Let A be a ring with 1, and let M be an A-module.

- 1. Let z be a central element of A. Show that zM is a submodule of M and that $\varphi: M \to M$ defined by $m \mapsto zm$ is an A-module endomorphism.
- 2. Let I be an ideal of A. Show that

$$IM = \left\{ \sum_{\text{fin.}} \alpha m_{\alpha} \mid \alpha \in I, m_{\alpha} \in M \right\}$$

is a submodule of M. Give an example where IM = M and an example where $0 \neq IM \subsetneq M$ Finally, give an example showing that if B is a subring of A, then BM is not necessarily a submodule of M.

- 3. Let $\varphi: M \to N$ be an A-module homomorphism. Prove that $\ker(\varphi)$ is a submodule of M and $\operatorname{img}(\varphi)$ is a submodule of N. Further, show that if M and N are simple, then either $\varphi = 0$ or φ is an isomorphism. [You may use anything that we have already proven for groups (since M and N are also additive groups.]
- 4. Let X and Y be submodules of M. Show that

$$0 \hookrightarrow X \cap Y \xrightarrow{f:x \mapsto (x,x)} X \oplus Y \xrightarrow{g:(x,y) \mapsto x+y} X + Y \to 0$$

is a short exact sequence of A-modules.

5. Let

$$0 \hookrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \quad \text{and} \quad 0 \hookrightarrow X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \to 0$$

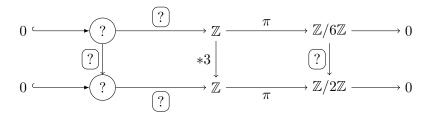
be short exact sequences. A collection of homomorphisms

$$\alpha: X \to X', \quad \beta: Y \to Y', \quad \text{and } \gamma: Z \to Z'$$

is a homomorphism of exact sequences if the following diagram commutes:

If α, β , and γ are isomorphisms, then this is an isomorphism or equivalence of exact sequences.

(a) Fill in the unknown modules and homomorphisms to make the following diagram into a homomorphism of short exact sequences



(b) If you have a homomorphism of short exact sequences as in (1), and you know something about the properties of α , β , or γ , what else (if anything) can you infer about the other two? Namely, fill in the blank (briefly justifying your answers):

If $(\alpha/\beta/\gamma)$ is (injective/surjective/bijective), then...

	injective	surjective	bijective
α			
β		(e.g. γ is surjective)	
γ			

[For example, if β is surjective, then you can infer that γ is surjective, but nothing can be said about α . (Why?)

To receive credit for this assignment, include the following in your solutions [edited appropriately]:

Academic integrity statement: I [violated/did not violate] the CUNY Academic Integrity Policy in completing this assignment. [enter your full name as a digital signature here]