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Math B4900
Homework 6
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Let A be a ring with 1, and let M be an A-module.

1. Investigate whether or not the left regular module for $\mathbb{F}_2 S_2$ is decomposable. If so, give the decomposition. If not, why not? What simple submodules does it contain, and what do the corresponding quotients look like?
   [Recall that you decomposed $\mathbb{C}S_2$ in Homework 5. Also note that $\mathbb{F}_2 S_2$ is small—it has only 4 elements.]

   Proof. Observe that A a group algebra with $A = FG$ where $F = \mathbb{F}_2$ and $G = S_2$. Hence, A is an algebra over a field (F-algebra). Thus, any A-module is a vector space. The dimension of our model $M = A$ is given by $\dim(M) = 2$ since the basis of M is given by $S_2 = \{1, (12)\}$. So if $M \cong U \oplus V$, then we must have that $\dim(U) = 1$ and $\dim(V) = 1$. Thus, our goal is to classify the 1-dimensional submodules of M. That is, we want to find $U = \mathbb{F}_2 u$ with $0 \neq u \in \mathbb{F}_2 S_2$ satisfying $(12) \cdot u = \lambda u$ for some $\lambda \in \mathbb{F}_2$. □

2. Prove that M is simple if and only if $Am = M$ for any non-zero $m \in M$.

   Proof. Suppose M is simple. Then the only submodules of M are 0 and itself. We have that Am is a submodule of M for any $m \neq 0 \in M$. Since M is simple, we have that either $Am = 0$ or $Am = M$. However, we know that $m \neq 0$. Since A is a ring with 1, we have that $1m = m \in Am$ and so $Am \neq 0$. Thus, we must have that $Am = M$ for any non-zero $m \in M$.

   Now suppose that $Am = M$ for any non-zero $m \in M$. Let $N \subset M$ be a submodule and suppose $N \neq 0$. Thus there is some $m \in N \setminus \{0\} \subset M \setminus \{0\}$. Now since $Am = M$ for every non-zero $m \in M$, we must have that $Am = M$ for this particular choice of m. Since N is a submodule and closed under the action of A on N, we must have that $N = M$. Thus, M is simple. □

3. Let M be a free module with basis $\mathcal{B}$ and let I be an ideal of A. We showed on Homework 4 that IM is a submodule of M, and it follows similarly that Im is a submodule of M for any $m \in M$.[1]

   Prove that as A-modules, we have
   $$M/IM \cong \bigoplus_{b \in \mathcal{B}} Ab/Ib.$$
   [*Hint:* Since $M \cong \mathcal{F}\mathrm{r}(\mathcal{B})$ is free, we know it decomposes similarly to the right-hand side of this expression. So what's the most natural homomorphism from M to $\bigoplus_{b \in \mathcal{B}} Ab/Ib$? What's its kernel? (You may assume the isomorphism theorems, as found in Lang §III.1 and D & F §10.2.) See also, D& F, Exercises 10.2.11 and 10.2.12.]

   Proof. Since M is a free module, we have that
   $$M \cong \bigoplus_{b \in \mathcal{B}} A$$

[1] This follows considering $M' = Am$, which is a submodule of M. Then since $1 \in A$, we have $IA = A$, so that $IM' = (IA)m = Im$ is also a submodule. Or, you know, just check it directly.

Let $I \subset A$ be an ideal of A. Recall that IM is a submodule of M, where

$$IM := \{ \sum_{a \in I, m \in M, \text{finite}} a_m \} = \{a_1 m_1 + \cdots + a_\ell m_\ell | a_i \in I, m_i \in M, \ell \in \mathbb{Z}_{\geq 0}\}$$

We want to show that $M/IM \cong \bigoplus_{b \in \mathcal{B}} Ab/Ib$. Note that

$$M \cong \bigoplus_{b \in \mathcal{B}} A \cong \bigoplus_{b \in \mathcal{B}} Ab$$

since $A \cong Ab$ because of the free action. Plugging this into the original equation we want to show gives us that we now want to show,

$$\left(\bigoplus_{b \in \mathcal{B}} A_b \right) / \left(I \bigoplus_{b \in \mathcal{B}} Ab \right) \cong \bigoplus_{b \in \mathcal{B}} Ab/Ib$$

We want to apply the First Isomorphism Theorem here, so let us construct a map $\varphi : \bigoplus_{b \in \mathcal{B}} Ab \to \bigoplus_{b \in \mathcal{B}} Ab/Ib$ which is an A-module homomorphism. Moreover, we wish to construct φ such that it is surjective and that $\ker(\varphi) = I \bigoplus_{b \in \mathcal{B}} Ab$. Note that we can write

$$\bigoplus_{b \in \mathcal{B}} Ab = \{(a_b)_{b \in \mathcal{B}} | a_b \in Ab, a_b = 0 \text{ for all but finitely many } b \in \mathcal{B}\}$$

So let us define $\varphi((a_b)_{b \in \mathcal{B}}) = (a_b + Ib)_{b \in \mathcal{B}}$.

□

4. **Ranks of free modules.**

   (a) **Rank is well-defined for commutative rings.**[2] Show that if A is a commutative ring with 1, that $A^m \cong A^n$ if and only if $n = m$.
   [Hint: Let I be a maximal ideal of A (what kind of ring does that make A/I?). Now consider A^m/IA^m, as in Problem 3. Recall that we showed earlier in the semester that two finite-dimensional *vector spaces* were isomorphic if and only if they had the same dimension.]

   Proof. Suppose A is is a commutative ring with 1 and suppose that $A^m \cong A^n$. Let I be a maximal ideal of A. Since $A^m \cong A^n$, we have that $IA^m \cong IA^n$, and so $A^m/IA^m \cong A^n/IA^n$. Moreover, on Q3, we proved that for a free module M over a ring A with basis $\mathcal{B}$ and ideal I, we have that,

   $$M/IM \cong \bigoplus_{b \in \mathcal{B}} Ab/Ib$$

   If we fix A as our ring with A^m, A^n as the free modules over this ring, we can apply this result as follows,

   $$A^m/IA^m \cong \oplus_{b \in \mathcal{B}} Ab/Ib$$
   $$A^n/IA^n \cong \oplus_{c \in \mathcal{C}} Ac/Ic$$

   Thus, we have that $\oplus_{b \in \mathcal{B}} Ab/Ib \cong \oplus_{c \in \mathcal{C}} Ac/Ic$. Since $\oplus_{b \in \mathcal{B}} Ab/Ib$ is $|\mathcal{B}|$-dimensional and $\oplus_{c \in \mathcal{C}} Ac/Ic$ is $|\mathcal{C}|$-dimensional, we must have that $|\mathcal{B}| = |\mathcal{C}|$. But note that $\mathcal{B}$ is a basis for A^m and $\mathcal{C}$ is a basis for A^n. We know that A^m must have a basis of size m and A^n must have a basis of size n, so let us take $\mathcal{B}$ and $\mathcal{C}$ to be of these sizes, respectively. Thus, we have that $|\mathcal{B}| = |\mathcal{C}|$ implies that $n = m$, as required.

[2] D&F Exercise 10.3.2.

Now suppose $n = m$. Then we must have $A^m = A^n$ and so $A^m \cong A^n$ by the identity map. $\square$

(b) **Rank is not well-defined in general.**[3] Let M be the $\mathbb{Z}$-module

$$M = \mathbb{Z} \times \mathbb{Z} \times \cdots = \{(n_1, n_2, \dots) \mid n_i \in \mathbb{Z}\}.$$

Note the difference here between this and $N = \bigoplus_{i \in \mathbb{Z}_{>0}} \mathbb{Z}$, which is the submodule of M where all but finitely many n_i are 0. In particular, N is free, but M is not (see D&F Exercise 10.3.24).

Let

$$A = \mathrm{End}_{\mathbb{Z}}(M) = \{\ \mathbb{Z}\text{-module homomorphisms } \varphi : M \to M\ \},$$

where, as usual, addition is defined point-wise (i.e. $(\varphi + \psi)(a) = \varphi(a) + \psi(a)$) and multiplication is defined by function composition (i.e. $\varphi\psi = \varphi \circ \psi$). See also D&F Exercise 7.1.30.

Define $\varphi_1, \varphi_2 \in A$ by

$$\varphi_1(n_1, n_2, n_3, \dots) = (n_1, n_3, n_5, \dots) \quad \text{and} \quad \varphi_2(n_1, n_2, n_3, \dots) = (n_2, n_4, n_6, \dots).$$

(i) **Show that $\{\varphi_1, \varphi_2\}$ is a free basis of the the left regular module of A.** [*Hnit:* Define $\psi_1, \psi_2 \in R$ by

$$\psi_1(n_1, n_2, n_3, \dots) = (n_1, 0, n_2, 0, n_3, \dots) \quad \text{and} \quad \psi_2(n_1, n_2, n_3, \dots) = (0, n_1, 0, n_2, 0, \dots).$$

Verify that

$$\varphi_i\psi_i = 1, \quad \varphi_1\psi_2 = 0 = \varphi_2\psi_1, \quad \text{and} \quad \psi_1\varphi_1 + \psi_2\varphi_2 = 1.$$

Use these relations to prove that φ_1, φ_2 are independent and generate A as a left A-module.]

Proof. We have that,

$$\begin{aligned} \varphi_1 \circ \psi_1(n_1, n_2, n_3, \dots) &= \varphi_1(n_1, 0, n_2, 0, n_3, \dots) \\ &= (n_1, n_2, n_3, \dots) \end{aligned}$$

and,

$$\begin{aligned} \varphi_2 \circ \psi_2(n_1, n_2, n_3, \dots) &= \varphi_2(0, n_1, 0, n_2, 0, \dots) \\ &= (n_1, n_2, n_3, \dots) \end{aligned}$$

Thus, $\varphi_i\psi_i = 1$. Moreover,

$$\begin{aligned} \varphi_1 \circ \psi_2(n_1, n_2, n_3, \dots) &= \varphi_1(0, n_1, 0, n_2, 0, \dots) \\ &= (0, 0, 0, \dots) \end{aligned}$$

and,

$$\begin{aligned} \varphi_2 \circ \psi_1(n_1, n_2, n_3, \dots) &= \varphi_2(n_1, 0, n_2, 0, n_3, \dots) \\ &= (0, 0, 0, \dots) \end{aligned}$$

[3] D&F Exercise 10.3.27.

As a result, we have $\varphi_1\psi_2 = 0 = \varphi_2\psi_1$. Lastly, we check,

$$\begin{aligned}(\psi_1\varphi_1 + \psi_2\varphi_2)(n_1, n_2, n_3, \ldots) &= \psi_1\varphi_1(n_1, n_2, n_3, \ldots) + \psi_2\varphi_2(n_1, n_2, n_3, \ldots) \\ &= \psi_1(n_1, n_3, n_5, \ldots) + \psi_2(n_2, n_4, n_6, \ldots) \\ &= (n_1, 0, n_3, 0, n_5, \ldots) + (0, n_2, 0, n_4, 0, n_6, \ldots) \\ &= (n_1, n_2, n_3, \ldots)\end{aligned}$$

Thus, we have $\psi_1\varphi_1+\psi_2\varphi_2 = 1$. Hence, φ_1 and φ_2 are independent and span $\mathrm{End}_{\mathbb{Z}}(M)$. Hence $\{\varphi_1, \varphi_2\}$ is a free basis of the left regular module of A. $\square$

(ii) Use the previous part to **prove that** $A \cong A^2$ **as** A**-modules**. Deduce that $A \cong A^n$ as A-modules for all $n \in \mathbb{Z}_{>0}$.

Proof. We have that $A = A1$ as an A-module and $A^2 = A\{\varphi_1, \varphi_2\}$ as an A-module. But 1 and $\{\varphi_1, \varphi_2\}$ are both free bases for the left regular module, and so $A \cong A^2$ as A-modules. $\square$

5. In class, we showed that for any A-module X, we have $\mathrm{Hom}_A(A, X) \cong A$ as groups (or as A-modules if A is commutative). It is *not* necessarily true that $\mathrm{Hom}_A(X, A) \cong A$.

Now suppose A is commutative and let $\mathcal{B}$ be a finite set of size n.

Prove that $\mathrm{Hom}_A(\mathcal{F}\mathrm{r}(\mathcal{B}), A) \cong \mathcal{F}\mathrm{r}(\mathcal{B})$ **as** A**-modules.**

(Namely, just like any finite-dimensional vector space is isomorphic its dual, we have any free module of finite rank is also isomorphic to its dual.) We say $\mathcal{F}\mathrm{r}(\mathcal{B})$ is *self-dual* (up to isomorphism). [*Hint:* Reasonable tactics include either showing that $\mathrm{Hom}_A(\mathcal{F}\mathrm{r}(\mathcal{B}), A)$ is free of rank n and using 4(a); or using the last proposition from Lecture 10.]

Proof. $\square$

To receive credit for this assignment, include the following in your solutions [edited appropriately]:

Academic integrity statement: I *did not violate* the CUNY Academic Integrity Policy in completing this assignment. *Christopher Hayduk*
