ID: **1651**
Math A4900
Proof portfolio draft, Round 1
October 4, 2020

Problem:	**1B**
No. stars:	**1**

Statement: Let G be a group. Prove that if $|x| \leqslant 2$ for all $x \in G$ then G is abelian.

use "this implies" every

↳ instead of $\Rightarrow$

Proof. Suppose for every $x \in G$, $|x| \leqslant 2$, which implies $n \leqslant 2$ for $x^n = 1$. This implies that for all $x \in G$, $x^2 = xx = 1 \implies x = x^{-1}$. This means that for $x, y \in G$, $xy = (xy)^{-1} = y^{-1}x^{-1} = yx = 1$. Not only that, but since the product $xy \in G$, we can also say $(xy)^2 = xyxy = 1$. For the inverse, $(xy)^{-1} = y^{-1}x^{-1} = yx = 1$ by the inverse axiom. Therefore, the group is abelian since all conditions hold. ☐

↳ "Moreover"

In line 5, you may want to write something like "which implies the smallest $n \in \mathbb{N}$ for which $x^n = 1$ is $n = 1$ or $n = 2$." The way it is written now implies $x^n \neq 1$ for $n > 2$, but if $|x| = 2$, then $x^4 = 1$ as well.

Overall nice proof though!

	Points Possible
complete	0 1 2 3 4 5
mathematically valid	0 1 2 3 4 5
readable/fluent	0 1 2 3 4 5
Total:	(out of 15)

Problem:	1C
No. stars:	2

Statement: Let G be a group and let $x \in G$. If $|x| = n < \infty$, prove that the elements $1, x, x^2, \ldots, x^{n-1}$ are all distinct. Deduce that $|x| = |\langle x \rangle|$

$a,b \in \{1,2,\dots,n-1\}$

Proof. Proof. Suppose $|x| = n < \infty \implies x^n = 1$. Suppose on the contrary that the elements $e, x, x^2, \ldots, x^{n-1}$ are not all distinct. Thus, there exists an $a, b \in G$ such that $x^a = x^b$. Through operations, we can see that $x^a = x^b \implies x^a x^{-b} = x^b x^{-b} \implies x^{a-b} = 1$. If this is true, $a - b = n$ but this is contradiction because there does not exist two integers such that $a - b = n$. The max value $(a - b)$ can be is $(n - 2)$ which does not equal n. Therefore, the elements are distinct by contradiction. If $G = \langle x \rangle$ where $|x| = n < \infty$, then G has n distinct elements including 1. Hence, $|x| = |G| = |\langle x \rangle| = n$. □

two integers a,b such that $1 \le a,b \le n-1$ and $a-b=n$

Try to write out statements rather than using "$\implies$". Also note that we need to assert $a \neq b$. Otherwise $x^{a-b} = x^0 = 1$ could be true without having $a-b=n$.

	Points Possible					
complete	0	1	2	3	4	5
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readable/fluent	0	1	2	3	4	5
Total:	(out of 15)					

Statement: Prove that if H and K are subgroups of G, then so is $H \cap K$. On the other hand, prove $H \cup K$ is a subgroup if and only if $H \subseteq K$ or $K \subseteq H$.

Problem:	**2A**
No. stars:	**2**

Proof. Suppose H and K are subgroups of G. This means $H \neq \varnothing$, $K \neq \varnothing$, $1 \in H$ and $1 \in K$. Therefore, $1 \in H \cap K \implies (H \cap K) \neq \varnothing$. If 1 is the only element in $H \cap K$, then already we can say that $H \cap K$ is a trivial subgroup of G. Since H and K are subgroups of G, we can say for all $a, b \in H$ and $x, y \in K$, $ab \in H$ and $xy \in K$. Thus, $ab \in (H \cap K)$ or $xy \in (H \cap K)$. If $z \in (H \cap K)$, then $z, z^{-1} \in H$ and K since both are subgroups, meaning $z^{-1} \in (H \cap K)$. With both subgroups closed under the same operations as G, we can conclude that $(H \cap K)$ is a subgroup of G.

One the other hand, we will now prove that the subgroups $H \subseteq K$ or $K \subseteq H \implies H \cup K$ is a subgroup. Without loss of generality, assume $H \subseteq K$. This implies that $H \cup K = K$ and since K is a subgroup of G, $H \cup K$ is a subgroup of G. Now, if $H \cup K$ is a subgroup of G. ← Not complete sentence

Now we need to prove $H \cup K \implies H \subseteq K$ or $K \subseteq H$ and we can do this by proving the contrapositive statement. We will prove $H \nsubseteq K$ *and* $K \nsubseteq H \implies H \cup K$ is not a subgroup. Suppose $x \in H$, $x \notin K$ and $y \in K$, $y \notin H$. This implies that the union contains x, y. For the sake of contradiction, suppose that $H \cup K$ was a group, implying $xy \in H \cup K$. Therefore, $xy \in H$ or K. Since H is subgroup of G, $xy \in H \implies x^{-1}xy \in H \implies y \in H$, a contradiction. We arrive at a similar contradiction for K, $x \in K$. Hence, $H \cup K$ is not a subgroup of G. By proving the contrapositve statement true, we have proved that $(H \subseteq K$ or $K \subseteq H) \implies H \cup K$ is a subgroup. ← Nice use of contrapositive! $\square$

Lines 3-4 are not necessarily true. We could have $a,b \in H$, $x,y \in K$ such that $ab, xy \notin (H \cap K)$.

You also need to show that for $x,y \in (H \cap K)$, we have $xy^{-1} \in (H \cap K)$

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In Line 10, need to prove $H \cup K \leq G \Rightarrow H \subseteq K$ or $K \subseteq H$.

Statement: Let G be a group. Show that the map

$$\varphi : G \to G \qquad \text{defined by} \quad \varphi : g \mapsto g^{-1}$$

is a homomorphism if and only if G is abelian. Now, verify that

$$\psi : D_{2n} \to D_{2n} \text{ defined by } \quad \psi(s) = s^{-1} \text{ and } \psi(r) = r^{-1}$$

extends to a well-defined homomorphism, and explain why this does not contradict the first statement.

Problem:	2B
No. stars:	2

Proof. Suppose $\varphi : G \to G$ definied by $\varphi : g \mapsto g^{-1}$ is a homomorphism. Therefore, $\varphi(gh) = \varphi(g) * \varphi(h) = g^{-1}h^{-1}$ and $\varphi(gh) = (gh)^{-1} = h^{-1}g^{-1} \implies h^{-1}g^{-1} = g^{-1}h^{-1}$. Through operations, we get that this implies that $hg = gh$. Since its commutative, the group G is abelian. Now, suppose group G is abelian. Then $\varphi(gh) = (gh)^{-1} = h^{-1}g^{-1} = g^{-1}h^{-1} = \varphi(g)\varphi(h)$, making the map a homomorphism.

may want to show these operations

← Nice!

Now, lets verifiy that $\psi : D_{2n} \to D_{2n}$ defined by $\psi(s) = s^{-1}$ and $\psi(r) = r^{-1}$ extends to a well defined homomorphism.

Since $r^i = r^{n+i}$, $s^j = s^{-j}$, $i, j \neq 0$

$\psi(r^i) = r^{-i} = r^{n-i}$, $\psi(r^{n+i}) = r^{-n-i} = r^{n-i}$

$\psi(s^j) = s^{-j} = sj = \psi(s^{-j})$

$\psi(r^i s) = (r^i s)^{-1} = (sr^{-i}) = \psi(s)\psi(r^i)$

$\psi(sr^i) = (sr^i)^{-1} = (r^{-i}s) = \psi(r^i)\psi(s)$

We may need to show that $\psi(s)\psi(r^i) = \psi(r^i)\psi(s)$ and $\psi(r^i)\psi(s) = \psi(s)\psi(r^i)$ via relations in D_{2n}

The homomorphism does not contradict the first statement because while D_{2n} is not abelian, both $\psi(s)\psi(r^i), \psi(r^i)\psi(s) \in D_{2n}$ image from the pre-image $r^i s, sr^i \in D_{2n}$ with a one-to-one relationship. Abelian groups in the original statement satisfies this requirement since the group is commutative, but the D_{2n} mapping satifies the well-defined homomorphism requirement due to unique non-communative characteristic of $r^i s$, sr^i and thier inverses. ∎

↑ May want to rephrase this paragraph. A bit confusing to follow

1

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Total:	(out of 15)

Statement: Let G act on a set A. Prove that the relation $\sim$ on A defined by

$$a \sim b \quad \text{when} \quad a = g \cdot b \text{ for some } g \in G$$

is an equivalence relation.

Problem:	3B
No. stars:	1

Proof. To prove ~~their~~ there is an equivalance relation, we need to prove reflextivity, symmetry, and transitivity.

For reflextivity $(a \sim a)$, $a = 1 * a = a$ since $1 \in G$. $1 * a = a$ implies $(a \sim a)$

For symmetry $(a \sim b \iff b \sim a)$, suppose for $a, b \in A$, there exists an $g \in G$ such that $a = gb$. Using group operations, since $g^{-1} \in G, a = gb \implies g^{-1}a = g^{-1}gb \implies b = g^{-1}a$. Hence $a \sim b \iff b \sim a$. Nice!

For transitivity $(a \sim b, b \sim c \implies a \sim c)$, if $a = gb, b = hc$ for some $g, h \in G$, then $a = g(hc) = (gh)c$ using substitution. Since $gh \in G$, then $a \sim c$. Nice! □

Nice work! Try to use fewer $\Rightarrow$, $\Leftrightarrow$ in your proof.

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