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Math A4900
Homework 5
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1. **Properties of quotient groups.**

   (a) Prove that in the quotient group G/N, (i) $(gN)^\alpha = g^\alpha N$ for all $\alpha \in \mathbb{Z}$, and (ii) that $|gN| = n$, where n is the smallest positive integer such that $g^n \in N$ (or is infinite if $g^\alpha \notin N$ for all α).

   Proof. Let G/N be a quotient group.

   (i)

   (ii)

   $\square$

   (b) Prove that if $G/Z(G)$ is cyclic, then G is abelian.
   [Hint: If $G/Z(G)$ is cyclic, with generator $xZ(G)$, show that every element of G can be written in the form $x^a z$ for some integer $a \in \mathbb{Z}$ and some element $z \in Z(G)$.]

   Proof. Note that $Z(G) = \{g \in G | gx = xg \text{ for all } x \in G\}$. Suppose $G/Z(G)$ is cyclic. That is, $G/Z(G) = \langle xZ(G) \rangle$ for some $x \in G$. $\square$

   (c) Let $N \trianglelefteq G$ and let $\overline{G} = G/N$. Prove that $\overline{x}$ and $\overline{y}$ commute in $\overline{G}$ if and only if $x^{-1}y^{-1}xy \in N$.

   Note: The element $x^{-1}y^{-1}xy$ is called the *commutator* of x and y, denoted $[x, y]$.

2. **Orders and indices.**

   (a) Show that if $|G| = pq$ for some primes p and q (not necessarily distinct) then either G is abelian, or $Z(G) = 1$. [Hint: See 1(b).]

   Proof. Suppose $|G| = pq$ where p and q are prime. Suppose that $p \neq q$. By Cauchy's Theorem, since $|G| = pq$ is finite and the prime p divides $|G|$, we have that there exists an element $x \in G$ such that $|x| = p$. In addition, we have that there exists an element $y \in G$ such that $|y| = q$. $\square$

   (b) Prove that if H and K are finite subgroups of G whose orders are relatively prime, then $H \cap K = 1$.

   Proof. Suppose H and K are finite subgroups of G where $|H| = p$, $|K| = q$ with q and p relatively prime. By Proposition 13 on page 93 of Dummit & Foote, we have that,

$$\begin{aligned} |HK| &= \frac{|H||K|}{|H \cap K|} \\ &= \frac{pq}{|H \cap K|} \end{aligned}$$

   Suppose without loss of generality that $p \geqslant q$. Then we have that $|H \cap K| \leqslant |K| = q$. Now note that $\frac{pq}{|H \cap K|}$ must yield an integer answer. However, we have that there are no common factors of p and q in the set $\{2, 3, 4, \cdots, q-1\}$. Thus, our choices for $|H \cap K|$ are 1 and q. We know that $|H \cap K| = q$ if $K \leqslant H$. However, if $K \leqslant H$, then by Lagrange's Theorem, $|K| = q$

divides $|H| = p$. Since p, q are relatively prime, this is not possible. Hence, $|H \cap K| = 1$.

Now, since both H and K are subgroups of G, we know that they must both contain the identity element 1. Hence, $1 \in H \cap K$. Since $|H \cap K| = 1$, we have that the identity must be the only element of $H \cap K$. Thus, $H \cap K = 1$. □

(c) Let $H \leqslant K \leqslant G$. Prove that $|G : H| = |G : K||K : H|$ (**do not** assume G is finite).

Proof. Suppose $H \leqslant K \leqslant G$. □

(d) Prove that if $H \trianglelefteq G$ and $|G : H| = p$ a prime, then for all $K \leqslant G$, either

$$K \leqslant H \qquad \text{or} \qquad G = HK \text{ and } |K : K \cap H| = p.$$

Proof. Assume K is not a subgroup of H. Note that since $H \trianglelefteq G$, we have that $N_G(H) = \{g \in G | gHg^{-1} = H\} = G$ by Theorem 6 on page 82. Since $K \leqslant G$, we have that $K \leqslant N_G(H)$. Now let $h \in H$ and $k \in K$. Then $khk^{-1} \in H$. Thus, we have $kh \in KH$, but also,

$$kh = (khk^{-1})k \in HK$$

Hence, $KH \subset HK$. We also have $hk = k(k^{-1}hk) \in KH$. Thus, $HK \subset KH$. Hence, $HK = KH$. We can then apply Proposition 14 on page 94, which states that HK is a subgroup of G.

Now fix $g \in G$. Note that since $H \trianglelefteq G$, we have that $N_G(H) = \{g \in G | gHg^{-1} = H\} = G$ by Theorem 6 on page 82. Since $K \leqslant G$, we have that $K \leqslant N_G(H)$. Hence, we can apply the Diamond Isomorphism Theorem, which states that $HK \leqslant G$ □

3. **Composition series.** In a group G, a sequence of subgroups

$$1 = N_0 \leqslant N_1 \leqslant N_2 \leqslant \cdots \leqslant N_{\ell-1} \leqslant N_\ell = G$$

is called a (finite) *composition series* for G if, for $1 \leqslant i \leqslant \ell$, we have

$$N_{i-1} \trianglelefteq N_i \quad \text{and} \quad N_i/N_{i-1} \text{ is simple.}$$

For a composition series, we call the quotient groups N_i/N_{i-1} *composition factors* of G.

[Note: $N_{i-1} \trianglelefteq N_i$ and $N_i \trianglelefteq N_{i+1}$ *does not* imply $N_{i-1} \trianglelefteq N_{i+1}$.]

(a) Briefly explain why N_i/N_{i-1} being simple means that N_{i-1} is "maximally" normal in N_i, i.e. there are no normal subgroups N such that $N_i \lneq N \lneq N_{i+1}$ and $N \trianglelefteq N_{i+1}$.

(b) The *Jordan-Hölder Theorem* (see Thm. 3.4.22) says that if G is finite, then compositions series exist and are essentially unique. Namely,

(I) G has a composition series, and

(II) the collection of composition factors is unique; i.e. if

$$1 = N_0 \leqslant N_1 \leqslant N_2 \leqslant \cdots \leqslant N_{\ell-1} \leqslant N_\ell = G$$

and

$$1 = M_0 \leqslant M_1 \leqslant M_2 \leqslant \cdots \leqslant M_{k-1} \leqslant M_k = G$$

are two composition series for G, then $k = \ell$ and there is some permutation σ of $\{1, \ldots, \ell\}$ such that

$$N_i/N_{i-1} \cong M_{\sigma(i)}/M_{\sigma(i)-1}, \quad \text{for } i = 1, \ldots, \ell.$$

Note that (I) is proven using a straightforward proof by (strong) induction on $|G|$.

(i) Check that

$$1 = N_0 \leqslant N_1 \leqslant N_2 \leqslant N_3 = D_8 \quad \text{and} \quad 1 = M_0 \leqslant M_1 \leqslant M_2 \leqslant M_3 = D_8,$$

where

$$N_1 = \langle s \rangle \ \&\ N_2 = \langle s, r^2 \rangle \qquad \text{and} \qquad M_1 = \langle r^2 \rangle \ \&\ M_2 = \langle r \rangle,$$

both define composition series of D_8. Then show that, as (multi)sets,

$$\{N_3/N_2, N_2/N_1, N_1/N_0\} = \{M_3/M_2, M_2/M_1, M_1/M_0\}$$

(up to isomorphism).

(ii) Prove the following special case of part (II) of Jordan-Hölder: Let G be a finite group, and assume that

$$1 = N_0 \leqslant N_1 \leqslant N_2 \leqslant \cdots \leqslant N_{\ell-1} \leqslant N_\ell = G \tag{$*$}$$

and

$$1 = M_0 \leqslant M_1 \leqslant M_2 = G. \tag{$**$}$$

are both composition series of G. Use the Diamond Isomorphism Theorem to show that $\ell = 2$ and that the collection of composition factors are the same.

[Note: The proof of the general version of part (II) now follows from this special case by induction on $\min\{k, \ell\}$.]