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Math B4900
Homework 10
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Let A be a ring with 1.

I **Lie algebras.**

1. For $x, y \in M_n(\mathbb{C})$, show that the binary operation $[x, y] = xy - yx$ is bilinear, skew symmetric, and satisfies the Jacobi identity.

2. Show that $\mathfrak{sl}_n(\mathbb{C}) \subseteq M_n(\mathbb{C})$ is, indeed, a closed Lie algebra.
[Namely, that if $\mathrm{tr}(x) = \mathrm{tr}(y) = 0$, then $\mathrm{tr}(xy - yx) = 0$ and $\mathrm{tr}(x + \alpha y) = 0$ for all $\alpha \in \mathbb{C}$.]

3. Let $L(d)$ be the simple $\mathfrak{sl}_2(\mathbb{C})$-module with dimension $d + 1$. Show that for all $d \geqslant 1$,
$$L(d) \otimes L(1) \cong L(d-1) \oplus L(d+1).$$
[Recall that we have a canonical action of a Lie algebra $\mathfrak{g}$ on the tensor product of two of its modules (see Hopf algebras).]

II **Characters.** Let G be a finite group, and let $A = \mathbb{C}G$.

1. **Characters and tensor products.** Let
$$\rho : A \to \mathrm{End}(U) \quad \text{and } \psi : A \to \mathrm{End}(V)$$
be finite-dimensional representations of A (so that U and V are A-modules). Let $\mathcal{B} = \{e_1, \dots, e_m\}$ and $\mathcal{B}' = \{f_1, \dots, f_n\}$ be ordered bases of U and V, respectively.

For $g \in G$, suppose
$$\rho(g) = \sum_{i,j=1}^{m} \alpha_{i,j} E_{i,j} = \begin{pmatrix} \alpha_{1,1} & \cdots & \alpha_{1,m} \\ \vdots & \ddots & \vdots \\ \alpha_{m,1} & \cdots & \alpha_{m,m} \end{pmatrix} \quad \text{and} \quad \psi(g) = \sum_{i,j=1}^{m} \beta_{i,j} E_{i,j} = \begin{pmatrix} \beta_{1,1} & \cdots & \beta_{1,n} \\ \vdots & \ddots & \vdots \\ \beta_{n,1} & \cdots & \beta_{n,n} \end{pmatrix},$$
with respect to the bases $\mathcal{B}$ and $\mathcal{B}'$, respectively. Namely,
$$g \cdot e_i = \sum_{\ell=1}^{m} \alpha_{\ell,i} e_\ell \quad \text{and} \quad g \cdot f_i = \sum_{\ell=1}^{n} \beta_{\ell,i} f_\ell,$$
for all i. Recall that $\mathcal{B} \times \mathcal{B}' = \{e_i \otimes f_j\}$ forms a basis of $U \otimes V$, and put the lexicographic order on it (i.e. $\mathcal{B} \times \mathcal{B}' = \{e_1 \otimes f_1, e_1 \otimes f_1, \dots, e_1 \otimes f_n, e_2 \otimes f_1, \dots, e_m \otimes f_n\}$).

   (a) Compute the action of g on each $e_i \otimes f_j$ for each i, j; and give the matrix for $(\rho \otimes \psi)(g)$, with respect to the ordered basis $\mathcal{B} \times \mathcal{B}'$, where $\rho \otimes \psi$ is the representation associated to the canonical action of FG on $U \otimes V$.

   (b) If

   χ_ρ is the character associated to ρ,

   χ_ψ is the character associated to ψ, and

   $\chi_{\rho \otimes \psi}$ is the character associated to $\rho \otimes \psi$,

   use the previous part to prove that $\chi_{\rho \otimes \psi} = \chi_\rho \chi_\psi$.

   (c) Conclude that the set of characters of $\mathbb{C}G$ forms a subring of $\mathrm{Cl}(\mathbb{C}G)$. Does this ring have an identity?

2. Let $\rho : \mathbb{C}G \to \mathrm{End}(U)$ be a finite-dimensional representation with character χ. Let $\rho_1, \ldots, \rho_r$ be the distinct simple representations of $\mathbb{C}G$, V_i be the corresponding simple modules, z_i be the corresponding primitive central idempotents, and let χ_i be their characters. Let $U_i = z_i U \cong V_i^{\oplus m_i}$ be the isotypic component of U corresponding to V_i. Show that $\dim(U_i) = \langle \chi, \chi_i \rangle$.
   [*Hint:* This should be a very short jump from things we proved in class.]

3. **Burnside's Lemma.** Let G act on a set Ω, and let $\Omega_1, \ldots, \Omega_\ell$ be the orbits of Ω. Define $U = \mathbb{C}\Omega$ (the vector space with basis Ω) and extend the action $G \circlearrowright \Omega$ linearly to an action $\mathbb{C}G \circlearrowright U$. Let $\rho : \mathbb{C}G \to \mathrm{End}(U)$ be the associated representation, and χ be the associated character.

   Fix $g \in G$. Define
   $$\mathrm{Fix}(g) = \{x \in \Omega \mid (12) \cdot x = x\}$$
   to be the number of fixed points under the action of g on Ω. [For example, if $G = S_4$ acts on naturally $\Omega = \{1, 2, 3, 4\}$, and $g = (12)$, then $\{x \in \Omega \mid (12) \cdot x = x\} = \{3, 4\}$. See below for more examples.]

   (a) Argue that $\chi(g) = |\mathrm{Fix}(g)|$.

   (b) Let $U_i = \mathbb{C}\Omega_i$. Argue briefly that $U \cong U_1 \oplus \cdots \oplus U_\ell$. For each i, let $v_i = \sum_{x \in \Omega_i} x$; show that if $gu = u$ for all $g \in G$ then $u \in \mathbb{C}\{v_1, \ldots, v_\ell\}$ (i.e. $\mathbb{C}\{v_1, \ldots, v_\ell\}$ is the isotypic component of U corresponding to the trivial module). [*Hint:* For the second statement, it suffices to look at one U_i at a time: show that for $u_i \in U_i$, if $g \cdot u_i = u_i$, then $u_i = \alpha v_i$. To do this, use the fact the G acts transitively on Ω_i.]

   (c) Prove that $\ell|G| = \sum_{g \in G} |\mathrm{Fix}(G)|$.
   [This is the statement that's called *Burnside's Lemma*, though it is due to Frobenius. *Hint:* Compute $\langle \chi, \mathrm{triv} \rangle$, where triv is the character corresponding to the trivial representation (i.e. $\mathrm{triv} : g \mapsto 1$ for all $g \in G$).]

   (d) Compare/contrast Burnside's Lemma to the Orbit-Stabilizer Theorem.

Examples for **??**:

(1) The group S_n acts naturally on $\Omega = \{1, \ldots, n\}$; the resulting representation is the permutation representation of S_n. Specifically, when $n = 3$, the fixed points of each element $g \in S_3$ are given by

h	1	(12)	(23)	(13)	(123)	(132)
$\mathrm{Fix}(h)$	$1, 2, 3$	3	2	1	none	none
$\lvert\mathrm{Fix}(h)\rvert$	3	1	1	1	0	0

Here, S_3 acts transitively on Ω, so has exactly $\ell = 1$ orbits. And indeed, $3 + 1 + 1 + 1 = 6 = \ell|S_3|$.

(2) Let $Z_6 = \langle g \rangle$ act on $\Omega = \{e_1, \ldots, e_6\}$ by

$$\begin{tikzcd} & \mathbf{e_2} \arrow[dr, bend left, "g"] & \\ \mathbf{e_1} \arrow[ur, bend left, "g"] & & \mathbf{e_3} \arrow[ll, bend left, "g"] \end{tikzcd}$$

$$\mathbf{e_4} \overset{g}{\longleftrightarrow} \mathbf{e_5} \quad . \qquad \text{Then} \quad \rho(g) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$g \circlearrowright \mathbf{e_6}$$

Here, the orbits are $\Omega_1 = \{e_1, e_2, e_3\}$, $\Omega_2 = \{e_4, e_5\}$, and $\Omega_3 = \{e_6\}$ (since the group is generated by g alone), and

$$U \cong U_1 \oplus U_2 \oplus U_3, \quad \text{where} \quad U_1 = \mathbb{C}\{e_1, e_2, e_3\},\ U_2 = \mathbb{C}\{e_4, e_5\},\ U_3 = \{e_6\}.$$

And $v_1 = e_1 + e_2 + e_3$, $v_2 = e_4 + e_5$, and $v_3 = e_6$ generate the isotypic component corresponding to the trivial module inside of U.

Further, the fixed points of each element of Z_6 are given by

h	1	g	g^2	g^3	g^4	g^5
$\mathrm{Fix}(h)$	Ω	e_6	e_4, e_5, e_6	e_1, e_2, e_3, e_6	e_4, e_5, e_6	e_6
$\lvert\mathrm{Fix}(h)\rvert$	6	1	3	4	3	1

Indeed, the number of orbits here is $\ell = 3$, and $6 + 1 + 3 + 4 + 3 + 1 = 3 * 6 = \ell|Z_6|$.