Chris Hayduk
Math B4900
Final proofs portfolio
May 25, 2020

Problem	$\star$**s**	**Points**	**Tot**
1C	2		
2B	2		
3B	1		
5A	2		
6A	2		
6C	2		
8B	1		
8C	2		
9A	2		

Problem:	**1C**
No. stars:	**2**

Statement: Let F be a field and V be a vector space over F. Fix $\varphi \in \operatorname{End}(V)$. For $\lambda \in F$, prove that the weight space V_λ and the generalized weight space V^λ are both subspaces of V.

Proof. Let F be a field and V be a vector space over F. Fix $\lambda \in F$. By definition, every $v \in V_\lambda$ is also an element of V. Hence, we have $V_\lambda \subset V$. Now observe that $\lambda 0 = 0$ for any $\lambda \in F$. Hence, $0 \in V_\lambda$ and thus V_λ is non-empty. Now, by the submodule criterion, we just need to show that $x + ry \in V_\lambda$ for all $r \in F$ and for all $x, y \in V_\lambda$. Let us start by applying φ to this element and using the properties of the linearity of φ,

$$\begin{aligned}\varphi(x + ry) &= \varphi(x) + r\varphi(y) \\ &= \lambda x + r\lambda y \\ &= \lambda(x + ry)\end{aligned}$$

Hence, $x + ry \in V_\lambda$ and so V_λ is a subspace of V.

By definition, every $v \in V^\lambda$ is also an element of V. Hence, we have $V^\lambda \subset V$. Now observe that, for any $\lambda \in F$,

$$\begin{aligned}&\varphi(v) = \lambda 0 = 0v = 0 \\ \iff &(\varphi - \lambda \cdot \mathrm{id})(0) = 0\end{aligned}$$

Hence, $0 \in V^\lambda$ and thus V^λ is non-empty. Again, by the submodule criterion, we just need to show that $x + ry \in V^\lambda$ for all $r \in F$ and for all $x, y \in V^\lambda$. Since, $x, y \in V^\lambda$, we have that

$$\begin{aligned}(\varphi - \lambda \cdot \mathrm{id})^\ell(x) = 0 \\ (\varphi - \lambda \cdot \mathrm{id})^m(y) = 0\end{aligned}$$

Let $k = \max\{\ell, m\}$. Then by the fact that if $(\varphi - \lambda \cdot \mathrm{id})^m(v) = 0$, then $(\varphi - \lambda \cdot \mathrm{id})^n v = 0$ for all integers $n \geqslant m$ and by the fact that linear combinations and compositions of linear functions are linear, we have that,

$$\begin{aligned}(\varphi - \lambda \cdot \mathrm{id})^k(x + ry) &= (\varphi - \lambda \cdot \mathrm{id})^k(x) + r(\varphi - \lambda \cdot \mathrm{id})^k(y) \\ &= 0 + r0 \\ &= 0\end{aligned}$$

Hence, $x + ry \in V^\lambda$ and so V^λ is also a subspace of V. $\square$

	Points Possible					
complete	0	1	2	3	4	5
mathematically valid	0	1	2	3	4	5
readable/fluent	0	1	2	3	4	5
Total:						(out of 15)

Statement: Prove that determinant is invariant under change of basis. [The details required in this proof are outlined in Homework 2; be sure to hit all the beats highlighted in that problem statement.]

Problem:	**2B**
No. stars:	**2**

Proof. Note that for I_n, we have that $\alpha_{ii} = 1$ for all $1 \leqslant i \leqslant n$ and $\alpha_{ij} = 0$ for all $i \neq j$. Hence, in the definition of the determinant, we must have that the only non-zero term in the summation is the one corresponding to the identity $\sigma = 1$. This gives us,

$$\begin{aligned}\det(I_n) &= \operatorname{sgn}(1)\alpha_{1,1}\alpha_{2,2}\cdots\alpha_{n,n}\\ &= 1\cdot 1\cdot 1\cdots 1\\ &= 1\end{aligned}$$

Now let $A \in \mathrm{GL}_n(F)$. Then A is invertible with inverse A^{-1}. But A^{-1} is also invertible with inverse A, so $A^{-1} \in \mathrm{GL}_n(F)$. Hence, by fact (2) on the Lecture 4 worksheet, we have $\det(A^{-1}), \det(A)^{-1} \neq 0$. Furthermore, since $\mathrm{GL}_n(F) \subset M_n(F)$, we have that $A, A^{-1} \in M_n(F)$ and so we can apply fact (3) from the Lecture 4 worksheet. Thus, we have that,

$$\begin{aligned}\det(AA^{-1}) &= \det(I_n)\\ &= 1\\ &= \det(A)\det(A^{-1})\end{aligned}$$

Since $\det(A)\det(A^{-1}) = 1$, we have that $\det(A^{-1}) = \det(A)^{-1}$.

Now let $B \in \mathrm{GL}_n(F)$. Consider $\det(ABA^{-1})$ and using fact (3) along with the associativity of matrix multiplication, we get,

$$\begin{aligned}\det((AB)A^{-1}) &= \det(AB)\det(A^{-1})\\ &= \det(A)\det(B)\det(A^{-1})\end{aligned}$$

By our initial derivation, we have that $\det(A^{-1}) = \det(A)^{-1}$ and so, by the fact that F is a field and hence commutative, we have,

$$\begin{aligned}\det((AB)A^{-1}) &= \det(A)\det(B)\det(A^{-1})\\ &= \det(A)\det(B)\det(A)^{-1}\\ &= \det(A)\det(A)^{-1}\det(B)\\ &= 1\cdot\det(B)\\ &= \det(B)\end{aligned}$$

Thus, if we let A be the matrix of the determinant under basis $\mathcal{A}$, and let P be the change of basis matrix from $\mathcal{A}$ to some basis $\mathcal{B}$. Then $P^{-1}AP = B$, where B is the matrix of the determinant under basis $\mathcal{B}$. However, from the above we get,

$$\begin{aligned}\det(P^{-1}AP) &= \det(A)\\ &= \det(B)\end{aligned}$$

Hence, the determinant is invariant under change of basis. $\square$

	Points Possible
complete	0 1 2 3 4 5
mathematically valid	0 1 2 3 4 5
readable/fluent	0 1 2 3 4 5
Total:	(out of 15)

Statement: Let $X \in M_n(\mathbb{C})$, let Λ be the set of eigenvalues for X, and let m_λ be the multiplicity of $\lambda \in \Lambda$. Show

$$\operatorname{tr}(X) = \sum_{\lambda \in \Lambda} \lambda m_\lambda \quad \text{and} \quad \det(X) = \prod_{\lambda \in \Lambda} \lambda^{m_\lambda}.$$

Problem:	**3B**
No. stars:	**1**

Proof. Since $\mathbb{C}$ is an algebraically closed field and $X \in M_n(\mathbb{C})$, we have that there is some J in Jordan canonical form such that $J \sim X$. That is, there is some choice of basis under which X can be written in Jordan canonical form. Trace is invariant under choice of basis, so we have that $\operatorname{tr}(X) = \operatorname{tr}(J)$. Note that in Jordan canonical form, the eigenvalues of X are placed along the diagonal. Thus, the diagonal of J contains all of the eigenvalues of X. Moreover, the multiplicity of an eigenvalue is given by the number of rows in which it appears in the matrix J. Hence, we must have that,

$$\begin{aligned} \operatorname{tr}(X) &= \operatorname{tr}(J) \\ &= \sum_{\lambda \in \Lambda} \lambda m_\lambda \end{aligned}$$

as required.

Similarly to the above, we have $\det(X) = \det(J)$. Now consider the characteristic polynomial of J. This is given by,

$$\begin{aligned} c_J(x) &= \det(J - x \cdot \mathrm{id}) \\ &= \prod_{\lambda \in \Lambda} (\lambda - x)^{m_\lambda} \end{aligned}$$

If we plug in 0 for x, we get,

$$\begin{aligned} c_J(0) &= \det(J - 0 \cdot \mathrm{id}) \\ &= \det(J) \\ &= \prod_{\lambda \in \Lambda} (\lambda)^{m_\lambda} \end{aligned}$$

Hence, we have that,

$$\begin{aligned} \det(X) &= \det(J) \\ &= \prod_{\lambda \in \Lambda} (\lambda)^{m_\lambda} \end{aligned}$$

as required. $\square$

	Points Possible					
complete	0	1	2	3	4	5
mathematically valid	0	1	2	3	4	5
readable/fluent	0	1	2	3	4	5
Total:					(out of 15)	

Problem:	**5A**
No. stars:	**2**

Statement: Prove that $\text{Hom}_A(*, M)$ is an exact functor. Namely, show that if $0 \hookrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is a split exact sequence of A-modules, then so is

$$0 \hookrightarrow \text{Hom}_A(M, X) \xrightarrow{F} \text{Hom}_A(M, Y) \xrightarrow{G} \text{Hom}_A(M, Z) \to 0,$$

where $F(\varphi) = f \circ \varphi$ and $G(\varphi) = g \circ \varphi$. [You may use any other propositions or theorems from Lecture 10 or before.]

Proof. By Proposition 2.2 in Section 3 of Lang, we have that,

$$0 \hookrightarrow \text{Hom}_A(M, X) \xrightarrow{F} \text{Hom}_A(M, Y) \xrightarrow{G} \text{Hom}_A(M, Z)$$

is exact. Moreover, we can assert that

$$0 \hookrightarrow \text{Hom}_A(M, X) \xrightarrow{F} \text{Hom}_A(M, Y) \xrightarrow{G} \text{Hom}_A(M, Z) \hookrightarrow 0$$

is a short exact sequence because $\text{Im}(G) = \text{Hom}_A(M, Z)$ and $\ker(0) = \text{Hom}_A(M, Z)$. Now define $\mu : \text{Hom}_A(M, Z) \to \text{Hom}_A(M, Y)$ by $\mu(\varphi) = \varphi^{-1} \circ g^{-1}$ and define $\lambda : \text{Hom}_A(M, Y) \to \text{Hom}_A(M, X)$ by $\lambda(\varphi) = \varphi^{-1} \circ f^{-1}$. Then

$$\begin{aligned} G\mu &= g \circ \varphi \circ \varphi^{-1} \circ g^{-1} \\ &= \text{id} \end{aligned}$$

and

$$\begin{aligned} \lambda f &= \varphi^{-1} \circ f^{-1} \circ f \circ \varphi \\ &= \text{id} \end{aligned}$$

Hence, by the Proposition from Lecture 10 part B, we have that our short exact sequence is also split, with μ and λ as the splitting homomorphisms. ☐

	Points Possible					
complete	0	1	2	3	4	5
mathematically valid	0	1	2	3	4	5
readable/fluent	0	1	2	3	4	5
Total:						(out of 15)

Problem:	**6A**
No. stars:	**2**

Statement: Prove that M is simple if and only if $Am = M$ for any non-zero $m \in M$.

Proof. Suppose M is simple. Then the only submodules of M are 0 and itself. By the footnote on Homework 6, we have that Am is a submodule of M. Since M is simple, we have that either $Am = 0$ and $Am = M$. However, we know that $m \neq 0$. Since A is a ring with 1, we have that $1m = m \in Am$ and so $Am \neq 0$. Thus, we must have that $Am = m$ for any non-zero $m \in M$.

Now suppose that $Am = M$ for any non-zero $m \in M$. Let $N \subset M$ be a submodule and suppose $N \neq 0$. Thus there is some $m \in N \backslash \{0\} \subset M \backslash \{0\}$. Now since $Am = M$ for every non-zero $m \in M$, we must have that $Am = M$ for this particular choice of m. Since N is a submodule and closed under the action of A on N, we must have that $N = M$. Thus, M is simple. □

	Points Possible
complete	0 1 2 3 4 5
mathematically valid	0 1 2 3 4 5
readable/fluent	0 1 2 3 4 5
Total:	(out of 15)

Problem:	**6C**
No. stars:	**2**

Statement: Show that if A is a commutative ring with 1, that $A^m \cong A^n$ if and only if $n = m$.

Proof. Suppose A is is a commutative ring with 1 and suppose that $A^m \cong A^n$. Let I be a maximal ideal of A. Since $A^m \cong A^n$, we have that $IA^m \cong IA^n$, and so $A^m/IA^m \cong A^n/IA^n$. Moreover, from Q3 on Homework 6, we have that,

$$A^m/IA^m \cong \oplus_{b \in \mathcal{B}} Ab/Ib$$
$$A^n/IA^n \cong \oplus_{c \in \mathcal{C}} Ac/Ic$$

Thus, we have that $\oplus_{b \in \mathcal{B}} Ab/Ib \cong \oplus_{c \in \mathcal{C}} Ac/Ic$.

Since I is a maximal ideal of A, by Proposition 12 in Section 7.4, we have that Ab/Ib and Ac/Ic are fields for every b, c.

Now suppose $n = m$. Then we must have $A^m = A^n$ and so $A^m \cong A^n$ by the identity map. □

	Points Possible
complete	0 1 2 3 4 5
mathematically valid	0 1 2 3 4 5
readable/fluent	0 1 2 3 4 5
Total:	(out of 15)

Problem:	**8B**
No. stars:	**1**

Statement: Classify the semisimple $\mathbb{Z}$-modules.

Proof. Note that a $\mathbb{Z}$ module is simple if it is of the form $\mathbb{Z}/I$ where I is a maximal ideal. The ideals of $\mathbb{Z}$ are precisely the sets of all integers divisible by a fixed integer n. That is, $n\mathbb{Z}$ is an ideal for all $n \in \mathbb{Z}$. Recall that an ideal $n\mathbb{Z}$ of $\mathbb{Z}$ is maximal if there are no other ideals of the form $k\mathbb{Z}$ such that $n\mathbb{Z} \subset k\mathbb{Z} \subset \mathbb{Z}$. Observe that if n is a composite integer, then we can write $n = p_1 p_2 \cdots p_\ell$ for primes in $\mathbb{Z}$. That is, for any p_j in that expansion, we have that p_j divides n and thus all multiples of n. Hence, $n\mathbb{Z} \subset p_j\mathbb{Z}$ for any prime p_j in that expansion. Moreover, for every prime we must have that there is no integer m such that $p_j\mathbb{Z} \subset m\mathbb{Z}$, otherwise m would divide p_j and hence p_j would not be prime. Thus, the maximal ideals of $\mathbb{Z}$ are precisely of the form $p\mathbb{Z}$ where p is a prime.

Now we have that the simple modules of $\mathbb{Z}$ are of the form $\mathbb{Z}/p\mathbb{Z}$ for all primes $p \in \mathbb{Z}$. Since semisimple modules are direct sums of simple modules, we have that any semisimple module of $\mathbb{Z}$ is of the form:

$$p_1\mathbb{Z} \oplus p_2\mathbb{Z} \oplus \cdots \oplus p_\ell\mathbb{Z}$$

for some primes $p_1, \ldots, p_\ell$ (not necessarily distinct). □

	Points Possible
complete	0 1 2 3 4 5
mathematically valid	0 1 2 3 4 5
readable/fluent	0 1 2 3 4 5
Total:	(out of 15)

Problem:	**8C**
No. stars:	**2**

Statement: Let M be a semisimple A-module. Prove that the following are equivalent:

(i) M is finitely-generated;
(ii) M is Noetherian;
(iii) M is Artinian;
(iv) M is a finite direct sum of simple modules.

Proof. First we will show that (i) is equivalent to (ii). Suppose M is finitely generated. Then there exist $m_1, m_2, \dots, m_n \in M$ such that for any $x \in M$, there exist $a_1, a_2, \dots, a_n \in A$ with $x = a_1 m_1 + a_2 m_2 + \cdots + a_n m_n$. Since every element of a submodule of M is also an element of M, then it must be true that every element of a submodule N of M is finitely generated as well. Hence, M is Noetherian. Now suppose M is Noetherian. Then every submodule of M is finitely generated. In particular, since M is a submodule of itself, it must be finitely generated. Thus, (i) and (ii) are equivalent.

Now we will show the equivalence of (i) and (iv). Suppose M is semisimple and finitely generated. Then M is the direct sum of simple modules and, since (i) is equivalent to (ii), each of those submodules is finitely generated. Since the generators of M are finite, they can old by combined in a finite number of ways. Hence, there must be finitely many submodules which are finitely generated. Hence, M is a finite direct sum of simple modules. Now let us assume that M is a finite direct sum of simple modules and work towards the other directions. Every simple module is cyclic and hence generated by one element. The union of these generators forms a basis for M since M is a direct sum of these simple modules. Since there are a finite number of these simple modules, then M is finitely-generated by this union as required. Hence, by this and our previous work, (i), (ii), and (iv) are equivalent.

Now we will show the equivalent of (iii) and (iv). Suppose M is semisimple and Artinian. Then the sequence of submodules of M

$$M_1 \supset M_2 \supset \dots$$

stabilizes. That is, there exists an integer N such that if $n \geqslant N$ then $M_n = M_{n+1}$. Since no simple module can have a submodule, then M_N is the only simple module in this chain. Observe that since M is semisimple, it must be the direct sum of simple submodules. There must be only finitely many of these simple submodules (why?), so M is a finite direct sum of simple modules. Now let us assume that M is semisimple and a finite direct sum of simple modules and work in reverse. □

	Points Possible
complete	0 1 2 3 4 5
mathematically valid	0 1 2 3 4 5
readable/fluent	0 1 2 3 4 5
Total:	(out of 15)

Problem:	**9A**
No. stars:	**2**

Statement: Let M be a completely reducible A-module. Show that for any submodule $N \subseteq M$, we have M/N is completely reducible as well. Moreover, if

$$M \cong \bigoplus_{\lambda \in \Lambda} M_\lambda, \quad \text{then} \quad M/N \cong \bigoplus_{\lambda \in \Gamma} M_\lambda,$$

for some $\Gamma \subseteq \Lambda$.

Proof. Since M is completely reducible, we have that $M \cong \bigoplus_{\lambda \in \Lambda} M_\lambda$ where M_λ is simple. This is equivalent to $M = \sum_{\lambda \in \Lambda} M_\lambda$ with $\left(\sum_{\lambda \in \Lambda - \mu} M_\lambda\right) \cap M_\mu = 0$. Hence, for any $m \in M$, we have that $m = \sum_{\lambda \in \Lambda} m_\lambda$ uniquely.

Let $N \subset M$. Then,

$$\begin{aligned} M/N &= \left(\sum_{\lambda \in \Lambda} M_\lambda\right)/N \\ &= \{m + N \mid m \in \sum_{\lambda \in \Lambda;\text{ finite}} M_\lambda\} \end{aligned}$$

Thus, for $m + N \in M/N$, we have,

$$\begin{aligned} m + N &= \sum_{\lambda;\text{ finite}} m_\lambda + N \\ &= \sum_{\lambda;\text{ finite}} m_\lambda + \sum_{\lambda; m_\lambda \neq 0} N \\ &= \{\sum_{\lambda;\ m_\lambda \neq 0} n_\lambda \mid n_\lambda \in N\} \\ &= N \\ &= \sum_{\lambda;\text{ finite}} (m_\lambda + N) \in (M_\lambda + N)/N \end{aligned}$$

The above derivation thus gives us that $M/N = \sum_\lambda (M_\lambda + N)/N$. Now, applying the second isomorphism theorem for modules, we have that,

$$\begin{aligned} M/N &= \sum_\lambda (M_\lambda + N)/N \\ &= \sum_\lambda M_\lambda/(M_\lambda \cap N) \end{aligned}$$

Observe that, since each M_λ is simple, we have that $M_\lambda \cap N = 0$ if $M_\lambda \not\subset N$ and $M_\lambda \cap N = M_\lambda$ if $M_\lambda \subset N$. These are the only two possible values for $M_\lambda \cap N$. Thus, for a fixed M_λ, we have either that,

$$\begin{aligned} M_\lambda/(M_\lambda \cap N) &= M_\lambda/0 \\ &= \{m_\lambda + 0 \mid m_\lambda \in M_\lambda\} \\ &= M_\lambda \end{aligned}$$

or,

$$\begin{aligned} M_\lambda/(M_\lambda \cap N) &= M_\lambda/M_\lambda \\ &= 0 \end{aligned}$$

Thus, we have that $\sum_\lambda M_\lambda/(M_\lambda \cap N)$ corresponds to some subset $\Gamma \subset \Lambda$, since the terms are either M_λ for some λ or 0. This gives us that,

$$\begin{aligned} M/N &= \sum_\lambda (M_\lambda + N)/N \\ &= \sum_\lambda M_\lambda/(M_\lambda \cap N) \\ &= \sum_{\lambda \in \Gamma} M_\lambda \\ &\cong \bigoplus_{\lambda \in \Gamma} M_\lambda \end{aligned}$$

as required. □

	Points Possible					
complete	0	1	2	3	4	5
mathematically valid	0	1	2	3	4	5
readable/fluent	0	1	2	3	4	5
Total:						(out of 15)