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Math A4900
Homework 6
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1. **Symmetric and alternating groups.**

   (a) In class we showed that S_n is generated by $T = \{(i\ j) \mid 1 \leqslant i < j \leqslant n\}$, the set of transpositions in S_n. Show by induction on $|j - i|$, that for $i < j$,

   $$(i\ j) = (i\ i+1)(i+1\ i+2)\cdots(j-2\ j-1)(j-1\ j)(j-2\ j-1)\cdots(i+1\ i+2)(i\ i+1),$$

   and conclude S_n is generated by $T' = \{(i\, i{+}1) \mid 1 \leqslant i < n\}$, the set of adjacent transpositions.

   (b) Let x, y be distinct 3-cycles in S_n.
   [*Hint:* Give their entries names so you can reference them. Like, if $x = (a\ b\ c)$ is a three-cycle, you know a, b, and c are distinct, and you know $x = (b\ c\ a) = (c\ a\ b) \neq (a\ c\ b)$. So you can assume without loss of generality things like a is the smallest of the three, but *not* things like $a < b < c$.]

      (i) Set $n = 4$ and assume $x \neq y^{-1}$. Show $\langle x, y\rangle = A_4$.
      [*Hint:* $x = (a\ b\ c)$ and $y = (\alpha\ \beta\ \gamma)$, then what does $x \neq y$ and $x \neq y^{-1}$ tell you about $\{a, b, c\} \cap \{\alpha, \beta, \gamma\}$?.]

      (ii) Set $n = 5$ and assume $x \neq y^{-1}$. Show that either

      x and y both fix some common elements of $[5]$
      (there is some $i \in [5]$ such that $x(i) = i$ and $y(i) = i$)
      and $\langle x, y\rangle \cong A_4$,

      or

      x and y do not fix any common elements of $[5]$
      (for all $i \in [5]$, if $x(i) = i$ then $y(i) \neq i$)
      and $\langle x, y\rangle = A_5$.

      [*Hint:* Try some examples.]

      (iii) Show, for all n, that $\langle x, y\rangle$ is isomorphic to one of Z_3, A_4, A_5, or $Z_3 \times Z_3$.
      [*Hint:* If a group is generated by two commuting elements x and y that otherwise satisfy no relations between them, then $\langle x, y\rangle \cong \langle x\rangle \times \langle y\rangle$.]

2. **Group actions.**

   (a) Let $G \circlearrowright A$. Prove that if $a, b \in A$ and $b = g \cdot a$ for some $g \in G$, then $G_b = gG_ag^{-1}$.
   Deduce that if G acts transitively on A, then the kernel of the action is $\bigcap_{g\in G} gG_ag^{-1}$.

   (b) Let S_3 act on the set of ordered triples $A = \{(i, j, k) \mid i, j, k \in [3]\}$.

      (i) Find the orbits of $S_3 \circlearrowright A$.
      [*Hint*: Break into cases like $i = j = k$, $i = j \neq k$, etc. *Avoid* writing out all the orbits explicitly.]

      (ii) For each orbit $\mathcal{O}$, choose one representative $a \in \mathcal{O}$ and calculate G_a. Verify that $|G : G_a| = |\mathcal{O}|$.

   (c) Suppose G acts transitively on a finite set A (i.e. $[a] = A$ for all $a \in A$), and let $H \trianglelefteq G$. Note that the action of G on A restricts to an action of H on A, which is not *necessarily* transitive anymore. [*Example*: $G = D_8$ acts transitively on $A = \{1, 2, 3, 4\}$, but $H = \langle r^2\rangle$ does not. The orbits under the action of H are $\{1, 3\}$ and $\{2, 4\}$.]

Let $\mathcal{O}_1 = [a_1]_H$, $\mathcal{O}_2 = [a_2]_H$, ..., $\mathcal{O}_r = [a_r]_H$ be the distinct orbits of the action of H on A. [*Hint*: It may be helpful to use set action notation. Namely, if $a \in A$, then the orbit of a under the action of H can be written as $H \cdot a = \{h \cdot a \mid h \in H\}$, whereas $G \cdot a$ is the orbit under the action of G.]

(i) Show that for each $a \in A$, $H_a = G_a \cap H$.

(ii) Prove that G permutes $\mathcal{O}_1$, $\mathcal{O}_2$, ..., $\mathcal{O}_r$, i.e.

- for each $g \in G$, $i \in [r]$, we have $g \cdot \mathcal{O}_i = \mathcal{O}_j$ for some $j \in [r]$ (where $g \cdot \mathcal{O}_i := \{g \cdot a \mid a \in \mathcal{O}_i\}$); and
- $\sigma_g : \{\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r\} \to \{\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r\}$ defined by $\mathcal{O}_i \mapsto g \cdot \mathcal{O}_i$ is a bijection for each $g \in G$.

(iii) Deduce that G acts on the set $\mathcal{A} = \{\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r\}$. Show that this action is transitive, and deduce that $|\mathcal{O}_i| = |\mathcal{O}_j|$ for all $i, j \in [r]$.

(iv) Fix $\mathcal{O} \in \mathcal{A}$, and let $a \in \mathcal{O}$ (so that $\mathcal{O} = H \cdot a$). Show that $|\mathcal{O}| = |H : H \cap G_a|$ and that $r = |G : HG_a|$ (where $r = |\mathcal{A}|$ as above).