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Math B4900
Homework 4
3/14/2021

Let A be a ring with 1, and let M be an A-module.

1. Let z be a central element of A. Show that zM is a submodule of M and that $\varphi : M \to M$ defined by $m \mapsto zm$ is an A-module endomorphism.

2. Let I be an ideal of A. Show that

$$IM = \left\{ \sum_{\text{fin.}} \alpha m_\alpha \;\middle|\; \alpha \in I, m_\alpha \in M \right\}$$

is a submodule of M. Give an example where $IM = M$ and an example where $0 \neq IM \subsetneq M$ Finally, give an example showing that if B is a subring of A, then BM is not necessarily a submodule of M.

3. Let $\varphi : M \to N$ be an A-module homomorphism. Prove that $\ker(\varphi)$ is a submodule of M and $\operatorname{img}(\varphi)$ is a submodule of N. Further, show that if M and N are simple, then either $\varphi = 0$ or φ is an isomorphism. [You may use anything that we have already proven for groups (since M and N are also additive groups.]

4. Let X and Y be submodules of M. Show that

$$0 \hookrightarrow X \cap Y \xrightarrow{f : x \mapsto (x,x)} X \oplus Y \xrightarrow{g:(x,y) \mapsto x+y} X + Y \to 0$$

is a short exact sequence of A-modules.

5. Let

$$0 \hookrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \quad \text{and} \quad 0 \hookrightarrow X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \to 0$$

be short exact sequences. A collection of homomorphisms

$$\alpha : X \to X', \quad \beta : Y \to Y', \quad \text{and } \gamma : Z \to Z'$$

is a *homomorphism of exact sequences* if the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r, hook] & X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r, hook] & X' \arrow[r, "f'"'] & Y' \arrow[r, "g"'] & Z' \arrow[r] & 0
\end{tikzcd} \tag{1}$$

If α, β, and γ are isomorphisms, then this is an *isomorphism* or *equivalence of exact sequences.*

(a) Fill in the unknown modules and homomorphisms to make the following diagram into a homomorphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r, hook] & ? \arrow[r, "\boxed{?}"] \arrow[d, "\boxed{?}"'] & \mathbb{Z} \arrow[r, "\pi"] \arrow[d, "*3"'] & \mathbb{Z}/6\mathbb{Z} \arrow[r] \arrow[d, "\boxed{?}"'] & 0 \\
0 \arrow[r, hook] & ? \arrow[r, "\boxed{?}"'] & \mathbb{Z} \arrow[r, "\pi"'] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

(b) If you have a homomorphism of short exact sequences as in (1), and you know something about the properties of α, β, or γ, what else (if anything) can you infer about the other two? Namely, fill in the blank (briefly justifying your answers):

If ($\alpha/\beta/\gamma$) is (injective/surjective/bijective), then...

	injective	surjective	bijective
α			
β		(e.g. γ *is surjective*)	
γ			

[For example, if β is surjective, then you can infer that γ is surjective, but nothing can be said about α. (Why?)]

To receive credit for this assignment, include the following in your solutions [edited appropriately]:

Academic integrity statement: I *[violated/did not violate]* the CUNY Academic Integrity Policy in completing this assignment. *[enter your full name as a digital signature here]*
