Chris Haydut 11/12

Lecture 19 - Ex. B

1. $100 = 2(36) + 28$

$36 = 1(28) + 8$

$28 = 3(8) + 4$

$8 = 2(4) + 0$

So $\gcd(100, 36) = 4$

2. Clearly $N(a) = 0$ $\forall a \in F$, so

$N: F \to \mathbb{Z}_{\geq 0}$ and $N(0) = 0$

as required

Now let $a, b \in F$, $b \neq 0$

$b^{-1} \in F$, so

$q = ab^{-1}$ and $r = 0$

$\Rightarrow$ $a = qb + r$

w/ $r = 0$

3. $N(p(x)) = \deg(p(x))$

$\deg(p(x)) \in \mathbb{Z}$ and $\deg(p(x)) \geq 0$
$\forall\, p(x)$, so

$N: R[x] \to \mathbb{Z}_{\geq 0}$

Also, $\deg(0) = 0$ so

$N(0) = 0$

and N defines a valid norm

Now to show that it's a Euclidean Function:

Let $a(x), b(x) \in R[x]$, $b(x) \neq 0$

4. Polynomial division might yield results in $\mathbb{Q}[x]$

5. $\gcd(a(x), b(x)) = x^2$

(from Wolfram Alpha)

6i $\frac{\alpha}{\beta} = \frac{(a+bi)(c-di)}{c^2+d^2}$

$= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i$

$x = \frac{ac+bd}{c^2+d^2}$ $y = \frac{bc-ad}{c^2+d^2}$

ii. Suppose $\frac{1}{2} < |m-x| < 1$

Then either

$|(m+1)-x| \leq \frac{1}{2}$

or

$|(m-1)-x| \leq \frac{1}{2}$

But m is closest integer to x, a contradiction.

So $|m-x| \leq \frac{1}{2}$

iii. $q\beta + r = (m + ni)\beta + \beta((x-m) + (y-n)i)$

$= (m + ni(x-m) + (y-n)i)\beta$

$= (x + iy)\beta$

$= \frac{\alpha}{\beta}\beta$

$= \alpha$

$r \in \mathbb{Z}[i]$

$N(\gamma) = (x-m)^2 + (y-n)^2$

$< (\frac{1}{2})^2 + (\frac{1}{2})^2$

$= \frac{2}{4} = \frac{1}{2}$

$N(r) = N(\beta)N(\gamma)$

$< \frac{1}{2}N(\beta)$

$< N(\beta$

Tk