# HOMEWORK 9
# MATH B4900
# DUE: 4/30/2021

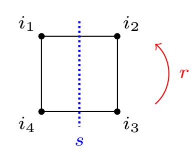
Let $A$ be a ring with 1.

1. Let $M$ be a completely reducible $A$-module. Show that for any submodule $N \subseteq M$, we have $M/N$ is completely reducible as well. Moreover, if
$$M \cong \bigoplus_{\lambda \in \Lambda} M_\lambda, \quad \text{then} \quad M/N \cong \bigoplus_{\lambda \in \Gamma} M_\lambda,$$
for some $\Gamma \subseteq \Lambda$.

   [*Hint:* if $M \cong \bigoplus_{\lambda \in \Lambda} M_\lambda$ (with $M_\lambda$ simple), then, more simply, $M = \sum_{\lambda \in \Lambda} M_\lambda$ (identifying $M_\lambda$ with $\hat{M}_\lambda$). Show that $M/N = \sum_\lambda (M_\lambda + N)/N$ (write out the cosets!), and then use the second isomorphism theorem on each piece. Finally, check that, for all $\mu \in \Lambda$, we have
$$(M_\mu + N)/N \cap \sum_{\lambda \neq \mu} (M_\lambda + N)/N = 0.]$$

2. Let $\{z_\lambda \mid \lambda \in \Lambda\}$ be the centrally primitive idempotents in a semisimple ring $A$, and let $U_\lambda$ be the simple $A$-module corresponding to $\lambda \in \Lambda$. Let $M$ be an $A$-module (not necessarily the left-regular module. Use Artin-Wedderburn to show that $z_\lambda M \cong \bigoplus_{i \in \mathcal{I}} U_\lambda$ (i.e. $z_\lambda$ projects onto a (not necessarily finite) direct sum of a bunch of copies of $U_\lambda$—called the $\lambda$-*isotypic component* of $M$).

3. Let $V = \mathbb{C}^2 = \mathbb{C}\{v_1, v_2\}$. Let $\mathbb{C}D_8$ act on $V^{\otimes 4} = V \otimes V \otimes V \otimes V$ by identifying the copies of $V$ with the vertices of the square, and applying the corresponding factor permutation:
$$r \cdot (v_{i_1} \otimes v_{i_2} \otimes v_{i_3} \otimes v_{i_4}) = v_{i_2} \otimes v_{i_3} \otimes v_{i_4} \otimes v_{i_1}$$
and
$$s \cdot (v_{i_1} \otimes v_{i_2} \otimes v_{i_3} \otimes v_{i_4}) = v_{i_2} \otimes v_{i_1} \otimes v_{i_4} \otimes v_{i_3}$$




   (where $i_1, i_2, i_3, i_4 \in \{1, 2\}$). For example, $r$ fixes $v_1 \otimes v_1 \otimes v_1 \otimes v_1$, but $r \cdot v_1 \otimes v_2 \otimes v_1 \otimes v_1 = v_2 \otimes v_1 \otimes v_1 \otimes v_1$.

   Use the primitive central idempotents of $\mathbb{C}D_8$ to decompose $V^{\otimes 4}$ into its isotypic components (you computed these idempotents in HW 5; you should also know which corresponds to which simple representations of $\mathbb{C}D_8$). Then make a dimension argument to classify the decomposition of $V^{\otimes 4}$ up to isomorphism—and make a complete decomposition if you can.

   [*See p. 2 for some help.*]

4. Let $S_3 \leq S_4$ in the usual way, and let $\mathcal{W}$ be the reflection representation. Compute the action of $s_1 = (12)$, $s_2 = (23)$, and $s_3 = (34)$ on $\mathrm{Ind}_{\mathbb{C}S_3}^{\mathbb{C}S_4}(\mathcal{W})$. [*Hint:* Stay organized!]

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*To receive credit for this assignment, include the following in your solutions [edited appropriately]:*

**Academic integrity statement:** I *[violated/did not violate]* the CUNY Academic Integrity Policy in completing this assignment. *[enter your full name as a digital signature here]*

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*Help with #3:* This is a big computational problem. But with a little bit of care, it won't be too bad. One tip is to encode a basis vector like $v_{i_1} \otimes v_{i_2} \otimes v_{i_3} \otimes v_{i_4}$ as $i_1 i_2 i_3 i_4$. For example, $v_1 \otimes v_2 \otimes v_1 \otimes v_1$ becomes 1211, and $r \cdot 1211 = 2111$. Another trick you have up your sleeve is action graphs; namely, the action of $\mathbb{C}D_8$ on $V^{\otimes 4}$ is a linear extension of the action of $D_8$ on $\{i_1 i_2 i_3 i_4 \mid i_\ell \in \{1, 2\}\}$. For example, one part of your action graph will look like

$$\begin{tikzcd}
1112 \arrow[r, "r"] \arrow[r, leftrightarrow, bend right, "s"'] & 1121 \arrow[d, "r"] \\
2111 \arrow[u, "r"] \arrow[r, leftrightarrow, bend left, "s"] & 1211 \arrow[l, "r"]
\end{tikzcd} .$$

Next, your job is to compute $z_j V^{\otimes 4}$ for each $j = 1, \ldots, 5$. But since the simple tensors $\{i_1 i_2 i_3 i_4 \mid i_\ell \in \{1, 2\}\}$ form a spanning set of $V^{\otimes 4}$; the action of $z_j$ on this set, $\{z_j \cdot i_1 i_2 i_3 i_4 \mid i_\ell \in \{1, 2\}\}$, sill form a spanning set of $z_j V^{\otimes 4}$. To compute $z_j V^{\otimes 4}$, you just need to compute $z_j \cdot i_1 i_2 i_3 i_4$ for each set of $i_\ell \in \{1, 2\}$, and taking the span of the result.

Now, recall that the coefficients in $z_1$ correspond to setting $r = 1$ and $s = 1$; the coefficients in $z_2$ correspond to setting $r = -1$ and $s = 1$; and so on. . . ; so the first four of these computations essentially amount to walking around the vertices of this graph, assigning $\pm 1$ coefficients by what edge we walk along, and then summing up the result. So for example, the computation of $z_2$ acting on 1112 looks like (starting from the upper-left corner, corresponding to the action of 1, and moving out)

$$\begin{tikzcd}
\text{start} \arrow[r] & \overset{1}{\boxed{1112}}_{+} \arrow[r, "r"] \arrow[d, bend right, "s"'] & \overset{r}{\boxed{1121}}_{-} \arrow[r, "r"] & \overset{r^2}{\boxed{1211}}_{+} \arrow[r, "r"] & \overset{r^3}{\boxed{2111}}_{-} \\
& \underset{s}{\boxed{1121}}^{+} \arrow[r, "r"'] & \underset{rs}{\boxed{1211}}^{-} \arrow[r, "r"'] & \underset{r^2 s}{\boxed{2111}}^{+} \arrow[r, "r"'] & \underset{r^3 s}{\boxed{1112}}^{-}
\end{tikzcd}$$

so $z_2 \cdot 1112 = \frac{1}{8}\Big(\boxed{1112} - \boxed{1121} + \boxed{1211} - \boxed{2111} + \boxed{1121} - \boxed{1211} + \boxed{2111} - \boxed{1112}\Big) = 0.$)

Continue computing the actions of the $z_j$ on the basis vectors, organize your computations by orbits. For example, setting

$$b_1 = 1112, \quad b_2 = 1121, \quad b_3 = 1211, \quad \text{and} \quad b_4 = 2111,$$

we have

$$\begin{aligned}
& z_1 b_i = \frac{1}{4}(b_1 + b_2 + b_3 + b_4) && \text{for } i = 1, 2, 3, 4; \\
& z_2 b_i = 0 \quad \text{and} \quad z_4 b_i = 0 && \text{for } i = 1, 2, 3, 4; \\
& z_3 b_1 = z_3 b_3 = -z_3 b_2 = -z_3 b_4 = \frac{1}{4}(b_1 - b_2 + b_3 - b_4); \\
& z_5 b_1 = -z_5 b_3 = \frac{1}{2}(b_1 - b_3); \quad \text{and} \quad z_5 b_2 = -z_5 b_4 = \frac{1}{2}(b_2 - b_4).
\end{aligned}$$

So

$$\begin{aligned}
& z_1 V^{\otimes 4} \text{ contains } b_1 + b_2 + b_3 + b_4; \\
& z_3 V^{\otimes 4} \text{ contains } b_1 - b_2 + b_3 - b_4; \quad \text{and} \\
& z_5 V^{\otimes 4} \text{ contains } b_1 - b_3 \text{ and } b_2 - b_4.
\end{aligned}$$

(We have accounted for 4 of 16 dim'ns in $V^{\otimes 4}$, so we're now 1/4 done with this computation!)