ID: **5152**

Math A4900

Proof portfolio draft, Round 1

October 4, 2020

Statement: Let G be a group and let $x \in G$. If $|x| = n < \infty$, prove that the elements $1, x, x^2, \cdots, x^{n-1}$ are all distinct. Deduce that $|x| = |\langle x \rangle|$

Problem:	**1C**
No. stars:	**2**

Proof. Let $|x| = n < \infty$. Now suppose that there are numbers $m, k \in \mathbb{Z}$ with $0 \leqslant k < m \leqslant n-1$ such that $x^m = x^k$.

Then we have that,

$$\begin{aligned} x^k \cdot x &= x^m \cdot x \\ x^k \cdot x^2 &= x^m \cdot x^2 \\ x^k \cdot x^3 &= x^m \cdot x^3 \\ &\vdots \\ x^k \cdot x^{n-m} &= x^m \cdot x^{n-m} \end{aligned}$$

However, on the right side of the equality, we have

$$\begin{aligned} x^m \cdot x^{n-m} &= x^{m+n-m} \\ &= x^n \\ &= e \end{aligned}$$

This implies that,

$$\begin{aligned} x^k \cdot x^{n-m} &= x^{k+n-m} \\ &= e \end{aligned}$$

where $k + n - m \in \mathbb{Z}$ and $0 < k + n - m < m + n - m = n$.

However, we know that the order of x is n, which is defined to be the smallest positive integer of x that yields the identity element. Hence, we have a contradiction and thus $e, x, x^2, \ldots, x^{n-1}$ are all distinct.

Now consider $\langle x \rangle$. We know that each x^c is distinct for every $c \in \mathbb{Z}$ such that $0 \leqslant k \leqslant n-1$. Now fix an $m \in \mathbb{Z}$ such that $m \geqslant n$. Choose $k \in \mathbb{N}$ as the greatest positive integer such that $m \geqslant kn$. Then we have,

$$\begin{aligned} x^m &= x^{kn+(m-kn)} \\ &= x^{kn}x^{m-kn} \\ &= x^{m-kn} \end{aligned}$$

Note that $0 \leqslant m-kn$ since $m \geqslant kn$. In addition, $m-kn < n$ because, if $m-kn \geqslant n$, it would mean that $(k+1)n \leqslant m$. But we chose k such that it was the greatest positive integer with $m \geqslant kn$, so this is not possible.

Hence we have that $0 \leqslant m-kn < n$, and so $x^{m-kn} \in \{1, x, x^2, \cdots, x^{n-1}\}$. Since $m \geqslant n$ was an arbitrary integer, this holds for any x^m with $m \geqslant n$. Thus, for any $a \in \mathbb{Z}$, we have that,

$$x^a \in \{1, x, x^2, \cdots, x^{n-1}\}$$

and so $|\langle x \rangle| = n = |x|$. □

	Points Possible
complete	0 1 2 3 4 5
mathematically valid	0 1 2 3 4 5
readable/fluent	0 1 2 3 4 5
Total:	(out of 15)

Problem:	**1D**
No. stars:	**2**

Statement: Show that if $s_1 = s$ and $s_2 = sr$, then those together with the relations

$$s_1^2 = s_2^2 = (s_1 s_2)^n = 1$$

forms and alternative presentation of D_{2n}

Proof. By the relations given above we have that,

$$s_1^2 = 1 = s^2$$

Moreover, we have that

$$s_2^2 = 1 = srsr$$

This implies that $sr = (sr)^{-1} = r^{-1}s^{-1}$.

However, we know that $s^{-1} = s$ since $s^2 = 1$, so we have

$$\begin{aligned} sr = (sr)^{-1} &= r^{-1}s^{-1} \\ &= r^{-1}s \end{aligned}$$

We are also given that $(s_1 s_2)^n = 1$. That is,

$$\begin{aligned} (s_1 s_2)^n &= (ssr)^n \\ &= r^n \\ &= 1 \end{aligned}$$

Hence, the elements s_1 and s_2 together with the relations shown above fully describe the initial presentation of D_{2n}. $\square$

	Points Possible
complete	0 1 2 3 4 5
mathematically valid	0 1 2 3 4 5
readable/fluent	0 1 2 3 4 5
Total:	(out of 15)

Statement: Prove that if H and K are subgroups of G, then so is $H \cap K$. On the other hand, prove $H \cup K$ is a subgroup if and only if $H \subseteq K$ or $K \subseteq H$. forms and alternative presentation of D_{2n}

Problem:	**2A**
No. stars:	**2**

Proof. Suppose $H, K \leqslant G$. Consider $H \cap K$. Note that $1 \in H, K$ by the definition of groups, so $1 \in H \cap K$. Hence, $H \cap K \neq \varnothing$. Now let $x, y \in H \cap K$. Then $x, y \in H$ and $x, y \in K$, both of which are groups. Hence, $y^{-1} \in H$ and $y^{-1} \in K$, which implies $xy^{-1} \in H$ and $xy^{-1} \in K$. Thus, $xy^{-1} \in H \cap K$. As a result, $H \cap K$ satisfies the subgroup criterion and is hence a subgroup of G.

Now consider $H \cup K$. Suppose for contraposition that $H \not\subset K$ and $K \not\subset H$. Then $\exists x \in H$ such that $x \notin K$ and $\exists y \in K$ such that $y \notin H$. Then we have $y^{-1} \notin H$ and $x \notin K$, so $xy^{-1} \notin H, K$. Hence $xy^{-1} \notin H \cup K$ and so $H \cup K$ does not satisfy the subgroup criterion. As a result, we have that if $H \cup K$ is a subgroup of G, then $H \subset K$ or $K \subset H$.

Now for the other direction of the proof. Suppose $H \subset K$. Then $\forall\ x \in H$ we have $x \in K$. Hence, $H \cup K = K$. Since $K \leqslant G$, we have $H \cup K \leqslant G$ as well.

Suppose $K \subset H$. Then $\forall\ x \in K$ we have $x \in H$. Hence, $H \cup K = H$. Since $H \leqslant G$, we have $H \cup K \leqslant G$ as well. Thus, we have proved that if $H \subset K$ or $K \subset H$, then $H \cup K$ is a subgroup of G. $\square$

	Points Possible
complete	0 1 2 3 4 5
mathematically valid	0 1 2 3 4 5
readable/fluent	0 1 2 3 4 5
Total:	(out of 15)

Statement: Prove that $\mathbb{R}^\times$ is not isomorphic to $\mathbb{C}^\times$, forms and alternative presentation of D_{2n}

Problem:	**2C**
No. stars:	**1**

Proof. There are only 2 elements in $\mathbb{R}^\times$ with order less than ∞: $|1| = 1$ and $|-1| = 2$. However, there are 4 in $\mathbb{C}^\times$: $|1| = 1$, $|-1| = 2$, $|i| = 4$, $|-i| = 4$.

Since there is no element $\mathbb{R}^\times$ with order 4, for any potential isomorphism $\varphi : \mathbb{C}^\times \to \mathbb{R}^\times$, we have

$$|i| \neq |\varphi(i)|$$

Hence, $\mathbb{R}^\times$ and $\mathbb{C}^\times$ are not isomorphic.

$\square$

	Points Possible					
complete	0	1	2	3	4	5
mathematically valid	0	1	2	3	4	5
readable/fluent	0	1	2	3	4	5
Total:	(out of 15)					

Problem:	**3B**
No. stars:	**1**

Statement: Let G act on a set A. Prove that the relation $\sim$ on A defined by

$$a \sim b \quad \text{if and only if} \quad a = g \cdot b \text{ for some } g \in G$$

is an equivalence relation.

Proof. We need to check that this relation is reflexive, symmetric, and transitive. We will start with reflexivity. Since G is a group, then $1 \in G$ and so we have

$$a = 1 \cdot a$$

Hence, we have $a \sim a$. Now let $a, b \in A$ and suppose $a \sim b$. Then,

$$a = g \cdot b$$

for some $g \in G$. Since G is a group, we have $g^{-1} \in G$ and hence

$$g^{-1} \cdot a = g^{-1} \cdot (g \cdot b)$$

By properties of group actions, we can write

$$\begin{aligned} g^{-1} \cdot a &= (g^{-1} g) \cdot b \\ &= b \end{aligned}$$

So we have that $b \sim a$ since $g^{-1} \in G$. Hence, the relation is symmetric.

Now let $a, b, c \in A$. Suppose $a \sim b$ and $b \sim c$. Then we have,

$$a = g_1 \cdot b$$

and

$$b = g_2 \cdot c$$

for some $g_1, g_2 \in G$. We can use our equation for b and the properties of group action to rewrite a as

$$a = (g_1 g_2) \cdot c$$

Since $g_1 g_2 \in G$, we have that $a \sim c$ and so the relation is transitive. Hence, this is an equivalence relation. ∎

1

	Points Possible
complete	0 1 2 3 4 5
mathematically valid	0 1 2 3 4 5
readable/fluent	0 1 2 3 4 5
Total:	(out of 15)