HOMEWORK 3
MATH A4900/44900
DUE: 9/25/2020

1. **Group actions**

   (a) For some fixed $g \in G$, prove that conjugation by g (i.e. the map $G \to G$ defined by $a \mapsto gag^{-1}$) is an automorphism of G. Deduce that a and gag^{-1} have the same order (by last week's work), and for any non-empty $S \subseteq G$, the map
   $$S \to gSg^{-1} \quad \text{defined by} \quad s \mapsto gsg^{-1}$$
   is also a bijection, so that $|gSg^{-1}| = |S|$.
   [Recall, even if A and/or B is infinite, we say $|A| = |B|$ exactly when there is a bijection $A \leftrightarrow B$]

   (b) Let A be a non-empty set and let $0 < k \leqslant |A|$. Check that the action of the symmetric group S_A on the set of size k subsets of A by
   $$\sigma \cdot \{a_1, \ldots, a_k\} = \{\sigma(a_1), \ldots, \sigma(a_k)\}$$
   satisfies the axioms of group actions. [Similar to the action of D_{2n} on sets from lecture.]

   (c) Let G act on a set A. Prove that the relation $\sim$ on A defined by
   $$a \sim b \quad \text{if and only if} \quad a = g \cdot b \text{ for some } g \in G$$
   is an equivalence relation.

   Note: the equivalence classes with respect to this relation are called **orbits**.

   (d) Describe the orbits of the action of S_4 on 2-element subsets of $\{1, 2, 3, 4\}$ (as in problem **??**).

2. **Cyclic groups**

   (a) If x is an element of a finite group G and $|x| = |G|$, prove that $G = \langle x \rangle$. Give an explicit example to show $|x| = |G|$ does not imply $G = \langle x \rangle$ if G is an infinite group.

   (b) Write $Z_{63} = \langle x \rangle$. For which integers a does the map ψ_a defined by
   $$\psi_a : \bar{1} \to x^a$$
   extend to a *well defined homomorphism* from $\mathbb{Z}/147\mathbb{Z}$ to Z_{63}? Can ψ_a ever be a surjective homomorphism? [Take care to remember that the binary operation on the left is $+$ and the binary operation on the right is $\times$: if the image of $\bar{1}$ is x^a, then the image of $\bar{1} + \bar{1} + \cdots + \bar{1} = \ell\bar{1}$ is $(x^a)^\ell$.]

   (c) For $a \in \mathbb{Z}$, define
   $$\sigma_a : Z_n \to Z_n \quad \text{by} \quad \sigma_a(x) = x^a \text{ for all } x \in Z_n.$$
   Show that σ_a is an automorphism of Z_n if and only if $(a, n) = 1$.

   (d) Under what circumstances does there exist a non-trivial homomorphism $\varphi : Z_n \to G$?
   [Note: φ need not be injective or surjective; just well-defined, and not the map $g \mapsto 1$ for all g.]

   (e) For which $n \in \mathbb{Z}_{\geqslant 1}$ is $(\mathbb{Z}/2^n\mathbb{Z})^\times$ cyclic? [Hint: Try to find more than one subgroup of order 2. Why would this prove $(\mathbb{Z}/2^n\mathbb{Z})^\times$ is *not* cyclic? Start by doing some examples.]

   (f) Prove that $\mathbb{Q} \times \mathbb{Q}$ is not cyclic.