ID: **5514**

Math B4900
Proof portfolio 1
03/07/2021

Problem:	**1C**
No. stars:	**2**

Statement: Let F be a field and V be a vector space over F. Fix $\varphi \in \mathrm{End}(V)$. For $\lambda \in F$, prove that the weight space V_λ and the generalized weight space V^λ are both subspaces of V.

Proof. Let F be a field and V be a vector space over F. Fix $\lambda \in F$. By definition, every $v \in V_\lambda$ is also an element of V. Hence, we have $V_\lambda \subset V$. Now observe that $\lambda 0 = 0$ for any $\lambda \in F$. Hence, $0 \in V_\lambda$ and thus V_λ is non-empty. Now, by the submodule criterion, we just need to show that $x + ry \in V_\lambda$ for all $r \in F$ and for all $x, y \in V_\lambda$. Let us start by applying φ to this element and using the properties of the linearity of φ,

$$\begin{aligned}\varphi(x + ry) &= \varphi(x) + r\varphi(y)\\ &= \lambda x + r\lambda y\\ &= \lambda(x + ry)\end{aligned}$$

Hence, $x + ry \in V_\lambda$ and so V_λ is a subspace of V.

By definition, every $v \in V^\lambda$ is also an element of V. Hence, we have $V^\lambda \subset V$. Now observe that, for any $\lambda \in F$,

$$\begin{aligned}&\varphi(v) = \lambda 0 = 0v = 0\\ \iff &(\varphi - \lambda \cdot \mathrm{id})(0) = 0\end{aligned}$$

Hence, $0 \in V^\lambda$ and thus V^λ is non-empty. Again, by the submodule criterion, we just need to show that $x + ry \in V^\lambda$ for all $r \in F$ and for all $x, y \in V^\lambda$. Since, $x, y \in V^\lambda$, we have that

$$\begin{aligned}(\varphi - \lambda \cdot \mathrm{id})^\ell(x) = 0\\ (\varphi - \lambda \cdot \mathrm{id})^m(y) = 0\end{aligned}$$

Let $k = \max\{\ell, m\}$. Then by the fact that if $(\varphi - \lambda \cdot \mathrm{id})^m(v) = 0$, then $(\varphi - \lambda \cdot \mathrm{id})^n v = 0$ for all integers $n \geqslant m$ and by the fact that linear combinations and compositions of linear functions are linear, we have that,

$$\begin{aligned}(\varphi - \lambda \cdot \mathrm{id})^k(x + ry) &= (\varphi - \lambda \cdot \mathrm{id})^k(x) + r(\varphi - \lambda \cdot \mathrm{id})^k(y)\\ &= 0 + r0\\ &= 0\end{aligned}$$

Hence, $x + ry \in V^\lambda$ and so V^λ is also a subspace of V. $\square$

	Points Possible					
complete	0	1	2	3	4	5
mathematically valid	0	1	2	3	4	5
readable/fluent	0	1	2	3	4	5
Total:					(out of 15)	

Statement: Define $\langle , \rangle : M_n(F) \times M_n(F) \to F$ by $\langle A, B\rangle = \mathrm{tr}(AB)$. Show that $\langle , \rangle$ is symmetric and nondegenerate.

Problem:	**2A.II**
No. stars:	**2**

Proof. Fix $A, B \in M_n(F)$. Then we have,

$$\begin{aligned}\langle A, B\rangle &= tr(AB)\\ &= \sum_{i=1}^{n} AB_{ii}\\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki}\\ &= \sum_{i=1}^{n} a_{i1}b_{1i} + a_{i2}b_{2i} + \cdots + a_{in}b_{ni}\\ &= a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} + a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2n}b_{n2} + \cdots + a_{n1}b_{1n} + a_{n2}b_{2n} + \cdots + a_{nn}b_{nn}\end{aligned}$$

and,

$$\begin{aligned}\langle B, A\rangle &= tr(BA)\\ &= \sum_{i=1}^{n} BA_{ii}\\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ik}a_{ki}\\ &= \sum_{i=1}^{n} b_{i1}a_{1i} + b_{i2}a_{2i} + \cdots + b_{in}a_{ni}\\ &= b_{11}a_{11} + b_{12}a_{21} + \cdots + b_{1n}a_{n1} + b_{21}a_{12} + b_{22}a_{22} + \cdots + b_{2n}a_{n2} + \cdots + b_{n1}a_{1n} + b_{n2}a_{2n} + \cdots + b_{nn}a_{nn}\end{aligned}$$

Note from the above two expressions, we can see that each term in $tr(BA)$ appears in $tr(AB)$, just with the order of a and b switched. Since F is a field, we have that multiplication is commutative and so, since the terms are the same in each expanded addition, we must have that

$$tr(AB) = tr(BA)$$

Hence, trace is symmetric.

Now we want to show that trace is a bilinear form. Let us fix $\alpha \in F$. We have that,

$$\begin{aligned}\langle \alpha A, B\rangle &= tr(\alpha AB)\\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} (\alpha a_{ik})b_{ki}\\ &= \alpha \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki}\\ &= \alpha tr(AB)\\ &= \alpha\langle A, B\rangle\end{aligned}$$

Moreover, we have that,

$$\begin{aligned}
\langle \alpha A, B\rangle &= tr(\alpha AB) \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n}(\alpha a_{ik})b_{ki} \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n}a_{ik}(\alpha b_{ki}) \\
&= tr(A(\alpha B)) \\
&= \langle A, \alpha B\rangle
\end{aligned}$$

Now in addition to A and B, let us fix $A', B' \in M_n(F)$. Then we have,

$$\begin{aligned}
\langle A + A', B\rangle &= tr((A + A')B) \\
&= tr(AB + A'B) \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n}\left(a_{ik}b_{ki} + a'_{ik}b_{ki}\right) \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n}a_{ik}b_{ki} + \sum_{i=1}^{n}\sum_{k=1}^{n}a'_{ik}b_{ki} \\
&= tr(AB) + tr(A'B) \\
&= \langle A, B\rangle + \langle A', B\rangle
\end{aligned}$$

In addition,

$$\begin{aligned}
\langle A, B + B'\rangle &= tr(A(B + B')) \\
&= tr(AB + AB') \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n}\left(a_{ik}b_{ki} + a_{ik}b'_{ki}\right) \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n}a_{ik}b_{ki} + \sum_{i=1}^{n}\sum_{k=1}^{n}a_{ik}b'_{ki} \\
&= tr(AB) + tr(AB') \\
&= \langle A, B\rangle + \langle A, B'\rangle
\end{aligned}$$

Thus, we have shown that trace is a symmetric bilinear form. Now we must show that it is nondegenerate. Fix $A \in M_n(F)$ and suppose

$$\langle A, B\rangle = 0$$

for all $B \in M_n(F)$. Then we have,

$$\begin{aligned}\langle A, B\rangle &= tr(AB)\\ &= \sum_{i=1}^{n} AB_{ii}\\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki}\\ &= \sum_{i=1}^{n} a_{i1}b_{1i} + a_{i2}b_{2i} + \cdots + a_{in}b_{ni}\\ &= a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} + a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2n}b_{n2} + \cdots + a_{n1}b_{1n} + a_{n2}b_{2n} + \cdots + a_{nn}b_{nn}\\ &= 0\end{aligned}$$

The above expression implies that every term of the form $a_{ij}b_{jk} = 0$. Since this holds for all $B \in M_n(F)$, we can assume that our matrix B has every entry greater than 0. Then, the fact that $a_{ij}b_{jk} = 0$ for every $1 \leqslant i, j, k \leqslant n$ implies that $a_{ij} = 0$ for all i, j. Hence, $\langle A, B\rangle = 0$ for all $B \in M_n(F)$ if and only if $A = \mathbf{0}$

Similarly, fix $B \in M_n(F)$ and suppose

$$\langle A, B\rangle = 0$$

for all $A \in M_n(F)$. Then we have,

$$\begin{aligned}\langle A, B\rangle &= tr(AB)\\ &= \sum_{i=1}^{n} AB_{ii}\\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki}\\ &= \sum_{i=1}^{n} a_{i1}b_{1i} + a_{i2}b_{2i} + \cdots + a_{in}b_{ni}\\ &= a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} + a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2n}b_{n2} + \cdots + a_{n1}b_{1n} + a_{n2}b_{2n} + \cdots + a_{nn}b_{nn}\\ &= 0\end{aligned}$$

The above expression implies that every term of the form $a_{ij}b_{jk} = 0$. Since this holds for all $A \in M_n(F)$, we can assume that our matrix A has every entry greater than 0. Then, the fact that $a_{ij}b_{jk} = 0$ for every $1 \leqslant i, j, k \leqslant n$ implies that $b_{jk} = 0$ for all i, j. Hence, $\langle A, B\rangle = 0$ for all $A \in M_n(F)$ if and only if $B = \mathbf{0}$

As a result, we have now shown that trace a symmetric, nondegenerate bilinear form. $\square$

	Points Possible					
complete	0	1	2	3	4	5
mathematically valid	0	1	2	3	4	5
readable/fluent	0	1	2	3	4	5
Total:						(out of 15)

Problem:	**2B**
No. stars:	**1**

Statement: Use the definition of determinant to verify that $\det(I_n) = 1$, where I_n is the identity matrix in $M_n(F)$.

Proof. Recall that the determinant function $\det : M_n(F) \to F$ is defined by

$$\det((\alpha_{i,j})) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\alpha_{1,\sigma(1)}\alpha_{2,\sigma(2)} \cdots \alpha_{n,\sigma(n)}.$$

Note that for I_n, we have that $\alpha_{ii} = 1$ for all $1 \leqslant i \leqslant n$ and $\alpha_{ij} = 0$ for all $i \neq j$. Hence, in the above definition of the determinant, we must have that the only non-zero term in the summation is the one corresponding to the identity $\sigma = 1$. This gives us,

$$\begin{aligned} \det(I_n) &= \operatorname{sgn}(1)\alpha_{1,1}\alpha_{2,2} \cdots \alpha_{n,n} \\ &= 1 \cdot 1 \cdot 1 \cdots 1 \\ &= 1 \end{aligned}$$

□

	Points Possible
complete	0 1 2 3 4 5
mathematically valid	0 1 2 3 4 5
readable/fluent	0 1 2 3 4 5
Total:	(out of 15)

Statement: Prove that determinant is invariant under change of basis. [The details required in this proof are outlined in Homework 2; be sure to hit all the beats highlighted in that problem statement.]

Problem:	**2C**
No. stars:	**2**

Proof. Note that for I_n, we have that $\alpha_{ii} = 1$ for all $1 \leqslant i \leqslant n$ and $\alpha_{ij} = 0$ for all $i \neq j$. Hence, in the definition of the determinant, we must have that the only non-zero term in the summation is the one corresponding to the identity $\sigma = 1$. This gives us,

$$\begin{aligned} \det(I_n) &= \operatorname{sgn}(1)\alpha_{1,1}\alpha_{2,2}\cdots\alpha_{n,n} \\ &= 1 \cdot 1 \cdot 1 \cdots 1 \\ &= 1 \end{aligned}$$

Now let $A \in \mathrm{GL}_n(F)$. Then A is invertible with inverse A^{-1}. But A^{-1} is also invertible with inverse A, so $A^{-1} \in \mathrm{GL}_n(F)$. Hence, by fact (2) on the Lecture 4 worksheet, we have $\det(A^{-1}), \det(A)^{-1} \neq 0$. Furthermore, since $\mathrm{GL}_n(F) \subset M_n(F)$, we have that $A, A^{-1} \in M_n(F)$ and so we can apply fact (3) from the Lecture 4 worksheet. Thus, we have that,

$$\begin{aligned} \det(AA^{-1}) &= \det(I_n) \\ &= 1 \\ &= \det(A)\det(A^{-1}) \end{aligned}$$

Since $\det(A)\det(A^{-1}) = 1$, we have that $\det(A^{-1}) = \det(A)^{-1}$.

Now let $B \in \mathrm{GL}_n(F)$. Consider $\det(ABA^{-1})$ and using fact (3) along with the associativity of matrix multiplication, we get,

$$\begin{aligned} \det((AB)A^{-1}) &= \det(AB)\det(A^{-1}) \\ &= \det(A)\det(B)\det(A^{-1}) \end{aligned}$$

By our initial derivation, we have that $\det(A^{-1}) = \det(A)^{-1}$ and so, by the fact that F is a field and hence commutative, we have,

$$\begin{aligned} \det((AB)A^{-1}) &= \det(A)\det(B)\det(A^{-1}) \\ &= \det(A)\det(B)\det(A)^{-1} \\ &= \det(A)\det(A)^{-1}\det(B) \\ &= 1 \cdot \det(B) \\ &= \det(B) \end{aligned}$$

Thus, if we let A be the matrix of the determinant under basis $\mathcal{A}$, and let P be the change of basis matrix from $\mathcal{A}$ to some basis $\mathcal{B}$. Then $P^{-1}AP = B$, where B is the matrix of the determinant under basis $\mathcal{B}$. However, from the above we get,

$$\begin{aligned} \det(P^{-1}AP) &= \det(A) \\ &= \det(B) \end{aligned}$$

Hence, the determinant is invariant under change of basis. $\square$

	Points Possible					
complete	0	1	2	3	4	5
mathematically valid	0	1	2	3	4	5
readable/fluent	0	1	2	3	4	5
Total:						(out of 15)

Statement: Let $X \in M_n(\mathbb{C})$, let Λ be the set of eigenvalues for X, and let m_λ be the multiplicity of $\lambda \in \Lambda$. Show

$$\operatorname{tr}(X) = \sum_{\lambda \in \Lambda} \lambda m_\lambda \quad \text{and} \quad \det(X) = \prod_{\lambda \in \Lambda} \lambda^{m_\lambda}.$$

Problem:	**3B**
No. stars:	**1**

Proof. Since $\mathbb{C}$ is an algebraically closed field and $X \in M_n(\mathbb{C})$, we have that there is some J in Jordan canonical form such that $J \sim X$. That is, there is some choice of basis under which X can be written in Jordan canonical form. Trace is invariant under choice of basis, so we have that $\operatorname{tr}(X) = \operatorname{tr}(J)$. Note that in Jordan canonical form, the eigenvalues of X are placed along the diagonal. Thus, the diagonal of J contains all of the eigenvalues of X. Moreover, the multiplicity of an eigenvalue is given by the number of rows in which it appears in the matrix J. Hence, we must have that,

$$\begin{aligned} \operatorname{tr}(X) &= \operatorname{tr}(J) \\ &= \sum_{\lambda \in \Lambda} \lambda m_\lambda \end{aligned}$$

as required.

Similarly to the above, we have $\det(X) = \det(J)$. Now consider the characteristic polynomial of J. This is given by,

$$\begin{aligned} c_J(x) &= \det(J - x \cdot \mathrm{id}) \\ &= \prod_{\lambda \in \Lambda} (\lambda - x)^{m_\lambda} \end{aligned}$$

If we plug in 0 for x, we get,

$$\begin{aligned} c_J(0) &= \det(J - 0 \cdot \mathrm{id}) \\ &= \det(J) \\ &= \prod_{\lambda \in \Lambda} (\lambda)^{m_\lambda} \end{aligned}$$

Hence, we have that,

$$\begin{aligned} \det(X) &= \det(J) \\ &= \prod_{\lambda \in \Lambda} (\lambda)^{m_\lambda} \end{aligned}$$

as required. $\square$

	Points Possible					
complete	0	1	2	3	4	5
mathematically valid	0	1	2	3	4	5
readable/fluent	0	1	2	3	4	5
Total:						(out of 15)