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Lecture 19, Ex. A

1a. $a + b\sqrt{D}$, $c + d\sqrt{D} \in \mathbb{Q}(\sqrt{D})$

$a + b\sqrt{D} - (c + d\sqrt{D}) = a + b\sqrt{D} - c - d\sqrt{D}$

$= (a-c) + (b-d)\sqrt{D}$

$\in \mathbb{Q}(\sqrt{D})$

$(a + b\sqrt{D})(c + d\sqrt{D}) = ac + ad\sqrt{D} + bc\sqrt{D} + bdD$

$= (ac + bdD) + (ad + bc)\sqrt{D}$

$\in \mathbb{Q}(\sqrt{D})$

b. Let $D = 2$ and $E = 8$

$a + b\sqrt{2} = a + \frac{b}{2}\sqrt{8} \in \mathbb{Q}(\sqrt{8})$

$a + b\sqrt{8} = a + 2b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$

$\mathbb{Q}(\sqrt{8}) = \mathbb{Q}(\sqrt{2})$

$\alpha = a + b\sqrt{D}$ $\quad \beta = c + d\sqrt{D}$

$\alpha\beta = (ac + bdD) + (bc + ad)\sqrt{D}$

c. $N(\alpha\beta) = (ac + bdD)^2 - D(bc + ad)^2$

$= a^2c^2 + 2acbdD + b^2d^2D^2 - Db^2c^2 - 2Dbcad - Da^2d^2$

$= a^2c^2 + D(\cancel{2acbd} + b^2d^2D - b^2c^2 - \cancel{2bcad} - a^2d^2)$

$= a^2c^2 + D^2a^2d^2 - Db^2c^2 + D^2b^2d^2$

$(a^2 - Db^2)(c^2 - Dd^2) = a^2c^2 - Da^2d^2 - Db^2c^2 + D^2b^2d^2$

$\shortparallel$

$N(\alpha)N(\beta)$

So $N(\alpha\beta) = N(\alpha)N(\beta)$

2a. $D \equiv_4 0 \Rightarrow 2^2 = 4 \mid D$. Hence D not square free

bi. $\omega = \sqrt{D} = \sqrt{-1} = i$

$\mathbb{Z}[\omega] = \{a + bi \mid a, b \in \mathbb{Z}\}$

ii. Need $N(\alpha) = \pm 1$

$a^2 - i^2b^2 = \pm 1$

$\Rightarrow a^2 + b^2 = \pm 1$

So we have,

$a=0,\ b=\pm 1$

OR $a=\pm 1,\ b>0$

$(2[w])^{\times} = \{1, -1, i, -i\}$

(i) $w = \frac{1+\sqrt{3}}{2} = \frac{1+\sqrt{3}}{2}$

$= \frac{1+i\sqrt{3}}{2}$

ii. $\alpha = a + b\left(\frac{1+i\sqrt{3}}{2}\right)$

$\beta = c + d\left(\frac{1+i\sqrt{3}}{2}\right)$

$\alpha\beta = ac + bd\left(\frac{1+i\sqrt{3}}{2}\right)^2 +$

$(ad + cb)\left(\frac{1+i\sqrt{3}}{2}\right)$

$= ac + bd\left(\frac{1+i\sqrt{3}}{2}\right) +$

$(ad + cb)\left(\frac{1+i\sqrt{3}}{2}\right)$

iii. Note $a + b\left(\frac{1+\sqrt{-3}}{2}\right)$

$= \frac{1}{2}((2a+b) + b\sqrt{-3})$

$N(a+bw) = a^2 - w^2b^2$

$= \frac{1}{4}((2a+b)^2 + 3b^2)$

iv. $N(a+bw) = \pm 1$

$\Rightarrow (2a+b^2) + 3b^2 = \pm 4$

d. $D \equiv 2, 3 \pmod 4$:

$N(a+bw) = N(a+b\sqrt{D})$

$= a^2 - b^2D \in \mathbb{Z}$