Chris Hayduk 9/23

Lecture 9, Exercise B

1a. Let $x \in G$.

Since $H \leq G$, $1 \in H$

Hence, $x1 = x \in xH$.

b. Let $x, y \in G$. Suppose $x \in yH$

Then

$$x = yh$$

for some $h \in H$.

Since H is a group, $h^{-1} \in H$. So,

$$xh^{-1} = y$$

and $xh^{-1} \in xH$.

Hence, $y \in xH$

c. Let $x, y \in G$. Suppose $x \in yH$ and $y \in zH$

Then $x = yh_1$, $y = zh_2$

for $h_1, h_2 \in H$

So we have

$$x = (z h_2) h_1$$
$$= z (h_2 h_1)$$

Since H group, $h_2 h_1 \in H$ and so

$$x \in zH$$

2a. Observe for i:

$$i(-1) = i(i)^2$$
$$= (i\, i\, i)$$
$$= (i^2)\, i$$
$$= (-1)\, i$$

and $1 i = i = i \cdot 1$

The same holds for k and j

So $iH = Hi = \{-i, i\}$

$jH = Hj = \{-j, j\}$

$kH = Hk = \{-k, k\}$

Now $1H = \{1, -1\} = H1$

and

$$-1H = \{-1, 1\} = H(-1)$$

So $H \trianglelefteq Q_8$

By Prop. 7, H is the kernel of some homomorphism

So by Theorem 3,

$$Q_8 / H$$

is a group

6.

	H	iH	jH	kH
H	H	iH	jH	kH
iH	iH	H	kH	jH
jH	jH	kH	H	iH
kH	kH	jH	iH	H

	$\bar{1}$	$\bar{i}$	$\bar{j}$	$\bar{k}$
$\bar{1}$	$\bar{1}$	$\bar{i}$	$\bar{j}$	$\bar{k}$
$\bar{i}$	$\bar{i}$	$\bar{1}$	$\bar{k}$	$\bar{j}$
$\bar{j}$	$\bar{j}$	$\bar{k}$	$\bar{1}$	$\bar{i}$
$\bar{k}$	$\bar{k}$	$\bar{j}$	$\bar{i}$	$\bar{1}$

c. $\mathbb{Z}_2 = \langle x \rangle = \{1, x\}$

$$(x,1) \mapsto \bar{i}$$
$$(1,x) \mapsto \bar{j}$$
$$(x,x) \mapsto \bar{k}$$
$$(1,1) \mapsto \bar{1}$$

ϕ is bijective as defined above

Now to check homomorphism:

$$\phi((x,1)\cdot(x,1)) = \phi((1,1))$$
$$= \bar{1}$$
$$= \phi((x,1))\,\phi((x,1))$$

$$\rho((x,1)\cdot(1,x)) = \rho((x,x))$$
$$= \bar{k}$$
$$= \rho((x,1))\cdot\rho((1,x))$$

$$\rho((x,1)\cdot(x,x)) = \rho((1,x))$$
$$= \bar{j}$$
$$= \rho((x,1))\cdot\rho((x,x))$$

$$\rho((x,1)\cdot(1,1)) = \rho((x,1))$$
$$= \bar{i}$$
$$= \rho((x,1))\cdot\rho((1,1))$$

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$$= \bar{1}$$
$$= \rho((1,x))\,\rho((1,x))$$

$$\rho((1,x)\cdot(x,x)) = \rho((x,1))$$
$$= \bar{i}$$
$$= \rho((1,x))\cdot\rho((x,x))$$

$$\rho((1,x)\cdot(1,1)) = \rho((1,x))$$
$$= \bar{J}$$
$$= \rho((1,x))\cdot\rho((1,1))$$

$$\rho((x,x)\cdot(x,x)) = \rho((1,1))$$
$$= \bar{I}$$
$$= \rho((x,x))\cdot\rho((x,x))$$

$$\rho((x,x)\cdot(1,1)) = \rho((x,x))$$
$$= \bar{K}$$
$$= \rho((x,x))\cdot\rho((1,1))$$

$$\rho((1,1)\cdot(1,1)) = \rho((1,1))$$
$$= \bar{1}$$
$$= \rho((1,1))\cdot\rho((1,1))$$

So $\rho(xy) = \rho(x)\rho(y) \ \forall x,y \in Z_2$

Hence ρ is an isomorphism

3a. $\langle r \rangle = \{1, r, r^2, r^3, r^4, r^5, r^6, r^7\}$