Coherent Diagrammatic Reasoning in Compositional Distributional Semantics

Gijs Jasper Wijnholds(✉)

Queen Mary University of London, London, UK
g.j.wijnholds@qmul.ac.uk

Abstract. The framework of Categorical Compositional Distributional models of meaning [3], inspired by category theory, allows one to compute the meaning of natural language phrases, given basic meaning entities assigned to words. Composing word meanings is the result of a functorial passage from syntax to semantics. To keep one from drowning in technical details, diagrammatic reasoning is used to represent the information flow of sentences that exists independently of the concrete instantiation of the model. Not only does this serve the purpose of clarification, it moreover offers computational benefits as complex diagrams can be transformed into simpler ones, which under coherence can simplify computation on the semantic side. Until now, diagrams for compact closed categories and monoidal closed categories have been used (see [2,3]). These correspond to the use of pregroup grammar [12] and the Lambek calculus [9] for syntactic structure, respectively. Unfortunately, the diagrammatic language of Baez and Stay [1] has not been proven coherent. In this paper, we develop a graphical language for the (categorical formulation of) the nonassociative Lambek calculus [10]. This has the benefit of modularity where extension of the system are easily incorporated in the graphical language. Moreover, we show the language is coherent with monoidal closed categories without associativity, in the style of Selinger's survey paper [17].

Keywords: Diagrammatic reasoning · Coherence theorem · Proof nets · Compositional distributional semantics

1 Background, Motivation

Having a form of visual representation of information flow is pervasive in the natural sciences: in physics, graphical languages have been developed coming from the ideas of Penrose [16], to formalise reasoning about matrix multiplication, for instance. Computer science and electronic engineering makes extensive use of diagrammatic representation of systems, circuits etc. In logic, there has been a great interest in graphical notation since the development of natural deduction and sequent calculi, but mostly after the introduction of proof nets [5]. Most of these languages can be greatly generalised; for instance, within physics, graphical

J. Kennedy and R.J.G.B. de Queiroz (Eds.): WoLLIC 2017, LNCS 10388, pp. 371–386, 2017.
DOI: 10.1007/978-3-662-55386-2_27

notation usually is describing the structure of a monoidal category; a similar situation occurs with electronic circuits, which have a notion of serial and parallel execution, categorically speaking a notion of composition and tensor, the basic principles of a monoidal category. In logic, different deductive systems capture different types of category, e.g. intuitionistic logic is captured by cartesian closed categories [13], a special instance of monoidal categories. Hence, proof nets for intuitionistic logic should bear a relation to a graphical language for cartesian closed categories (though the latter, as far as the author knows, does not exist).

While graphical languages are visually appealing, they will not convince the practicing scientist to be useful unless they are also precise. In other words, we want a graphical language to be coherent (i.e. sound and complete) with respect to the category it is describing. In recent work on categorical compositional distributional semantics [2], the clasp language of [1] was assumed, in order to describe the morphisms of a monoidal biclosed category. The reason is that the authors relied on a functorial passage from the Lambek calculus (the logic of monoidal biclosed categories) to finite dimensional vector spaces (an instance of a compact closed category). Here, the diagrammatic notation does not only describe the combination of those two systems, it has an additional functional role in simplifying calculations: by rewriting a diagram as much as possible, one will obtain the same results with smaller computational effort. Sadly, no attempt at proving the coherence of the clasp language is known to the author; moreover, one of the inventors of the language has stated to have no interest in doing so[1]. Hence, there is a need to obtain either a coherence result for this language, or to introduce another graphical language and show its coherence. In this paper, we will take the latter option and motivate it by pointing out some potential problems with the clasp language.

Originally out of interest in proof nets and their relation to categorical diagrammatic reasoning, we will introduce a graphical notation for morphisms in a biclosed monoidal category without associativity/units (!) which will give a fairly easy way of showing coherence. This comes from considerations on structural rules in the Lambek calculus, and the need for modularity in said systems. We will then argue how the addition of associativity and units can be easily incorporated graphically, so that we obtain a modular way of describing categories, moreover giving a coherent language for monoidal biclosed categories, which would be first successful attempt.

The rest of this paper is organised as follows: in Sect. 2 we describe some related work on graphical languages for monoidal categories and the clasp language for monoidal closed categories. Then Sect. 3 discusses definitions and notation for a graphical language for non-associative non-unital monoidal closed categories. Section 4 contains our main result, and we give some extensions in Sect. 5, after which we conclude in Sect. 6 with some avenues for future work.

[1] John Baez, personal communication, 2014.

2 Related Work: Visualising Monoidal Categories

There has been a fair body of research devoted to displaying several variants of monoidal categories using a graphical language. For instance, diagrammatic reasoning for compact closed categories was introduced and shown to be coherent by Kelly and Laplaza [8], and the research of Joyal and Street has led to graphical languages for, amongst others, planar monoidal and braided monoidal categories (see [6,7]). For some cases of autonomous categories (monoidal categories with dual objects) there are some results as well [4]. In this paper, however, we follow the presentation style of the survey of Selinger [17], as it is uniform, and in our opinion a gentle introduction to graphical languages. We will introduce the basic constructs of a graphical language for monoidal categories and then review the clasp language for the general case of monoidal closure, proposed by Baez and Stay [1]. As we will consider monoidal categories without associativity too, which for lack of a better name we will refer to as magmatic categories[2], we split the definition of a monoidal category:

Definition 1. *We say that a category* $\mathbf{C}$ *with objects* A, B *and morphisms* $f : A \to B$ *between objects has a* ***magma structure*** *if it has*

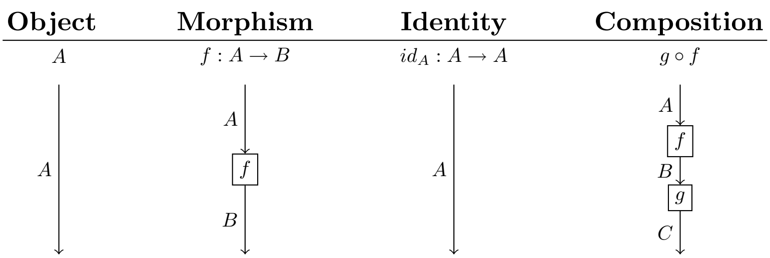
1. *An object* $A \otimes B$ *for all objects* A, B *and a morphims* $f \otimes g : A \otimes B \to C \otimes D$ *for any pair of morphisms* $f : A \to C, g : B \to D$

And has a ***monoidal structure*** *if for this magma structure it has*

1. *A natural isomorphism* $\alpha : (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$,
2. *A unit object* I *with natural isomorphisms* $\lambda : I \otimes A \cong A$ *and* $\rho : A \otimes I \cong A$.

In addition, a monoidal category also satisfies the so-called pentagon and triangle identities, expressing the behavior of associativity and the interaction of λ, ρ *with associativity.*

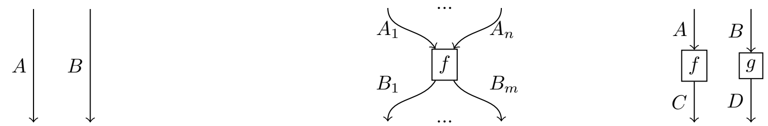
Representing any category visually is fairly straightforward, as is shown in the following diagram:



[2] This terminology comes from the algebraic concept of a magma, a monoid with no associativity or unit properties. We refer the reader to a blog post that advocates the name *magmatic*: https://bartoszmilewski.com/2014/09/29/how-to-get-enriched-over-magmas-and-monoids/.

To obtain a graphical language for a monoidal category, one represents the tensor of two objects by juxtaposing their arrows, and similar for the tensor product of two morphisms. The unit is represented as an empty arrow, and in general a morphism with a tensor of objects in its domain and codomain is represented by allowing several wires to be input/output to the box of that morphism:

Tensor	**Unit**	**Morphism**	**Tensor**
$A \otimes B$	I	$f : A_1 \otimes ... \otimes A_n \to B_1 \otimes ... \otimes B_m$	$f \otimes g$



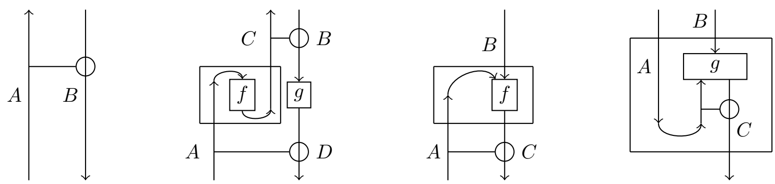
Because we do not draw the unit explicitly, and there is no bracketing around the wires, the associativity and unit equations get automatically satisfied. Bifunctoriality is satisfied by the fact that horizontal and vertical composition in any order will result in the same diagram. A full coherence theorem for this language can be found in [6].

We recall that a closed monoidal category is obtained by considering bifunctors that form an adjunction with the tensor when one of the arguments of the bifunctor is fixed. In this way we may obtain two right adjoints for the tensor. Note that so far we have not considered symmetric monoidal categories (ones in which $A \otimes B \cong B \otimes A$), so that there is a difference between a left closed and a right closed monoidal category.

Definition 2. *If a monoidal category* $\mathbf{C}$ *has a contravariant-covariant bifunctor* $\backslash$*, that is, for* $f : A \to C, g : B \to D$ *we get* $f\backslash g : B\backslash C \to A\backslash D$*, and when there is additionally a natural isomorphism* $\beta : Hom_{\mathbf{C}}(A \otimes B, C) \cong Hom_{\mathbf{C}}(B, A\backslash C)$ *for a fixed* A*, we say that* $\mathbf{C}$ *is* ***left closed****. In a similar fashion we can define a* ***right closed*** *monoidal category by requiring a covariant-contravariant bifunctor* $/$ *that gives* $f/g : A/D \to C/B$ *for* $f : A \to C, g : B \to D$ *and a natural isomorphism* $\gamma : Hom_{\mathbf{C}}(A \otimes B, C) \cong Hom_{\mathbf{C}}(A, C/B)$ *for a fixed* B*. Objects of the form* $A\backslash B$ *and* B/A *are often referred to as the* ***internal hom*** *of the category.*

To represent closure on monoidal categories, a language was introduced by Baez and Stay in [1]. Their proposal amounts to representing internal homs with upward pointing arrows that are attached with a "clasp" as in the table below (we only include left closure here, as right closure is symmetric):

Closure	**Closure**	**Currying**	**Uncurrying**
$A\backslash B$	$f\backslash g$	$\beta(f)$	$\beta^{-1}(g)$



The reason to draw clasps and bend arrows around is that biclosed monoidal categories in general do not allow "dual behavior". Hence, arrows pointing upwards need to be attached to a downward pointing arrow, and arrow bending must be containing within a box. Unfortunately, the clasp language has not been proved to be coherent with respect to biclosed monoidal categories. We will point out why we doubt whether the clasp language can be shown to be coherent.

Problem: Yanking is required. In order to satisfy the categorical equations $id_A \backslash id_B = id_{A \backslash B}$ one has to allow yanking inside a box. However, one draws boxes every time an arrow is bent around, which implies that yanking inside a box means the same as having yanking everywhere, in turn rendering the clasp language a graphical language for compact closed categories instead of monoidal closed categories! The situation is similar when one wants to show isomorphicity of β, i.e. $\beta(\beta^{-1}(g)) = g$ and $\beta^{-1}(\beta(f)) = f$.

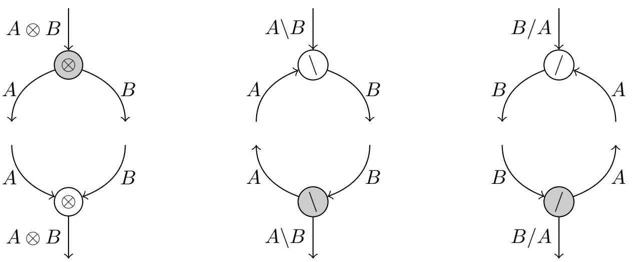
In the next sections, we define an alternative graphical language that does not suffer from the issue described above. It is a language defined for magmatic closed categories, following the type logical grammar philosophy. Starting out with such a restricted system has the benefit of modularity; different categorical concepts can be added by simply extending the diagrams. We will shed a light on this in Sect. 5, after establishing the base language and showing its coherence.

3 A Graphical Language for Closed Magmatic Categories

As noted near the end of Selingers survey, once one goes beyond a single tensor product in a category, simply juxtaposing arrows presents an ambiguity in a graphical language. So it is the case when one takes out associativity from monoidal categories. Within research on *proof nets* (see [5,15]) every bifunctor will get its own representation by means of *links*. Usually proof nets are defined restrictively: once the links are defined, correctness criteria decide whether a graph built by said links is a proof net or not. We will follow the philosophy of proof nets, but instead of defining proof nets by correctness criteria, we give an inductive definition since it is equivalent to the restrictive definition ([18], p. 42) and easier to work with.

For every bifunctor we define two labelled links that make the merging and unmerging of objects explicit:

Definition 3 (Links). *For every bifunctor we define constructor and destructor links that respect the variance of the bifunctor:*



Because the list of correction criteria on these proof nets is quite extensive, we will skip to the inductive definition of proof nets for reasons of space.

Definition 4 (Proof Nets). *Any diagram from Fig. 1 (see appendix) is a proof net, given that the boxes for N_1, N_2 are proof nets. N^* refers to a proof net that has been drawn upside down.*

Besides having a definition for proof nets themselves, we also define equations on proof nets to establish the categorical equations we want to capture. Interestingly enough, these equations correspond to *cut elimination* in sequent calculi in logic!

Definition 5 (Proof Net Equations). *All the equations from Fig. 2 hold on proof nets. Note that N^* is a proof net drawn upside down, which explains why we have equations for "sliding" a net upside down.*

4 Main Result

Following along the lines of [17], we prove coherence by means of a freeness theorem. It works as follows: first, we define for any biclosed magmatic Σ the associated proof net language $\mathbf{PN}(\Sigma)$ and show that it is in turn biclosed. Then, we show coherence by proving that $\mathbf{PN}(\Sigma)$ is the free biclosed magmatic category over Σ.

Definition 6. *A **biclosed magmatic signature** $\Sigma = (\Sigma_0, \Sigma_1, dom, cod)$ has:*

1. *a set Σ_0 of object variables,*
2. *a set Σ_1 of morphism variables,*
3. *two maps $dom, cod : \Sigma_1 \to CT(\Sigma_0)$.*

where $CT(\Sigma_0)$ is the free $(\otimes, \backslash, /)$-algebra generated by Σ_0.

Definition 7. *Given a biclosed magmatic signature Σ and a biclosed magmatic category $\mathbf{C}$, an **interpretation** $i : \Sigma \to \mathbf{C}$ consists of:*

1. *an object map $i_0 : \Sigma_0 \to Ob(\mathbf{C})$ such that*

$$\begin{aligned} i_0(A \otimes B) &= i_0(A) \otimes i_0(B) \\ i_0(A \backslash B) &= i_0(A) \backslash i_0(B) \\ i_0(B/A) &= i_0(B)/i_0(A), \end{aligned}$$

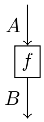
2. *for every $f \in \Sigma_1$ a morphism $i_1(f) : i_0(dom(f)) \to i_0(cod(f))$.*

Definition 8. *A biclosed magmatic category $\mathbf{C}$ is a **free biclosed magmatic category** over a biclosed magmatic signature Σ if there is an interpretation $i : \Sigma \to \mathbf{C}$ such that for any biclosed magmatic category $\mathbf{D}$ and biclosed magmatic interpretation $j : \Sigma \to \mathbf{D}$, there is a unique biclosed magmatic functor $F : \mathbf{C} \to \mathbf{D}$ such that $j = F \circ i$.*

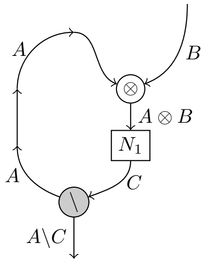
Given the preceding definitions, we can define the category of proof nets over a signature, and show that it is in fact the *free* category over that signature.

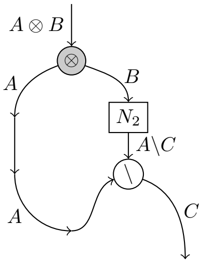
Definition 9. *Given $\Sigma = (\Sigma_0, \Sigma_1, dom, cod)$ a biclosed magmatic signature, the* ***proof net category*** $\mathbf{PN}(\Sigma)$ *over Σ is defined as follows:*

1. *The objects of $\mathbf{PN}(\Sigma)$ are the elements of Σ_0,*
2. *For every object A of $\mathbf{PN}(\Sigma)$, the identity net is defined as in Definition 4,*
3. *For every element f in Σ_1 with $dom(f) = A$, $cod(f) = B$, we stipulate a proof net*

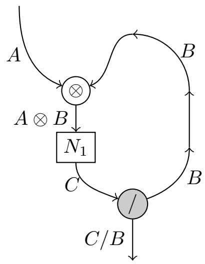


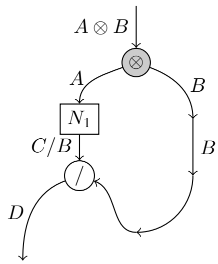
4. *Composition of morphisms and applying bifunctors $\otimes, \backslash, /$ to morphisms are given by composition and monotonicity in Definition 4,*
5. *Left closure $\beta(N_1) : B \to A\backslash C$ for a morphism $N_1 : A \otimes B \to C$ and its inverse $\beta^{-1}(N_2) : A \otimes B \to C$ for $N_2 : B \to A\backslash C$ are given by*





Right closure is treated similarly, where $\gamma(N_1) : A \to C/B$ for a morphism $N_1 : A \otimes B \to C$ and its inverse $\gamma^{-1}(N_2) : A \otimes B \to C$ for $N_2 : A \to C/B$ are given by

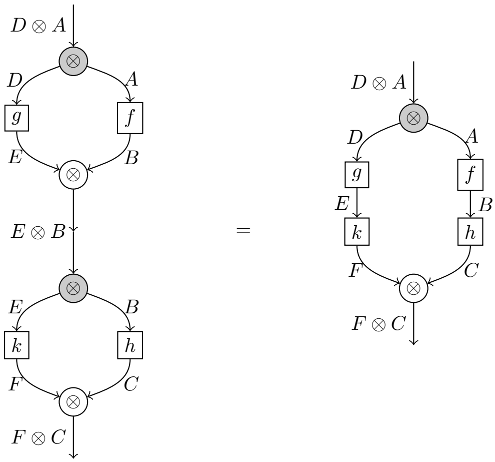




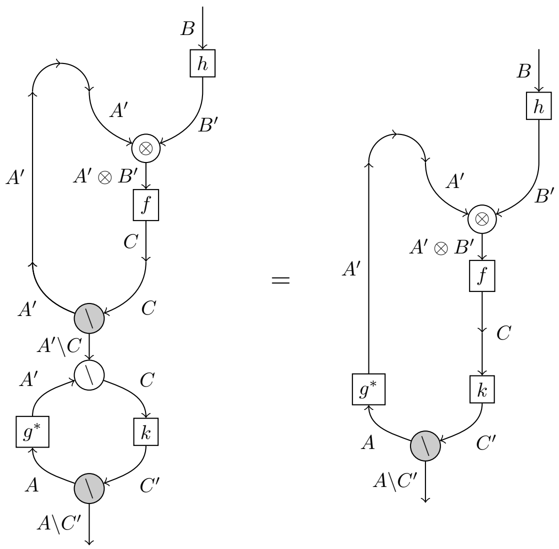
6. *All the proof net equations from Definition 5 hold.*

Proposition 1. *For any biclosed magmatic signature Σ, $\mathbf{PN}(\Sigma)$ is a biclosed magmatic category.*

Proof. The basic categorical axioms are trivially satisfied: the associativity of gluing gives associativity of composition, and for any morphism $N_1 : A \to B$ we have that $id_B \circ N_1$ is the result of gluing an extra piece of wire on the bottom and $N_1 \circ id_A$ is the result of gluing an extra piece of wire on the top, which is just the same as the original morphism. $id_A \otimes id_B = id_{A\otimes B}$ is also trivially satisfied by the identy cut equation for $\otimes$ (similarly for $\backslash$ and $/$. Bifunctioriality of $\otimes$, is also satisfied because $(k \otimes h) \circ (g \otimes f) = (k \circ g) \otimes (h \circ f)$ translates to



which definitely is a valid equation by virtue of the general cut equation. Bifunctoriality of $\backslash$ and $/$ follows similarly. Isomorphicity of β and γ is easily verified using the snake equations. Finally, we need to show naturality of β and γ. We show (half of) the naturality of β as naturality for γ follows similarly. For $(g\backslash k) \circ ((\beta(f)) \circ h)$ we have



and these nets are obviously equal, given that we can bend around g to g^* and vice versa.

In order to show freeness of the proof net category, we need a way to obtain a categorical morphism from a diagram. In previous work we achieved such a translation as a two step process: diagrams are translated to sequent proofs, which then get translated to morphisms, which is the reason this process is called *sequentialisation*. In this paper we simply give the composed translation to save space. This translation is given in Fig. 3, where we use two extra notations, defined below:

Definition 10. *We write, in sequent calculus style, $\Gamma[B]$ ($\Gamma[\Delta]$) for a tensor of objects Γ that "contains" the object B (Δ). Formally, we can define $\Gamma[]$ as any object of a biclosed category with a "hole" in it, that, is:*

$$\Gamma[] := [] \mid \Gamma'[] \otimes \Delta \mid \Delta \otimes \Gamma'[]$$

Then, we say that $\Gamma[B]$ ($\Gamma[\Delta]$) is the object with its "hole" replace by B (Δ). Given a morphism $f : A \to B$, we construct $op(f) : \Gamma[A] \to \Gamma[B]$ for the morphism that acts as the identity on $\Gamma[]$ but applies f to A. Formally, we have

1. *$op(f) = f$ for $\Gamma[] = []$,*
2. *$op(f) = op'(f) \otimes id_\Delta$ for $\Gamma[] = \Gamma'[] \otimes \Delta$,*
3. *$op(f) = id_\Delta \otimes op'(f)$ for $\Gamma[] = \Delta \otimes \Gamma'[]$.*

Now that we have fully defined the translation from diagrams to categorical morphisms, we can state the sequentialisation property:

Proposition 2. *Every proof net sequentialises using the translation in Fig. 3 and all the equalities between diagrams are preserved under this sequentialisation.*

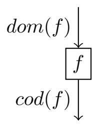
Proof. This is shown for the two step translation by Wijnholds ([18], pp. 50–61).

Given that any diagram can be transformed back into a categorical morphism, we are ready to prove freeness and thus coherence of the proof net category.

Theorem 1. *For any biclosed magmatic signature Σ, the proof net category $\mathbf{PN}(\Sigma)$ is the free biclosed category over Σ.*

Proof. We need to give an interpretation $i : \Sigma \to \mathbf{PN}(\Sigma)$ and for any biclosed magmatic category $\mathbf{D}$ and biclosed magmatic interpretation $j : \Sigma \to \mathbf{D}$ give a unique biclosed magmatic functor $F : \mathbf{PN}(\Sigma) \to \mathbf{D}$ such that $j = F \circ i$.

First, we define $i = \langle i_0, i_1 \rangle$ with i_0 the identity on Σ_0 and i_1 the map that sends morphism variables f to



Now let $\mathbf{D}$ be any biclosed magmatic category and let $j : \Sigma \to \mathbf{D}$ be an interpretation. We define $F : \mathbf{PN}(\Sigma) \to \mathbf{D}$ as follows:

1. On objects we define $F(A) = j_0(A)$,
2. On morphisms/nets we define $F(N : A \to B) = \hat{j_1} \circ S$ where
   - S is the sequentialization that turns a proof net into a categorical morphism module morphism variables,
   - $\hat{j_1}$ sends all objects A to $j_0(A)$, and on morphisms $\hat{j_1}$ sends morphism variables $f : A \to B$ to $j_1(f) : j_0(A) \to j_0(B)$ but otherwise preserves the bifunctors $\otimes, \backslash, /$ and composition.

We need to show that F is well defined as a biclosed magmatic functor, that $j = F \circ i$ and that F is unique.

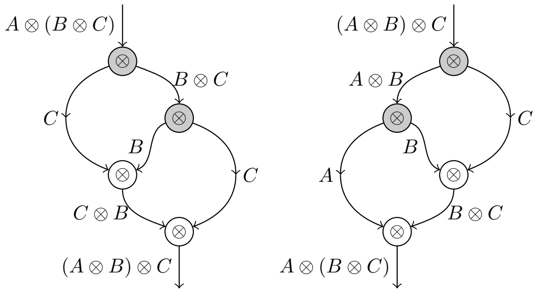
1. F preserves the closed structure of $\mathbf{PN}(\Sigma)$ in $\mathbf{D}$. This is easy to see since F acts as j_0 on objects and therefore strictly preserves the object structure of $\mathbf{PN}(\Sigma)$. On morphisms, we first note that $\hat{j_1}$ preserving morphism structure and only maps morphism variables to their corresponding variables in $\mathbf{D}$. We then note that for β and β^{-1} in $\mathbf{PN}(\Sigma)$, the proof net $\beta^{-1}(\beta(N)) : A \otimes B \to C$ is equal to N by the proof net equations, hence F will send it to the morphism $F(N) : F(A) \to F(B)$. The case for the converse composition and for γ with γ^{-1} is similar.
2. $j = F \circ i$. We note that i_0 is the identity and F on objects is j_0, hence $j_0 = F \circ i_0$. On morphism variables $f : A \to B$, we note that these simply are encoded in $\mathbf{PN}(\Sigma)$ as a box labelled with f with one ingoing arrow labelled with A and an outgoing arrow labelled with B by i. Then, F will send this net to the morphism $j_1(f) : j_0(A) \to j_0(B)$, and so we have that $j_1 = F \circ i_1$.
3. F is unique. Let $G : \mathbf{PN}(\Sigma) \to \mathbf{D}$ be a biclosed magmatic functor such that $j = G \circ i$. As i_0 is the identity, we must have that (on objects) $G = j_0 = F$. On morphism variables, note that S and i_1 are inverse as i_1 turns a morphism variable $f : A \to B$ into its graphical version, whereas S recovers the morphism itself. Hence, we can state that

$$F = F \circ i_1 \circ S = G \circ i_1 \circ S = G$$

By showing the freeness of the proof net category, we have shown that any equation of a biclosed magmatic category will hold if and only if it holds in the proof net category. In other words, this proof net category allows us to reason about biclosed magmatic categories graphically in a coherent way. One may consider the relevance of biclosed magmatic categories in particular and wonder whether the proof net category is more general. In the next section we highlight two extensions of the graphical language, one of which is shown to easily lead to a coherent graphical language for biclosed monoidal categories, the kind of categories that the above-mentioned clasp language was supposed to capture.

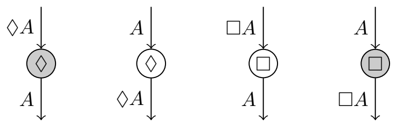
5 Extensions

Associativity. The graphical language we have considered so far is inspired by proof nets for the nonassociative Lambek calculus. Of course, a logical step is to consider associativity as well, in order to coherently capture monoidal closed categories, as the language of Baez and Stay tries to do. A simple solution is to add the associators as two hardcoded diagrams:

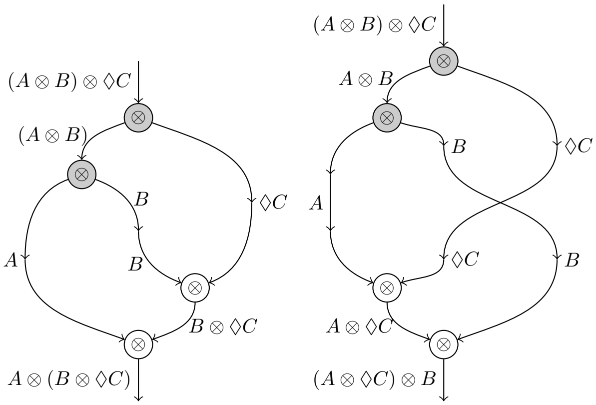


If we define the proof nets using correctness criteria, the so-called *operator balance* criterium would fail. However, when we stick to the inductive proof net definition, we can easily verify that the resulting proof net category is in fact monoidal biclosed (soundness), which we leave as an exercise to the reader. The completeness part requires a bit more thought, but basically amounts to adapting the sequentialisation such that every time a destructor link for $\otimes$ is rewritten, one composes with the associator α when it happens that the domain of f is not compatible with the domain that is obtained when A and B are conjoined into $A \otimes B$. Once we do this, one can show freeness and thus coherence of the graphical language.

Modalities. So far we have argued for modularity in defining diagrammatic calculi. Much in the spirit of Lambek's deductive systems view [11], and the development of proof nets for linguistic analysis [15], the language we developed so far enjoys a modular approach by adding structural properties as diagrams that satisfy coherence under the equations imposed on those diagrams. We wish to go a bit further in this approach by arguing that modalities as used for linguistic purposes [14] can also be incorporated in our proof net language. In a nutshell, one adds two unary connectives to the nonassociative Lambek calculus that exhibit residuating behavior. Categorically speaking, this corresponds to covariant adjunction. Then, one adds structural rules governing special behavior of the unary connectives. Again, these are added in the form of extra diagrams, much like the case of associativity. In pictures, we define the links for $\Diamond, \Box$ as follows:



Then, structural rules take the form



6 Conclusion

In this paper, we argued for a modular approach to diagrammatic reasoning in the context of categorical compositional distributional semantics. We developed a graphical language for closed magmatic categories, inspired by proof nets for Lambek calculi. We then showed coherence for this *proof net category* and argued that associativity can be recovered by adding two diagrams to the language, effectively obtaining a graphical language for monoidal closed categories. Finally, we offered some thoughts on adding diagrams that in logical terminology are called unary residuated connectives but categorically can be thought of as pairs of adjoint functors. Working out coherence for these unary modalities, as well as incorporating bialgebras and Frobenius algebras graphically, constitutes future work.

Acknowledgements. The author is greatly indebted for many fruitful discussions with Michael Moortgat during the writing of the MSc thesis on which this paper is largely based. Also, a thanks goes out to Mehrnoosh Sadrzadeh for discussions culminating in the existence of this paper. A thanks as well to John Baez and Peter Selinger for giving some advice a long time ago on the topic of diagrammatic reasoning. Finally, the author would like to thank the two anonymous referees of this paper. The author was supported by a Queen Mary Principal's Research Studentship during the writing of this paper.

References

1. Baez, J., Stay, M.: Physics, topology, logic and computation: a rosetta stone. In: Coecke, B. (ed.) New Structures for Physics, pp. 95–172. Springer, Heidelberg (2010). doi:10.1007/978-3-642-12821-9_2
2. Coecke, B., Grefenstette, E., Sadrzadeh, M.: Lambek vs. Lambek: functorial vector space semantics and string diagrams for lambek calculus. Ann. Pure Appl. log. **164**(11), 1079–1100 (2013)

3. Coecke, B., Sadrzadeh, M., Clark, S.: Mathematical foundations for a compositional distributional model of meaning. arXiv preprint arXiv:1003.4394 (2010)
4. Freyd, P., Yetter, D.N.: Coherence theorems via knot theory. J. Pure Appl. Algebr. **78**(1), 49–76 (1992)
5. Girard, J.Y.: Linear logic. Theor. Comput. Sci. **50**(1), 1–101 (1987)
6. Joyal, A., Street, R.: The geometry of tensor calculus, I. Adv. Math. **88**(1), 55–112 (1991)
7. Joyal, A., Street, R.: Braided tensor categories. Adv. Math. **102**(1), 20–78 (1993)
8. Kelly, G.M., Laplaza, M.L.: Coherence for compact closed categories. J. Pure Appl. Algebr. **19**, 193–213 (1980)
9. Lambek, J.: The mathematics of sentence structure. Am. Math. Mon. **65**(3), 154–170 (1958)
10. Lambek, J.: On the calculus of syntactic types. Struct. Lang. Math. Asp. **166**, C178 (1961)
11. Lambek, J.: Deductive systems and categories. Theory Comput. Syst. **2**(4), 287–318 (1968)
12. Lambek, J.: Type grammar revisited. In: Lecomte, A., Lamarche, F., Perrier, G. (eds.) LACL 1997. LNCS, vol. 1582, pp. 1–27. Springer, Heidelberg (1999). doi:10.1007/3-540-48975-4_1
13. Lambek, J., Scott, P.J.: Introduction to Higher-Order Categorical Logic, vol. 7. Cambridge University Press, Cambridge (1988)
14. Moortgat, M.: Multimodal linguistic inference. J. Log. Lang. Inf. **5**(3–4), 349–385 (1996)
15. Moot, R.: Proof Nets for Linguistic Analysis. Ph.D. thesis, Utrecht University (2002)
16. Penrose, R.: Applications of negative dimensional tensors. In: Combinatorial Mathematics and its Applications, vol. 1, pp. 221–244 (1971)
17. Selinger, P.: A survey of graphical languages for monoidal categories. In: Coecke, B. (ed.) New Structures for Physics, pp. 289–355. Springer, Heidelberg (2010)
18. Wijnholds, G.J.: Categorical foundations for extended compositional distributional models of meaning. MSc. thesis, University of Amsterdam (2014)

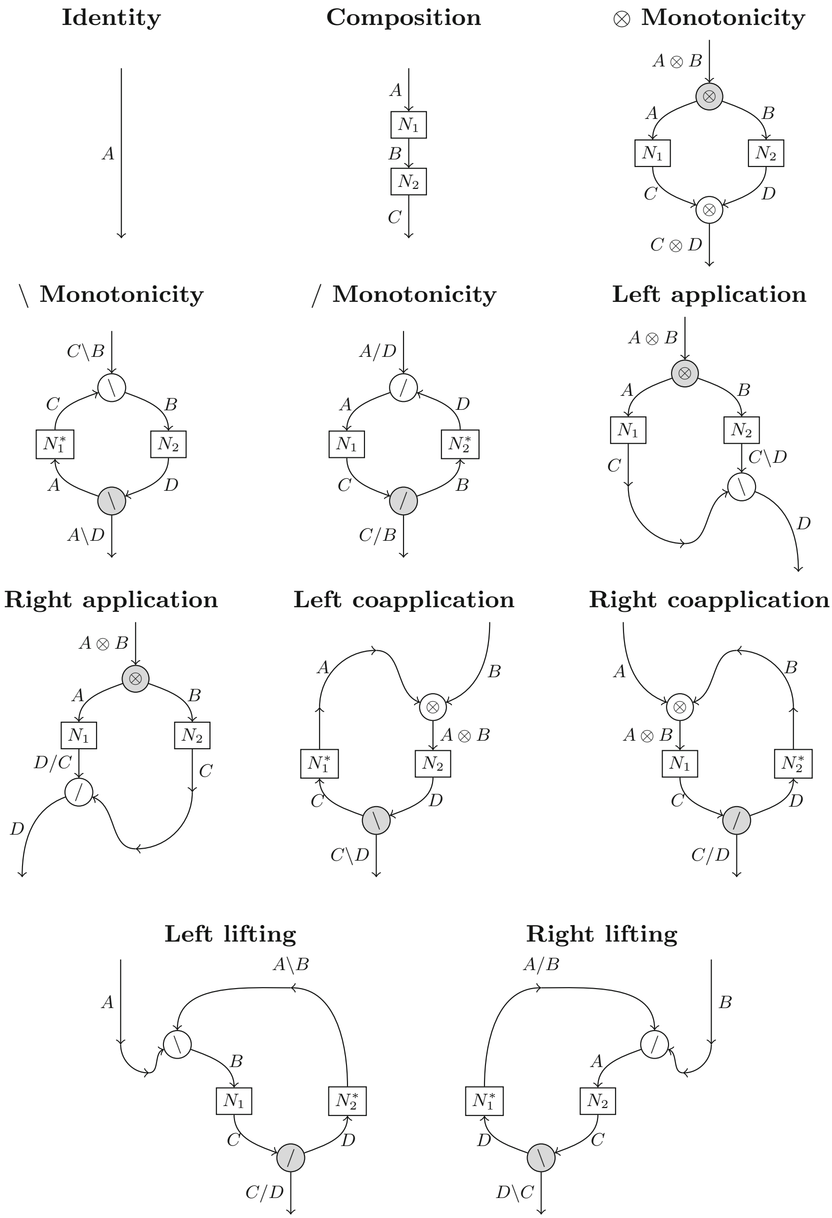


Fig. 1. Inductive definition of proof nets

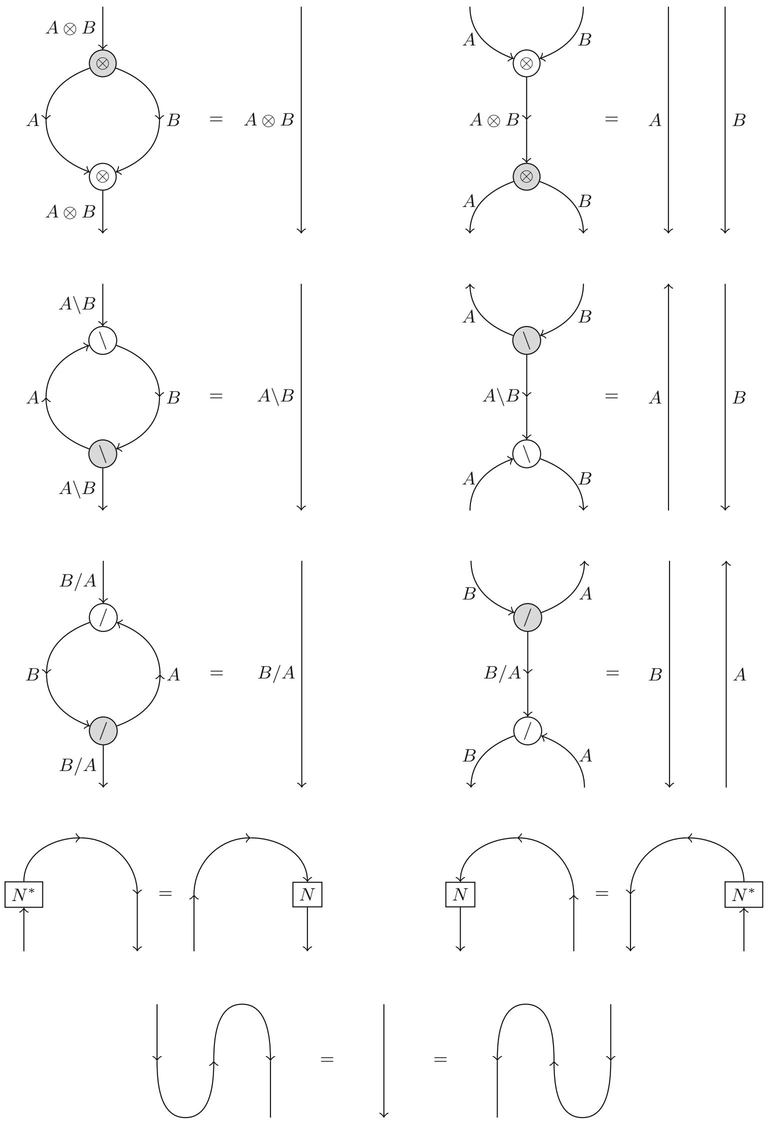


Fig. 2. Equations on proof nets

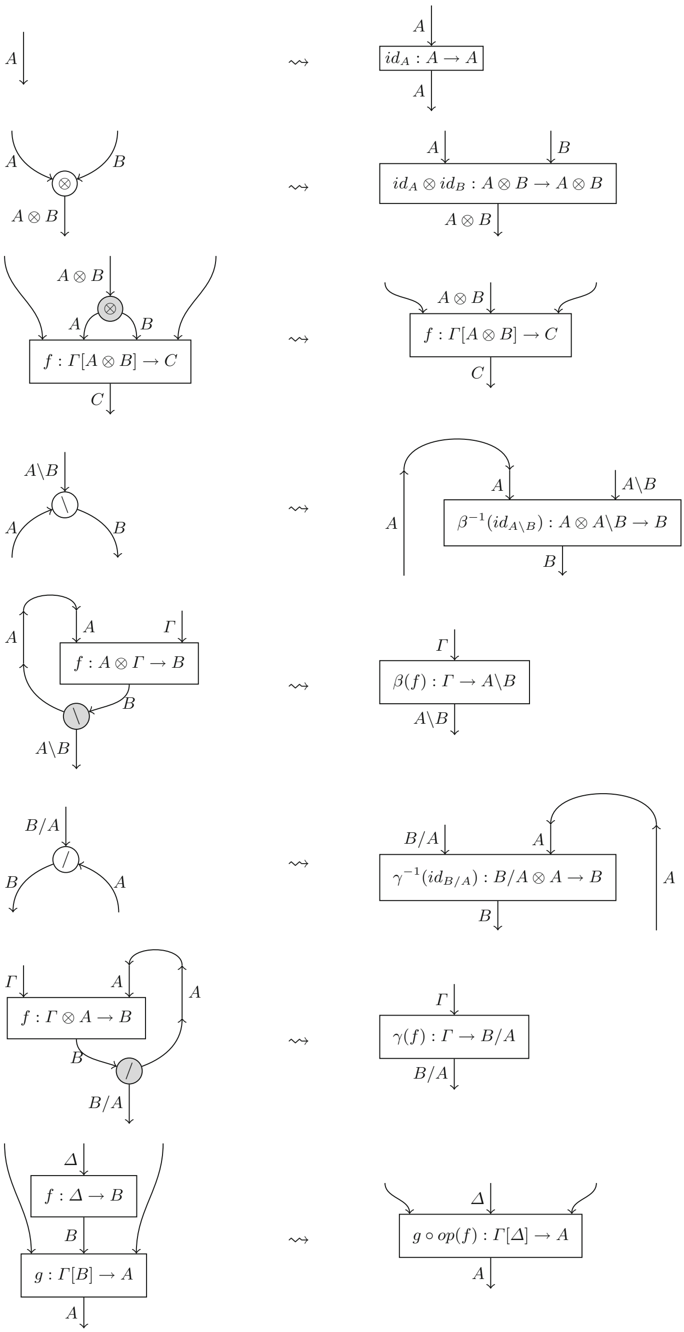


Fig. 3. Translating proof nets into categorical morphisms