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Lec. 6, Ex. A

1. Polynomials are commutative with 1 and $M_2(\mathbb{R})$ is a ring

$$f(p+q) = (p+q)(0)\, I_2$$

$$= (\alpha_n x^n + \cdots + \alpha_1 x + \alpha_0 + \beta_m x^m + \cdots + \beta_1 x + \beta_0)(0)\, I_2$$

$$= (\alpha_0 + \beta_0)\, I_2$$

$$= \begin{bmatrix} \alpha_0 + \beta_0 & 0 \\ 0 & \alpha_0 + \beta_0 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_0 & 0 \\ 0 & \alpha_0 \end{bmatrix} + \begin{bmatrix} \beta_0 & 0 \\ 0 & \beta_0 \end{bmatrix}$$

$$= \alpha_0 I_2 + \beta_0 I_2$$

$$= p(0)\, I_2 + q(0)\, I_2$$

$$= f(p) + f(q) \checkmark$$

Note only coefficient without x variable is 2 coefficients without x term multiplied together

$$f(pq) = \left[(pq)(0) \right] I_2$$

$$= (\alpha\beta) I_2$$

$$= \begin{bmatrix} \alpha\beta & 0 \\ 0 & \alpha\beta \end{bmatrix}$$

$$= \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \cdot \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix}$$

$$= \alpha I_2 \cdot \beta I_2$$

$$= f(p) f(q)$$

2ai) A ring $\Rightarrow$ additive group

Let $r, s \in R$, $m, n \in A$

$$(r+s)m = f(r+s)m$$

$$= (f(r) + f(s))m$$

$$= f(r)m + f(s)m$$

$$(rs)m = f(rs)m$$

$$= f(r) f(s) m$$

$$r(m+n) = f(r)(m+n)$$

$$= f(r)m + f(r)n$$

$$1m = f(1)m = 1_A m$$
$$= m$$

ii) $r \cdot (ab) = f(r) \cdot (ab)$

$\hookrightarrow \in Z(A)$

associativity

$= f(r)\, ab$

$= a\, f(r)\, b$

$= a\,(f(r)\, b)$

$= a\,(r \cdot b)$

b. $f: R \to Z(A)$, $f(r) = r \cdot 1$

$a f(r) = a\,(r \cdot 1)$

$= r \cdot a \cdot 1$

$= r \cdot 1 \cdot a$

$= f(r) \cdot a$

$f(r) \in Z(A)$

$r, s \in R$

$$f(r+s) = (r+s)\cdot 1$$
$$= r\cdot 1 + s\cdot 1$$
$$= f(r) + f(s)$$

$$f(rs) = (rs)\cdot 1$$
$$= r\cdot(s\cdot 1)$$
$$= r\cdot f(s)$$
$$= r\cdot(1 f(s))$$
$$= (r\cdot 1)\cdot f(s)$$
$$= f(r)f(s)$$

3. β basis of A (as a vec. space over F)