = n(x) > 0$ with $T^n(X) \in A$.

Fix $x_0 \in \mathbb{R}^2$ and suppose $f(T(x_0)) > f(x_0)$. Since f, T are continuous, for points x_1 near x_0, we would expect to see $f(T(x_1)) > f(x_1)$ as well.

Problem 3.

(a) Let $C_{m,n}$ be the set of points $x \in X$ for which m and n are consecutive visit times to A. Suppose $C_{m,n}$ is not measurable. Then $C_{m,n} \notin \mathcal{S}$. But since T is measurable and $T^{-1}(T^{m+1}(X)) = T^m(X) \in \mathcal{S}$. Furthermore, since A measurable, we must have $A \cap T^m(X)$ is measurable as well. The same applies to $T-1(T^{n+1}(X)) = T^n(x)$. Hence, $A \cap T^n(X)$ is measurable as well. Then we have that

$$A \cap T^n(X) \cup A \cap T^m(X)$$

must be measurable. Finally, we have,

$$A \cap T^m(X) \cup A \cap T^n(X) \setminus \left(\bigcup_{i=1}^{n-m} T^{m+i}(X) \cap A \right)$$

is measurable since we are taking a finite union of measurable sets and then set minusing this finite union from another measurable set. However, note that this is exactly $C_{m,n}$, and so $C_{m,n}$ is measurable.

(b) Now let us take $C_{m,n}$ and define the following set:

$$C_{m,n} \cap \left(\bigcup_{i=1}^{n-m-1} T^{m+i}(X) \cap B \right)$$

That is, we have taken the intersection of $C_{m,n}$ with the union of the set of points in X such that $T^i(X) \in B$ for some i with $m < i < n$. Note that, by the same arguments as above $T^{m+i}(X) \cap B$ is measurable for every i. Furthermore, we are taking a finite union of measurable sets, so $\bigcup_{i=1}^{n-m-1} T^{m+i}(X) \cap B$ is measurable as well. And lastly, $C_{m,n}$ is measurable by (a), so

$$D_{m,n} = C_{m,n} \cap \left(\bigcup_{i=1}^{n-m-1} T^{m+i}(X) \cap B \right)$$

is measurable.

(c) We have that $x \in E$ if and only if $x \in C_{m,n}$ implies $x \in D_{m,n}$ for all integers m and n with $0 \leq m < n$. That is,

Problem 4.

Let us show that T is continuous. That is, we want to show that for each $x \in \Sigma_2^+$, we have that for all $\epsilon > 0$, there exists $\delta > 0$ such that $d(T(x), T(y)) < \epsilon$ whenever $y \in \Sigma_2^+$ and $d(x, y) < \delta$. First let us fix $x \in \Sigma_2^+$ and $\epsilon > 0$. Let m be the length of the initial constant sequence in x. Let $\delta = \frac{1}{2^m}\epsilon$. Then for any $y \in \Sigma_2^+$ with $d(x, y) < \delta$, we have that the first position where $x \neq y$ is some k such that

$$\begin{aligned} & 1/2^k < \delta = \frac{1}{2^m}\epsilon \\ \implies & 2^m 2^{-k} < \epsilon \\ \implies & 2^{m-k} < \epsilon \\ \implies & 1/2^{k-m} < \epsilon \end{aligned}$$

That is, after removing the first m elements of x and y (call these new elements x', y'), we still have that $d(x', y') < \epsilon$. Now note that $T(x)$ will remove exactly the first m elements. Since $k \geq m$ and $x_i = y_i$ for all $0 \leq i \leq k-1$, we thus also have that $T(y)$ removes exactly the first m elements of y. Hence, by what we have just shown, for any fixed $x \in \Sigma_2^+$ and $\epsilon > 0$, we have that for all $y \in B(x, \delta)$, $T(y) \in B(T(x), \epsilon)$. Thus, T is continuous.