ID: **5152**
Math A4900
Proof portfolio draft, Round 2
November 15, 2020

Problem:	**4A**
No. stars:	**1**

Statement: Prove that every finitely generated subgroup of $\mathbb{Q}$ is cyclic.

Proof. Let H be a finitely generated subgroup of $\mathbb{Q}$ and suppose that there is a finite set $\mathbb{Q}$ such that $H = \langle A \rangle$. Now consider k, the product of all the denominators that appear in A. Then every element $a/b \in A$ can be re-written as $\frac{a \cdot k/b}{b \cdot k/b} = \frac{a \cdot k/b}{k}$ since b is in the product that yields k and hence is a divisor of k. Thus, we can rewrite every fraction in A as a fraction with denominator k. That is, every fraction in A can be written as n/k for some $n \in \mathbb{Z}$. This lets us conclude that,

$$H = \langle A \rangle \leqslant \langle 1/k \rangle$$

Thus, by Theorem 7 in §2.3 of DF, we have that H is cyclic since $\langle 1/k \rangle$ is cyclic. $\square$

	Points Possible
complete	0 1 2 3 4 5
mathematically valid	0 1 2 3 4 5
readable/fluent	0 1 2 3 4 5
Total:	(out of 15)

Problem:	**5A**
No. stars:	**2**

Statement: Prove that if $G/Z(G)$ is cyclic, then G is abelian.

Proof. Suppose $G/Z(G)$ is cyclic. That is, there is an element $x \in G$ such that $G/Z(G) = \langle xZ(G) \rangle$. That is, for every $g \in G$, we can rewrite $gZ(G)$ as $x^\ell Z(G)$ for some $\ell \in \mathbb{Z}$. Hence, $gZ(G) = x^\ell Z(G)$. By Proposition 4, we then have,

$$x^{-\ell} g \in Z(G)$$

But if $x^{-\ell} g \in Z(G)$, then,

$$x^\ell (x^{-\ell} g) = g \in x^\ell Z(G)$$

Hence, $g = x^\ell z$ for some $z \in Z(G)$ (in particular, $z = x^\ell g$). Since g and ℓ were arbitrary, this holds for every element $g \in G$. Now let us fix $g_1, g_2 \in G$ such that $g_1 = x^a z_1$ and $g_2 = x^b z_2$ where $z_1, z_2 \in Z(G)$. Since $Z(G)$ is the set of elements that commute with everything in G, we have,

$$\begin{aligned} g_1 g_2 &= x^a z_1 \cdot x^b z_2 \\ &= x^a x^b z_1 z_2 \\ &= x^{a+b} z_2 z_1 \\ &= x^{b+a} z_2 z_1 \\ &= x^b x^a z_2 z_1 \\ &= x^b z_2 x^a z_1 \\ &= g_2 g_1 \end{aligned}$$

Thus, G is an abelian group. $\square$

	Points Possible
complete	0 1 2 3 4 5
mathematically valid	0 1 2 3 4 5
readable/fluent	0 1 2 3 4 5
Total:	(out of 15)

Problem:	**5B**
No. stars:	**1**

Statement: Prove that if H and K are finite subgroups of G whose orders are relatively prime, then $H \cap K = 1$.

Proof. Suppose H and K are finite subgroups of G where $|H| = p$, $|K| = q$ with q and p relatively prime. By Proposition 13 on page 93 of Dummit & Foote, we have that,

$$\begin{aligned} |HK| &= \frac{|H||K|}{|H \cap K|} \\ &= \frac{pq}{|H \cap K|} \end{aligned}$$

Suppose without loss of generality that $p \geqslant q$. Then we have that $|H \cap K| \leqslant |K| = q$. Now note that $\frac{pq}{|H \cap K|}$ must yield an integer answer. However, we have that there are no common factors of p and q in the set $\{2, 3, 4, \cdots, q - 1\}$. Thus, our choices for $|H \cap K|$ are 1 and q. We know that $|H \cap K| = q$ if $K \leqslant H$. However, if $K \leqslant H$, then by Lagrange's Theorem, $|K| = q$ divides $|H| = p$. Since p, q are relatively prime, this is not possible. Hence, $|H \cap K| = 1$.

Now, since both H and K are subgroups of G, we know that they must both contain the identity element 1. Hence, $1 \in H \cap K$. Since $|H \cap K| = 1$, we have that the identity must be the only element of $H \cap K$. Thus, $H \cap K = 1$. □

	Points Possible
complete	0 1 2 3 4 5
mathematically valid	0 1 2 3 4 5
readable/fluent	0 1 2 3 4 5
Total:	(out of 15)

Problem:	**6B.I**
No. stars:	**2**

Statement: Let x, y be distinct 3-cycles in S_5. Show that $\langle x, y \rangle$ is isomorphic to one of Z_3, A_4, or A_5.

Proof. Let x, y be distinct 3-cycles in S_5. There are 3 possible cases: $y = x^{-1}$, $y \neq x^{-1}$ and they overlap at one element, in which case they permute all 5 elements, or $y \neq x^{-1}$ and they overlap at two elements, in which case they permute 4 elements and fix the fifth element.

Let us examine the first case, $y = x^{-1}$. Since $y = x^{-1}$, we must have that $y^{-1} \in \langle x \rangle$. Hence, $\langle x, y \rangle = \langle x \rangle$. Note that if we let $x = (a\ b\ c)$, we have that,

$$x^2 = (a\ c\ b)$$
$$x^3 = 1$$

So $\langle x \rangle$ is a cyclic group of order 3. Z_3 is also a cyclic group of order 3 and, by Theorem 4 in §2.3 of Dummit and Foote, we have that any two cyclic groups of the same order are isomorphic. Hence, in this case, $\langle x, y \rangle = \langle x \rangle \cong Z_3$.

Now assume $y \neq x^{-1}$ and that x, y overlap at one element. Hence, x and y permute all 5 elements in $\{1, 2, 3, 4, 5\}$.

□

	Points Possible
complete	0 1 2 3 4 5
mathematically valid	0 1 2 3 4 5
readable/fluent	0 1 2 3 4 5
Total:	(out of 15)

Problem:	**7A**
No. stars:	**2**

Statement: Show that if H has finite index n in G, then there is a normal subgroup $K \trianglelefteq G$ with $K \leqslant H$ and $|G : K| \leqslant n!$.

Proof. Suppose H has finite index n in G. That is, $|G : H| = n$ or, equivalently, $|G/H| = n$. Note that G/H is the set of left cosets of H in G. Now, since $|G : H| = n$, we can label the distinct left cosets of H in G by $a_1H, a_2H, \ldots, a_nH$. For each $g \in G$, we can then think of the action of left multiplication as a permutation σ_g of the indices $1, \ldots, n$. That is, $\sigma_g \in S_n$. Then if we define the homomorphism $\varphi : g \mapsto \sigma_g$, we have that $\varphi : G \to S_n$. Now the kernel of φ is given by $K = \{g \in G \mid gaH = aH \text{ for all } a \in G\}$.

Suppose $g \in K$. Then we have $g1H = gH = H$, and hence $g \in H$. This gives us that $K \subset H$. Moreover, by the First Isomorphism Theorem, we now have that $K \trianglelefteq G$ and $G/K \cong \varphi(G)$. Since $\varphi : G \to S_n$, we have that $\varphi(G) \leqslant S_n$. In addition, $|S_n| = n!$, so we have that $\varphi(G) \leqslant |S_n| = n!$. Lastly, by Corollary 17 in §3.3 of Dummit and Foote, we have $|G/K| = |\varphi(G)| \leqslant n!$, as required. □

	Points Possible
complete	0 1 2 3 4 5
mathematically valid	0 1 2 3 4 5
readable/fluent	0 1 2 3 4 5
Total:	(out of 15)