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lecture 18, Ex. A

1. $(n) = n\mathbb{Z}$

$1 = (\frac{1}{n})n \in (n)$, so

$(n) = \mathbb{Q}$ (Proposition 9)

2. $(n) = (4) \in \mathbb{Z}$

$\Rightarrow (n) = (-4)$

3a. Suppose $2p(x) + xq(x) = 1$

Then,

$$q(x) = \frac{1 - 2p(x)}{x}$$

But then $q(x)$ has term with degree < 0, which is not allowed in our definition of $\mathbb{Z}[x]$

So $1 \notin (2, x)$

6. Suppose $(2, x) = (a(x))$ for some $a(x) \in \mathbb{Z}[x]$

We know

$$(a(x)) = \{a(x)\, p(x) \mid p(x) \in \mathbb{Z}[x]\}$$

Since $2, x \in (2, x) = (a(x))$, $\exists\, p(x), q(x) \in \mathbb{Z}[x]$ s.t.

$$a(x)\, p(x) = 2, \quad a(x)\, q(x) = x$$

i. $a(x)\, p(x) = 2$

$$\underbrace{\deg(a(x)\, p(x))}_{= 0} = \deg(a(x)) + \deg(p(x))$$

$$\Rightarrow \deg(a(x)) = 0$$

ii. $a(x)\, q(x) = x$

$$\underbrace{\deg(a(x)\, q(x))}_{= 1} = \deg(a(x)) + \deg(q(x))$$

$$\Rightarrow \deg(a(x)) = 1 \text{ or } 0$$

iii $\deg(a(x)) = 0$ to satisfy i, iii

But since 2 is prime, we must

$a(x) = \pm 1$ or ± 2

$(a(x))$ is a proper ideal so we have

$$a(x) = \pm 2$$

But then there is no $q(x) \in \mathbb{Z}[x]$ s.t.

$$2q(x) = x$$

$$\text{or } -2q(x) = x$$

a contradiction, so $(2, x)$ is not principal.

4. Suppose $1 \in I$. Since I is an ideal, it is closed under multiplication. Hence $\forall r \in R$, we have

$$r = r \cdot 1 \in I$$

So $R \subseteq I$. We know $I \subseteq R$ since it's an ideal, so $I = R$

b Let u be a unit in R. Suppose $u \in I$

Since I is ideal,

$$ru \in I \quad \forall r \in R$$

Since u is a unit in R, $\exists v \in R$ s.t.

$$uv = vu = 1$$

By both the above, we have

$$vu = 1 \in I$$