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Math B4900
Homework 2
2/19/2021

Let F denote a field, and V denote a vector space over F of dimension n. Fix $\mathcal{E} = \{e_1, \ldots, e_n\}$ an ordered basis of V, and use that basis to identify V with F^n or $M_{n,1}(F)$ and V^* with F^n or $M_{1,n}(F)$ as needed, and to identify $\mathrm{End}(V)$ with $M_n(F)$.

See page 2 for hints.

1. **Bilinear forms.** Let $\langle , \rangle_J : V \times V \to F$ associated to a matrix J.

   (a) In lecture exercises, we proved that if J is symmetric, then so is $\langle , \rangle_J$.
   Prove the converse: if $\langle , \rangle_J$ is symmetric, then so is J.

   Proof. Suppose that $\langle , \rangle_J$ is symmetric. Then we have that, for $u, v \in V$,

$$\begin{aligned} \langle u, v \rangle_J &= u^t J v \\ &= v^t J u \\ &= \langle v, u \rangle_J \end{aligned}$$

   Recall that J is symmetric if $J = J^t$. That is, for every m, n, we have $j_{mn} = j_{nm}$. Let us now employ $e_m, e_n \in V$ in order to show this. We have that,

$$\begin{aligned} \langle e_m, e_n \rangle_J &= e_m^t J e_n \\ &= j_{mn} \end{aligned}$$

   But, since $\langle , \rangle_J$ is symmetric, we know that,

$$\begin{aligned} \langle e_m, e_n \rangle_J &= j_{mn} \\ &= \langle e_n, e_m \rangle_J \\ &= e_n^t J e^m \\ &= j_{nm} \end{aligned}$$

   Since m, n were arbitrary, this holds for every entry of J and hence J is symmetric. $\square$

   (b) Prove that $\langle , \rangle_J$ is nondegenerate if and only if J is invertible.

   Proof. Suppose $\langle , \rangle_J$ is nondegenerate.

   Suppose J is invertible. $\square$

2. **Trace form.** Define $\langle , \rangle : M_n(F) \times M_n(F) \to F$ by $\langle A, B \rangle = \mathrm{tr}(AB)$. Briefly verify that this is a symmetric bilinear form. Then prove that $\langle , \rangle$ is nondegenerate.

Proof. Fix $A, B \in M_n(F)$. Then we have,

$$\begin{aligned}\langle A, B\rangle &= tr(AB)\\ &= \sum_{i=1}^{n} AB_{ii}\\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki}\end{aligned}$$

and,

$$\begin{aligned}\langle B, A\rangle &= tr(BA)\\ &= \sum_{i=1}^{n} BA_{ii}\\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ik}a_{ki}\end{aligned}$$

□

3. **Determinants.** Recall that the determinant function $\det : M_n(F) \to F$ is defined by

$$\det((\alpha_{i,j})) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\alpha_{1,\sigma(1)}\alpha_{2,\sigma(2)}\cdots\alpha_{n,\sigma(n)}.$$

(a) Use this definition to verify that $\det(I_n) = 1$, where I_n is the identity matrix.

Proof. Note that for I_n, we have that $\alpha_{ii} = 1$ for all $1 \le i \le n$ and $\alpha_{ij} = 0$ for all $i \neq j$. Hence, in the above definition of the determinant, we must have that the only non-zero term in the summation is the one corresponding to the identity $\sigma = 1$. This gives us,

$$\begin{aligned}\det(I_n) &= \operatorname{sgn}(1)\alpha_{1,1}\alpha_{2,2}\cdots\alpha_{n,n}\\ &= 1\cdot 1\cdot 1\cdots 1\\ &= 1\end{aligned}$$

□

(b) Use this definition to show that if, for some fixed i, $\alpha_{i,1} = \alpha_{i,2} = \cdots = \alpha_{i,n} = 0$, then $\det((\alpha_{i,j})) = 0$.

Proof. Suppose that, for some $i \in \mathbb{Z}$ such that $1 \le i \le n$, we have that $\alpha_{i,1} = \alpha_{i,2} = \cdots = \alpha_{i,n} = 0$. Now fix some $\sigma \in S_n$ and let us denote $\sigma(i) = j$ for some $j \in \mathbb{Z}$ such that $1 \le j \le n$. We can see from the previous statements that,

$$\begin{aligned}\alpha_{i,\sigma(i)} &= \alpha_{i,j}\\ &= 0\end{aligned}$$

Hence, in the definition of the determinant, we have,

$$\begin{aligned}\det((\alpha_{i,j})) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\alpha_{1,\sigma(1)}\alpha_{2,\sigma(2)}\cdots\alpha_{i,\sigma(i)}\cdots\alpha_{n,\sigma(n)} \\ &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\alpha_{1,\sigma(1)}\alpha_{2,\sigma(2)}\cdots 0 \cdots\alpha_{n,\sigma(n)} \\ &= 0\end{aligned}$$

as required. □

(c) Show that, for any $A \in \mathrm{GL}_n(F)$, we have $\det(A^{-1}) = \det(A)^{-1}$, and that $\det(B) = \det(ABA^{-1})$. Conclude that determinant is independent of change of basis, so that

$$\det : \mathrm{End}(V) \to F \quad \text{defined by} \quad \det(\varphi) = \det(M_{\mathcal{B}}^{\mathcal{B}}(\varphi))$$

is well-defined. You may use the facts established in our worksheet about determinants.

Proof. Let $A \in \mathrm{GL}_n(F)$. Then A is invertible with inverse A^{-1}. But A^{-1} is also invertible with inverse A, so $A^{-1} \in \mathrm{GL}_n(F)$. Hence, by fact (2) we have $\det(A^{-1}), \det(A)^{-1} \neq 0$. Furthermore, since $\mathrm{GL}_n(F) \subset M_n(F)$, we have that $A, A^{-1} \in M_n(F)$ and so we can apply fact (3). Thus, we have that,

$$\begin{aligned}\det(AA^{-1}) &= \det(I_n) \\ &= 1 \\ &= \det(A)\det(A^{-1})\end{aligned}$$

Since $\det(A)\det(A^{-1}) = 1$, we have that $\det(A^{-1}) = \det(A)^{-1}$.

Now let $B \in \mathrm{GL}_n(F)$. Consider $\det(ABA^{-1})$ and using fact (3) along with the associativity of matrix multiplication, we get,

$$\begin{aligned}\det((AB)A^{-1}) &= \det(AB)\det(A^{-1}) \\ &= \det(A)\det(B)\det(A^{-1})\end{aligned}$$

By our initial derivation, we have that $\det(A^{-1}) = \det(A)^{-1}$ and so, by the fact that F is a field and hence commutative, we have,

$$\begin{aligned}\det((AB)A^{-1}) &= \det(A)\det(B)\det(A^{-1}) \\ &= \det(A)\det(B)\det(A)^{-1} \\ &= \det(A)\det(A)^{-1}\det(B) \\ &= 1 \cdot \det(B) \\ &= \det(B)\end{aligned}$$

□

(d) Pick one of facts (1), (2), or (3) from p. 4 of the Lecture 4 worksheet, and spell out the details (more so than the proof sketches already given in the worksheet). Cite your sources.

Proof. Fact (2)

Let $A \in M_n(F)$. Suppose $A \in \mathrm{GL}_n$. Then A is invertible and thus the columns of A are linearly independent. Hence, by fact 1, $\det(A) \neq 0$.

Now suppose $\det(A) \neq 0$. Then again by fact (1), we have that the columns of A are linearly independent. Hence, A is invertible and $A \in \mathrm{GL}_n$ as required. $\square$

To receive credit for this assignment, include the following in your solutions [edited appropriately]:

Academic integrity statement: I *did not violate* the CUNY Academic Integrity Policy in completing this assignment. *Christopher Hayduk*

Some hints.

1a: Consider $e_i^t J e_j$.

1b: If $\langle V, u\rangle = 0$ for some $u \in V$, then in particular, $\langle e_i, u\rangle = 0$ for all i (and similarly the coordinates reversed). Try to avoid "proof by contradiction"–you don't need it!

2: Let $A \in M_n(F)$ be a non-zero matrix, and let $\alpha_{i,j}$ be a non-zero entry in A. For each such A, what is your goal? (Go back to the definition of degenerate for the answer.)