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Lecture 2, Exercise A

1a. $\det(x) = 1 \cdot 1 - 2 \cdot 0$

$= 1$

$\det(y) = 2 \cdot 1 - (-1 \cdot 4)$

$= 2 - (-4)$

$= 6$

b. $xx^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$x^{-1}x = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$yy^{-1} = \begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix} \cdot \frac{1}{6} \begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix}$

$= \frac{1}{6} \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$Y^{-1}Y = \frac{1}{6}\begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

c. $$XY = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 6 \\ -1 & 1 \end{bmatrix}$$

$$YX = \begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix}$$

$$X^{-1}Y^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \cdot \frac{1}{6}\begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -1 & -8 \\ 1 & 2 \end{bmatrix}$$

$$Y^{-1}X^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 1 & -6 \\ 1 & 0 \end{bmatrix}$$

d. $(XY)^{-1} = \left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix}\right)^{-1}$

$$= \begin{bmatrix} 0 & 6 \\ -1 & 1 \end{bmatrix}^{-1}$$

$$= \frac{1}{6}\begin{bmatrix} 1 & -6 \\ 1 & 0 \end{bmatrix}$$

$Y^{-1}X^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -6 \\ 1 & 0 \end{bmatrix}$ (from above)

So $(XY)^{-1} = Y^{-1}X^{-1}$

$(YX)^{-1} = \begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix}^{-1}$

$$= \frac{1}{6}\begin{bmatrix} -1 & -8 \\ 1 & 2 \end{bmatrix}$$

$$x^{-1}y^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \cdot \frac{1}{6}\begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -1 & -8 \\ 1 & 2 \end{bmatrix}$$

So $(yx)^{-1} = x^{-1}y^{-1}$

2. Let $x, y \in A$.

Then $\exists\, a, k, b, l$ s.t.

$$x = \frac{a}{2k+1} \quad y = \frac{b}{2l+1}$$

and $\gcd(a, 2k+1) = 1 = \gcd(b, 2l+1)$

Thus,

$$x + y = \frac{a}{2k+1} + \frac{b}{2l+1}$$

$$= \frac{a(2l+1) + b(2k+1)}{(2k+1)(2l+1)}$$

Since $(2k+1)(2l+1)$ is odd, this fraction will be odd even in lowest terms.

Hence, if $x, y \in A$, $x + y \in A$

Now + is associate because addition is associative

0 is in A because $\frac{0}{1} = 0$, so A has an identity element

For any $x \in A$, $-x \in A$ and $x + (-x) = 0$, so every element has an additive inverse

3 $x = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$$x^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$x^3 = x^2 \cdot x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

$|x| = 3$

4. $\bar{0} = \bar{0}$, so $|\bar{0}| = 1$

$\underbrace{\bar{1} + \bar{1} + \bar{1} + \cdots + \bar{1}}_{11 \text{ times}} = \bar{0}$, so $|\bar{1}| = 12$

$\bar{2} + \bar{2} + \bar{2} + \bar{2} + \bar{2} + \bar{2} = \bar{0}$, so $|\bar{2}| = 6$

$\bar{3} + \bar{3} + \bar{3} + \bar{3} = \bar{0}$, so $|\bar{3}| = 4$

$\bar{4} + \bar{4} + \bar{4} = \bar{3}$, so $|\bar{4}| = 3$

$\underbrace{\bar{5} + \bar{5} + \cdots + \bar{5}}_{12 \text{ times}} = \bar{0}$, so $|\bar{5}| = 12$

$\bar{6} + \bar{6} = \bar{0}$, so $|\bar{6}| = 2$

$\underbrace{\bar{7} + \cdots + \bar{7}}_{12 \text{ times}} = \bar{0}$, so $|\bar{7}| = 12$

$\bar{8} + \bar{8} + \bar{8} = \bar{0}$, so $|\bar{8}| = 3$

$\bar{9} + \bar{9} + \bar{9} + \bar{9} = \bar{0}$, so $|\bar{9}| = 4$

$\overline{10} + \overline{10} + \overline{10} + \overline{10} + \overline{10} + \overline{10} = \overline{0}$,

so $|\overline{10}| = 6$

$\underbrace{\overline{11} + \cdots + \overline{11}}_{11 \text{ times}} = \overline{0}$, so $|\overline{11}| = 11$

5a. Suppose $n | m$.

Then $m = nk$ for some $k \in \mathbb{Z}$

Hence,

$$x^m = x^{nk} = (x^n)^k$$
$$= 1^k$$
$$= 1$$

b. Let $x^m = 1$

By the division algorithm, $\exists q, r \in \mathbb{Z}$ s.t.

$$m = nq + r, \quad 0 \leq r < |n|$$

So, $x^m = x^{nq+r} = 1$

Note that

$$x^{nq+r} = x^{nq}\,x^{r}$$

$$= (x^n)^q\,x^r$$

$$= 1^q\,x^r$$

$$= x^r = 1$$

So we have $x^r = 1$.

However, we know $|x| = n$ and $0 \leq r < |n|$.

If $0 < r < |n|$, then $x^r = 1$
$\Rightarrow |x| = r$, a contradiction.

So $r = 0$ and thus $m = nq$.

Hence $n \mid m$.