Chris Haydak 9/8

Lecture 4, Exercise B

1a. i) H can't be empty because a group must contain an identity element

ii) $y^{-1} \in H$ because, since H is a group, every element has an inverse

$xy^{-1} \in H$ because H must be closed under its operation in order to be a group.

b. i) If $x = y$, then

$$xy^{-1} = yy^{-1}$$
$$= 1 \in H$$

ii) Suppose $x = 1$. Then $\forall y \in H$, we have

$$xy^{-1} \in H$$
$$\Longrightarrow 1 \cdot y^{-1} \in H$$
$$\Longrightarrow y^{-1} \in H$$

iii) We know $\forall xy \in H$,

$$xy^{-1} \in H$$

Now take $x, y^{-1} \in H$. Then

$$x(y^{-1})^{-1} \in H$$

Note that $(y^{-1})^{-1} = y$ since

$$yy^{-1} = y^{-1}y = 1$$

So we have

$$xy \in H$$

iv) If H is the empty set then the statement is vacuously true

We know the empty set isn't a group because it has no identity.

2a. Let $A, B \in SL_n(F)$

Then $\det(A) \cdot \det(B) = 1$

$= \det(AB)$

So $AB \in SL_n(F)$

We know $\det(I) = 1$, so

$I \in SL_n(F)$

For $A \in SL_n(F)$, we have

$\det(A) = 1$

From linear algebra, we know

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$= 1$$

So $A^{-1} \in SL_n(F)$

6. Consider the set $SL_2(\mathbb{F}_2)$. That is:

$$SL_2(\mathbb{F}_2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \begin{array}{l} a,b,c,d \in \mathbb{F}_2 = \{\bar{0},\bar{1}\} \\ ad-bc = \bar{1} \end{array} \right\}$$

We know $\det(I) = \bar{1}$, so

$$I \in SL_2(\mathbb{F}_2)$$

Furthermore, based on modular arithmetic, we know that for $ad-bc = \bar{1}$, we need

$$ad = \bar{1} \quad \text{and} \quad bc = \bar{0}$$

Hence the other matrices in $SL_2(\mathbb{F}_2)$ are:

$$\begin{bmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{1} \end{bmatrix} \quad \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{1} \end{bmatrix}$$

So there are 3 matrices in $SL_2(\mathbb{F}_2)$

3. $H_1 = \{1\}$ is the first subgroup.

Let H_2 be subgroup generated by r.

Then we have

$$H_2 = \{1, r, r^2\}$$

Note that $r^2 = r^{-1} \in H_2$

and $r = (r^2)^{-1} \in H$

Generated by s:

$$H_3 = \{1, s\}$$

$s = s^{-1}$ since $s^2 = 1$

Generated by sr:

$$H_4 = \{1, sr\}$$

$sr = (sr)^{-1}$ since $(sr)^2 = 1$

Generated by sr^2:

$$H_5 = \{1, sr^2\}$$

$$sr^2 = (sr^2)^{-1} \quad \text{since } (sr^2)^2 = 1$$

And let $H_6 = D_6$ since any group is a subgroup of itself