Dynamical Systems II: Homework 3

Chris Hayduk

February 25, 2021

1 Questions from Silva

1.1 Section 2.5

Problem 14.

We have that $E = \{\emptyset\} \subset X$. We also have that $\{\emptyset\} \in \mathcal{S}$. Hence, with $A, B = \{\emptyset\}$ we trivially have that

$$A \subset E \subset B$$

and

$$\mu(B \setminus A) = 0$$

Thus, $E = \{\emptyset\} \in \mathcal{S}_\mu$ and $\mathcal{S}_\mu$ is non-empty. Now fix $E \in \mathcal{S}_\mu$ and consider E^C. Since $E \in S_\mu$, we have $A, B \in \mathcal{S}$ such that

$$A \subset E \subset B$$

and

$$\mu(B \setminus A) = 0$$

Applying the complement to the above expression gives us that $A^C \supset E^C \supset B^C$. So we want to show that $\mu(A^C \setminus B^C) = 0$. But note that $A^C \setminus B^C = A^C \cap B$ and $B \setminus A = B \cap A^C$, so we have that,

$$\begin{aligned}\mu(A^C \setminus B^C) &= \mu(A^C \cap B)\\ &= \mu(B \setminus A)\\ &= 0\end{aligned}$$

Hence, $\mathcal{S}_\mu$ is closed under complements.

Now let $E_n \in \mathcal{S}_\mu$, $n \geq 1$ and consider $\cup_{n=1}^\infty E_n$. For each n, we have that there exists $A_n, B_n \in S$ such that $A_n \subset E_n \subset B_n$ and $\mu(B_n \setminus A_n) = 0$. Since $E_n \subset B_n$ for all n, we must have that $\cup E_n \subset \cup B_n$. Similarly, we have $\cup A_n \subset \cup E_n$.

Now we need to show that $\mu(\cup B_n \setminus \cup A_n) = 0$. By Corollary 2.4.2 and countable subadditivity, we have,

$$\begin{aligned}
\mu(\cup B_n \setminus \cup A_n) &= \mu(\cup B_n) - \mu(\cup A_n) \\
&\leq \mu(B_1) + \mu(B_2) + \cdots - \mu(A_1) - \mu(A_2) - \mu(A_3) - \cdots \\
&= \mu(B_1) - \mu(A_1) + \mu(B_2) - \mu(A_2) + \mu(B_3) - \mu(A_3) + \cdots \\
&= \mu(B_1 \setminus A_1) + \mu(B_2 \setminus A_2) + \mu(B_3 \setminus A_3) + \cdots \\
&= 0 + 0 + 0 + \cdots \\
&= 0
\end{aligned}$$

Hence, $\cup E_n \in \mathcal{S}_\mu$ and so $\mathcal{S}_\mu$ is closed under countable unions. Hence, $\mathcal{S}_\mu$ is a σ-algebra. Now fix $E \in \mathcal{S}$. Then if we set $A, B = E \in \mathcal{S}$, we have that $A \subset E \subset B$ and $\mu(B \setminus A) = \mu(E \setminus E) = \mu(\emptyset) = 0$ as required. Thus, every element in $\mathcal{S}$ is also in $\mathcal{S}_\mu$, and so $\mathcal{S}_\mu$ is a σ-algebra containing $\mathcal{S}$.

Problem 15.

Define $\overline{\mu}$ on elements of S_μ by $\overline{\mu}(E) = \mu(A)$ for any $A \in \mathcal{S}$ such that there is a $B \in \mathcal{S}$ with $A \subset E \subset B$ and $\mu(B \setminus A) = 0$.

Fix $E \in S_\mu$ and let $A, B \in \mathcal{S}$ be any two sets such that $A \subset E \subset B$ and $\mu(B \setminus A) = 0$.

Since $E \subset B$, we have that $E \setminus A \subset B \setminus A$. Thus,

$$\mu(E \setminus A) \leq \mu(B \setminus A) = 0$$

For any other choice of $A', B' \in \mathcal{S}$ such that $A' \subset E \subset B'$ and $\mu(B' \setminus A') = 0$, we also have that $\mu(E \setminus A') = 0$. Since $A, A' \subset E$, we have,

$$\begin{aligned}
\mu(A' \triangle A) &\leq \mu(E \setminus A) \\
&= 0
\end{aligned}$$

Hence, we have that $\mu(A' \setminus A) = 0 = \mu(A \setminus A')$. Now note that,

$$\begin{aligned}
\mu(A') &= \mu(A \setminus (A \setminus A') \\
&= \mu(A) - \mu(A \setminus A') \\
&= \mu(A) - 0 \\
&= \mu(A)
\end{aligned}$$

Thus, we have that $\overline{\mu}(E)$ is independent of the choice of A and B.

Now fix $E_1 \in \mathcal{S}_\mu$ with $\mu(E_1) = 0$ and consider $E_2 \subset E_1$.

1.2 Section 2.6

Problem 2.

Note that the rational numbers are countable and that for each $q \in \mathbb{Q}$, we have that $\{q\}$ is closed. Thus, $\{q\}^C$ is open for all $q \in \mathbb{Q}$. Hence, if we take,

$$\bigcap_{q \in \mathbb{Q}} \{q\}^C$$

we see that we are taking a countable intersection of open sets. This is precisely the definition of a G_δ set. Moreover, this set is the intersection of the complements of the rational numbers. That is, the only numbers in $\mathbb{R}$ not included in this set are the rationals. Hence, we have constructed the irrational numbers.

Note the irrational numbers are not open. If we fix an irrational number x, observe that for any value of $\epsilon > 0$, we will find a rational number q such that $q \in B(x, \epsilon)$. This follows directly from the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$. Moreover, the irrational numbers are not closed. The irrationals are also dense in $\mathbb{R}$, so by definition we can find a sequence of irrational numbers converging to any real number. If we pick $q \in \mathbb{Q} \subset \mathbb{R}$, then we see that we can construct a sequence of irrational numbers which converge to $q \notin \bigcap_{q \in \mathbb{Q}} \{q\}^C$. Thus, the irrational numbers are neither open nor closed as required.

Problem 5.

Let $\mathcal{C}$ be the collection of all closed intervals with rational endpoints. That is, intervals of the form $\left[\frac{p_1}{q_1}, \frac{p_2}{q_2}\right]$ with $p_1, q_1, p_2, q_2 \in \mathbb{Z}$. Note that $\mathcal{B}$ contains all the closed sets and open sets. Moreover, since $\mathcal{B}$ is a σ-algebra, it must also include all complements and countable unions of the closed and open sets. In particular, all complements and countable unions of the closed intervals with rational endpoints must be included in $\mathcal{B}$. Thus, $\sigma(\mathcal{C}) \subset \mathcal{B}$.

Now we need to show that $\mathcal{B} \subset \mathcal{C}$. Since $\mathcal{C}$ contains all closed intervals, any σ algebra containing $\mathcal{C}$ must contain the complements of these elements. That is, it must contain all open intervals with rational endpoints. This is true because, for an open interval $\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)$, we have that it must equal,

$$\left(\left(-\infty, \frac{p_1}{q_1}\right] \bigcup \left[\frac{p_2}{q_2}, \infty\right)\right)^C$$

Note that $\left(-\infty, \frac{p_1}{q_1}\right], \left[\frac{p_2}{q_2}, \infty\right) \in \mathcal{C}$ and any σ-algebra containing these intervals must contain their union and the complement of this union. Thus, $\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)$ must be in any σ-algebra containing $\mathcal{C}$ for any $p_1, q_1, p_2, q_2 \in \mathbb{Z}$. We know from the properties of $\mathbb{R}$ that any open set can be written as a countable union of disjoint open intervals. Since any σ-algebra containing $\mathbb{C}$ must contain all open intervals with rational endpoints and must be closed under countable unions, we have that all open sets must be contained in any σ-algebra containing $\mathbb{C}$. Lastly, since any σ-algebra containing $\mathbb{C}$ contains all the open sets and is closed under complements and countable unions, it must contain the σ-algebra generated by the open sets. Thus, we have that,

$$\mathcal{B} \subset \sigma(\mathcal{C})$$

By the above arguments, we thus have that $\mathcal{B} = \sigma(\mathcal{C})$.

1.3 Section 2.7

Problem 1.

Suppose that $\mathcal{C}$ is a semi-ring of subsets of a nonempty set X and $\emptyset \neq Y \subset X$. Consider the collection $\{A \cap Y \ : \ A \in \mathcal{C}\}$ and suppose that this collection is non-empty.

Suppose $C, D \in \{A \cap Y : A \in \mathcal{C}\}$. We have $C = A \cap Y$ and $D = B \cap Y$ for some $A, B \in \mathcal{C}$. Consider the following,

$$\begin{aligned} C \cap D &= (A \cap Y) \cap (B \cap Y) \\ &= (A \cap B) \cap Y \end{aligned}$$

Since $\mathcal{C}$ is a semi-ring, we have that $A \cap B \in \mathcal{C}$ and so $C \cap D = (A \cap B) \cap Y$ is in our collection.

Now consider $C \setminus D$. We can express this as the following,

$$\begin{aligned} C \setminus D &= (A \cap Y) \setminus (B \cap Y) \\ &= (A \setminus B) \cap Y \end{aligned}$$

Since $\mathcal{C}$ is a semi-ring, we have that $A \setminus B$ can be expressed as

$$\sqcup_{j=1}^{n} E_j$$

where $E_j \in \mathcal{C}$ for every j. Thus, we can rewrite the previous statement as,

$$\begin{aligned} C \setminus D &= (A \setminus B) \cap Y \\ &= (\sqcup_{j=1}^{n} E_j) \cap Y \\ &= \sqcup_{j=1}^{n} (E_j \cap Y) \end{aligned}$$

Since $E_j \in \mathcal{C}$, we have that $E_j \cap Y \in \{A \cap Y \ : \ A \in \mathcal{C}\}$. Furthermore, since $E_j \in \mathcal{C}$ are disjoint, $E_j \cap Y \in \{A \cap Y \ : \ A \in \mathcal{C}\}$ are disjoint as well. Therefore, $C \setminus D$ satisfies all the required properties for a semi-ring.

Now let us extend to the case where $\mathcal{C}$ is a ring. We now need to show that our new collection is closed under finite unions. It suffices to show that the collection is closed for a single union (i.e. $C \cup D$) because we can then extend this union a finite number of times n. So now let us consider $C \cup D$,

$$\begin{aligned} C \cup D &= (A \cap Y) \cup (B \cap Y) \\ &= (A \cup B) \cap Y \end{aligned}$$

Since $\mathcal{C}$ is a ring, we have that $A \cup B \in \mathcal{C}$ and so $(A \cup B) \cap Y$ is in our collection. Hence, $\{A \cap Y \ : \ A \in \mathcal{C}\}$ is a ring.

Problem 2.

Let $(X, \mathcal{L}, \lambda)$ be a canonical Lebesgue measure space and $\mathcal{C}$ a sufficient semi-ring. Now for any nonempty measurable set $X_0 \subset X$, consider $\mathcal{C} \cap X_0 = \{C \cap X_0 : C \in \mathcal{C}\}$. We want to show that this set is a sufficient semi-ring for $(X_0, \mathcal{L}(X_0), \lambda)$.

Fix $Y \in \mathcal{C} \cap X_0$. Then $Y = C \cap X_0$ for some $C \in \mathcal{C}$ and so $y \in Y \iff y \in X_0$ and $y \in C$. Thus, $Y \subset X_0$ by definition and, as a result, every set in $\mathcal{C} \cap X_0$ is a subset of X_0.

Now since $\mathcal{C}$ is a semi-ring, we have $\emptyset \in \mathcal{C}$. Hence, $\emptyset \cap X_0 = \emptyset \in \mathcal{C} \cap X_0$, and so the collection is non-empty.

Fix $A, B \in \mathcal{C} \cap X_0$. Then $A = C_1 \cap X_0$, $B = C_2 \cap X_0$ for some $C_1, C_2 \in \mathcal{C}$. We have,

$$\begin{aligned} A \cap B &= (C_1 \cap X_0) \cap (C_2 \cap X_0) \\ &= (C_1 \cap C_2) \cap X_0 \end{aligned}$$

Since $\mathcal{C}$ is a sufficient semi-ring, we have that $C_1 \cap C_2 \in \mathcal{C}$ and thus $(C_1 \cap C_2) \cap X_0 \in \mathcal{C} \cap X_0$.

Now consider $A \setminus B$. Note that this is given by $(C_1 \cap X_0) \setminus (C_2 \cap X_0)$. This is equivalent to $(C_1 \setminus C_2) \cap X_0$. Since $C_1, C_2 \in \mathcal{C}$, we have,

$$C_1 \setminus C_2 = \sqcup_{j=1}^{n} E_j$$

where $E_j \in \mathcal{C}$ are disjoint. Rewriting our formulation for $A \setminus B$ yields,

$$\begin{aligned} A \setminus B &= (C_1 \setminus C_2) \cap X_0 \\ &= (\sqcup_{j=1}^{n} E_j) \cap X_0 \\ &= \sqcup_{j=1}^{n} (E_j \cap X_0) \end{aligned}$$

Note that $E_j \cap X_0 \in \mathcal{C} \cap X_0$ for every j. Also note that the sets $E_j \cap X_0$ are disjoint. To prove this, fix i and j and suppose $x \in E_j \cap X_0$ and $x \in E_i \cap X_0$. Then by the definition of intersection, we have that $x \in E_j$, $x \in E_i$, and $x \in X_0$. But we know that E_i, E_j are disjoint, so x cannot be in both sets. Thus we have a contradiction and the sets $E_j \cap X_0$ are disjoint as required.

Hence we have shown that $\mathcal{C} \cap X_0$ is a semi-ring. Now we need to show that it satisfies the sufficient semi-ring property. First we need to show that every set contained in $\mathcal{C} \cap X_0$ has finite measure. Consider A as defined above. We have,

$$\begin{aligned} \lambda(A) &= \lambda(C_1 \cap X_0) \\ &\leq \lambda(C_1) < \infty \end{aligned}$$

Hence, A has finite measure. Recall that $\mathcal{C}$ is a sufficient semi-ring. Then

$$\lambda(C_1) = \inf\left\{\sum_{j=1}^{\infty}\lambda(I_j) \;:\; C_1 \subset \cup_{j=1}^{\infty} I_j \text{ and } I_j \in \mathcal{C} \text{ for } j \geq 1\right\}$$

Applying this to the definition of A yields,

$$\begin{aligned}\lambda(A) &= \lambda(C_1 \cap X_0) \\ &= \inf\left\{\sum_{j=1}^{\infty}\lambda(I_j \cap X_0) \;:\; A \subset \cup_{j=1}^{\infty} I_j \cap X_0 \text{ and } I_j \cap X_0 \in \mathcal{C} \cap X_0 \text{ for } j \geq 1\right\}\end{aligned}$$

as required.

Problem 6.

Let $\mathcal{C}$ be a collection of sets. Denote $r(\mathcal{C})$ as the intersection of all rings containing $\mathcal{C}$. We want to show that this collection is a ring.

Firstly, since $\mathcal{C} \subset \mathcal{R}$ for every ring containing $\mathcal{C}$ by definition, we have that $\mathcal{C}$ must be contained in their intersection. That is, $\mathcal{C} \subset r(\mathcal{C})$.

Every ring containing $\mathcal{C}$ must also contain the empty set. Hence, $\emptyset \in r(\mathcal{C})$.

Now fix $A, B \in r(\mathcal{C})$. Then A, B are in every ring containing $\mathcal{C}$. Since rings are closed under intersection, we must have that $A \cap B$ is in every ring containing $\mathcal{C}$ as well. Hence, $A \cap B \in r(\mathcal{C})$.

Now consider $A \setminus B$. Since $A, B \in r(\mathcal{C})$, then A, B are in every ring containing $\mathcal{C}$. Thus, there exist disjoint sets E_j such that

$$A \setminus B = \sqcup_{j=1}^{n} E_j$$

Hence, $E_j \in r(\mathcal{C})$ for $j = 1, \ldots, n$.

Lastly, since A, B in every ring containing $\mathcal{C}$ and each ring is closed under finite unions, we must have that $A \cup B$ is in every ring containing $\mathcal{C}$. Hence, $A \cup B \in r(\mathcal{C})$.

Problem 7.

Suppose that $\mathcal{R}$ is a semi-ring. Suppose we take finite unions of all disjoint elements in $\mathcal{R}$. Fix $A \in \mathcal{R}$. Since $\emptyset \in \mathcal{R}$, we can use property (3) of semi-rings to state that, for disjoint sets $E_j \in \mathcal{R}$ with $j = 1, \ldots, n$, we have,

$$\begin{aligned}A &= A \setminus \emptyset \\ &= \sqcup_{j=1}^{n} E_j\end{aligned}$$

Thus, we can express the union of any two elements $A, B \in \mathcal{R}$ as follows,

$$A \cup B = (\sqcup_{j=1}^{n} E_j^A) \cup (\sqcup_{j=1}^{m} E_j^B)$$

where $A = \sqcup_{j=1}^{n} E_j^A$ and $B = \sqcup_{j=1}^{m} E_j^A$. Hence, taking finite unions of disjoint sets in $\mathcal{R}$ suffices for expressing finite unions of all sets in $\mathcal{R}$. Thus, this constitutes a ring.

Now suppose $r(\mathcal{R})$ is contained in the ring we have just described. Then there exists at least one element in the above ring which is not in $r(\mathcal{R})$. The only new elements introduced in the above ring were done by taking union of disjoint sets. Thus, this is where the difference must lie. Then there must exist $E_1, E_2 \in \mathcal{R}$ such that E_1 and E_2 are disjoint with $E_1 \sqcup E_2 \notin r(\mathcal{R})$. However, if $E_1 \sqcup E_2 \notin r(\mathcal{R})$, then there is sets $A \in \mathcal{R}$ such that,

$$A = E_1 \sqcup E_2 \sqcup_{j=3}^{n} E_3 \notin r(\mathcal{R})$$

Hence, there must exist $B \in \mathcal{R}$ such that $A \cup B \notin r(\mathcal{R})$. Thus, this would not define a ring and, as a result, the ring which we defined at the start of this problem must be $r(\mathcal{R})$.

1.4 Section 3.3

Problem 1.

Define $T : [0,1] \to [0,1]$ by $T(x) = 2x$ if $0 \leq x \leq 1/2$ and $T(x) = 2 - 2x$ if $1/2 < x < 1$. Define $S_1 : [0,1] \to [0,1/2]$ by $S_1(y) = y/2$ and $S_2 : [0,1] \to [1/2,1]$ by $S_2(y) = y/2 + 1/2$. For a measurable set $A \subset [0,1]$. We have that,

$$T^{-1}(A) = S_1(A) \sqcup (S_2(A \setminus \{0\}))$$

Since we are taking a singleton away from A and S_2 is a well-defined function, this will not change the measure of $S_2(A)$. Thus, from the above and from the results in Chapter 2, we have,

$$\begin{aligned}
\lambda(T^{-1}(A)) &= \lambda(S_1(A)) + \lambda(S_2(A \setminus \{0\})) \\
&= \lambda(\frac{1}{2}A) + \lambda(\frac{1}{2}A + \frac{1}{2}) \\
&= \frac{1}{2}\lambda(A) + \frac{1}{2}\lambda(A) \\
&= \lambda(A)
\end{aligned}$$

Thus, T is measure preserving.