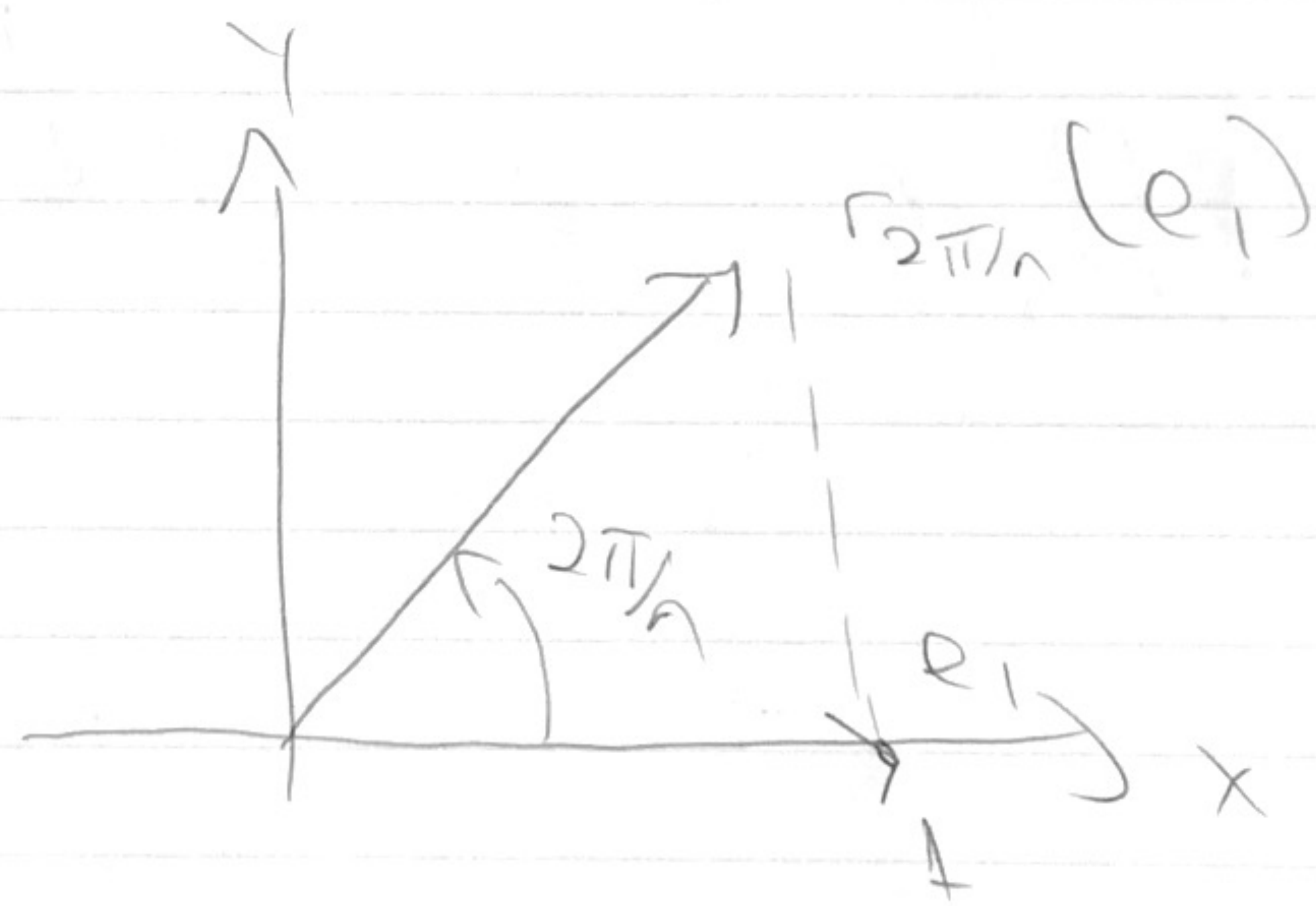
Chris Haydnk 9/4

Lecture 5, Exercise A

1a. We are given that $r_{2\pi/n}$ is the counter-clockwise rotation by $2\pi/n$.

Note that $r_{2\pi/n}^{-1}(r_{2\pi/n}(e_1)) = e_1$

So,



So we need to rotate $r_{2\pi/n}(e_1)$ in the opposite direction.

Hence it is the clockwise rotation by $2\pi/n$. Thus,

$$r_{2\pi/n}^{-1} = r_{-2\pi/n}$$

Now let's compare to the matrix inverse:

$$\begin{pmatrix} \cos\left(\frac{2\pi}{n}\right) & -\sin\left(\frac{2\pi}{n}\right) \\ \sin\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) \end{pmatrix}^{-1}$$

$$= \frac{1}{\cos^2\left(\frac{2\pi}{n}\right) + \sin^2\left(\frac{2\pi}{n}\right)} \begin{pmatrix} \cos\left(\frac{2\pi}{n}\right) & \sin\left(\frac{2\pi}{n}\right) \\ -\sin\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\left(\frac{2\pi}{n}\right) & \sin\left(\frac{2\pi}{n}\right) \\ -\sin\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) \end{pmatrix}$$

6. i) We have $r_{2\pi/n}$ is a 2×2 matrix and

$$\det(r_{2\pi/n}) = \cos^2\left(\frac{2\pi}{n}\right) + \sin^2\left(\frac{2\pi}{n}\right)$$
$$= 1 \neq 0$$

So $r_{2\pi/n} \in GL_2(\mathbb{R})$

s_y also is a 2×2 matrix and

$$\det(s_y) = -1 - 0 = -1 \neq 0$$

So $s_y \in GL_2(\mathbb{R})$

Since $GL_2(\mathbb{R})$ is a group any combination of $r_{2\pi/n}$ and s_y is in $GL_2(\mathbb{R})$ as well

Hence

$$\varphi(D_{2n}) \subseteq GL_2(\mathbb{R})$$

ii) $$s_y^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$r_{2\pi/n} s_y = \begin{pmatrix} \cos\left(\frac{2\pi}{n}\right) & -\sin\left(\frac{2\pi}{n}\right) \\ \sin\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\cos\left(\frac{2\pi}{n}\right) & -\sin\left(\frac{2\pi}{n}\right) \\ -\sin\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) \end{pmatrix}$$

$$S_Y r_{2\pi/n}^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos(\frac{2\pi}{n}) & \sin(\frac{2\pi}{n}) \\ -\sin(\frac{2\pi}{n}) & \cos(\frac{2\pi}{n}) \end{pmatrix}$$

$$= \begin{pmatrix} -\cos(\frac{2\pi}{n}) & -\sin(\frac{2\pi}{n}) \\ -\sin(\frac{2\pi}{n}) & \cos(\frac{2\pi}{n}) \end{pmatrix}$$

So $r_{2\pi/n} S_Y = S_Y r_{2\pi/n}^{-1}$ as required

c. We have that D_{2n} is finite, but GL_n is infinite.

So φ cannot be surjective and is not an isomorphism

2a. Since $n > m$, then if

$$(i \;\; i+1) \in S_m \Rightarrow (i \;\; i+1) \in S_n$$

So $\varphi(\sigma_i) = \sigma_i \in S_n$

and φ is a homomorphism

φ is injective because $\varphi(\sigma_i) = \sigma_i$ $\forall$ $\sigma_i \in S_m$ from the above.

φ is not surjective because:

$\sigma_m = (m, m+1) \notin S_m$, but
$\sigma_m \in S_n$

b. Consider σ_{m-1} and $\sigma_m \in S_n$

3a. Let $\varphi = id$. Then $\varphi(x) = x$
$\forall x \in G$.

Hence,

$\varphi(xy) = xy$

and $\varphi(xy) = \varphi(x)\varphi(y) = xy$

Thus φ is a homomorphism

ϕ is injective because

$$\phi(x_1) = \phi(x_2)$$

$$\Rightarrow x_1 = x_2$$

and surjective because

$$\phi(x) \in G \Rightarrow x \in G$$

So ϕ is an isomorphism from $G \to G$

b. Let $\phi: G \to H$

c.