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Math B4900
Homework 3
2/26/2021

1. Prove that for finite dimensional vector spaces U and V over F, and $\varphi \in \mathrm{End}(U)$, $\psi \in \mathrm{End}(V)$, we have
$$\det(\varphi \oplus \psi) = \det(\varphi)\det(\psi).$$
[Hint: Explain why $(\varphi \oplus \mathrm{id})(\mathrm{id} \oplus \psi) = \varphi \oplus \psi$ and $\det(\varphi \oplus \mathrm{id}) = \det(\varphi)$. Recall that while determinant is independent of choice of basis, choosing a basis helps us actually compute it.]

Proof. We have that,
$$\varphi \oplus \mathrm{id} = (\varphi(u), v)$$
for any $u \in U$ and $v \in V$. Moreover, we have,
$$(\mathrm{id} \oplus \psi) = (u, \psi(v))$$
for any $u \in U$ and $v \in V$. Hence, multiplying these two elements of $U \oplus V$ yields,
$$\begin{aligned}(\varphi \oplus \mathrm{id})(\mathrm{id} \oplus \psi) &= (\varphi \cdot \mathrm{id}, \mathrm{id} \cdot \psi) \\ &= \varphi(\mathrm{id}(u)), \mathrm{id}(\psi(v)) \\ &= (\varphi(u), \psi(v))\end{aligned}$$
for all $u \in U$ and $v \in V$.

$\square$

2. Let $X \in M_n(\mathbb{C})$, let Λ be the set of eigenvalues for X, and let m_λ be the multiplicity of $\lambda \in \Lambda$. For any of the following, do not assume that X is in Jordan form, but you may use the *existence* of Jordan form over $\mathbb{C}$.

   (i) Show that $\{\lambda^k \mid \lambda \in \Lambda\}$ are the eigenvalues of X^k.

   Proof. Fix $\lambda \in \Lambda$. Then there exists a nonzero $v \in \mathbb{C}$ such that
$$Xv = \lambda v$$
   Now consider $X^k v$. We have that,
$$\begin{aligned}X^k v &= (X^{k-1}X)v \\ &= X^{k-1}(Xv) \\ &= X^{k-1}(\lambda v) \\ &= \lambda(X^{k-1}v)\end{aligned}$$
   Proceeding inductively, we get,
$$X^k v = \lambda^k v$$
   Hence, λ^k is an eigenvalue of X^k. Now suppose there exists an eigenvalue α of X^k which is not in the set $\{\lambda^k \mid \lambda \in \Lambda\}$. $\square$

   (ii) If $X^k = I$, what are the possible eigenvalues of X?

Proof. If $X^k = I$, we must have that for any eigenvalue λ of X, there exists a nonzero $v \in \mathbb{C}$ such that,

$$\begin{aligned} X^k v &= Iv \\ &= v \\ &= \lambda v \end{aligned}$$

That is,

$$v = \lambda v$$

Multiplying by v^{-1} on the right on both sides of the equality yields

$$1 = \lambda$$

Since by part (i) we have that all of the eigenvalues of X^k are characterized by the eigenvalues of X raised to the kth power, this must be the only possible eigenvalue of X. □

(iii) Show

$$\operatorname{tr}(X) = \sum_{\lambda \in \Lambda} \lambda m_\lambda \quad \text{and} \quad \det(X) = \prod_{\lambda \in \Lambda} \lambda^{m_\lambda}.$$

Proof. Let us denote X in Jordan form by the matrix Y. We know that the eigenvalues will be placed along the diagonal of Y. □

3. Let F be a field. The *(first) Weyl algebra* W is the F-algebra *generated by* a *and* b, with the *relation*

$$ba = ab - 1. \tag{1}$$

Specifically, this means start with the (free) monoid generated by a and b,

$$\langle\!\langle a, b \rangle\!\rangle = \{1, a, b, a^2, ab, b^2, a^3, a^2b, aba, ab^2, ba^2, bab, \cdots\},$$ [1]

use it to build the *monoid algebra* (just like the group algebra, only spanned by a monoid)

$$F\langle\!\langle a, b \rangle\!\rangle = \left\{ \sum_{\substack{w \in \langle\!\langle a,b \rangle\!\rangle \\ \text{(fin.)}}} \alpha_w w \;\middle|\; \alpha_w \in F \right\},$$

and finally, impose the additional relation $ba = ab - 1$ (which is equivalent to taking a quotient by the principal ideal $(ab - ba - 1)$). Some examples of elements of this algebra include

$$1, \quad 32 + a - 17ab, \quad \text{and} \quad a + a^2 + a^{52} - b - 8abab^{10}.$$

However, in the presence of the relation (1), there may be more than one way to write any given element (i.e. $\langle\!\langle a, b \rangle\!\rangle$ is a basis of $F\langle\!\langle a, b \rangle\!\rangle$ and spans W, but is not a basis of W because it's not linearly independent). For example,

$$ba = ab - 1 \quad \text{and} \quad bab = (ba)b = (ab - 1)b = ab^2 - b. \tag{2}$$

[1] Being the monoid generated by a and b, rather than the group generated by a and b, means that we don't include **inverses** by default.

(a) **Claim:** W is spanned (over F) by the set $S = \{a^m b^n \mid m, n \in \mathbb{Z}_{\geq 0}\}$.[2]

The main idea of the proof is that we can use the the relation (1) to rewrite any element of W as a linear combination of terms of the form $a^m b^n$ (where $a^0 = b^0 = 1$), just like we did in (2).

(i) Rewrite aba, a^2bab, and ab^2a as a linear combination of terms of the form $a^m b^n$.

Answer. We have,

$$\begin{aligned}
&aba = a(ab-1) = a^2b - a \\
&a^2bab = a^2(ab-1)b = a^3b^2 - b \\
&ab^2a = ab(ba) = ab(ab-1) = a(bab - b) = a((ab-1)b - b) = a^2b^2 - 2ab
\end{aligned}$$

. . . .

(ii) For any word $w \in \langle\langle a, b \rangle\rangle$, define the *length* $\ell(w)$ as the number of terms in w; e.g. $\ell(aba) = \ell(b^2a) = 3$, $\ell(1) = 0$. Define the *height* $\text{ht}(w)$ as the sum over the b's in w of the number of a's their right: for example,

$$\text{ht}(a^7ba^2ba) = 3 + 1 = 4, \quad \text{ht}(bab) = 1 + 0, \quad \text{and} \quad \text{ht}(1) = 0.$$[3]

Verify that in each step of moving b's to the right in your calculations in part (i) (i.e. replacing 'ba' with '$ab - 1$' and expanding) that lengths of the corresponding terms weakly decreased and the heights strictly decreased.

Answer. We have,

$$\begin{aligned}
\ell(aba) &= 3 \\
\ell(a^2b - a) &= 4
\end{aligned}$$

. . . .

(iii) Prove the claim.
[Hint: Since W is spanned by $\langle\langle a, b \rangle\rangle$, it suffices to show that any element $w \in \langle\langle a, b \rangle\rangle$ can be expressed as a linear combination of terms in S of length less than or equal to the length $\ell(w)$. Prove this by induction on $\ell(w)$ and $\text{ht}(w)$. Be careful not to get too bogged down in the details though!]

(b) The definition of the Weyl algebra was motivated by studying endomorphisms polynomials, $\text{End}(F[x]) = \text{End}_F(F[x])$ (thinking of $F[x]$ as a vector space over F, not as a ring). In particular, define

$$L : F[x] \to F[x] \quad \text{by} \quad f(x) \mapsto xf(x)$$

and

$$D : F[x] \to F[x] \quad \text{by} \quad f(x) \mapsto f'(x) := \frac{d}{dx} f(x)$$

(L for "left multiplication" and D for "derivative").

(i) Verify that L and D are both elements of $\text{End}(F[x])$. [Again, we're thinking of $F[x]$ as a vector space, not as a ring, so your job is to prove that these are both linear.]

[2] In fact, S is a basis, but I won't make you prove linear independence.

[3] In a^7ba^2ba, the first b has 3 a's to its right in total, even though they're separated by another b.

Answer. Fix $f, g \in F[x]$. Then,

$$\begin{aligned} L(f(x)+g(x)) &= x(f(x)+g(x)) \\ &= xf(x)+xg(x) \\ &= L(f(x))+L(g(x)) \end{aligned}$$

Now fix $\alpha \in F$. Then we have,

$$\begin{aligned} L(\alpha f(x)) &= x(\alpha f(x)) \\ &= (x\alpha)f(x) \\ &= (\alpha x)f(x) \\ &= \alpha(xf(x)) \\ &= \alpha L(f(x)) \end{aligned}$$

Hence, $L \in \mathrm{End}(F[x])$.

Now let us consider D. We can verify that, for $f, g \in F[x]$ and $\alpha \in F$, we get

$$\begin{aligned} D(f(x)+g(x)) &= \frac{d}{dx}(f(x)+g(x)) \\ &= \frac{d}{dx}f(x)+\frac{d}{dx}g(x) \\ &= D(f(x))+D(g(x)) \end{aligned}$$

and,

$$\begin{aligned} D(\alpha f(x)) &= \frac{d}{dx}(\alpha f(x)) \\ &= \alpha\frac{d}{dx}f(x) \\ &= \alpha D(f(x)) \end{aligned}$$

Thus, we have that $D \in \mathrm{End}(F[X])$ as well.

. . . .

(ii) Show that $\varphi : W \to \mathrm{End}(F[x])$ defined by $a \mapsto D$ and $b \mapsto L$ is an F-algebra homomorphism.[4] [Hint: As usual, if you want to show two maps in $\mathrm{End}(F[x])$ are equal, the best way to do this is point-wise, i.e. by applying them to the same polynomial.]

Proof. We need to show that f is a homomorphism, φ maps 1_W to $1_{\mathrm{End}(F[x])}$, and that $\varphi(W) \subset Z(\mathrm{End}(F[x]))$. □

To receive credit for this assignment, include the following in your solutions [edited appropriately]:

[4]What is more is that φ is an isomorphism in the case where F is of characteristic 0. This, together with part (a), proves that $\mathrm{End}_F(F[x])$ is equal to the set of operators of the form $\displaystyle\sum_{\substack{f \in F[x] \\ n \in \mathbb{Z}_{\geq 0} \\ \text{(fin)}}} f(L)D^n$.

Academic integrity statement: I *did not violate* the CUNY Academic Integrity Policy in completing this assignment. *Christopher Hayduk*
