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Homework 7
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1. **Left actions of groups on themselves**

(a) Show that if H has finite index n in G, then there is a normal subgroup $K \trianglelefteq G$ with $K \leqslant H$ and $|G : K| \leqslant n!$.

Proof. Suppose H has finite index n in G. As described in DF, §4.2, p. 119, let us label the left cosets of H with integers $1, 2, \ldots, n$. Thus, we can list the distinct left cosets of H in G as $a_1H, a_2H, \ldots, a_mH$ and for each $g \in G$ the permutation σ_g may be described as a permutation of the indices $1, 2, \ldots, n$ as follows:

$$\sigma_g(i) = j \text{ if and only if } ga_iH = a_jH$$

Hence if we let $\pi_H = \{\sigma_g \mid g \in G\}$. That is, π_H is the permutation representation afforded by the action. Then, by DF, §4.2, Theorem 3, we have that the kernel of the action (i.e. the kernel of π_H) is $\cap_{x \in G} xHx^{-1}$, and $\ker \pi_H$ is the largest normal subgroup of G contained in H. Hence, we know that $\ker \pi_H \trianglelefteq G$ and $\ker \pi_H \leqslant H$. Thus, this normal subgroup exists and we will call it K. Now we need to show that $|G : K| \leqslant n!$.

We know $|G : H| = n$. □

(b) Prove that every non-abelian group of order 6 has a non-normal subgroup of order 2. Use this to classify all groups over order 6 (up to isomorphism). [Hint: What are the possible orders of elements? For the second part, what are the abelian groups of order 6? For a non-abelian order-6 group, produce an injective homomorphism into S_3.]

Proof. Let G be a group such that G is non-abelian and $|G| = 6$. Since $|G| < \infty$, we have that G is a finite group. In addition, note $2 \mid 6$ and 2 is prime. Hence, we can apply Cauchy's Theorem which states that G has an element of order 2. Let us call this element $x \in G$. Then,

$$H = \langle x \rangle = \{e, x\}$$

Note that $e, x \in H, G$, so $H \subset G$ and $H \neq \varnothing$. In addition,

$$\begin{aligned} xe^{-1} &= xe \\ &= x \in H \end{aligned}$$

and,

$$\begin{aligned} ex^{-1} &= ex \\ &= x \in H \end{aligned}$$

Hence, for every $x, y \in H$, we have that $xy^{-1} \in H$. Thus, H satisfies the subgroup criterion and so $H \leqslant G$. Now fix $y \in G$ such that $y \neq x, e$. We know such an element exists since $|G| = 6$. Furthermore, since G is non-abelian, we can fix y such that $xy \neq yx$. If we could not, then it would mean that x commutes with all elements of G. Consider yxy^{-1} and suppose $yxy^{-1} \in H$. Then $yxy^{-1} = x$ or $yxy^1 = e$. In the first case, note that $yxy^{-1} = x$ implies that $yx = xy$, a contradiction. In the second case, we have that $yxy^1 = e$ implies that $yx = y$, which implies that $x = e$. However, we know $x \neq e$, so this also a contradiction. Hence, $yxy^{-1} \notin H$ and thus H is a non-normal subgroup of order 2. □

2. **Conjugacy classes**

(a) If the center of G is of index n, prove that every conjugacy class has at most n elements.

Proof. Recall that the center of G is defined as $Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}$. We have that $|G : Z(G)| = n$. Now fix $s \in G$. Then $C_G(s) = \{g \in G \mid gsg^{-1} = s\}$. If $gsg^{-1} = s$, then we can multiply by g on the right and get $gs = sg$. Thus, we can reformulate the above definition as $C_G(s) = \{g \in G \mid gsg^{-1} = s\}$. Now observe that if $g \in Z(G)$, then $gx = xg$ for all $x \in G$. Hence, we have that $g \in C_G(s$ and so $Z(G) \subset C_G(S)$. Thus, $|Z(G)| \leqslant |C_G(s)|$. Thus, by Lagrange's Theorem we have,

$$|G : C_G(s)| = \frac{|G|}{|C_G(s)|} \leqslant \frac{|G|}{|Z(G)|} = |G : Z(G)| = n$$

Thus, by DF, §4.3, Proposition 6, we have that the number of conjugates of s in G is less than or equal to n. Hence, the conjugacy class of s (i.e. orbit of s in G acting on itself of conjugation) has at most n elements. Since s was arbitrary, this holds for every $s \in G$. □

(b) A subgroup $M \leqslant G$ is *maximal* if $M \neq G$ and if $M \leqslant H \leqslant G$, then $H = M$ of $H = G$. (Assume G is finite).

(i) Prove that if $M < G$ is maximal, then $N_G(M) = M$ or G. (ii) Deduce that if a maximal subgroup M is not normal, then the number of non-identity elements of G that are contained in conjugates of M is at most $(|M| - 1)|G : M|$, i.e.

$$\left| \left(\bigcup_{g \in G} gMg^{-1} \right) - \{1\} \right| \leqslant (|M| - 1)|G : M|$$

(iii) Deduce further that for any $H < G$, the set $\{gHg^{-1} \mid g \in G\}$ does *not* partition G (in contrast to the set of left cosets).

Proof. **(i)** Suppose $M < G$ is maximal and consider $N_G(M) = \{g \in G \mid gMg^{-1} = M\}$. Suppose M is normal. Then by DF, §3.1, Theorem 6, we have that $N_G(M) = G$. Now suppose M is not normal.

(ii)

(iii) □

(c) Suppose G is a finite group with r conjugacy classes. Let $g_1, \ldots, g_r$ be representatives of those r distinct conjugacy classes. Show that if these representatives all pairwise commute, then G is abelian.

Proof. Suppose G is a finite group with r conjugacy classes. Suppose that the representatives for these classes, $g_1, g_2, \ldots, g_r$ pairwise commute. Fix $i, j \in \{1, \ldots, r\}$ with $i \neq j$. Then, $g_i g_j = g_j g_i$. Now fix a in the conjugacy class of g_i and b in the conjugacy class of g_j. Then there exists $g, g' \in G$ such that $gag^{-1} = g_i$ and $g'bg'^{-1} = g_j$. Hence, we have that $g_i g_j = g_j g_i$ implies,

$$gag^{-1}g'bg'^{-1} = g'bg'^{-1}gag^{-1}$$

□

(d) Let G be a finite group of odd order. Prove that if $x \neq 1$, then x and x^{-1} are not conjugate.

Proof. Suppose that G is a finite group of odd order and that $x \in G$ such that $x \neq 1$. Suppose x and x^{-1} are conjugate in G. That is, there is some $g \in G$ such that $x^{-1} = gxg^{-1}$. This implies that $x^{-1}g = gx$ and hence $x^{-1}gx^{-1} = g$ $\square$