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Math B4900
Homework 8
4/23/2021

Let A be a ring with 1.

1. Prove that if every A-module is free, then A is a division ring. [*Caution:* this proof may involve a lot of machinery. Give it time, and ample brainstorming of the tools you have so far.]

   Proof. Suppose A is a ring with 1 and that every A-module is free. $\square$

2. Classify the semisimple $\mathbb{Z}$-modules. [*Hint.* What are the simple $\mathbb{Z}$-modules?]

3. Let M be a semisimple A-module. Prove that the following are equivalent:

   (i) M is finitely-generated;

   (ii) M is Noetherian;

   (iii) M is Artinian; M is a finite direct sum of simple modules.

4. (a) Let R and S be rings such that $M_m(R) \cong M_n(S)$ for some $m, n \in \mathbb{Z}_{\geq 1}$. For this imply that $m = n$ and $R \cong S$? If so, why? If not, give a counter-example.

   (b) We call A a *full matrix ring* if $A \cong M_n(R)$ for some ring R and some $n \in \mathbb{Z}_{\geq 1}$. Is the homomorphic image of a matrix ring necessarily itself a matrix ring? If so, prove it. If not, give a counterexample.

To receive credit for this assignment, include the following in your solutions [edited appropriately]:

Academic integrity statement: I *did not violate* the CUNY Academic Integrity Policy in completing this assignment. *Chris Hayduk*
