Chris Hayduk
Math A4900
Homework 10
11/20/2020

1. **Prime and maximal ideals**

   (a) Prove that if M is an ideal such that R/M is a field then M is a maximal ideal (do not assume R is commutative).

   (b) Let $\varphi : R \to S$ be a homomorphism of commutative rings.

   (i) Prove that if P is a prime ideal of S then either $\varphi^{-1}(P) = R$ or $\varphi^{-1}(P)$ is a prime ideal of R. Apply this to the special case when R is a subring of S and φ is the inclusion homomorphism to deduce that if P is a prime ideal of S, then $P \cap R$ is either R or a prime ideal of R.

   (ii) Prove that if M is a maximal ideal of S and φ is surjective then $\varphi^{-1}(M)$ is a maximal ideal of R. Give an example to show that this need not be the case if φ is not surjective.

2. **Principal ideal domains.**

   (a) Prove that if R is a PID, then so is R/I for any ideal $I \subseteq R$.

   (b) Let R be an integral domain and suppose that every prime ideal in R is principal. This exercise proves that R must be a PID.

   (i) Assume that the set of ideals of R that are not principal is nonempty and prove that this set has *at least one* maximal element under inclusion (which by hypothesis is not prime). [Use Zorn's lemma.]

   (ii) Let I be an ideal that is maximal with respect to being nonprincipal, and let $a, b \in R$ with $ab \in I$ but $a, b \notin I$ (which exist because prime ideals are all principal, but I is not principal and hence is not prime). Let $I_a = (I, a)$ and $I_b = (I, b)$, and define $J = \{r \in R \mid rI_a \subseteq I\}$.

   Prove that I_a and I_b are principal, writing
   $$I_a = (\alpha) \quad \text{and} \quad J = (\beta),$$
   and that they satisfy $I \subsetneq I_b \subseteq J$ and $I_a J = (\alpha\beta) \subseteq I$.

   (iii) If $x \in I$, show that $x = s\alpha$ for some $s \in J$. Deduce that $I = I_a J$ is principal, a contradiction, and conclude that R is a PID.

3. **Euclidean domains.**

   (a) Let R be a Euclidean domain with norm N. Let
   $$\Lambda = N(R - 0) = \{n \in \mathbb{Z}_{\geqslant 0} \mid N(r) = n \text{ for some non-zero } r \in R\},$$
   and let $m = \min(\Lambda)$ (which exists by the well-ordering of $\mathbb{Z}$).
   [For example, in $\mathbb{Z}$ with $N(a) = |a|$ for each $a \in \mathbb{Z}$, then $m = 1$; or in a field F with $N(a) = 0$ for all $a \in F$, then $m = 0$.]

   Prove that if $r \in R$ with $N(r) = m$, then a is a unit. Deduce that if $r \in R - 0$ and $N(r) = 0$, then r is a unit.

   (b) Let $D = 2$ (so that $\omega_D = \sqrt{2}$). Prove that $\mathbb{Z}[\sqrt{2}]$ is a Euclidean domain (with norm $N(a + b\sqrt{2}) = |a^2 - b^2\sqrt{2}|$.