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Math B4900
Homework 3
2/26/2021

1. Prove that for finite dimensional vector spaces U and V over F, and $\varphi \in \mathrm{End}(U)$, $\psi \in \mathrm{End}(V)$, we have

$$\det(\varphi \oplus \psi) = \det(\varphi)\det(\psi).$$

[Hint: Explain why $(\varphi \oplus \mathrm{id})(\mathrm{id} \oplus \psi) = \varphi \oplus \psi$ and $\det(\varphi \oplus \mathrm{id}) = \det(\varphi)$. Recall that while determinant is independent of choice of basis, choosing a basis helps us actually compute it.]

2. Let $X \in M_n(\mathbb{C})$, let Λ be the set of eigenvalues for X, and let m_λ be the multiplicity of $\lambda \in \Lambda$. For any of the following, do not assume that X is in Jordan form, but you may use the *existence* of Jordan form over $\mathbb{C}$.

   (i) Show that $\{\lambda^k \mid \lambda \in \Lambda\}$ are the eigenvalues of X^k.

   (ii) If $X^k = I$, what are the possible eigenvalues of X?

   (iii) Show

$$\mathrm{tr}(X) = \sum_{\lambda \in \Lambda} \lambda m_\lambda \quad \text{and} \quad \det(X) = \prod_{\lambda \in \Lambda} \lambda^{m_\lambda}.$$

3. Let F be a field. The *(first) Weyl algebra* W is the F-algebra *generated by* a *and* b, with the *relation*

$$ba = ab - 1. \tag{1}$$

Specifically, this means start with the (free) monoid generated by a and b,

$$\langle\langle a, b\rangle\rangle = \{1, a, b, a^2, ab, b^2, a^3, a^2b, aba, ab^2, ba^2, bab, \cdots\},$$ [1]

use it to build the *monoid algebra* (just like the group algebra, only spanned by a monoid)

$$F\langle\langle a, b\rangle\rangle = \left\{ \sum_{\substack{w \in \langle\langle a,b\rangle\rangle \\ (\text{fin.})}} \alpha_w w \;\middle|\; \alpha_w \in F \right\},$$

and finally, impose the additional relation $ba = ab - 1$ (which is equivalent to taking a quotient by the principal ideal $(ab - ba - 1)$). Some examples of elements of this algebra include

$$1, \quad 32 + a - 17ab, \quad \text{and} \quad a + a^2 + a^{52} - b - 8abab^{10}.$$

However, in the presence of the relation (**??**), there may be more than one way to write any given element (i.e. $\langle\langle a, b\rangle\rangle$ is a basis of $F\langle\langle a, b\rangle\rangle$ and spans W, but is not a basis of W because it's not linearly independent). For example,

$$ba = ab - 1 \quad \text{and} \quad bab = (ba)b = (ab - 1)b = ab^2 - b. \tag{2}$$

[1] Being the monoid generated by a and b, rather than the group generated by a and b, means that we don't include **inverses** by default.

(a) **Claim:** W is spanned (over F) by the set $S = \{a^m b^n \mid m, n \in \mathbb{Z}_{\geq 0}\}$.[2]

The main idea of the proof is that we can use the the relation (**??**) to rewrite any element of W as a linear combination of terms of the form $a^m b^n$ (where $a^0 = b^0 = 1$), just like we did in (**??**).

(i) Rewrite aba, a^2bab, and ab^2a as a linear combination of terms of the form $a^m b^n$.

(ii) For any word $w \in \langle\langle a, b \rangle\rangle$, define the *length* $\ell(w)$ as the number of terms in w; e.g. $\ell(aba) = \ell(b^2 a) = 3$, $\ell(1) = 0$. Define the *height* $\mathrm{ht}(w)$ as the sum over the b's in w of the number of a's their right: for example,

$$\mathrm{ht}(a^7ba^2ba) = 3 + 1 = 4, \quad \mathrm{ht}(bab) = 1 + 0, \quad \text{and} \quad \mathrm{ht}(1) = 0.^3$$

Verify that in each step of moving b's to the right in your calculations in part (i) (i.e. replacing 'ba' with '$ab - 1$' and expanding) that lengths of the corresponding terms weakly decreased and the heights strictly decreased.

(iii) Prove the claim.
[Hint: Since W is spanned by $\langle\langle a, b \rangle\rangle$, it suffices to show that any element $w \in \langle\langle a, b \rangle\rangle$ can be expressed as a linear combination of terms in S of length less than or equal to the length $\ell(w)$. Prove this by induction on $\ell(w)$ and $\mathrm{ht}(w)$. Be careful not to get too bogged down in the details though!]

(b) The definition of the Weyl algebra was motivated by studying endomorphisms polynomials, $\mathrm{End}(F[x]) = \mathrm{End}_F(F[x])$ (thinking of $F[x]$ as a vector space over F, not as a ring). In particular, define

$$L : F[x] \to F[x] \quad \text{by} \quad f(x) \mapsto xf(x)$$

and

$$D : F[x] \to F[x] \quad \text{by} \quad f(x) \mapsto f'(x) := \frac{d}{dx} f(x)$$

(L for "left multiplication" and D for "derivative").

(i) Verify that L and D are both elements of $\mathrm{End}(F[x])$. [Again, we're thinking of $F[x]$ as a vector space, not as a ring, so your job is to prove that these are both linear.]

(ii) Show that $\varphi : W \to \mathrm{End}(F[x])$ defined by $a \mapsto D$ and $b \mapsto L$ is an F-algebra homomorphism.[4] [Hint: As usual, if you want to show two maps in $\mathrm{End}(F[x])$ are equal, the best way to do this is point-wise, i.e. by applying them to the same polynomial.]

To receive credit for this assignment, include the following in your solutions [edited appropriately]:

Academic integrity statement: I *[violated/did not violate]* the CUNY Academic Integrity Policy in completing this assignment. *[enter your full name as a digital signature here]*

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[2] In fact, S is a basis, but I won't make you prove linear independence.

[3] In a^7ba^2ba, the first b has 3 a's to its right in total, even though they're separated by another b.

[4] What is more is that φ is an isomorphism in the case where F is of characteristic 0. This, together with part (a), proves that $\mathrm{End}_F(F[x])$ is equal to the set of operators of the form $\displaystyle\sum_{\substack{f \in F[x] \\ n \in \mathbb{Z}_{\geq 0} \\ \text{(fin)}}} f(L)D^n$.