Chris Hayduk
Math A4900
Homework 8
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1. **Direct products.**

(a) Let $G = A_1 \times \cdots \times A_n$, and for each $i$, let $B_i$ be a normal subgroup of $A_i$. Prove that $B_1 \times \cdots \times B_n \trianglelefteq G$ and that

$$(A_1 \times \cdots \times A_n)/(B_1 \times \cdots \times B_n) \cong (A_1/B_1) \times \cdots \times (A_n/B_n).$$

[Hint: You may use (without proof) the fact that, for groups $C_1, \dots, C_n$, if for each $i$, you have a homomorphism $\varphi_i : A_i \to C_i$, then

$$\varphi_1 \times \cdots \times \varphi_n : A_1 \times \cdots \times A_n \to C_1 \times \cdots \times C_n$$

defined by

$$(a_1, \dots, a_n) \mapsto (\varphi(a_1), \dots, \varphi(a_n))$$

is a homomorphism as well.]

*Proof.* Let $(a_1, a_2, \dots, a_n) \in G$. We have that $a_i^{-1} \in A_i$ for every $i$ such that $1 \leqslant i \leqslant n$, so $a^{-1} = (a_1^{-1}, a_2^{-1}, \dots, a_n^{-1}) \in G$. Now fix $b = (b_1, b_2, \cdots, b_n) \in B_1 \times \cdots \times B_n$. We know that, for every $i$ such that $1 \leqslant i \leqslant n$, we have that $B_i \trianglelefteq A_i$. Hence, $a_i b a_i^{-1} \in B_i$ by the definition of normal subgroups. Thus, this gives us that,

$$aba^{-1} = (a_1 b_1 a_1^{-1}, \dots, a_n b_n a_n^{-1}) \in B$$

Since $a$ was arbitrary in $G$ and $b$ was arbitrary in $B$, we have that $B \trianglelefteq G$.

Now we have that,

$$(A_1 \times \cdots \times A_n)/(B_1 \times \cdots \times B_n) = \{a(B_1 \times \cdots \times B_n) \mid a \in G\}$$

and,

$$(A_1/B_1) \times \cdots \times (A_n/B_n) = \{a_1 B_1 \mid a_1 \in A_1\} \times \cdots \times \{a_n B_n \mid a_n \in A_n\}$$

Define $\varphi : \{a(B_1, \dots, B_n) \mid a \in G\} \to \{a_1 B_1 \mid a_1 \in A_1\} \times \cdots \times \{a_n B_n \mid a_n \in A_n\}$ such that $\varphi(a(B_1 \times \cdots \times B_n))) = a_1 B_1 \times \cdots \times a_n B_n$ where $a = (a_1, \dots, a_n)$.

Now let $a, a' \in A$ and suppose $\varphi(a(B_1 \times \cdots \times B_n))) = \varphi(a'(B_1 \times \cdots \times B_n)))$. Then,

$$a_1 B_1 \times \cdots \times a_n B_n = a_1' B_1 \times \cdots \times a_n' B_n$$

That is,

$$\begin{aligned} a_1 B_1 &= a_1' B_1 \\ a_2 B_2 &= a_2' B_2 \\ &\vdots \\ a_n B_n &= a_n' B_n \end{aligned}$$

Since $B_i \trianglelefteq A_i$, we have that $a_i = a_i'$ for every $i$. Thus, $\varphi$ is injective. Now suppose $a_1' \in A_1, \ldots, a_n' \in A_n$ and consider $a_1'B_1, \ldots, a_n'B_n$. Since $a_1' \in A_1, \ldots, a_2' \in A_2$, we can define $a' = (a_1', \ldots, a_n') \in G$. Hence, it is clear that,

$$\varphi(a'(B_1 \times \cdots \times B_n)) = a_1'B_1 \times \cdots a_n'B_n$$

Thus, $\varphi$ is surjective and hence is a bijection. $\square$

(b) Let $G$ be a group, and let $G^n = G \times \cdots \times G$ ($n$ factors). Define an action of $S_n$ by

$$\sigma \cdot (g_1, g_2, \ldots, g_n) = (g_{\sigma^{-1}(1)}, g_{\sigma^{-1}(2)}, \cdots, g_{\sigma^{-1}(n)}),$$

for $g_i \in G$ and $\sigma \in S_n$. For example, for $n = 4$, we have

$$(124) \cdot (g_1, g_2, g_3, g_4) = (g_4, g_1, g_3, g_2).$$

(i) Check that this, indeed, defines a group action of $S_n$ on $G^n$. (Make a particular note here of why you need $\sigma^{-1}$ in the subscript, rather than $\sigma$.)

*Proof.* Let $\sigma, \tau \in S_n$ and let $g = (g_1, g_2, \ldots, g_n) \in G^n$. We have,

$$\begin{aligned}
\tau(\sigma(g)) &= \tau(g_{\sigma^{-1}(1)}, g_{\sigma^{-1}(2)}, \ldots, g_{\sigma^{-1}(n)}) \\
&= (g_{\tau^{-1}(\sigma^{-1}(1))}, g_{\tau^{-1}(\sigma^{-1}(2))}, \ldots, g_{\tau^{-1}(\sigma^{-1}(n))}) \\
&= (\sigma\tau) \circ g
\end{aligned}$$

Now let $1 \in S_n$ be the identity element. Observe that $1^{-1} = 1$. Then,

$$\begin{aligned}
1(g) &= (g_1, \ldots, g_n) \\
&= g
\end{aligned}$$

Hence, this defines a valid group action. $\square$

(ii) For $g \in G$, let

$$g^{(i)} = (g_1, g_2, \ldots, g_n), \quad \text{where } \begin{cases} g_j = g & \text{if } j = i, \\ g_j = 1 & \text{otherwise.} \end{cases}$$

For example, if $n = 4$, then $g^{(3)} = (1, 1, g, 1)$. Show $\sigma \cdot g^{(i)} = g^{(\sigma(i))}$.

*Proof.* Fix $g \in G$ and consider $g^{(i)} = (g_1, g_2, \ldots, g_n)$ where $g_i = g$ and all other entries are 1. Now fix $\sigma \in S_n$. We have,

$$\sigma \cdot g^{(i)} = (g_{\sigma^{-1}(1)}, g_{\sigma^{-1}(2)}, \ldots, g_{\sigma^{-1}(n)})$$

Hence, we see that the element $g$ is moved to the position $\sigma^{-1}(i)$. All of the other entries remain 1. Hence,

$$\sigma \cdot g^{(i)} = g^{\sigma^{-1}(i)}$$

$\square$

(iii) Show that

$$\varphi_\sigma : G^n \to G^n \quad \text{defined by} \quad \bar{g} \mapsto \sigma \cdot \bar{g}$$

is an automorphism of $G^n$. Deduce that $\varphi : S_n \to \mathrm{Aut}(G^n)$ is an injective homomorphism.

2. **Semidirect products.** Let $A$ and $B$ be groups, and let $B \circlearrowright A$ via automorphisms of $A$. Define $A \rtimes B$ with respect to this action.

   (a) Prove that $C_B(A) = \ker$, where $C_B(A) = \{b \in B \mid ba = ab \text{ for all } b \in B\}$ (and ker is the kernel of the action of $B$ on $A$).

   *Proof.* Let $b \in C_B(A)$. Then $ab = ba$ for for every $a \in A$. Hence, we have
   $$b^{-1}ab = a$$
   $\square$

   (b) We showed that $A \trianglelefteq A \rtimes B$. Show $(A \rtimes B)/A \cong B$.
   [Hint: Consider the map $(a, b) \mapsto b$ and set up a 1st isomorphism theorem argument.]

   (c) Let $Z = \langle x \rangle$ be the infinite cyclic group.

      (i) Compute $\mathrm{Aut}(Z)$. [*Hint.* $x$ must map to a generator of $Z$.]

      (ii) Classify all actions of $Z$ on itself that correspond to automorphisms (i.e. actions where for each $z \in Z$, the map $\varphi_z : Z \to Z$ defined by $a \mapsto z \cdot a$ is an automorphism.)

      (iii) Classify all semidirect products of $Z$ with itself. [i.e. How many examples are there of $Z \rtimes Z$, which depends intrinsically on the action of $Z$ on itself.]