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Lec 4, Ex. A

1a. Homomorphism:

0 map module homomorphism

Inclusion map $(x \mapsto (x,0))$ module homomorphism

$(x,y) \mapsto y$: (call ϕ)

$\phi((x+x', y+y'))$

$= y+y'$

$= \phi((x,y)) + \phi((x,y'))$

$\phi((rx, ry)) = ry$

$= r\phi((x,y))$ ✓

$\mathbb{Z}$: (Show $\mathbb{Z} \xrightarrow{x \mapsto (x,0)} \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ injec.)
Call ψ

Suppose $\psi(x) = \psi(x')$

$\Rightarrow (x,0) = (x',0)$

$\Rightarrow x = x'$

So ψ injective $\Rightarrow$ exact at $\mathbb{Z}$

$\mathbb{Z}/n\mathbb{Z}$: (Show $\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} \xrightarrow{(x,y) \mapsto y} \mathbb{Z}/n\mathbb{Z}$ surj.)

Suppose $\exists y' \in \mathbb{Z}/n\mathbb{Z}$ s.t. there is no $(x,y) \in \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ with

$$\phi((x,y)) = y'$$

$$\Rightarrow (x, y') \notin \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} \quad \forall x \in \mathbb{Z}$$

But since $y' \in \mathbb{Z}/n\mathbb{Z}$, must have

$$(x, y') \in \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} \quad \forall x \in \mathbb{Z}$$

Hence, we have contradiction and map is surjective

$\Rightarrow$ Exact at $\mathbb{Z}/n\mathbb{Z}$

$\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$: (Show exactness properties)

Image of $\mathbb{Z} \xrightarrow{x \mapsto (x,0)} \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$

$= \mathbb{Z} \oplus 0$

Kernel of $\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} \xrightarrow{(x,y) \mapsto y} \mathbb{Z}/n\mathbb{Z}$

$= \{(x,0) \mid x \in \mathbb{Z}\}$

$= \mathbb{Z} \oplus 0$

So exact at $\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$

$\Rightarrow$ Sequence is an SES

b. Homomorphism:

$$\varphi(x) = nx$$

$$\psi(x) = x + n\mathbb{Z}$$ ← natural projection onto quotient (homomorphism) ✓

$$\varphi(x+y) = n(x+y)$$
$$= nx + ny$$
$$= \varphi(x) + \varphi(y)$$

$$\varphi(mx) = n(mx)$$
$$= (nm)x$$
$$= (mn)x$$
$$= m(nx)$$
$$= m\,\varphi(x)$$ ✓

0 map homomorphism ✓

All maps in sequence homomorphisms

Exact at first $\mathbb{Z}$: (Show $\mathbb{Z} \xrightarrow{x \mapsto nx} \mathbb{Z}$ injective)

Suppose $\phi(x) = \phi(x')$

$\Rightarrow nx = nx'$

$\Rightarrow n^{-1}(nx) = n^{-1}(nx')$

$\Rightarrow x = x'$

Injective ✓

$\Rightarrow$ Exact at first $\mathbb{Z}$

Exact at $\mathbb{Z}/n\mathbb{Z}$: (Show $\mathbb{Z} \xrightarrow{x \mapsto x + n\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ surjective)

Suppose $\exists x' + n\mathbb{Z} \in \mathbb{Z}/n\mathbb{Z}$ s.t.
$\nexists x \in \mathbb{Z}$ with $\psi(x) = x' + n\mathbb{Z}$

$\Rightarrow x' \notin \mathbb{Z}$

But by def. of $\mathbb{Z}/n\mathbb{Z}$, $x' \in \mathbb{Z}$.

Thus we have contradiction and ψ surjective

$\Rightarrow$ Exact at $\mathbb{Z}/n\mathbb{Z}$

Exact at second $\mathbb{Z}$: (Check exact props.)

Image of $x \mapsto nx$:

$n\mathbb{Z}$

Kernel of $x \mapsto x + n\mathbb{Z}$:

$n\mathbb{Z}$

$\Rightarrow$ Exact at second $\mathbb{Z}$

$\Rightarrow$ Sequence is SES

2a Inclusions are injective homomorphisms

$\Rightarrow$ exact at $\ker(\varphi)$

$X \xrightarrow{\varphi} \varphi(X)$ must be surjective by def.

$\Rightarrow$ exact at $\varphi(X)$

X exact by def.

(image of $\ker(\varphi)$ in X is $\ker(\varphi)$)

b. Inclusions are injective $\Rightarrow$ exactness at $p(X)$

$Y \xrightarrow{\pi} Y/\ker(y)$ surjective homomorphism by definition

$\Rightarrow$ exact at $Y/\ker(y)$

By fact that $X \xrightarrow{p} Y \xrightarrow{y} Z$ exact,

$p(X) = \ker(y)$

$\Rightarrow$ exact at Y

c. Homomorphisms:

Inclusion map and 0 maps are module homomorphism

Natural projection onto component of direct sum is a homomorphism

Exact at A.

$A \xrightarrow{a \mapsto (a,0)} A \oplus B$ injective

Exact at B.

$A \oplus B \xrightarrow{(a,b) \mapsto b} B$ surjective

Exact at $A \oplus B$:

Image of $A \xrightarrow{a \mapsto (a,0)} A \oplus B$:

$A \oplus 0$

Kernel of $A \oplus B \xrightarrow{(a,b) \mapsto b} B$:

$(a, 0) = A \oplus 0$