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Math B4900
Homework 5
3/21/2021

1. Let A be a ring with 1, and let X, Y, and Z be A-modules. In class, we showed that if $0 \hookrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is a split sequence, then so is $\mathrm{Hom}_A(*, M)$ of this sequence for any A-module M. Complete the proof of that theorem by showing that $\mathrm{Hom}_A(M, *)$ of this sequence is split for any A-module M, i.e.

$$0 \hookrightarrow \mathrm{Hom}_A(M, X) \xrightarrow{F} \mathrm{Hom}_A(M, Y) \xrightarrow{G} \mathrm{Hom}_A(M, Z) \to 0$$

is a split exact sequence, where $F(\varphi) = f \circ \varphi$ and $G(\varphi) = g \circ \varphi$. [You may use any other propositions or theorems from class, even if we didn't explicitly prove the cases you need.]

Proof. By Proposition 2.2 in Section 3 of Lang, we have that,

$$0 \hookrightarrow \mathrm{Hom}_A(M, X) \xrightarrow{F} \mathrm{Hom}_A(M, Y) \xrightarrow{G} \mathrm{Hom}_A(M, Z)$$

is exact. Moreover, we can assert that

$$0 \hookrightarrow \mathrm{Hom}_A(M, X) \xrightarrow{F} \mathrm{Hom}_A(M, Y) \xrightarrow{G} \mathrm{Hom}_A(M, Z) \hookrightarrow 0$$

is a short exact sequence because $\mathrm{Im}(G) = \mathrm{Hom}_A(M, Z)$ and $\ker(0) = \mathrm{Hom}_A(M, Z)$. Now define $\mu : \mathrm{Hom}_A(M, Z) \to \mathrm{Hom}_A(M, Y)$ by $\mu(\varphi) = \varphi^{-1} \circ g^{-1}$ and define $\lambda : \mathrm{Hom}_A(M, Y) \to \mathrm{Hom}_A(M, X)$ by $\lambda(\varphi) = \varphi^{-1} \circ f^{-1}$. Then

$$\begin{aligned} G\mu &= g \circ \varphi \circ \varphi^{-1} \circ g^{-1} \\ &= \mathrm{id} \end{aligned}$$

and

$$\begin{aligned} \lambda f &= \varphi^{-1} \circ f^{-1} \circ f \circ \varphi \\ &= \mathrm{id} \end{aligned}$$

Hence, by the Proposition from Lecture 10 part B, we have that our short exact sequence is also split, with μ and λ as the splitting homomorphisms. $\square$

2. Completely decompose the left-regular representation of $\mathbb{C}S_2$ using the central elements $z_1 = \frac{1}{2}(1+x)$ and $z_2 = \frac{1}{2}(1-x)$, where $x = (12)$. [You'll need to check that z_1 and z_2 are actually the tools you need. You'll also need to check that the result *is* actually a *complete* decomposition.]

Proof. Observe that,

$$\begin{aligned} z_1 z_2 &= \frac{1}{2}(1+x) \cdot \frac{1}{2}(1-x) \\ &= \frac{1}{4} - \frac{1}{4} x \cdot x \\ &= \frac{1}{4} - \frac{1}{4} \cdot 1 \\ &= 0 \end{aligned}$$

and,

$$\begin{aligned} z_2 z_1 &= \frac{1}{2}(1-x)\cdot\frac{1}{2}(1+x) \\ &= \frac{1}{4} - \frac{1}{4}x\cdot x \\ &= \frac{1}{4} - \frac{1}{4}\cdot 1 \\ &= 0 \end{aligned}$$

Moreover, we have that,

$$z_1 = \frac{1}{2}(\kappa_1 + \kappa_{(12)})$$

and

$$z_2 = \frac{1}{2}(\kappa_1 - \kappa_{(12)})$$

Hence, $z_1, z_2 \in Z(\mathbb{C}S_2)$. Now fix some $\sigma \in S_2$ and $x \in z_1 M$. Then $x = z_1 m$ for some $m \in M$ and,

$$\begin{aligned} \sigma x &= \sigma(z_1 m) \\ &= (\sigma z_1)m \\ &= z_1 m \\ &= x \end{aligned}$$

Now instead fix $x \in z_2 M$. Then $x = z_2 m$ for some $m \in M$ and,

$$\begin{aligned} \sigma x &= \sigma(z_2 m) \\ &= (\sigma z_2)m \\ &= \operatorname{sgn}(\sigma) z_2 m \\ &= \operatorname{sgn}(\sigma) x \end{aligned}$$

Now let us define,

$$\begin{aligned} \varphi_1 :& M \to M \\ & m \mapsto z_1 m \end{aligned}$$

and

$$\begin{aligned} \varphi_2 :& M \to M \\ & m \mapsto z_2 m \end{aligned}$$

Then we have,

$$\begin{aligned} (\varphi_1 + \varphi_2)(m) &= z_1 m + z_2 m \\ &= (z_1 + z_2)m \\ &= (\frac{1}{2}(1+x) + \frac{1}{2}(1-x))m \\ &= (\frac{1}{2} + \frac{1}{2}x + \frac{1}{2} - \frac{1}{2}x)m \\ &= (1)m \\ &= m \end{aligned}$$

Hence, $\varphi_1 + \varphi_2$ maps $m \mapsto m$ and so $\varphi_1 + \varphi_2 = \mathrm{id}_M$. Moreover,

$$\begin{aligned}\varphi_1\varphi_2 &= z_1 m z_2 m \\ &= z_1 z_2 m m \\ &= 0\end{aligned}$$

Similarly,

$$\varphi_2\varphi_1 = 0$$

$\square$

3. Consider $A = \mathbb{C}D_8$ and recall that the conjugacy classes of D_8 are given by

$$\{1\}, \quad \{r^2\}, \quad \{r, r^3\}, \quad \{s, r^2 s\}, \quad \text{and} \quad \{rs, r^3 s\}.$$

Let $\mathcal{R} = \{1, r^2, r, s, rs\}$ be our favorite set of representatives of these classes, so that $\{\kappa_g \mid g \in \mathcal{R}\}$ is a basis of the center of $\mathbb{C}D_8$ (where κ_g is the class sum corresponding to g.

For $i = 1, \ldots, 5$, define

$$z_i = \frac{1}{8}\sum_{g\in\mathcal{R}} \chi_i(g)\kappa_g,$$

where χ_i is given by the following table.

$\chi_i(g)$	1	r^2	r	s	rs
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	-1	-1	1
χ_4	1	1	1	-1	-1
χ_5	2	-2	0	0	0

(1)

For example,

$$\begin{aligned}z_2 &= \frac{1}{8}(\kappa_1 + \kappa_{r^2} - \kappa_r + \kappa_s - \kappa_{rs}) \\ &= \frac{1}{8}(1 + r_2 - r - r^3 + s + r^2 s - rs - r^3 s).\end{aligned}$$

Since $z_i \in \mathbb{C}\{\kappa_g \mid g \in \mathcal{R}\}$, it is immediate that the z_i are all central.

(a) Verify that $z_1 + \cdots + z_5 = 1$.

Answer. We have that,

$$\begin{aligned} z_1 + z_2 + z_3 + z_4 + z_5 &= \frac{1}{8}(\kappa_1 + \kappa_{r^2} + \kappa_r + \kappa_s + \kappa_{rs} \\ &+ \kappa_1 + \kappa_{r^2} - \kappa_r + \kappa_s - \kappa_{rs} \\ &+ \kappa_1 + \kappa_{r^2} - \kappa_r - \kappa_s + \kappa_{rs} \\ &+ \kappa_1 + \kappa_{r^2} + \kappa_r - \kappa_s - \kappa_{rs} \\ &+ 4\kappa_1 - 4\kappa_{r^2}) \\ &= \frac{1}{8}(1 + r^2 + r + s + rs \\ &+ 1 + r^2 - r + s - rs \\ &+ 1 + r^2 - r - s + rs \\ &+ 1 + r^2 + r - s - rs \\ &+ 4 - 4r^2) \\ &= \frac{1}{8}(8) \\ &= 1 \end{aligned}$$

. . . .

(b) Compare the values in the first 4 rows of the table in (1) to the 1-dimensional representations that we classified in Lecture 7, Exercise B # 2 (which generalized the computation that we did on p. 5 of the Lecture 7 notes). Now, for each $i = 1, \ldots, 5$, extend χ_i to a function (not a homomorphism) $\chi_i : D_8 \to \mathbb{C}$ by defining

$$\chi_i(h) = \chi_i(g) \quad \text{whenever } h \text{ is conjugate to } g.$$

(For example, $\chi_2(r^3) = \chi_2(r) = -1$ since r^3 is in the same conjugacy class as r. This is well-defined since the conjugacy classes partition D_8. We call χ_i a *class function.*)

Verify that, as functions $D_8 \to \mathbb{C}^\times$, we have

$$\chi_1 = \rho_{+,+}, \quad \chi_2 = \rho_{-,+}, \quad \chi_3 = \rho_{-,-}, \quad \text{and} \quad \chi_4 = \rho_{+,-},$$

but that χ_5 isn't a homomorphism. (Careful! You can't assume that the χ_i are homomorphisms for $i = 1, \ldots, 4$.)

Answer. We have that $\chi_1(r) = 1$ and $\chi_1(s) = 1$, which matches $\rho_{+,+}$

We have that $\chi_2(r) = -1$ and $\chi_2(s) = 1$, which matches $\rho_{-,+}$

We have that $\chi_3(r) = -1$ and $\chi_3(s) = -1$, which matches $\rho_{-,-}$

And we have that $\chi_4(r) = 1$ and $\chi_4(s) = -1$, which matches $\rho_{-,-}$

. . . .

(c) Show that for any $h \in D_8$ and $i = 1, 2, 3$, or 4, we have

$$h \cdot z_i = \chi_i(h) z_i.$$

[*Hint:* Use the previous problem. Note that just as with S_3, when you act on any $\sum_{g \in D_8} \alpha_g g \in \mathbb{C}D_8$ by some $h \in D_8$, you're just permuting the terms—the work that you have to do is in keeping track of where the coefficients end up in that permutation. If you're lost, I highly recommend trying the examples of $h = r, s$, or rs acting on z_2. Just like with $\mathrm{sgn} : S_3 \to \mathbb{C}$, you'll want to establish the fact that $\rho_{\epsilon_1,\epsilon_2}$ is a homomorphism (for $\epsilon_1, \epsilon_2 = \pm$), and that $\rho_{\epsilon_1,\epsilon_2}(h^-1) = \rho_{\epsilon_1,\epsilon_2}(h)^{-1} = \rho_{\epsilon_1,\epsilon_2}(h)$ for all $h \in D_8$.]

(d) Verify that $z_i z_j = 0$ for all $i \neq j$.
[*Hint:* Use the previous problem. It may be helpful to fill out the rest of the table

$\chi_i(g)$	1	r^2	r	r^3	s	$r^2 s$	rs	$r^3 s$
χ_1	1	1	1		1		1	
χ_2	1	1	-1		1		-1	
χ_3	1	1	-1		-1		1	
χ_4	1	1	1		-1		-1	
χ_5	2	-2	0		0		0	

for your own uses and consider the dot product of rows—but you'll need to convince yourself (and your reader!), using the previous parts, that the dot product of rows is interesting.]

(e) Decompose the left regular module for $\mathbb{C}D_8$ using $z_1, \ldots, z_5$. [This will not be a complete decomposition–no worries.]

To receive credit for this assignment, include the following in your solutions [edited appropriately]:

Academic integrity statement: I *did not violate* the CUNY Academic Integrity Policy in completing this assignment. *Christopher Hayduk*
