Chris Haydук 2/2/21

Lecture 2, Ex A

1a. We have $\phi, \psi \in \mathrm{Hom}_F(V, W)$, so

$$\phi, \psi : V \to W$$

Thus,

$$(\phi + \psi)(v) = \phi(v) + \psi(v) \in W$$

And

$$(\alpha \phi)(v) = \alpha(\phi(v)) \in W$$

since $F \times W \to W$ by W being vector space

Now, ϕ, ψ are linear (since $\in \mathrm{Hom}_F(V, W)$):

$$(\phi + \psi)(v_1 + v_2) = \phi(v_1 + v_2) + \psi(v_1 + v_2)$$
$$= \phi(v_1) + \phi(v_2) + \psi(v_1) + \psi(v_2)$$
$$= (\phi + \psi)(v_1) + (\phi + \psi)(v_2)$$

$$(\phi + \psi)(\beta v) = \phi(\beta v) + \psi(\beta v)$$
$$= \beta \phi(v) + \beta \psi(v)$$
$$= \beta(\phi(v) + \psi(v))$$
$$= \beta((\phi + \psi)(v))$$

$\Rightarrow \phi + \varphi \in Hom_F(V, W)$

$u, v \in V$, $\beta \in F$

$$(\alpha \phi)(u+v) = \alpha(\phi(u+v))$$
$$= \alpha(\phi(u) + \phi(v))$$
$$(\alpha \phi)(\beta v) = \alpha((\phi(\beta v))$$
$$= \alpha \beta \phi(v)$$
$$= \beta \alpha(\phi(v))$$
$$= \beta(\alpha \phi(v))$$

So $\alpha \phi \in Hom_F(V, W)$

b. $\phi(v)$, $\varphi(v) \in W$ $\forall v \in V$

Since W is additive group, then $Hom_F(V, W)$ is additive.

Identity: $\phi(v) = 0$ $\forall v$

Inverse: $\phi(v) + (-\phi(v)) = 0$

W is vector space over F.

Since $\phi(v) \in W$ $\forall v, \phi$, have that F-action is associative and distributive here too.

2. The additive identity $\phi(v) = 0$ $\forall v$ has no well-defined inverse so not in $GL(V)$

3 $(3,-1) = \alpha_1 v_1 + \alpha_2 v_2$

$= \alpha_1(1,-1) + \alpha_2(1,1)$

$3 = \alpha_1 + \alpha_2, \quad -1 = -\alpha_1 + \alpha_2$

$\alpha_1 = 2, \quad \alpha_2 = 1$

$$\begin{aligned}
\rho((3,-1)) &= \rho(2(1,-1) + 1\cdot(1,1)) \\
&= 2\rho(1,-1) + \rho(1,1) \\
&= 2(1,1,2) + (0,3,0) \\
&= (2,2,4) + (0,3,0) \\
&= (2,5,4)
\end{aligned}$$

4. Assume $\phi: \mathbb{R} \to \mathbb{R}$ sends $1 \to 1$, $2 \to 1$

Then,

$$\phi(1+1) = \phi(2) = 1$$

But

$$\phi(1) + \phi(1) = 1 + 1 = 2$$

So $\phi(1+1) \neq \phi(1) + \phi(1)$

5.
$$(\alpha + \beta)\overline{v} = \overline{(\alpha+\beta)v}$$
$$= \overline{\alpha v + \beta v}$$
$$= \overline{\alpha v} + \overline{\beta v}$$
$$= \alpha \overline{v} + \beta \overline{v}$$

$$(\alpha\beta)\overline{v} = \overline{(\alpha\beta)v}$$
$$= \overline{\alpha(\beta v)}$$
$$= \alpha(\overline{\beta v})$$
$$= \alpha(\beta \overline{v})$$

$$\alpha(\bar{v}+\bar{w}) = \alpha\,(\overline{v+w})$$
$$= \overline{\alpha(v+w)}$$
$$= \overline{\alpha v + \alpha w}$$
$$= \overline{\alpha v} + \overline{\alpha w}$$
$$= \alpha\bar{v} + \alpha\bar{w}$$

6 $\alpha \in F$, $v \in V$, $\phi \in \mathrm{Ker}(\phi)$

$$\phi(\alpha v) = \alpha\phi(v) = 0$$

So $\alpha v \in \mathrm{Ker}(\phi)$

7. Independent

$$\Rightarrow 0 = \phi(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1\phi(v_1) + \alpha_2\phi(v_2)$$
$$\Longrightarrow \alpha_1 = \alpha_2 = 0$$

$\mathrm{Ker}(\phi) = 0$, $\dim(\mathrm{Ker}(\phi)) = 0$