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Lecture 1, Ex. B

1. R commutative ring with 1, G be a group

a. $M = RG$

$$1 \cdot \sum r_g g = \sum (1 \cdot r_g) g$$
$$= \sum r_g g \quad \checkmark$$

$$r_1 \cdot (r_2 \cdot \sum r_g g) = r_1 \cdot (\underbrace{\sum r_2 r_g g}_{\in RG})$$
$$= \sum (r_1 r_2) r_g g \quad \checkmark$$
$$= (r_1 r_2) \cdot \sum r_g g \quad \checkmark$$

$$(r + s) \cdot \sum r_g g = \sum (r+s) r_g g$$
$$= \sum (r_g r + r_g s) \cdot g$$
$$= \sum [r_g r g + r_g s g]$$
$$= \sum r r_g g + \sum s r_g g$$
$$= r \sum r_g g + s \sum r_g g \quad \checkmark$$

$$r \cdot \left(\sum r_g g + \sum s_h h\right)$$

$$= r \cdot \sum (r_g + s_g) g$$

$$= \sum r(r_g + s_g) g$$

$$= r \sum r_g g + r \sum s_h h \quad \checkmark$$

b. R ring, so M additive ✓

$m \in R$, $\sum_g r_g g \in RG$

$$\sum r_g g \cdot m = \sum r_g (gm)$$

$$= \sum r_g m \in RG \quad \checkmark$$

$$\sum r_g g \cdot \left(\sum s_h h \cdot m\right)$$

$$= \sum r_g s_h m$$

$$= \sum (r_g s_h)(gh) \cdot m \quad \checkmark$$

$$1 \cdot m = 1m = m \quad \checkmark$$

c. $\left(1 + 2(13) + 2(123)\right) \cdot 4$

$= 1 \cdot 4 + 2(13) \cdot 4 + 2(123) \cdot 4$

$= 4 + 2(4) + 2(4)$

$= 4 + 8 + 8$

$= 20 = 0$

since $20 \equiv_5 0$

2. R ring w/ 1, M R-module

Let $m \in M$.

$$\cancel{0_R} m + 0_R m = \cancel{0_R} m$$
$$-0_R m \qquad\qquad -0_R m$$
$$\overline{\qquad 0_R m = 0_m \qquad}$$

$-1m + -1m = -1 \cdot (m + m)$

3. $M_n(F) \to$ ring of all $n \times n$ matrices with entries in F, with the usual addition and multiplication of matrices

Check $M_n(F)$ is vector space over F

Need to check

$$F \times M_n(F) \to M_n(F)$$

$$1v = v, \quad \alpha(\beta v) = (\alpha\beta)v$$

$$(\alpha + \beta)v = \alpha v + \beta v, \quad \alpha(u+v) = \alpha u + \alpha v$$

$$\underset{\in F}{x} \cdot \begin{bmatrix} f_{11} & f_{12} & f_{13} & \cdots \\ f_{21} & f_{22} & \cdots & \\ \vdots & & & \end{bmatrix} = \begin{bmatrix} xf_{11} & xf_{12} & xf_{13} \\ xf_{21} & xf_{22} & \cdots \\ \vdots & & \end{bmatrix}$$

$\in M_n(F)$ ✓

$$1 \cdot \begin{bmatrix} f_{11} & f_{12} & \cdots \\ \vdots & & \end{bmatrix} = \begin{bmatrix} 1f_{11} & 1f_{12} & \cdots \\ \vdots & & \end{bmatrix}$$

$$= \begin{bmatrix} f_{11} & f_{12} & \cdots \\ \vdots & & \end{bmatrix}$$ ✓

Lecture 1 $E_x B$ (cont.)

3. $$x(yF) = x\begin{bmatrix} yf_{11} & yf_{12} & \cdots \\ \vdots & \ddots & \\ & & \end{bmatrix}$$

$$= \begin{bmatrix} x(yf_{11}) & x(yf_{12}) & \cdots \\ \vdots & \ddots & \\ & & \end{bmatrix}$$

$$= \begin{bmatrix} (xy)f_{11} & (xy)f_{12} & \cdots \\ \vdots & \ddots & \\ & & \end{bmatrix}$$

$$= (xy)F \checkmark$$

$$(x+y)F = \begin{bmatrix} (x+y)f_{11} & (x+y)f_{12} & \cdots \\ \vdots & \ddots & \\ & & \end{bmatrix}$$

$$= \begin{bmatrix} xf_{11} + yf_{11} & xf_{12} + yf_{12} & \cdots \\ \vdots & \ddots & \\ & & \end{bmatrix}$$

$$= \begin{bmatrix} xf_{11} & xf_{12} & \cdots \\ \vdots & \ddots & \end{bmatrix} + \begin{bmatrix} yf_{11} & yf_{12} & \cdots \\ \vdots & \ddots & \end{bmatrix}$$

$$= xF + yF \checkmark$$

$$x(F + F') =$$

$$x\left(\begin{bmatrix} f_{11} & f_{12} & \cdots \\ \vdots & \ddots & \end{bmatrix} + \begin{bmatrix} f'_{11} & f'_{12} & \cdots \\ \vdots & \ddots & \end{bmatrix}\right)$$

$$= x \begin{bmatrix} f_{11}+f'_{11} & f_{12}+f'_{12} & \cdots \\ \vdots & \ddots & \end{bmatrix}$$

$$= \begin{bmatrix} x(f_{11}+f'_{11}) & x(f_{12}+f'_{12}) & \cdots \\ \vdots & \ddots & \end{bmatrix}$$

$$= \begin{bmatrix} xf_{11} + xf'_{11} & xf_{12} + xf'_{12} & \cdots \\ \vdots & \ddots & \end{bmatrix}$$

$= xF + xF'$ ✓

So $M_n(F)$ is vector space over F

4. F field, $V = F$ is vector space over F under left regular action

Let G be nontrivial subgroup of V (i.e. $G \neq V$, $G \neq \{1\}$)

Show that G not closed under F-action

Since $G \neq V$, $\exists v \in V$ s.t. $v \notin G$

Since $V = F$, we have $v \in F$.

Since G subgroup, $1 \in G$.

Thus, $1 \in G$, $v \in F$ but

$v \cdot 1 = v \notin G$