

# Dynamical Systems II: Final

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## Problem 1.

Recall that a rotation of the plane is a linear map of the form

$$R : \mathbb{R}^2 \rightarrow \mathbb{R}^2; \quad R \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

This map preserves Lebesgue measure, but we want to show that it is never ergodic.

We have that,

$$\begin{aligned} R \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{pmatrix} \end{aligned}$$

$$\begin{aligned} R^{-1} \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{\cos^2 \theta + \sin^2 \theta} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} x \cos \theta + y \sin \theta \\ -x \sin \theta + y \cos \theta \end{pmatrix} \end{aligned}$$

Let us now consider the unit disk in  $\mathbb{R}^2$ , denoted by  $A = \{(x, y) \in \mathbb{R}^2 : -1 \leq x^2 + y^2 \leq 1\}$ . We have that Lebesgue measure is a generalization of area in  $\mathbb{R}^2$  and, since the unit disk has a well-defined area, we know its Lebesgue measure must be equal to 1.

Let us fix  $(x, y) \in A$ . Then  $-1 \leq x^2 + y^2 \leq 1$ . Applying  $R$  to this point, we get,

$$\begin{aligned} x' &= x \cos \theta - y \sin \theta \\ y' &= x \sin \theta + y \cos \theta \end{aligned}$$

And note that,

$$\begin{aligned}
(x')^2 + (y')^2 &= (x \cos \theta - y \sin \theta)^2 + (x \sin \theta + y \cos \theta)^2 \\
&= x^2 \cos^2 \theta - 2xy \cos \theta \sin \theta + y^2 \sin^2 \theta + x^2 \sin^2 \theta + 2xy \sin \theta \cos \theta + y^2 \cos^2 \theta \\
&= x^2 \cos^2 \theta + y^2 \sin^2 \theta + x^2 \sin^2 \theta + y^2 \cos^2 \theta \\
&= x^2(\cos^2 \theta + \sin^2 \theta) + y^2(\cos^2 \theta + \sin^2 \theta) \\
&= x^2 + y^2
\end{aligned}$$

And so we have  $-1 \leq (x')^2 + (y')^2 \leq 1$  as well. Thus,  $R \begin{pmatrix} x \\ y \end{pmatrix} \in A$  and, since  $(x, y) \in A$  was arbitrary, we have that  $(x, y) \in A \implies R \begin{pmatrix} x \\ y \end{pmatrix} \in A$ .

Now applying  $R^{-1}$  to an arbitrary  $(x, y)$ , we get,

$$\begin{aligned}
x' &= x \cos \theta + y \sin \theta \\
y' &= -x \sin \theta + y \cos \theta
\end{aligned}$$

And note that,

$$\begin{aligned}
(x')^2 + (y')^2 &= (x \cos \theta + y \sin \theta)^2 + (-x \sin \theta + y \cos \theta)^2 \\
&= x^2 \cos^2 \theta + 2xy \cos \theta \sin \theta + y^2 \sin^2 \theta + x^2 \sin^2 \theta - 2xy \sin \theta \cos \theta + y^2 \cos^2 \theta \\
&= x^2 \cos^2 \theta + y^2 \sin^2 \theta + x^2 \sin^2 \theta + y^2 \cos^2 \theta \\
&= x^2(\cos^2 \theta + \sin^2 \theta) + y^2(\cos^2 \theta + \sin^2 \theta) \\
&= x^2 + y^2
\end{aligned}$$

And so we have  $-1 \leq (x')^2 + (y')^2 \leq 1$  as well. Thus,  $R^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \in A$  and, since  $(x, y) \in A$  was arbitrary, we have that  $(x, y) \in A \implies R^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \in A$ .

Thus, from the above derivations, we must have that  $A$  is strictly  $R$ -invariant. However, as we said above, note that  $A$  has measure 1. Moreover,  $A^c = \{(x, y) \in \mathbb{R}^2 : -1 \leq x^2 + y^2 \leq 1\}$ , which is a set of infinite measure (the area of the plane minus the unit disk). Hence, although  $A$  is strictly  $R$ -invariant, we do not have that  $\mu(A) = 0$ , nor that  $\mu(A^c) = 0$ . As a result,  $R$  is not ergodic for any value of  $\theta$ .

**Problem 2.**

Let  $(X, \mathcal{S}, \mu)$  be a measure space and suppose that  $S : X \rightarrow X$  and  $T : X \rightarrow X$  are measure-preserving transformations. We want to show that their composition,  $S \circ T$ , preserves measure as well.

Since  $S$  and  $T$  are measure-preserving transformations, for any  $A \in \mathcal{S}(X)$  we have that  $\mu(S^{-1}(A)) = \mu(A)$  and  $\mu(T^{-1}(A)) = \mu(A)$ . Let us consider  $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$ . We have,

$$(T^{-1} \circ S^{-1})(A) = T^{-1}(S^{-1}(A))$$

We have that, since  $S$  is measure-preserving, then  $\mu(S^{-1}(A)) = \mu(A)$ . Moreover, since  $S^{-1}(A)$  is measurable, we have that  $S^{-1}(A) \in \mathcal{S}(X)$  and hence. Let us call this set  $B$ . Then we have,

$$T^{-1}(S^{-1}(A)) = T^{-1}(B)$$

and, since  $T$  is measure-preserving, we have

$$\begin{aligned} \mu(T^{-1})(B) &= \mu(B) \\ &= \mu(S^{-1}(A)) \\ &= \mu(A) \end{aligned}$$

Since  $A \in \mathcal{S}(X)$ , this holds for every set in  $\mathcal{S}(X)$  and so  $S \circ T$  is measure-preserving.

**Problem 3.**

- a) Let us begin by showing that  $D$  is measurable using the Dyadic squares. Fix a set  $A$  in the Dyadic squares and consider  $D^{-1}(A)$ . Observe that for  $x, y \in [0, 1/2)$ , we have that  $2x \in [0, 1)$  and  $y \in [0, 1)$ . Hence, we can consider  $D(x, y)$  on  $[0, 1/2) \times [0, 1/2)$  to be just  $D(x, y) = (2x, 2y)$ . Now, for  $x, y \in (1/2, 1)$ , we have that  $D(x, y) \in (0, 1)$ . Note that for  $x, y$  in this interval, we have  $2x \bmod 1 = 2x - 1$  and  $2y \bmod 1 = 2y - 1$ . Thus, for  $x, y \in (1/2, 1)$  we have  $D(x, y) = (2x - 1, 2y - 1)$ . Hence, rephrasing  $D$ , we have,

$$D(x, y) = \begin{cases} (2x, 2y) & x, y \in [0, 1/2) \\ (2x, 2y - 1) & x \in [0, 1/2), y \in (1/2, 1) \\ (2x - 1, 2y) & x \in (1/2, 1), y \in [0, 1/2) \\ (2x - 1, 2y - 1) & x, y \in (1/2, 1) \end{cases}$$

Hence, we can think of  $D(x, y)$  as a cross product of the tent map on  $[0, 1)$  with  $\mu = 2$ . We can now formulate the inverse of  $D(x, y)$  as,

$$D^{-1}(x, y) = \begin{cases} (x/2, y/2) & x, y \in [0, 1/2) \\ (x/2, y/2 + 1/2) & x \in [0, 1/2), y \in (1/2, 1) \\ (x/2 + 1/2, y/2) & x \in (1/2, 1), y \in [0, 1/2) \\ (x/2 + 1/2, y/2 + 1/2) & x, y \in (1/2, 1) \end{cases}$$

Denote each of the four cases of  $D^{-1}(x, y)$  as  $D_i^{-1}(x, y)$  for  $i \in \{1, \dots, 4\}$  and observe that  $\lambda(A) = 1/2^k \cdot 1/2^k = 1/4^k$ . Now let us check  $D^{-1}(A)$ . We have,

$$\lambda(D^{-1}(A)) = \lambda(D_1^{-1}(A)) + \lambda(D_2^{-1}(A)) + \lambda(D_3^{-1}(A)) + \lambda(D_4^{-1}(A))$$

Let us now split up each  $D_i$  into  $D_{i_x}$  and  $D_{i_y}$  for the  $x$  and  $y$  components. In addition, write  $A_x$  for the  $x$  component of  $A$  and  $A_y$  for the  $y$  component. Then,

$$\lambda(D_i^{-1}(A)) = \lambda(D_{i_x}^{-1}(A_x)) \cdot \lambda(D_{i_y}^{-1}(A_y))$$

Further, observe that  $\lambda(D_{i_x}^{-1}(A_x)) = \frac{1}{2}\lambda(A_x)$  and  $\lambda(D_{i_y}^{-1}(A_y)) = \frac{1}{2}\lambda(A_y)$  by the proof of Theorem 3.3.1 in the text. Note that  $\lambda(A_x) = 1/2^k$  and  $\lambda(A_y) = 1/2^k$ , so

$$\begin{aligned} \lambda(D_i^{-1}(A)) &= \frac{1}{2}\lambda(A_x) \cdot \frac{1}{2}\lambda(A_y) \\ &= \frac{1}{4} \cdot \frac{1}{4^k} \end{aligned}$$

This is true for every  $i$  and thus,

$$\begin{aligned} \lambda(D^{-1}(A)) &= \lambda(D_1^{-1}(A)) + \lambda(D_2^{-1}(A)) + \lambda(D_3^{-1}(A)) + \lambda(D_4^{-1}(A)) \\ &= \frac{1}{4} \cdot \frac{1}{4^k} + \frac{1}{4} \cdot \frac{1}{4^k} + \frac{1}{4} \cdot \frac{1}{4^k} + \frac{1}{4} \cdot \frac{1}{4^k} \\ &= \frac{1}{4^k} \\ &= \lambda(A) \end{aligned}$$

Since  $A$  was an arbitrary Dyadic square, we thus have that  $D$  is measure-preserving.

- b) We want to show that  $D$  is ergodic. Let  $A_1$  and  $B_1$  be any sets of positive measure in  $\mathcal{L}(X)$ . Then, since the Dyadic squares form a sufficient semiring for Lebesgue measure, there are squares  $I$  and  $J$  such that,

$$\lambda(A_1 \cap I) > \frac{3}{4}\lambda(I) \quad \text{and} \quad \lambda(B_1 \cap J) > \frac{3}{4}\lambda(J)$$

Furthermore, we may assume that  $I$  and  $J$  are of the same measure. Write  $A = A_1 \cap I$  and  $B = B_1 \cap J$ .

Now define  $I = [\frac{p_1}{2^k}, \frac{p_1+1}{2^k}) \times [\frac{q_1}{2^k}, \frac{q_1+1}{2^k})$  and  $J = [\frac{p_2}{2^m}, \frac{p_2+1}{2^m}) \times [\frac{q_2}{2^m}, \frac{q_2+1}{2^m})$ . Since the doubling map is chaotic in 1 dimension, we have that its orbit is dense in both the  $x$  and  $y$  axes. Thus, there is an integer  $n$  such that  $\lambda(D^n(I) \cap J) > \frac{3}{4}\lambda(J)$ . Thus, we have,

$$\begin{aligned} \lambda(D^n(A) \cap B) &\geq \lambda(D^n(I) \cap J) - \lambda(I \setminus A) - \lambda(J \setminus B) \\ &> \frac{3}{4}\lambda(J) - \frac{1}{4}\lambda(I) - \frac{1}{4}\lambda(J) > 0 \end{aligned}$$

and, by Lemma 3.7.2(5), we have that  $D$  is ergodic.

**Problem 4.**

Recall that  $\mathcal{B}(\mathbb{R})$  is the smallest  $\sigma$ -algebra containing the open sets. Since  $X \in \mathcal{B}(\mathbb{R})$ , we have that  $\mathcal{B}(X) = \{A \cap X : A \in \mathcal{B}(\mathbb{R})\}$ .

We are given that  $T : X \rightarrow X$  is a Borel-measurable transformation for a set  $X \in \mathcal{B}(\mathbb{R})$  (that is,  $X$  is a Borel set). Since  $T$  is Borel measurable, we have  $T^{-1}(B) \in \mathcal{B}(X)$  for all  $B \in \mathcal{B}(X)$ .

Define  $p : X \rightarrow [-\infty, +\infty]$  by,

$$x \mapsto \begin{cases} k & \text{if } x \text{ is periodic and has least period } k \\ +\infty & \text{if } x \text{ is not periodic} \end{cases}$$

where *least period*  $k$  means that  $k \geq 1$  is the minimal integer such that  $T^k(x) = x$ .

We want to prove that  $p$  is Borel measurable. Recall that  $p : X \rightarrow \mathbb{R}^*$  is  $\mathcal{B}(X)$  measurable iff  $p^{-1}([-\infty, a)) \in \mathcal{B}(X)$  for every  $a \in \mathbb{R}$ . Equivalently,  $p : X \rightarrow \mathbb{R}^*$  is  $\mathcal{B}(X)$ -measurable iff  $p^{-1}(C) \in \mathcal{B}(X)$  for every  $C \in \mathcal{B}(\mathbb{R}^*) = \mathcal{B}(\mathbb{R}) \sqcup P(\{\pm\infty\})$ .

Note that  $x \in X$  is period  $k$  if and only if  $T^k(x) - x = 0$ . So consider the set  $[-\infty, a)$  for some  $a \in \mathbb{R}$ . We have  $p^{-1}([-\infty, a))$  is the set of  $x \in X$  with  $T^k(x) - x = 0$  for some  $k \in \mathbb{N}$  with  $k < a$ . Since we know  $k \geq 1$  and  $p(x) \mapsto +\infty$  when  $x$  is not periodic, we have that

$$p^{-1}([-\infty, a)) = p^{-1}([1, a))$$

for  $a \geq 1$  and

$$p^{-1}([-\infty, a)) = \emptyset$$

for  $a < 1$ . Since the empty set is trivially measurable, let us only consider  $a \geq 1$ .

For  $a \geq 1$ , we have that,

$$\begin{aligned} p^{-1}([-\infty, a)) &= p^{-1}([1, a)) \\ &= \bigcup_{i=1}^a \{x \in X : T^i(x) - x = 0\} \\ &= \bigcup_{i=1}^a \{x \in X : T^i(x) = 0\} \end{aligned}$$

Now, note that by Lemma 4.2.6, since  $T : X \rightarrow X$  is a measurable transformation and  $X \subset \mathbb{R}$ , we then have that  $T \circ T = T^2$  is measurable. Proceeding inductively, we have that  $T^j$  is measurable for any  $j \geq 1$ . Moreover, by Exercise 4.4.9, we have that since  $T^j$  is Borel measurable (and hence Lebesgue measurable), then  $\{x \in X : T^j(x) = a\}$  is measurable for any  $a$ . In particular,  $\{x \in X : T^j(x) = x\}$  is measurable. Hence, for  $a \geq 1$ , we have that

$p^{-1}([-\infty, a)) = p^{-1}([1, a))$  is just a countable union of elements in  $\mathcal{B}(X)$  and hence, since  $\mathcal{B}(X)$  is a  $\sigma$ -algebra, we have,

$$\begin{aligned} p^{-1}([-\infty, a)) &= p^{-1}([1, a)) \\ &= \left( \bigcup_{i=1}^a \{x \in X : T^i(x) = 0\} \right) \in \mathcal{B}(X) \end{aligned}$$

Thus,  $p^{-1}([-\infty, a))$  is measurable for any  $a \geq 1$ . We have now covered every case of  $a \in \mathbb{R}$  and so, we have that  $p^{-1}([-\infty, a)) \in \mathcal{B}(X)$  for every  $a \in \mathbb{R}$ . Thus,  $p$  is measurable.

**Problem 5.**

- a) Recall that a point  $x$  is periodic if  $T^k(x) = x$  for some  $k \geq 1$ . Since almost every point of  $X$  is periodic under  $T$ , there is a set  $N$  with  $\lambda(N) = 0$  which contains all of the points which are *not* periodic under  $T$ . Now fix a measurable set  $A \subset X$ . Consider the set  $A \setminus N$ . By the definitions of  $T$  and  $N$ , we have that  $A \setminus N$  must consist only of points which are recurrent. Thus, for every  $x \in A \setminus N$ , there exists  $k = k(x) > 0$  such that  $T^k(x) = x \in A \setminus N$ . Since  $\lambda(N) = 0$  and  $A$  was arbitrary, we have that  $T$  is recurrent.
- b) Fix  $A \subset X$  and a point  $x \in A \setminus N$ . Hence,  $x$  is periodic with period  $k \geq 1$ . Clearly we have that  $\{x, T(x), T^2(x), \dots, T^{k-1}(x)\}$  is a  $T$ -invariant set. Applying  $T^{-1}$  gives us,

$$\begin{aligned} \{T^{-1}(x), x, T(x), \dots, T^{k-2}(x)\} &= \{T^{k-1}(x), x, T(x), \dots, T^{k-2}(x)\} \\ &= \{x, T(x), T^2(x), \dots, T^{k-1}(x)\} \end{aligned}$$

Observe that, since there are a finite number of elements in this orbit, we have that  $\lambda(A) = 0$  and  $\lambda(X \setminus (A \cup N)) = \lambda(X)$ . Since  $\lambda(X) > 0$ , there must be “more” of these orbits outside of  $A$  in some sense. Moreover, observe that the orbits of each point are either disjoint or equal to each other. To show this, suppose that the orbits of two points  $x$  and  $y$  have a non-empty intersection. Then  $T^j(x) = T^i(y)$  for some  $i, j \geq 1$ . But note that, if  $x$  has period  $k$ , then  $T^{k-j}(T^j(x)) = T^k(x) = x$  and thus  $T^{k-j}(T^i(y)) = T^{k-j+i}(y) = x$ . Thus we must have,

$$\begin{aligned} T^{k-j+1}(T^j(x)) &= T^{k+1}(x) = T^2(x) \\ &= T^{k-j+1}(T^i(y)) \\ &= T^{k-j+1+i}(y) \end{aligned}$$

Continuing by induction, we see that the orbit of  $y$  equals the orbit of  $x$  for every point after  $T^i(y)$ . But note that  $y$  is also periodic with period  $\ell$ . Then there is some point in the orbit of  $x$  such that  $T^m(x) = y$ . Hence, for all  $n$  such that  $1 \leq n < i$ , we must have that  $T^n(y)$  is equal to an element of the orbit of  $x$  as well. Thus, these are the same orbit.

In the case where the intersection of the orbits is empty, it is clear that the two orbits are disjoint.

Let us now divide  $X$  into two sets,  $A$  containing some of the orbits and  $B$  containing orbits which are disjoint from all of the orbits in  $A$ . Observe that it is possible to construct two such sets of positive measure. If it were not, it would mean that every element, up to a set of 0 measure, would be contained within a single orbit. But note that each orbit has finite elements and thus has measure 0. Hence, this is a contradiction and there must be more than 1 orbit.

Thus, we have  $\lambda(A) > 0$  and  $\lambda(B) > 0$ , with  $A \cap B = \emptyset$ . Observe further that  $A^c = B \sqcup N$  because  $B$  contains all the orbits which are not in  $A$  and  $N$  contains all the points which are not part of any orbit (we know the points of  $N$  are not part of any orbit because they are not periodic and every point of an orbit is periodic).

Thus, we have that  $\lambda(A) > 0$  and  $\lambda(A^c) = \lambda(B \sqcup N) = \lambda(B) > 0$  as well. In addition, by our discussion above, we have that every orbit in  $A$  is  $T$ -invariant and thus  $T^{-1}(A) = A$ . Moreover, all of the orbits in  $B$  are  $T$ -invariant. Lastly, as previously stated, no element of  $N$  can be an element of an orbit. Since the orbits are exactly the set  $N^c$ , we have that  $N$  is  $T$ -invariant and so  $T^{-1}(N) = N$  as well. Thus,

$$T^{-1}(A) = A$$

and

$$T^{-1}(A^c) = T^{-1}(B \sqcup N) = B \sqcup N$$

but  $\lambda(A) > 0$  and  $\lambda(A^c) > 0$ . As a result, we have violated the definition of ergodicity and  $T$  is not ergodic.

### Problem 6.

- a) We have that the set of intervals with rational endpoints forms a sufficient semi-ring for  $\mathbb{R}$ . Hence, in order to show that  $S$  preserves Lebesgue measure, we must show that  $S^{-1}(I)$  is measurable and that  $\lambda(S^{-1}(I)) = \lambda(I)$  for any interval  $I$  with rational endpoints.

Observe that,

$$S^{-1}(y_1) = \{y \in \mathbb{R} : y = x + n \text{ for some } x \in [0, 1), n \in \mathbb{Z} \text{ with } T(x) + n + \phi(x) = x_1 + n_1 = y_1\}$$

Now let us fix an interval with rational endpoints  $I = (n_1 + p_1/q_1, n_2 + p_2/q_2)$  with  $n_1, n_2, p_1, q_1, p_2, q_2 \in \mathbb{Z}$  and  $0 \leq p_1 < q_1$  and  $0 \leq p_2 < q_2$ . Assume  $n_2 \geq n_1$  and  $p_2/q_2 \geq p_1/q_1$  without loss of generality. We have that,

$$\lambda(A) = n_2 + p_2/q_2 - (n_1 + p_1/q_1)$$

and we must show that  $\lambda(S^{-1}(I)) = \lambda(I)$  in order to show that  $S$  preserves Lebesgue measure. Observe that  $T$  is measure-preserving and is also the only part of  $S$  which

maps onto the fractional part of any  $x \in I$ . Hence if we consider  $T^{-1}(I)$ , the range of values upon which  $T$  is acting is only the fractional components,  $(p_1/q_1, p_2/q_2)$  and so, we must have that,

$$\begin{aligned}\lambda(T^{-1}(I)) &= \lambda(T^{-1}((p_1/q_1, p_2/q_2))) \\ &= \lambda((p_1/q_1, p_2/q_2)) \\ &= p_2/q_2 - p_1/q_1\end{aligned}$$

- b) Suppose  $T$  is ergodic and  $\phi$  is integrable with  $\int_{[0,1)} \phi d\lambda \neq 0$ . We want to show that  $S$  is not recurrent. That is, we want to show that there is a measurable set  $A$  of positive measure such that there is no null set  $N \subset A$  with the property that for all  $x \in A \setminus N$  there is  $n = n(x) > 0$  with  $T^n(x) \in A$ .

Let  $A$  be some set of positive measure and fix  $y \in A$  with  $y = x + n$  where  $x$  is the fractional part and  $n$  is the integral part. We have that  $S^k(y) = T^k(x) + n + \sum_{i=0}^{k-1} \phi(T^i x)$ . Then we have,

$$\frac{1}{k} S^k(y) = \frac{1}{k} (T^k(x) + n + \sum_{i=0}^{k-1} \phi(T^i x))$$

and so, by the Birkhoff Ergodic Theorem, we have,

$$\begin{aligned}\lim_{k \rightarrow \infty} \frac{1}{k} S^k(y) &= \lim_{k \rightarrow \infty} \frac{1}{k} (T^k(x) + n + \sum_{i=0}^{k-1} \phi(T^i x)) \\ &= \lim_{k \rightarrow \infty} \frac{1}{k} T^k(x) + \lim_{k \rightarrow \infty} \frac{1}{k} n + \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \phi(T^i x) \\ &= 0 + 0 + \int \phi d\lambda \\ &> 0\end{aligned}$$

But note then that  $\lim_{k \rightarrow \infty} \frac{1}{k} S^k(y) > 0$  implies that  $S^k(y) \rightarrow \infty$ . Thus, if we fix  $y$  in any interval with rational endpoints  $(a, b)$ , then  $S^k(y)$  will leave that interval as  $k \rightarrow \infty$ . But by Lemma 3.5.1, if  $S$  were recurrent, then there would be an increasing sequence  $m_i > 0$  with  $S^{m_i}(y) \in A \setminus N$  for some null set  $N$  for all  $i \geq 0$ . However, if this were the case, then  $\lim_{k \rightarrow \infty} \frac{1}{k} S^k(y)$  would equal 0. Thus, we have a contradiction and so  $S$  cannot be recurrent.