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Math B4900
Homework 1
2/12/2021

1. **Class sums.** Let R be a commutative ring with 1, and let G be a finite group. Before starting, review our work considering groups acting on themselves by conjugation (Section 4.3 in D&F). In particular, the conjugacy classes of group elements partition the group. For example, in S_n, the conjugacy classes are in bijection with cycle type; in S_3 in particular, the classes are

$$\{1\}, \quad \{(12),(13),(23)\}, \quad \text{and} \quad \{(123),(132)\}.$$

   (a) In RG, a *class sum* corresponding to a conjugacy class

$$\mathcal{K}_g = \{h \in G \mid h = aga^{-1} \text{ for some } a \in G\} \qquad \text{is} \qquad \kappa_g = \sum_{h \in \mathcal{K}_g} h.$$

   For example, the class sums in RS_3 are

$$\kappa_1 = 1, \quad \kappa_{(12)} = (12) + (13) + (23), \quad \text{and} \quad \kappa_{(123)} = (123) + (132).$$

   Compute the class sums in RD_8 and RA_4.

   Answer. Recall that $D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$ with $|r| = 4, |s| = 2, s \neq r^i$ for every i, and $rs = sr^{-1}$. The equivalency classes of D_8 under conjugacy are

$$\{1\} \quad \{r, r^3\} \quad \{r^2\} \quad \{s, sr^2\} \quad \{sr, sr^3\}$$

   Hence, the class sums of RD_8 are,

$$\kappa_1 = 1, \quad \kappa_r = r + r^3 \quad \kappa_{r^2} = r^2 \quad \kappa_s = s + sr^2 \quad \kappa_{sr} = sr + sr^3$$

   Now note that $A_4 =$

   ....

   (b) For each of the class sums κ in RD_8, compute $r\kappa r^{-1}$ and $s\kappa s^{-1}$. Use your results to argue that $g\kappa = \kappa g$ for all $g \in D_8$.

   Answer. We have the following for $r\kappa r^{-1}$,

$$\begin{aligned}
r\kappa_1 r^{-1} &= r1r^{-1} = 1 \\
r\kappa_r r^{-1} &= r(r + r^3)r^{-1} = (r^2 + r^4)r^{-1} = r + r^3 \\
r\kappa_{r^2} r^{-1} &= r(r^2)r^{-1} = r^2 \\
r\kappa_s r^{-1} &= r(s + sr^2)r^{-1} = (rs + rsr^2)r^{-1} = (sr^{-1} + sr)r^{-1} = sr^{-2} + s = s + sr^2 \\
r\kappa_{sr} r^{-1} &= r(sr + sr^3)r^{-1} = (rsrrsr^3)r^{-1} = (s + sr^2)r^{-1} = sr^{-1} + sr = sr + sr^3
\end{aligned}$$

   Now for $s\kappa s^{-1}$,

$$\begin{aligned}
s\kappa_1 s^{-1} &= s1s^{-1} = 1 \\
s\kappa_r s^{-1} &= s(r + r^3)s^{-1} = (sr + sr^3)s^{-1} = r^{-1} + r^{-3} = r + r^3 \\
s\kappa_{r^2} s^{-1} &= s(r^2)s^{-1} = r^{-2}ss^{-1} = r^2 \\
s\kappa_s s^{-1} &= s(s + sr^2)s^{-1} = (s^2 + s^2r^2)s^{-1} = s^{-1} + r^2s^{-1} = s + sr^2 \\
s\kappa_{sr} s^{-1} &= s(sr + sr^3)s^{-1} = (s^2 + s^2r^3)s^{-1} = s^{-1} + r^3s^{-1} = s + sr^3
\end{aligned}$$

Now we know that D_8 is generated by $\{r, s\}$. That is, the elements r, s along with the rules mentioned in part (a) allow us to express any element of D_8. Hence, for any $g \in D_8$, we can write

$$g = r^i s^j$$

with $1 \leq i \leq 8$ and $1 \leq j \leq 2$. Thus, by the previous reasoning and above derivations, for any class sum κ, we have,

$$\begin{aligned} g\kappa g^{-1} &= r^i s^j \kappa s^{-j} r^{-i} \\ &= r^i \kappa r^{-i} \\ &= \kappa \end{aligned}$$

Now, multiplying by g on the right side of both sides of the equation yields,

$$\begin{aligned} &(g\kappa g^{-1})g = \kappa g \\ \iff &g\kappa = \kappa g \end{aligned}$$

as required.

. . . .

(c) **Claim:** the center of the group algebra RG is the R-span of the class sums of G,

$$Z(RG) = R\{\kappa_g \mid g \in G\} = \{r_1\kappa_1 + \cdots r_\ell \kappa_\ell \mid r_i \in R\},$$

where $\kappa_1, \ldots, \kappa_\ell$ denote the ℓ class sums of G.

Let's prove it:

(i) For each $g \in G$, show that for all $h \in G$, we have $h\kappa_g h^{-1} = \kappa_g$. Conclude that $a\kappa_g = \kappa_g a$ for all $a \in RG$ (showing that $\kappa_g \in Z(RG)$).

Proof. Let $g \in G$ and suppose κ_g is the class sum of g. Then $\kappa_g = g + g_1 + \cdots + g_k$ for $g, g_1, \ldots, g_k \in \mathcal{K}_g$. Now fix $h \in G$. Then we have,

$$\begin{aligned} h\kappa_g h^{-1} &= h(g + g_1 + \cdots + g_k)h^{-1} \\ &= hgh^{-1} + hg_1h^{-1} + \cdots + hg_kh^{-1} \end{aligned}$$

Since R is commutative, we have,

$$\begin{aligned} h\kappa_g h^{-1} &= hgh^{-1} + hg_1h^{-1} + \cdots hg_kh^{-1} \\ &= h(gh^{-1}) + h(g_1h^{-1}) + \cdots + h(g_kh^{-1}) \\ &= hh^{-1}g + hh^{-1}g_1 + \cdots + hh^{-1}g_k \\ &= 1g + 1g_1 + \cdots + 1g_k \\ &= g + g_1 + \cdots + g_k \\ &= \kappa_g \end{aligned}$$

Since $g, h \in G$ were arbitrary, this holds for all such $g, h \in G$ as required.

Now fix $a \in RG$. Then $a = a_1g_1 + a_2g_2 + \cdots + a_ng_n$ for $a_i \in R$, $g_j \in G$. $\square$

(ii) Use the previous part to show that $r_1\kappa_1 + \cdots r_\ell\kappa_\ell \in Z(RG)$ for all $r_i \in R$ (showing that $R\{\kappa_i \mid i = 1, \ldots \ell\} \subseteq Z(RG)$).

Proof. By part (i), we have that for each $g \in G$, $\kappa_g \in Z(RG)$. Hence, $a\kappa_g = \kappa_g a$ for all $a \in RG$. So let us consider,

$$a(r_1\kappa_1 + \cdots + r_\ell\kappa_\ell) = ar_1\kappa_1 + \cdots + ar_\ell\kappa_\ell$$

Since R is commutative and $\kappa_g \in Z(RG)$ for all g, we have,

$$\begin{aligned} ar_1\kappa_1 + \cdots + ar_\ell\kappa_\ell &= r_1\kappa_1 a + \cdots + r_\ell\kappa_\ell a \\ &= (r_1\kappa_1 + \cdots + r_\ell\kappa_\ell)a \end{aligned}$$

As a result, $(r_1\kappa_1 + \cdots + r_\ell\kappa_\ell) \in Z(RG)$. □

(iii) Conversely, show that for $a = \sum_{g\in G} s_g g \in RG$, if $hah^-1 = a$ for all $h \in G$, then $s_g = s_{g'}$ whenever g is conjugate to g (i.e. the coefficients are constant across conjugacy classes). [Hint: Start one at a time: if $hah^-1 = a$, then compare both sides to get $s_g = s_{h^{-1}gh}$. Try on your examples in part (b) to get started if you need help.]

Proof. □

(iv) Let $a \in RG$. Show that if $ha = ah$ for all $h \in G$, then $ba = ab$ for all $b \in RG$.

Proof. Let $a \in RG$ and suppose $ha = ah$ for all $h \in G$. Let $b \in RG$ and let us write b as $b = b_1g_1 + b_2g_2 + \cdots + b_ng_n$ for $b_i \in R$, $g_j \in G$ with $1 \leq i, j \leq n$. Let us also write $a \in RG$ as $a = a_1g_1 + a_2g_2 + \cdots + a_ng_n$ for $a_i \in R$, $g_j \in G$ with $1 \leq i, j \leq n$. Then we have,

$$\begin{aligned} ba &= (b_1g_1 + b_2g_2 + \cdots + b_ng_n) \cdot (a_1g_1 + a_2g_2 + \cdots + a_ng_n) \\ &= \sum_{g_ig_j=g_k} a_ib_jg_k \end{aligned}$$

We have that $ha = ah$ for all $h \in G$ and that □

(d) Let F be a field with $n! \neq 0$ in F.[1] Show that

$$e_+ = \sum_{\sigma\in S_n} \sigma \qquad \text{and} \qquad e_- = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\sigma$$

are essential idempotents in FS_n and are central, and compute the corresponding (pure) idempotents. [*Hint:* Do e_+ first, using the fact that any group acts transitively on itself by left multiplication. For e_-, do some small examples first, and modify your proof for e_+ appropriately.]

2. **Vector spaces.** U, V, and W denote vector spaces over a common field F; φ and ψ denote linear transformations; $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ denote bases; A, B, and C denote matrices in $M_n(F)$.

(a) Let $\varphi : V \to V$ be a linear map. An element $v \in V$ satisfying $\varphi(v) = \lambda v$ for some $\lambda \in F$ is called a *weight vector* of *weight* λ (otherwise known as an *eigenvector* of *eigenvalue* λ). More restrictively, element $\lambda \in F$ is called a *weight* or *eigenvalue* of φ if it is the weight of some non-zero weight vector of φ. Given a weight of φ, the *weight space* of V associated to λ is

$$V_\lambda = \{v \in V \mid \varphi(v) = \lambda v\}$$

[1] As usual, as an element of F, $n!$ means $1 + 1 + \cdots 1$ ($n!$ terms).

(the set of weight vectors in V of weight λ).

Show that V_λ is a subspace of V.

Proof. By definition, every $v \in V_\lambda$ is also an element of V. Hence, we have $V_\lambda \subset V$. Now observe that $\lambda 0 = 0$ for any $\lambda \in F$. Hence, $0 \in V_\lambda$ and thus V_λ is non-empty. Now, by the submodule criterion, we just need to show that $x + ry \in V_\lambda$ for all $r \in F$ and for all $x, y \in V_\lambda$. Let us start by applying φ to this element and using the properties of the linearity of φ,

$$\begin{aligned}\varphi(x + ry) &= \varphi(x) + r\varphi(y) \\ &= \lambda x + r\lambda y \\ &= \lambda(x + ry)\end{aligned}$$

Hence, $x + ry \in V_\lambda$ and so V_λ is a subspace of V. □

(b) Check briefly that $\varphi(v) = \lambda v$ is equivalent to $(\varphi - \lambda \cdot \mathrm{id})(v) = 0$.

Proof. We have that $\varphi(v) \in \mathrm{Hom}_F(V, V)$. Now, $\mathrm{id}(rx + y) = rx + y = r\mathrm{id}(x) + \mathrm{id}(y)$ for all $x, y \in V$ and $r \in F$, so $\mathrm{id} \in \mathrm{Hom}_F(V, V)$ as well. Thus, we can apply Proposition 2(2) from Section 10.2 of Dummit and Foote to get,

$$\begin{aligned}&(\varphi - \lambda \cdot \mathrm{id})(v) = 0 \\ \iff &\varphi(v) - \lambda \cdot \mathrm{id}(v) = 0 \\ \iff &\lambda v - \lambda v = 0 \\ \iff &\lambda v = \lambda v\end{aligned}$$

□

(c) Given a weight λ of φ, the *generalized weight space* associated to λ is

$$V^\lambda = \{v \in V \mid (\varphi - \lambda \cdot \mathrm{id})^m(v) = 0 \text{ for some } m \in \mathbb{Z}_{>0}\}.^2$$

(i) Let

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Check that $v \in V^2$ but $v \notin V_2$.

Answer. Consider,

$$\begin{aligned}\varphi(v) &= Av \\ &= \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\end{aligned}$$

Note that,

$$2v = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

[2] Here, ψ^m means $\psi \circ \psi \circ \cdots \circ \psi$ (m terms).

Hence, we have that $\varphi(v) \neq 2v$, and so $v \notin V_2$. Now we have,

$$Av - 2v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

And thus,

$$\begin{aligned}
(\varphi - 2 \cdot \mathrm{id})(Av - 2v) &= A(Av - 2v) - 2(Av - 2v) \\
&= \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0
\end{aligned}$$

As a result, we have that $(\varphi - 2 \cdot \mathrm{id})^2(v) = 0$, and so $v \in V^2$

. . . .

(ii) Briefly argue that $V_\lambda \subseteq V^\lambda$.

Answer. Suppose $v \in V_\lambda$. Then $\varphi(v) = \lambda v$ and, by part (b), we have that

$$(\varphi - \lambda \cdot \mathrm{id})(v) = (\varphi - \lambda \cdot \mathrm{id})^1(v) = 0$$

. Hence, v also satisfies the definition of V^λ with $m = 1$, so $v \in V^\lambda$. Since v was arbitrary, this holds for every $v \in V_\lambda$ and so $V_\lambda \subset V^\lambda$.

. . . .

(iii) Show that V^λ is a subspace of V.
[*Hint:* If $(\varphi - \lambda \cdot \mathrm{id})^m(v) = 0$, then $(\varphi - \lambda \cdot \mathrm{id})^n v = 0$ for all integers $n \geq m$.]

Proof. By definition, every $v \in V^\lambda$ is also an element of V. Hence, we have $V^\lambda \subset V$. Now observe that, for any $\lambda \in F$,

$$\begin{aligned}
&\varphi(v) = \lambda 0 = 0v = 0 \\
\iff &(\varphi - \lambda \cdot \mathrm{id})(0) = 0
\end{aligned}$$

Hence, $0 \in V^\lambda$ and thus V^λ is non-empty. Now, by the submodule criterion, we just need to show that $x + ry \in V^\lambda$ for all $r \in F$ and for all $x, y \in V^\lambda$. Since, $x, y \in V^\lambda$, we have that

$$\begin{aligned}
(\varphi - \lambda \cdot \mathrm{id})^\ell(x) = 0 \\
(\varphi - \lambda \cdot \mathrm{id})^m(y) = 0
\end{aligned}$$

Let $k = \max\{\ell, m\}$. Then by the fact that if $(\varphi - \lambda \cdot \mathrm{id})^m(v) = 0$, then $(\varphi - \lambda \cdot \mathrm{id})^n v = 0$ for all integers $n \geq m$ and by the fact that linear combinations and compositions of linear functions are linear, we have that,

$$\begin{aligned}
(\varphi - \lambda \cdot \mathrm{id})^k(x + ry) &= (\varphi - \lambda \cdot \mathrm{id})^k(x) + r(\varphi - \lambda \cdot \mathrm{id})^k(y) \\
&= 0 + r0 \\
&= 0
\end{aligned}$$

Hence, $x + ry \in V^\lambda$ and so V^λ is a subspace of V. $\square$

To receive credit for this assignment, include the following in your solutions [edited appropriately]:

Academic integrity statement: I *did not violate* the CUNY Academic Integrity Policy in completing this assignment. *Christopher Hayduk*

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