Dynamical Systems II: Homework 10

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1 Questions from Silva

1.1 Section 4.6

Problem 2.

Observe that $|f|$ is defined as,

$$|f| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

In addition, we have,

$$f^+ = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0 \end{cases}$$

and,

$$f^- = \begin{cases} -f(x) & \text{if } f(x) \leq 0 \\ 0 & \text{if } f(x) > 0 \end{cases}$$

Now let us consider $f^+(x) + f^-(x)$ for 3 cases: when $f(x) > 0$, when $f(x) = 0$, and when $f(x) < 0$. When $f(x) > 0$, we have that $f^+(x) = f(x)$ and $f^-(x) = 0$. Hence, $f^+(x) + f^-(x) = f(x)$ in this case, just as in the case of $|f|$. Now suppose $f(x) = 0$. Then $f^+(x) = 0$ and $f^-(x) = 0$, so $f^+(x) + f^-(x) = 0$. Again, $|f(x)| = 0$ when $f(x) = 0$, so they coincide in this case as well. Now suppose $f(x) < 0$. Then $f^+(x) = 0$ and $f^-(x) = -f(x)$. Thus, $f^+(x) + f^-(x) = -f(x)$. This is precisely the same as $|f|$. Hence, in all 3 possible cases for $f(x)$, we have that $f^+ + f^-$ coincides with $|f|$, and so $f^+ + f^- = |f|$.

Problem 3.

Suppose f is integrable. Then $\int f^+ d\mu < \infty$ and $\int f^- d\mu < \infty$. Hence, $\int f^+ d\mu + \int f^- d\mu < \infty$. By Lemma 4.6.2 Part 2, we have that,

$$\begin{aligned} \int f^+ d\mu + \int f^- d\mu &= \int (f^+ + f^-) d\mu \\ &< \infty \end{aligned}$$

By Exercise 2, we thus have that,

$$\begin{aligned} \int (f^+ + f^-) d\mu &= \int |f| d\mu \\ &< \infty \end{aligned}$$

Now suppose $|f|$ is integrable. Then $\int |f|^+ d\mu < \infty$ and $\int |f|^- d\mu < \infty$. But observe that, since $|f| \geq 0$ everywhere, then $|f|^- = 0$ everywhere. Hence, we from this and our work in Exercise 2 that,

$$\begin{aligned} \int |f| d\mu &= \int |f|^+ d\mu - \int |f|^- d\mu \\ &= \int |f|^+ d\mu \\ &= \int (f^+ - f^-) d\mu \end{aligned}$$

Then, by Lemma 4.6.2 Part 2, we have that f^+ and f^- are integrable and thus, $\int f^+ d\mu < \infty$ and $\int f^- d\mu < \infty$. Hence,

$$\int f^+ d\mu - \int f^- d\mu < \infty$$

And thus, we have that $\int f d\mu = \int f^+ d\mu - \int f^- d\mu$ is integrable, as required.

Problem 4.

Suppose f is an integrable function and fix $a \in \mathbb{R}$. We have that,

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu$$

with $\int f^+ d\mu < \infty$ and $\int f^- d\mu < \infty$. Observe that f^+, f^- are thus nonnegative integrable functions. Thus, applying Theorem 4.4.5, we have that, af^+ and af^- are integrable. Thus, $\int af^+ d\mu < \infty$ and $\int af^- d\mu < \infty$ and so,

$$\int af^+ d\mu - \int af^- d\mu < \infty$$

But the above is precisely the definition of $\int af d\mu$, and so we must have that af is integrable.

Problem 5.

Suppose that $f \leq g$ a.e. Then of course $f^+ \leq g^+$ a.e. If this were not the case, then there would be a set of positive measure on which $f^+ > g^+$, which, by the definition of f^+ and g^+, would imply that there is a set of positive measure on which $f > g$, a contradiction.

In addition, we have that $f^- \geq g^-$ a.e. Suppose that this is not the case. Then there is a set of positive measure on which $f^- < g^-$. But this implies that $f(x) > g(x)$ for x in this set (either $f(x) \geq 0$ or $f(x)$ is a negative number greater than $g(x)$). This is a contradiction, and so we must have $f^- \geq g^-$ a.e. Hence, we have that,

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu$$

where $f^+ \leq g^+$ a.e. and $f^- \geq g^-$ a.e. Now let us consider $\int f^+ d\mu$ and $\int g^+ d\mu$. We have,

$$\int f^+ d\mu = \sup\{\int s d\mu \ : \ s \text{ is simple and } 0 \leq s \leq f^+\}$$
$$\int g^+ d\mu = \sup\{\int s d\mu \ : \ s_g \text{ is simple and } 0 \leq s \leq g^+\}$$

We have that $f^+ \leq g^+$ a.e. Let s_g be the supremum of simple function in the above set, and the same for s_f. Then we must have $s_f \leq s_g$ a.e. as well. Observe that the set X where $s_f > s_g$ is measure 0, and so it does contribute at all to the value of the integral by Corollary 4.3.3. Thus, we can disregard X when calculating the integral and so apply Theorem 4.3.2(2) which states that,

$$\int s_f d\mu \leq \int s_g d\mu$$

Since these were the supremum of simple functions approximating f^+ and g^+, we have that,

$$\int f^+ d\mu \leq \int g^+ d\mu$$

Similarly, we have that,

$$\int f^- d\mu \geq \int g^- d\mu$$

These two inequalities and the fact that all of these integrals are nonnegative give us that,

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu \leq \int g^+ d\mu - \int g^- d\mu = \int g d\mu$$

as required.

Problem 6.

Let f be an integrable function and suppose that $\int_A f d\mu = 0$ for all measurable sets A. Then we have,

$$\int_A f^+ d\mu - \int_A f^- d\mu = 0$$

Since both of the above integrals are nonnegative, we must have that $\int_A f^+ d\mu = 0 = \int_A f^- d\mu$. Since f^+ and f^- are both nonnegative measurable functions, by Problem 4.4.2 (solved on the previous HW), we have that $f^+ = 0$ a.e. and $f^- = 0$ a.e. on A. Let X be the set where $f^+ > 0$ and Y be the set where $f^- > 0$ and let $Z = X \cup Y$. Then $\mu(Z) \leq \mu(X) + \mu(Y) = 0 + 0 = 0$. Note that Z is precisely the set where $f \neq 0$ (since when $f = 0$ we have $f^+ = f^- = 0$ and when $f \neq 0$, one of f^+ and f^- is greater than 0). Hence, the set of values where $f \neq 0$ on A has measure 0 and so $f = 0$ a.e. on A.

Problem 7.

Suppose f is a nonnegative integrable function and that $\{E_p\}_{p>0}$ is a sequence of decreasing ($E_{p+1} \subset E_p$) measurable sets. Furthermore, suppose $\lim_{p\to\infty} \mu(E_p) = 0$. We want to show that

$$\int_{\cap_{p>0} E_p} f d\mu = 0$$

We know that since $\lim_{p\to\infty} \mu(E_p) = 0$ that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, we have that $|E_n| < \epsilon$. Observe that since we have a decreasing sequence of sets, for any finite subset $\{E_0, E_1, \ldots, E_k\}$, we have that $\cap_{i=0}^k E_i = E_k$ and so $\mu\left(\cap_{i=0}^k E_i\right) = \mu(E_k)$. Thus, by Proposition 2.5.2(2), we have

$$\begin{aligned} \mu\left(\cap_{p>0} E_p\right) &= \lim_{p\to\infty} \mu\left(E_p\right) \\ &= 0 \end{aligned}$$

Note that since $\mu\left(\cap_{p>0} E_p\right)$, any measurable subset of $\cap_{p>0} E_p$ must have measure 0 as well. Thus, for any simple function s defined on $\mu\left(\cap_{p>0} E_p\right)$, we have that $s = 0$. But this implies that $\int_{\cap_{p>0} E_p} s d\mu$ for all simple functions s by the formulation of the integral given by Corollary 4.3.3. But if the integral of any simple function on $\cap_{p>0} E_p$ is 0, then we have that the integral of non-negative simple function must be 0 as well because, for a nonnegative function g,

$$\begin{aligned} \int_{\cap_{p>0} E_p} g d\mu &= \sup\{\int_{\cap_{p>0} E_p} s d\mu \ : \ s \text{ is simple and } 0 \leq s \leq g\} \\ &= \sup\{0\} \\ &= 0 \end{aligned}$$

Since f^+ and f^- are nonnegative measurable functions defined on $\cap_{p>0} E_p$, we have that

$$\int_{\cap_{p>0} E_p} f^+ d\mu = 0 = \int_{\cap_{p>0} E_p} f^- d\mu$$

And, hence,

$$\begin{aligned}\int_{\cap_{p>0}E_p} f d\mu &= \int_{\cap_{p>0}E_p} f^+ d\mu - \int_{\cap_{p>0}E_p} f^- d\mu \\ &= 0 - 0 = 0\end{aligned}$$

as required.

Problem 9.

Let $f : X \to \mathbb{R}^*$ be a measurable function and f is integrable. Now suppose $|f(x)| = \infty$ on a set X with $\mu(X) > 0$. Then $f = \infty$ or $f = -\infty$ (or both) on a set of positive measure. Thus, we have that either $f^+ = \infty$ or $f^- = \infty$ (or both) on a set of positive measure. Thus, a maximal s approximating simple function on f^+ or f^- (or both) must attain ∞ on a set of positive measure. Hence, we have $s = \sum^n a_i\mu(A_i) = a_0\mu(A_0) + a_1\mu(A_1) + \cdots + \infty\mu(A_k) + \cdots + a_n\mu(A_n) = \infty$. Since ∞ is the max value attainable in $\mathbb{R}^*$ it must be the supremum of any subset of $\mathbb{R}^*$ containing it, and so,

$$\begin{aligned}\int f^+ d\mu &= \sup\{\int s d\mu \ : \ s \text{ is simple and } 0 \leq s \leq f^+\} \\ &= \infty\end{aligned}$$

or,

$$\begin{aligned}\int f^- d\mu &= \sup\{\int s d\mu \ : \ s \text{ is simple and } 0 \leq s \leq f^-\} \\ &= \infty\end{aligned}$$

or both. But note that f is only Lebesgue integrable if both $\int f^+ < \infty$ and $\int f^- < \infty$. Hence, we have that f is not integrable, a contradiction. Thus, we must have that the set where $|f(x)| = \infty$ has measure 0. That is, $|f(x)| < \infty$ a.e.