Chris Haydnk 8/30

Lecture 1, Exercise A

1a. i) Well-defined

Not injective:

$$1^2 = (-1)^2 = 1$$

Not surjective:

$-1 \in \mathbb{R}$, but $\sqrt{-1} = i \notin \mathbb{R}$

So there is no $x \in \mathbb{R}$ s.t. $x^2 = -1$

ii) Well-defined

Injective:

$$x_1^2 = x_2^2 \quad (x_1, x_2 \in A)$$

$$\Rightarrow \pm x_1 = \pm x_2$$

Since $A = \mathbb{R}_{\geq 0}$, we can ignore negative.

Hence, $x_1 = x_2$

Not surjective:

No $x \in A$ s.t. $x^2 = y$
with $y \in B$ and $y < 0$

iii) Well-defined

Injective:

Same reasoning as ii)

Surjective:

Let $y \in B$.

Then $\sqrt{y} \in \mathbb{R}_{>0}$ $(= A)$

and $(\sqrt{y})^2 = y$

iv) Well-defined

Injective:

Similar to ii)

Not surjective:

$\nexists x \in A$ s.t. $x^2 = 3$

6. i) Not well-defined:

$\sqrt{-1} = ?$ $(\in \mathbb{R})$

ii) Not well-defined:

Same as above

iii) Not well-defined:

$\sqrt{1} = \pm 1$ (both in B)

iv) Not well-defined:

$\sqrt{3} = ?$ $(\in \mathbb{Z}_{\geq 0})$

v) Well-defined

Injective:

Let $\sqrt{x_1} = \sqrt{x_2}$ $(x_1, x_2 \in A)$

Then,

$$(\sqrt{x_1})^2 = (\sqrt{x_2})^2$$

and,

$$x_1 = x_2$$

Surjective:

Let $y \in B$.

Then $y \in \mathbb{R}_{>0}$

Hence $y^2 \in \mathbb{R}_{>0}$

Thus, $y^2 \in A$ and

$\sqrt{y^2} = y$

c. i) Not well-defined:

$\frac{1}{\frac{0}{1}} = ?$

ii) Not well-defined:

$\frac{1}{2} = ?$ in $\mathbb{Z}$

iii) Well-defined

Injective:

Let $\frac{x_1}{2} = \frac{x_2}{2}$ $(x_1, x_2 \in A)$

Then, $2\left(\frac{x_1}{2}\right) = 2\left(\frac{x_2}{2}\right)$

$\Rightarrow x_1 = x_2$

Not surjective:

$\frac{?}{2} = 3$ (in $\mathbb{Z}$)

iv) Well-defined

Injective:

Let $\frac{x_1}{1} = \frac{x_2}{1}$ $(x_1, x_2 \in A)$

Then $x_1 = x_2$

Let $\frac{x_1}{-1} = \frac{x_2}{-1}$

Then $-x_1 = -x_2$

$\Rightarrow x_1 = x_2$

Surjective:

Let $x \in B$.

We have $x \in \mathbb{Z}$ and $1 \in \{1, -1\}$.

Hence, $(x, 1) \in A$ and

$f(x, 1) = x$

v) Well-defined

Not injective:

$2 = \frac{2}{1} = \frac{4}{2}$

Surjective:

Let $y \in B$

Then $y = \frac{y}{1}$.

Note $y \in \mathbb{R}$, $1 \in \mathbb{R}_{>0}$

Hence $(y, 1) \in A$

and $f(y, 1) = y$

2. Suppose f is invertible.

Then $\forall b \in B$, $f^{-1}(b)$ maps to exactly one element in A.

That is, $\exists$ a unique $a \in A$ s.t.

$f^{-1}(b) = a$

Note that $f^{-1}(b) = \{a \in A \mid f(a) = b\}$

Hence, since $f^{-1}(b) = a$, we have

$f(a) = b \quad \forall b \in B$. Thus, f is surjective

Moreover, since a is the only element in the set $f^{-1}(b)$, a is the only element in A s.t.

$$f(a) = b.$$

Since b was arbitrary, this holds for every $b \in B$.

Hence f is injective as well and thus bijective

Now suppose f is bijective.

Then $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

That is, for each $y \in B$, there is at most one element $x \in A$

$$f(x) = y$$

Moreover, since f is surjective, there is actually <u>exactly</u> one element $x \in A$ s.t.

$$f(x) = y$$

That is, $\forall y \in B$,

$$f^{-1}(y) = x$$

ie. each fiber of over y for any $y \in B$ is a singleton

Thus, f is invertible

3a. Not reflexive

Symmetric

Not transitive:

$5-2=3$, $2-1=1$,

but $5-1=4$

Not an equivalence relation

b. Not reflexive

Symmetric

Transitive

Not an equivalence relation

c. Reflexive:

$\forall a \in A$, $f(a) = f(a)$ so

$a \sim a$

Symmetric:

$a \sim b \Rightarrow f(a) = f(b)$

$\Rightarrow f(b) = f(a)$

$\Rightarrow b \sim a$

Transitive:

$a \sim b \Rightarrow f(a) = f(b)$

$b \sim c \Rightarrow f(b) = f(c)$

So we have,

$f(a) = f(b) = f(c)$

Hence $f(a) = f(c)$ and

$a \sim c$

Equivalence relation ✓

Each equivalence class represents the fiber of f over a point $x \in \mathbb{Z}$

d. Reflexive:

$$f(1) = f(1) \quad \forall f$$

Symmetric:

$$f(1) = g(1) \Rightarrow g(1) = f(1)$$

Transitive:

$$f(1) = g(1) \text{ and } g(1) = h(1)$$
$$\Rightarrow f(1) = h(1)$$

Equivalence relation ✓

An equivalence class C_n is given by

$$\{ f : \mathbb{Z} \to \mathbb{Z} \mid f(1) = n \}$$

These exist for every $n \in \mathbb{Z}$, hence there is a bijection between the equivalence classes and integers.

4. Modulo 4:

$+$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$
$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{0}$
$\overline{2}$	$\overline{2}$	$\overline{3}$	$\overline{0}$	$\overline{1}$
$\overline{3}$	$\overline{3}$	$\overline{0}$	$\overline{1}$	$\overline{2}$

$*$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$
$\overline{2}$	$\overline{0}$	$\overline{2}$	$\overline{0}$	$\overline{2}$
$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{2}$	$\overline{1}$

Modulo 5:

$+$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{0}$
$\overline{2}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{0}$	$\overline{1}$
$\overline{3}$	$\overline{3}$	$\overline{4}$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{4}$	$\overline{4}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$

$*$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{2}$	$\overline{0}$	$\overline{2}$	$\overline{4}$	$\overline{1}$	$\overline{3}$
$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{1}$	$\overline{4}$	$\overline{2}$
$\overline{4}$	$\overline{0}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$

5a. $\gcd(2^3, 2^5) = 2^3 = 8$

$\operatorname{lcm}(2^3, 2^5) = 2^5 = 32$

$8 \cdot 32 = 256 = 2^8 = 2^5 \cdot 2^3$

b. $\gcd(2 \cdot 3, -2 \cdot 5) = 2$

$\operatorname{lcm}(2 \cdot 3, -2 \cdot 5) = 30$

$30 \cdot 2 = 60 = -(2 \cdot 3 \cdot -2 \cdot 5)$

c. $\gcd(-2 \cdot 3, -5 \cdot 7) = 1$

$\operatorname{lcm}(-2 \cdot 3, -5 \cdot 7) = 210$

$1 \cdot 210 = 210 = -2 \cdot 3 \cdot -5 \cdot 7$