Dynamical Systems II: Homework 4

Chris Hayduk

March 16, 2021

1 Questions from Silva

1.1 Section 3.2

Problem 10.

1.2 Section 3.12

Problem 2.

Let Ω be the subset Σ_3^+ consisting of all sequences x that do not have the word 010 at any place. Suppose that z is a limit point of Ω that is not contained within the set Ω. That is, since z not in Ω, it has the sequence 010 starting at position k for some $k \in \mathbb{Z}$ such that $k \geq 0$. Moreover, since z is a limit point of Ω, there is a sequence $(x_n) \in \Omega$ such that for every $\epsilon > 0$, there exists $N > 0$ such that $d(x_N, z) < \epsilon$. However, note that every element x_j in the above sequence does not have the sequence 010. Thus, for any $j > 0$, we have that,

$$d(x_j, z) \geq 2^{-k}$$

Hence, for any $n \in \mathbb{N}$ and any $\epsilon < 2^{-k}$, we have,

$$d(x_n, z) \geq 2^k > \epsilon$$

Thus, we have a contradiction and so z must be contained in Ω. Since z was an arbitrary limit point of the set, we must have that Ω contains all its limit points and is thus closed.

1.3 Section 3.13

Problem 1.

Note that alternative definition for density of a set $D \subset \Sigma_N^+$ is that, for every non-empty open set $U \subset \Sigma_N^+$, we have $D \cap U \neq \emptyset$.

For a point $x \in \Sigma_N^+$ to be periodic, it must be the repetition for some block of symbols in $\{1, \cdots, N-1\}$. That is, there is some $n \in \mathbb{N}$ such that $x_{[0,n-1]} = x_{[n,2n-1]} = x_{[2n,3n-1]} = \cdots$.

Now fix $y \in \Sigma_N^+$ and fix $\epsilon > 0$. Then consider the set,

$$B(y, \epsilon) = \{z \in \Sigma_N^+ : d(z, y)\} < \epsilon$$

So for every $z \neq y \in B(y, \epsilon)$, we have $d(z, y) = 1/2^{\min\{i \geq 0 : z_i \neq y_i\}} < \epsilon$

Now let $k \in \mathbb{N}$ be the smallest natural number such that $1/2^{\min\{i \geq 0 : x_i \neq y_i\}} < \epsilon$. Then define the block $y_{[0,k-1]}$ using elements of y. Next, define $x_k = y_k + 1 \mod N$. Let $x_y = (y_{[0,k-1]} x_k y_{[0,k-1]} x_k y_{[0,k-1]} x_k \cdots) = (y_0 y_1 \cdots y_{k-1} x_k y_0 y_1 \cdots y_{k-1} x_k \cdots)$. Clearly we have,

$$d(x_y, y) = 1/2^k < \epsilon$$

Moreover, we have that $x_{[0,k]} = x_{[k+1,2k+1]} = x_{[2k+2,3k+2]} = \cdots$. So we have that,

$$\sigma^{k+1}(x) = x$$

Hence, x is a periodic point with period $k + 1$.

Since $\epsilon > 0$ and $y \in \Sigma_N^+$ were arbitrary, this holds for any $y \in \Sigma_N^+$ with any choice of $\epsilon > 0$. Hence, we have that the set of periodic points of σ intersects every non-empty open subset of Σ_N^+, and so the periodic points of σ are dense in Σ_N^+.

Problem 3.

First, let us show that τ is continuous. That is, we want to show that for each $x \in \Sigma_N^+$, we have that for all $\epsilon > 0$, there exists $\delta > 0$ such that $d(\tau(x), \tau(y)) < \epsilon$ whenever $y \in \Sigma_N^+$ and $d(x, y) < \delta$. First let us fix $x \in \Sigma_N^+$ and $\epsilon > 0$. Let $\delta = \epsilon$. Then for any $y \in \Sigma_N^+$ with $d(x, y) < \delta$, we have that the first position where $x \neq y$ is k such that $1/2^k < \delta = \epsilon$. Now let us consider $\tau(x)$ and $\tau(y)$. Since the first $k - 1$ terms in x and y are the same, the modulo addition and carrying over process must be exactly the same on these $k - 1$ terms. Hence, we have that

$$d(\tau(x), \tau(y)) < 1/2^k < \epsilon$$

Hence, since x and ϵ were arbitrary and y was an arbitrary point with distance less than δ from x, we have that τ is continuous.

Recall that $\tau : \Sigma_N^+ \to \Sigma_N^+$ is minimal if the positive orbit $\{\tau^n(x)\}_{n \geq 0}$ is dense in Σ_N^+ for all $x \in \Sigma_N^+$. Let us fix $x \in \Sigma_N^+$ and consider some arbitrary $y \in \Sigma_N^+$ such that $y \neq x$. Then there exists some $k \in \mathbb{Z}$ such that $k \geq 0$ and $x_k \neq y_k$. Now fix some $\epsilon > 0$ and consider the set,

$$B(y, \epsilon) = \{z \in \Sigma_N^+ : d(z, y)\} < \epsilon$$

So for every $z \neq y \in B(y, \epsilon)$, we have $d(z, y) = 1/2^{\min\{i \geq 0 : z_i \neq y_i\}} < \epsilon$. If we fix j to be the smallest integer such that $1/2^j < \epsilon$, then we need to apply τ enough times until the distance between y and the iterated image of x is $1/2^j$. That is, the first difference between y and the iterated image of x must be at position j.

Observe that, since the addition is modulo N, the first digit x_0 is periodic with period N. The second digit x_1 is periodic with period N^2, and so on. Hence, let us apply τ until $x_0 = y_0$ and then fix that "cycle" around that point. Next let us, apply τ until $x_1 = y_1$ and cycle around $x_0 x_1$. Let us continue this process until we have that $x_j = y_j$ and cycle until $x_0 x_1 \cdots x_{j-1} = y_0 y_1 \cdots y_{j-1}$ as well. Then we have that,

$$d(\tau^m(x), y) < 1/2^j < \epsilon$$

for some $m \in \mathbb{N}$. Hence, the forward orbit of x is dense in Σ_N^+. Since x was arbitrary, this hold for every $x \in \Sigma_N^+$ and so τ is minimal.