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Lecture 12 - Exercise A

1a.

$$\begin{tikzcd} (1,1) \arrow[loop left, "(23)"] \arrow[r, mapsto, "(12)"] & (2,2) \arrow[r, mapsto, "(23)"] & (3,3) \arrow[loop right, "(12)"] \end{tikzcd}$$

$$\begin{tikzcd} (1,2) \arrow[r, mapsto, "(23)"] \arrow[d, leftrightarrow, "(12)"] & (1,3) \arrow[r, mapsto, "(12)"] \arrow[d, leftrightarrow, "(13)"] & (2,3) \arrow[d, leftrightarrow, "(23)"] \\ (2,1) \arrow[r, mapsto, "(23)"] & (3,1) \arrow[r, mapsto, "(12)"] & (3,2) \end{tikzcd}$$

b. $[(1,1)] = \{(1,1), (2,2), (3,3)\}$

$[(1,2)] = \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}$

c. Stabilizer of a is $\{g \in G \mid ga = a\}$

Stabilizer of $(1,1)$:

$\{1, (23)\}$

Stabilizer of $(1,2)$:

$\{1\}$

d. Kernel $= \{g \in G \mid g \cdot s = s \ \forall s \in S\}$

$Ker = \{1\}$

e. $(123) = (12)(23)$

$[(1,1)] = \{(1,1), (2,2), (3,3)\}$

$[(1,2)] = \{(1,2), (2,3), (3,1)\}$

$[(1,3)] = \{(1,3), (2,1), (3,2)\}$

f. $[(1,1)] = \{(1,1), (2,2)\}$

$[(3,3)] = \{(3,3)\}$

$[(1,2)] = \{(1,2), (2,1)\}$

$[(1,3)] = \{(1,3), (2,3)]$

$[(3,1)] = \{(3,1), (3,2)\}$

2a. Stabilizers:

$\{1,2\}: \{1, s\}$

$\{1,4\}: \{1, sr^2\}$

$\{2,3\}: \{1, sr^2\}$

$\{3,4\}: \{1, s\}$

$\{1,3\}: \{1, sr, r^2, sr^3\}$

$\{2,4\}: \{1, sr, r^2, sr^3\}$

b. Kernel = $\{1\}$

3a. Faithful. Not transitive or trivial

b. Faithful and transitive. Not trivial

c. Faithful. Not transitive or trivial

d. Faithful. Not transitive or trivial

4a. It is transitive if $\forall x, y \in [n]$, $\exists z \in \mathbb{Z}$ s.t.

$$x = z \cdot y$$

Suppose $x = y$. Then

$$0 \cdot y = (0 + y) \pmod{n}$$
$$= y = x$$

Since $0 \in \mathbb{Z}$

Now suppose $y > x$ w.l.o.g. Then,

$$1 \leq y - x \leq n - 1$$

So $(y - x) \in [n]$

Thus, $y - x$ is an integer and hence in $\mathbb{Z}$. Thus we also have $-(y - x) = x - y \in \mathbb{Z}$ since $\mathbb{Z}$ is closed under additive inverses. So we have,

$$(x - y) \cdot y = (x - y + y) \pmod{n}$$
$$= x$$

So this action is transitive

It is faithful if

$$\ker = \{z \in \mathbb{Z} \mid z \cdot a = a \ \forall a \in [n]\}$$
$$= \{0\}$$

Now consider n. We have:

$$n \cdot 1 \equiv n+1 \pmod n$$
$$= 1$$

$$n \cdot 2 = n+2 \pmod n$$
$$= 2$$

$$\vdots$$

$$n \cdot n \equiv n+n \pmod n$$

So $n \in \ker$ and the action is not faithful.

6. Suppose $|A| = 1$ and $|G| > 1$

Since there is only one element in A, there is only one orbit for any action. Hence it is transitive.

Now recall that actions are permutation of the set A. Hence

$$g \cdot a \in A$$

$\forall g \in G$, $\forall a \in A$

Let $A = \{a\}$. Then $\forall g \in G$,

$$g \cdot a = a$$

is the only possibility that remains in A.

So $\ker = G$ and $|G| > 1$, so $\exists g \in G$ s.t. $g \neq 1$.

Hence the action is not faithful.

c. The action is not transitive if for some $x, y \in [n] \times [n]$, there is no $g \in G$ s.t. $x = g \cdot y$.

Let $x = (1,2)$ and $y = (1,1)$ and let $\bar{z} \in \mathbb{Z}$.

We need,

$$z + 1 \pmod n = 1 \qquad (1)$$

and

$$z + 2 \pmod n = 1 \qquad (2)$$

Note that (1) is only true if $z = 0$ or $z = kn$ for some $k \in \mathbb{Z}$. But (2) is only true if $z = k(n-1)$ for some $k \in \mathbb{Z} \setminus \{0\}$.

These are mutually exclusive, so no such $\bar{z}$ exists and the action is not transitive.

It is faithful if

$$\ker = \{\bar{z} \in \mathbb{Z}/n\mathbb{Z} \mid \bar{z} \cdot a = a \ \forall a \in [n] \times [n]\}$$

$$= \{\bar{0}\}$$

If

$$\bar{z} \cdot (j,k) = (z + j \pmod n),\ z + k \pmod n)$$
$$= (j,k)$$

Then $z+j \cong_n j$ and hence $z \cong_n 0$.

Thus $\bar{z} = \bar{0}$ and $\ker = \{\bar{0}\}$

d. Let $|G| = 1$ $(G = \{1\})$ and $|A| > 1$

$\ker \leq G = \{1\}$. So $\ker = \{1\}$ and action is faithful

Since $|A| > 1$, $\exists\, a,b \in A$ with $a \neq b$. Thus

$$[a] = \{g \cdot a \mid g \in G\}$$
$$= \{1 \cdot a\} = \{a\} \not\ni b$$

So $[a] \neq A$ and it is not transitive