

Dynamical Systems II: Homework 1

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1 Notebook Question

Problem 1.

Let $G = \{g_0, \dots, g_{k-1}\}$ be a contracting iterated function system with contracting similarity ratio $c < 1$. Fix $i \in \{0, \dots, k-1\}$. We have $d(g_i(x), g_i(y)) = cd(x, y)$ for all $x, y \in \mathbb{R}^n$.

Problem 2.

The limit set given by Definition 1 is defined as $\Lambda = \bigcap_{n=0}^{\infty} \Lambda_n$ where $\Lambda_0 = K$ and Λ_n for $n \geq 0$ is defined as $\Lambda_{n+1} = \bigcup_{i=0}^{k-1} g_i(\Lambda_n)$.

The limit set for Definition 2 is

$\{z \in \mathbb{R}^n : \text{for any neighborhood } N \text{ of } z \text{ there are infinitely many finite words } w$
such that the image $\Phi(w)$ lies in $N\}$

Problem 3.

Section 1.2 of Edgar states that the Sierpinski gasket can be constructed using three dilation with ratio $1/2$ and centers at the three vertices of the triangle S_0 . Let one of these vertices be at the origin, $(0, 0)$. We are given that S_0 has side length 1, so the other two vertices must be at $(1, 0)$ and $(1/2, 1/2)$. Hence, the three dilations are given by,

$$\begin{aligned} f_1(x, y) &= \frac{(x, y)}{2} \\ f_2(x, y) &= \frac{(x, y) - (1, 0)}{2} \\ f_3(x, y) &= \frac{(x, y) - (1/2, 1/2)}{2} \end{aligned}$$

2 Questions from Silva

2.1 Section 2.1

Problem 3.

Suppose we have some interval from the definition of outer measure. That is, an interval I_j such that I_j is open and bounded. Then we can write $I_j = (a, b)$ where $a, b \in \mathbb{R}$. Let us fix $k \in \mathbb{Z}$ such that $k/2^j \leq a$. We will construct a sequence of dyadic intervals $[k/2^j, (k+1)/2^j), [(k+1)/2^j, (k+2)/2^j), \dots, [(k+\ell)/2^j, (k+\ell+1)/2^j)$ such that $(a, b) \subset \bigcup_{i=0}^{\ell} [(k+i)/2^j, (k+i+1)/2^j)$.

From Lemma 2.1.2, we have,

$$\begin{aligned} b - a &\leq \sum_{i=0}^{\ell} |[(k+i)/2^j, (k+i+1)/2^j)| = \sum_{i=0}^{\ell} 1/2^j \\ &= (\ell+1)/2^j \end{aligned}$$

Problem 4.

Let $A \subset \mathbb{R}$ and let $t \in \mathbb{R}$. Let us define $A + t = \{a + t : a \in A\}$. Suppose that $C = \bigcup I_j$ is an open cover for A .

Now suppose that $C + t = \{c + t : c \in C\}$ is not an open cover for $A + t$. Then there exists some $a \in A$ such that $a \in C$ but $a + t \notin C + t$. Note that since C is a union of open intervals, $\exists j_a \in \mathbb{N}$ such that $a \in I_{j_a} = (x_{j_a}, y_{j_a})$. By our assumption that $a + t \notin C + t$, we must have that $a + t \notin (x_{j_a} + t, y_{j_a} + t)$. However, we know that

$$x_{j_a} < a < y_{j_a} \iff x_{j_a} + t < a + t < y_{j_a} + t$$

Thus, we have a contradiction and so $C + t$ is an open cover for $A + t$.

Now take some countable collection $D = \bigcup I_j$ of open intervals. Suppose $D + t$ is an open cover of $A + t$ but D is not an open cover of A . Then there exists some $a \in A$ such that $a + t \in D + t$ but $a \notin D$. Note that since D is a union of open intervals, $\exists k_a \in \mathbb{N}$ such that $a + t \in I_{k_a} + t = (c_{k_a} + t, d_{k_a} + t)$. By our assumption that $a \notin D$, we must have that $a \notin (c_{k_a}, d_{k_a})$. However, we know that

$$c_{k_a} + t < a + t < d_{k_a} + t \iff c_{k_a} < a < d_{k_a}$$

Thus, we have a contradiction and so D is an open cover for A .

From the above, we have shown that the unions of countable open intervals which cover A and cover $A + t$ are precisely the same covers, up to a shift by t . Now note that for $I = (b, a)$, $I + t = (a + t, b + t)$, we have

$$|I| = |b - a|$$

and

$$|I + t| = |b + t - (a + t)| = |b - a|$$

Hence, the intervals that make up each countable union retain their length when shifted by t . Thus,

$$\left\{ \sum_{j=1}^{\infty} |I_j| : A \subset \bigcup_{j=1}^{\infty} I_j, \text{ where } I_j \text{ are bounded open intervals} \right\} = \left\{ \sum_{j=1}^{\infty} |I_j + t| : A + t \subset \bigcup_{j=1}^{\infty} I_j + t, \text{ where } I_j \text{ are open bounded intervals} \right\}$$

As a result, the above two sets must have the same infimum and so $\lambda^*(A) = \lambda^*(A + t)$.

Problem 5.

Suppose N is a null set. Then $\lambda^*(N) = 0$ by definition. Now fix some set $A \subset \mathbb{R}$. By countable subadditivity, we have that

$$\lambda^*(A \cup N) \leq \lambda^*(A) + \lambda^*(N) = \lambda^*(A)$$

Now observe that $A \subset A \cup N$. Hence, by Proposition 2.1.1 (3), we have that $\lambda^*(A) \leq \lambda^*(N \cup A)$. Thus, by both of the above statements, we have

$$\lambda^*(A \cup N) = \lambda^*(A)$$

Problem 7.

Suppose we have countably many null sets N_1, N_2, \dots . By countable subadditivity, we have that,

$$\lambda^* \left(\bigcup_{k=1}^{\infty} N_k \right) \leq \sum_{k=1}^{\infty} \lambda^*(N_k) = 0$$

In addition, we know that $\lambda^* \left(\bigcup_{k=1}^{\infty} N_k \right)$ is bounded below by 0 because we are taking the infimum of a sum of interval lengths, where length is defined to be a non-negative real number. Hence, $\lambda^* \left(\bigcup_{k=1}^{\infty} N_k \right) = 0$ and thus is a null set, as required.

Problem 8.

Let $A \subset \mathbb{R}$ and $t \in \mathbb{R}$. Define $tA = \{ta : a \in A\}$. If $t = 0$, then $tA = \{0\}$ and so

$$\lambda^*(tA) = \lambda^*(\{0\}) = 0 = |t|\lambda^*(A)$$

as required.

Now suppose $t \neq 0$. Suppose that $C = \bigcup I_j$ is an open cover for A .

Now suppose that $tC = \{tc : c \in C\}$ is not an open cover for tA . Then there exists some $a \in A$ such that $a \in C$ but $ta \notin tC$. Note that since C is a union of open intervals, $\exists j_a \in \mathbb{N}$ such that $a \in I_{j_a} = (x_{j_a}, y_{j_a})$. By our assumption that $ta \notin tC$, we must have that $ta \notin (tx_{j_a}, ty_{j_a})$. However, we know that when t is positive

$$x_{j_a} < a < y_{j_a} \iff tx_{j_a} < ta < ty_{j_a}$$

and when t is negative

$$x_{j_a} < a < y_{j_a} \iff tx_{j_a} > ta > ty_{j_a}$$

Thus, we have a contradiction and so tC is an open cover for tA .

Now take some countable collection $D = \bigcup I_j$ of open intervals. Suppose tD is an open cover of tA but D is not an open cover of A . Then there exists some $a \in A$ such that $ta \in tD$ but $a \notin D$. Note that since D is a union of open intervals, $\exists k_a \in \mathbb{N}$ such that $ta \in tI_{k_a} = (tc_{k_a}, td_{k_a})$. By our assumption that $a \notin D$, we must have that $a \notin (c_{k_a}, d_{k_a})$. However, we know that when t is positive

$$tc_{k_a} < a < td_{k_a} \iff c_{k_a} < a < d_{k_a}$$

and when t is negative

$$tc_{k_a} < a < td_{k_a} \iff c_{k_a} > a > d_{k_a}$$

Thus, we have a contradiction and so D is an open cover for A .

From the above, we have shown that the unions of countable open intervals which cover A and cover tA are precisely the same covers, up to a scaling by t . Now note that for $I = (b, a)$, $tI = (ta, tb)$, we have

$$|I| = |b - a|$$

and

$$|tI| = |tb - ta| = |t| \cdot |b - a|$$

Hence, the intervals that make up each cover for A have their lengths scaled by $|t|$. Thus, $\sum |tI_j| = \sum |t| |I_j| = |t| \sum |I_j|$. Thus, from the reasoning above about the length of each interval and the fact that the covers for tA are precisely the covers for A scaled by t , we have

$$\begin{aligned} \lambda^*(tA) &= \inf \left\{ \sum_{j=1}^{\infty} |tI_j| : tA \subset \bigcup_{j=1}^{\infty} tI_j, \text{ where } I_j \text{ are open bounded intervals} \right\} \\ &= \inf |t| \cdot \left\{ \sum_{j=1}^{\infty} |I_j| : A \subset \bigcup_{j=1}^{\infty} I_j, \text{ where } I_j \text{ are open bounded intervals} \right\} \\ &= |t| \inf \left\{ \sum_{j=1}^{\infty} |I_j| : A \subset \bigcup_{j=1}^{\infty} I_j, \text{ where } I_j \text{ are open bounded intervals} \right\} \\ &= |t| \cdot \lambda^*(A) \end{aligned}$$

as required.

2.2 Section 2.2

Problem 5.

Problem 6.

Note that $K \subset [0, 1]$, and for any $x, y \in [0, 1]$, we have that $0 \leq x + y \leq 2$. Thus, $K + K \subset [0, 2]$. Now we need to show that $[0, 2] \subset K + K$