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Math B4900
Homework 10
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Let A be a ring with 1.

I Lie algebras.

1. For $x, y \in M_n(\mathbb{C})$, show that the binary operation $[x, y] = xy - yx$ is bilinear, skew symmetric, and satisfies the Jacobi identity.

Proof. We have,

$$\begin{aligned} [rx, y] &= (rx)y - y(rx) \\ &= r(xy) - r(yx) \\ &= r[x, y] \end{aligned}$$

and,

$$\begin{aligned} [x, y \cdot s] &= x(y \cdot s) - (y \cdot s)x \\ &= (xy) \cdot s - (yx) \cdot s \\ &= [x, y] \cdot s \end{aligned}$$

Thus, the operator is bilinear. Now note that,

$$\begin{aligned} [x, y] &= xy - yx \\ &= -(yx - xy) \\ &= [y, x] \end{aligned}$$

Hence, the operator is skew symmetric. Lastly, for $x, y, z \in M_n(\mathbb{C})$,

$$\begin{aligned} [x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= [x, yz - zy] + [y, zx - xz] + [z, xy - yx] \\ &= x(yz - zy) - (yz - zy)x + y(zx - xz) - (zx - xz)y \\ &+ z(xy - yx) - (xy - yx)z \\ &= xyz - xzy - yzx + zyx + yzx - yxz - zxy + xzy + zxy - zyx - xyz + yxz \\ &= 0 \end{aligned}$$

Thus the operator satisfies the Jacobi identity. □

2. Show that $\mathfrak{sl}_n(\mathbb{C}) \subseteq M_n(\mathbb{C})$ is, indeed, a closed Lie algebra.
[Namely, that if $\mathrm{tr}(x) = \mathrm{tr}(y) = 0$, then $\mathrm{tr}(xy - yx) = 0$ and $\mathrm{tr}(x + \alpha y) = 0$ for all $\alpha \in \mathbb{C}$.]

Proof. Let $x, y \in \mathfrak{sl}_n(\mathbb{C})$. Note that $\mathrm{tr}(xy) = \sum_{i,j} x_{ij} y_{ij}$ and $\mathrm{tr}(xy) = tr(yx)$. We have that, for fixed i, j, that $(xy)_{ij} = \sum_k = x_{ik} y_{kj}$. Lastly, we also have that $\mathrm{tr}(xy - yx) = \mathrm{tr}(xy) - \mathrm{tr}(yx)$. Thus, using all of the above formulas, we have,

$$\begin{aligned} \mathrm{tr}(xy - yx) &= \mathrm{tr}(xy) - \mathrm{tr}(yx) \\ &= \mathrm{tr}(xy) - \mathrm{tr}(xy) \\ &= 0 \end{aligned}$$

□

3. Let $L(d)$ be the simple $\mathfrak{sl}_2(\mathbb{C})$-module with dimension $d+1$. Show that for all $d \geqslant 1$,

$$L(d) \otimes L(1) \cong L(d-1) \oplus L(d+1).$$

[Recall that we have a canonical action of a Lie algebra $\mathfrak{g}$ on the tensor product of two of its modules (see Hopf algebras).]

II **Characters.** Let G be a finite group, and let $A = \mathbb{C}G$.

1. **Characters and tensor products.** Let

$$\rho : A \to \mathrm{End}(U) \quad \text{ and } \psi : A \to \mathrm{End}(V)$$

be finite-dimensional representations of A (so that U and V are A-modules). Let $\mathcal{B} = \{e_1, \ldots, e_m\}$ and $\mathcal{B}' = \{f_1, \ldots, f_n\}$ be ordered bases of U and V, respectively.

For $g \in G$, suppose

$$\rho(g) = \sum_{i,j=1}^{m} \alpha_{i,j} E_{i,j} = \begin{pmatrix} \alpha_{1,1} & \cdots & \alpha_{1,m} \\ \vdots & \ddots & \vdots \\ \alpha_{m,1} & \cdots & \alpha_{m,m} \end{pmatrix} \quad \text{and} \quad \psi(g) = \sum_{i,j=1}^{m} \beta_{i,j} E_{i,j} = \begin{pmatrix} \beta_{1,1} & \cdots & \beta_{1,n} \\ \vdots & \ddots & \vdots \\ \beta_{n,1} & \cdots & \beta_{n,n} \end{pmatrix},$$

with respect to the bases $\mathcal{B}$ and $\mathcal{B}'$, respectively. Namely,

$$g \cdot e_i = \sum_{\ell=1}^{m} \alpha_{\ell,i} e_\ell \quad \text{and} \quad g \cdot f_i = \sum_{\ell=1}^{n} \beta_{\ell,i} f_\ell,$$

for all i. Recall that $\mathcal{B} \times \mathcal{B}' = \{e_i \otimes f_j\}$ forms a basis of $U \otimes V$, and put the lexicographic order on it (i.e. $\mathcal{B} \times \mathcal{B}' = \{e_1 \otimes f_1, e_1 \otimes f_1, \ldots, e_1 \otimes f_n, e_2 \otimes f_1, \ldots, e_m \otimes f_n\}$).

(a) Compute the action of g on each $e_i \otimes f_j$ for each i, j; and give the matrix for $(\rho \otimes \psi)(g)$, with respect to the ordered basis $\mathcal{B} \times \mathcal{B}'$, where $\rho \otimes \psi$ is the representation associated to the canonical action of FG on $U \otimes V$.

Proof. Fix $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. Then we have,

$$\begin{aligned} g \cdot e_i \otimes f_j &= (g \cdot e_i) \otimes (g \cdot f_j) \\ &= \left(\sum_{\ell=1}^{m} \alpha_{\ell,i} e_\ell \right) \otimes \left(\sum_{\ell=1}^{n} \beta_{\ell,i} f_\ell \right) \end{aligned}$$

In addition, the matrix for $(\rho \otimes \psi)(g)$ is given by,

$$\begin{aligned}
(\rho \otimes \psi)(g) &= \rho(g) \otimes \psi(g) \\
&= \begin{pmatrix} \alpha_{1,1} & \cdots & \alpha_{1,m} \\ \vdots & \ddots & \vdots \\ \alpha_{m,1} & \cdots & \alpha_{m,m} \end{pmatrix} \otimes \begin{pmatrix} \beta_{1,1} & \cdots & \beta_{1,n} \\ \vdots & \ddots & \vdots \\ \beta_{n,1} & \cdots & \beta_{n,n} \end{pmatrix} \\
&= \begin{pmatrix} \alpha_{1,1}\psi(g) & \cdots & \alpha_{1,m}\psi(g) \\ \vdots & \ddots & \vdots \\ \alpha_{m,1}\psi(g) & \cdots & \alpha_{m,m}\psi(g) \end{pmatrix} \\
&= \begin{pmatrix} \alpha_{1,1}\beta_{1,1} & \alpha_{1,1}\beta_{1,2} & \cdots & \alpha_{1,1}\beta_{1,n} & \alpha_{1,2}\beta_{1,1} & \cdots & \alpha_{1,m}\beta_{1,n} \\ \vdots & \ddots & & & & & \vdots \\ \alpha_{m,1}\beta_{1,1} & \alpha_{m,1}\beta_{1,2} & \cdots & \alpha_{m,1}\beta_{1,n} & \alpha_{m,2}\beta_{1,1} & \cdots & \alpha_{m,m}\beta_{1,n} \\ \vdots & & & & & & \vdots \\ \alpha_{m,1}\beta n, 1 & \alpha_{m,1}\beta_{n,2} & \cdots & \alpha_{m,1}\beta_{n,n} & \alpha_{m,2}\beta_{n,1} & \cdots & \alpha_{m,m}\beta_{n,n} \end{pmatrix}
\end{aligned}$$

So our new matrix has dimension mn. □

(b) If

χ_ρ is the character associated to ρ,

χ_ψ is the character associated to ψ, and

$\chi_{\rho\otimes\psi}$ is the character associated to $\rho \otimes \psi$,

use the previous part to prove that $\chi_{\rho\otimes\psi} = \chi_\rho \chi_\psi$.

Proof. Fix $g \in G$. Then,

$$\begin{aligned}
\chi_\rho(g) &= \operatorname{tr}(\rho(g)) \\
&= \sum_{i=1}^{m} \alpha_{ii}
\end{aligned}$$

and,

$$\begin{aligned}
\chi_\psi(g) &= \operatorname{tr}(\psi(g)) \\
&= \sum_{i=1}^{n} \beta_{ii}
\end{aligned}$$

Thus, we have,

$$\begin{aligned}
\chi_\rho(g)\chi_\psi(g) &= (\chi_\rho \chi_\psi)(g) \\
&= \sum_{i=1}^{m} \alpha_{ii} \cdot \sum_{j=1}^{n} \beta_{jj} \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} = \alpha_{ii}\beta_{jj}
\end{aligned}$$

Lastly, using the matrix derived in part (a), we have,

$$\begin{aligned}
\chi_{\rho\otimes\psi}(g) &= \mathrm{tr}(\alpha_{1,1}\psi(g) + \alpha_{2,2}\psi(g) + \cdots + \alpha_{m,m}\psi(g)) \\
&= \mathrm{tr}(\alpha_{1,1}\psi(g)) + \mathrm{tr}(\alpha_{2,2}\psi(g)) + \cdots + \mathrm{tr}(\alpha_{m,m}\psi(g)) \\
&= \alpha_{1,1}\mathrm{tr}(\psi(g)) + \alpha_{2,2}\mathrm{tr}(\psi(g)) + \cdots + \alpha_{m,m}\mathrm{tr}(\psi(g)) \\
&= \alpha_{1,1}\sum_{i=1}^{n}\beta_{ii} + \alpha_{2,2}\sum_{i=1}^{n}\beta_{ii} + \cdots + \alpha_{m,m}\sum_{i=1}^{n}\beta_{ii} \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} = \alpha_{ii}\beta_{jj} \\
&= \chi_\rho(g)\chi_\psi(g)
\end{aligned}$$

as required. □

(c) Conclude that the set of characters of $\mathbb{C}G$ forms a subring of $\mathrm{Cl}(\mathbb{C}G)$. Does this ring have an identity?

Proof. Since ρ, ψ were arbitrary, we have that the set of characters of $\mathbb{C}G$ is closed under multiplication. □

2. Let $\rho : \mathbb{C}G \to \mathrm{End}(U)$ be a finite-dimensional representation with character χ. Let $\rho_1, \ldots, \rho_r$ be the distinct simple representations of $\mathbb{C}G$, V_i be the corresponding simple modules, z_i be the corresponding primitive central idempotents, and let χ_i be their characters. Let $U_i = z_iU \cong V_i^{\oplus m_i}$ be the isotypic component of U corresponding to V_i. Show that $\dim(U_i) = \langle \chi, \chi_i \rangle$.
[*Hint:* This should be a very short jump from things we proved in class.]

3. **Burnside's Lemma.** Let G act on a set Ω, and let $\Omega_1, \ldots, \Omega_\ell$ be the orbits of Ω. Define $U = \mathbb{C}\Omega$ (the vector space with basis Ω) and extend the action $G \circlearrowright \Omega$ linearly to an action $\mathbb{C}G \circlearrowright U$. Let $\rho : \mathbb{C}G \to \mathrm{End}(U)$ be the associated representation, and χ be the associated character.

Fix $g \in G$. Define

$$\mathrm{Fix}(g) = \{x \in \Omega \mid g \cdot x = x\}$$

to be the number of fixed points under the action of g on Ω. [For example, if $G = S_4$ acts on naturally $\Omega = \{1, 2, 3, 4\}$, and $g = (12)$, then $\{x \in \Omega \mid (12) \cdot x = x\} = \{3, 4\}$. See below for more examples.]

(a) Argue that $\chi(g) = |\mathrm{Fix}(g)|$.

Proof. Let ρ be the representation of the action on U with respect to its basis ω. Then ρ is a permutation matrix. Let $e_i, e_j \in \Omega$. Then the ij-th entry in ρ is nonzero iff $e_j = g \cdot e_i$. Thus, the nonzero diagonal elements correspond to the fixed points of g on Ω because, if a diagonal element i, i is nonzero, then we have $e_i = g \cdot e_i$ as required. Thus, the trace of ρ will be equal to the number of fixed points in g. But χ is exactly $\mathrm{tr}(\rho)$, and so we have $\chi(g) = |\mathrm{Fix}(g)|$ as required. □

(b) Let $U_i = \mathbb{C}\Omega_i$. Argue briefly that $U \cong U_1 \oplus \cdots \oplus U_\ell$. For each i, let $v_i = \sum_{x\in\Omega_i} x$; show that if $gu = u$ for all $g \in G$ then $u \in \mathbb{C}\{v_1, \ldots, v_\ell\}$ (i.e. $\mathbb{C}\{v_1, \ldots, v_\ell\}$ is the isotypic component of U corresponding to the trivial module). [*Hint:* For the second statement, it suffices to look at one U_i at a time: show that for $u_i \in U_i$, if $g \cdot u_i = u_i$, then $u_i = \alpha v_i$. To do this, use the fact the G acts transitively on Ω_i.]

Proof. We have that,

$$\begin{aligned} U_1 \oplus \cdots \oplus U_\ell &= \mathbb{C}\Omega_1 \oplus \cdots \oplus \mathbb{C}\Omega_\ell \\ &= \mathbb{C}\Omega_1 \times \cdots \times \mathbb{C}\Omega_\ell \\ &= \mathbb{C}(\Omega_1 \times \cdots \times \Omega_\ell) \\ &= \mathbb{C}\Omega \end{aligned}$$

Thus, we have that $U \cong U_1 \oplus \cdots U_\ell$.

Now fix i and let $u_i \in U_i$. Suppose $g \cdot u_i = u_i$. □

(c) Prove that $\ell|G| = \sum_{g\in G} |\text{Fix}(G)|$.
[This is the statement that's called *Burnside's Lemma*, though it is due to Frobenius. *Hint:* Compute $\langle \chi, \text{triv} \rangle$, where triv is the character corresponding to the trivial representation (i.e. triv : $g \mapsto 1$ for all $g \in G$).]

Proof. □

(d) Compare/contrast Burnside's Lemma to the Orbit-Stabilizer Theorem.

Answer. Burnside's Lemma is the complement to the Orbit Stabilizer Theorem.
. . . .

Examples for II3:

(1) The group S_n acts naturally on $\Omega = \{1, \ldots, n\}$; the resulting representation is the permutation representation of S_n. Specifically, when $n = 3$, the fixed points of each element $g \in S_3$ are given by

h	1	(12)	(23)	(13)	(123)	(132)
$\mathrm{Fix}(h)$	$1, 2, 3$	3	2	1	none	none
$\lvert\mathrm{Fix}(h)\rvert$	3	1	1	1	0	0

Here, S_3 acts transitively on Ω, so has exactly $\ell = 1$ orbits. And indeed, $3 + 1 + 1 + 1 = 6 = \ell|S_3|$.

(2) Let $Z_6 = \langle g \rangle$ act on $\Omega = \{e_1, \ldots, e_6\}$ by

$$\begin{tikzcd} & \mathbf{e_2} \arrow[dr, bend left, "g"] & \\ \mathbf{e_1} \arrow[ur, bend left, "g"] & & \mathbf{e_3} \arrow[ll, bend left, "g"] \end{tikzcd}$$

$$\mathbf{e_4} \overset{g}{\longleftrightarrow} \mathbf{e_5} \quad . \qquad \text{Then} \quad \rho(g) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$g \circlearrowright \mathbf{e_6}$$

Here, the orbits are $\Omega_1 = \{e_1, e_2, e_3\}$, $\Omega_2 = \{e_4, e_5\}$, and $\Omega_3 = \{e_6\}$ (since the group is generated by g alone), and

$$U \cong U_1 \oplus U_2 \oplus U_3, \quad \text{where} \quad U_1 = \mathbb{C}\{e_1, e_2, e_3\},\ U_2 = \mathbb{C}\{e_4, e_5\},\ U_3 = \{e_6\}.$$

And $v_1 = e_1 + e_2 + e_3$, $v_2 = e_4 + e_5$, and $v_3 = e_6$ generate the isotypic component corresponding to the trivial module inside of U.

Further, the fixed points of each element of Z_6 are given by

h	1	g	g^2	g^3	g^4	g^5
$\mathrm{Fix}(h)$	Ω	e_6	e_4, e_5, e_6	e_1, e_2, e_3, e_6	e_4, e_5, e_6	e_6
$\lvert\mathrm{Fix}(h)\rvert$	6	1	3	4	3	1

Indeed, the number of orbits here is $\ell = 3$, and $6 + 1 + 3 + 4 + 3 + 1 = 3 * 6 = \ell|Z_6|$.