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Lecture 4, Exercise A

1a. $\det(I) = \bar{1} \cdot \bar{1} - \bar{0} \cdot \bar{0}$

$= \bar{1} \neq \bar{0}$

b. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot I$

$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{bmatrix}$

$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$I \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

c.i) $\det(A) = \bar{1} \cdot \bar{1} - \bar{1} \cdot \bar{0}$

$= \bar{1}$

$\det(B) = \bar{1} \cdot \bar{1} - \bar{1} \cdot \bar{0}$

$= \bar{1}$

All of the entries of A and B are in $\mathbb{F}_2$ and both determinants are non-zero so

$$A, B \in GL_2(\mathbb{F}_2)$$

ii) $$AB = \begin{bmatrix} \bar{1} & \bar{1} \\ 0 & \bar{1} \end{bmatrix} \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{1} \end{bmatrix}$$

$$= \begin{bmatrix} \bar{0} & \bar{1} \\ \bar{1} & \bar{1} \end{bmatrix}$$

$$BA = \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{1} \end{bmatrix} \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{1} \end{bmatrix}$$

$$= \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{0} \end{bmatrix}$$

So $AB \neq BA$ and $GL_2(\mathbb{F}_2)$ is not Abelian

iii) $$A^2 = \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{1} \end{bmatrix} \cdot \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{1} \end{bmatrix}$$

$$= \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{bmatrix} = I$$

$|A| = 2$

iv) $A^{-1} = A$ because $AA = I$

2. Let $a, d = 2$ and $c, b = 1$

Then we have

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

and

$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 \neq 0$$

Now consider

$$\begin{bmatrix} \bar{2} & \bar{1} \\ \bar{1} & \bar{2} \end{bmatrix}$$

We have:

$$\begin{vmatrix} \bar{2} & \bar{1} \\ \bar{1} & \bar{2} \end{vmatrix} = \bar{4} - \bar{1}$$

$$= \bar{3} = \bar{0}$$

So

$$\begin{bmatrix} \bar{2} & \bar{1} \\ \bar{1} & \bar{2} \end{bmatrix} \notin GL_2(\mathbb{F}_3)$$

but

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \in GL_2(\mathbb{R})$$