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Math B4900
Homework 8
4/23/2021

Let A be a ring with 1.

1. Prove that if every A-module is free, then A is a division ring. [*Caution:* this proof may involve a lot of machinery. Give it time, and ample brainstorming of the tools you have so far.]

   Proof. Note that A is a division ring if $A^x = A - \{0\}$. That is, every non-zero element is a unit.

   Suppose A is a ring with 1 and that every A-module is free. Properties of A: every A-module is projective, every A-module is injective, A is semisimple. Since A is semisimple, then the left regular A-module is completely decomposible (semisimple). In addition, $A \cong M_{n_1}(\Delta_1) \times \cdots \times M_{n_\ell}(\Delta_\ell)$.

   Now we will show that A is simple as a left regular A-module in order to show that every non-zero element of A has a multiplicative inverse.

   We have that A is a submodule of A. Then we must have that A/A is semisimple. But $A/A = 0_A$. □

2. Classify the semisimple $\mathbb{Z}$-modules. [*Hint.* What are the simple $\mathbb{Z}$-modules?]

   Proof. Note that a $\mathbb{Z}$ module is simple if it is of the form $\mathbb{Z}/I$ where I is a maximal ideal. The ideals of $\mathbb{Z}$ are precisely the sets of all integers divisible by a fixed integer n. That is, $n\mathbb{Z}$ is an ideal for all $n \in \mathbb{Z}$. Recall that an ideal $n\mathbb{Z}$ of $\mathbb{Z}$ is maximal if there are no other ideals of the form $k\mathbb{Z}$ such that $n\mathbb{Z} \subset k\mathbb{Z} \subset \mathbb{Z}$. Observe that if n is a composite integer, then we can write $n = p_1p_2 \cdots p_\ell$ for primes in $\mathbb{Z}$. That is, for any p_j in that expansion, we have that p_j divides n and thus all multiples of n. Hence, $n\mathbb{Z} \subset p_j\mathbb{Z}$ for any prime p_j in that expansion. Moreover, for every prime we must have that there is no integer m such that $p_j\mathbb{Z} \subset m\mathbb{Z}$, otherwise m would divide p_j and hence p_j would not be prime. Thus, the maximal ideals of $\mathbb{Z}$ are precisely of the form $p\mathbb{Z}$ where p is a prime.

   Now we have that the simple modules of $\mathbb{Z}$ are of the form $\mathbb{Z}/p\mathbb{Z}$ for all primes $p \in \mathbb{Z}$. Since semisimple modules are direct sums of simple modules, we have that any semisimple module of $\mathbb{Z}$ is of the form:

   $$p_1\mathbb{Z} \oplus p_2\mathbb{Z} \oplus \cdots \oplus p_\ell\mathbb{Z}$$

   for some primes $p_1, \ldots, p_\ell$ (not necessarily distinct). □

3. Let M be a semisimple A-module. Prove that the following are equivalent:

   (i) M is finitely-generated;

   (ii) M is Noetherian;

   (iii) M is Artinian;

(iv) M is a finite direct sum of simple modules.

Proof. First we will show that (i) is equivalent to (ii). Suppose M is finitely generated. Then there exist $m_1, m_2, \ldots, m_n \in M$ such that for any $x \in M$, there exist $a_1, a_2, \ldots, a_n \in A$ with $x = a_1m_1 + a_2m_2 + \cdots + a_nm_n$. Since every element of a submodule of M is also an element of M, then it must be true that every element of a submodule N of M is finitely generated as well. Hence, M is Noetherian. Now suppose M is Noetherian. Then every submodule of M is finitely generated. In particular, since M is a submodule of itself, it must be finitely generated. Thus, (i) and (ii) are equivalent.

Now we will show the equivalence of (i) and (iv). Suppose M is semisimple and finitely generated. Then M is the direct sum of simple modules and, since (i) is equivalent to (ii), each of those submodules is finitely generated. Since the generators of M are finite, they can old by combined in a finite number of ways. Hence, there must be finitely many submodules which are finitely generated. Hence, M is a finite direct sum of simple modules. Now let us assume that M is a finite direct sum of simple modules and work towards the other directions. Every simple module is cyclic and hence generated by one element. The union of these generators forms a basis for M since M is a direct sum of these simple modules. Since there are a finite number of these simple modules, then M is finitely-generated by this union as required. Hence, by this and our previous work, (i), (ii), and (iv) are equivalent.

Now we will show the equivalent of (iii) and (iv). Suppose M is semisimple and Artinian. Then the sequence of submodules of M

$$M_1 \supset M_2 \supset \ldots$$

stabilizes. That is, there exists an integer N such that if $n \geq N$ then $M_n = M_{n+1}$. Since no simple module can have a submodule, then M_N is the only simple module in this chain. Observe that since M is semisimple, it must be the direct sum of simple submodules. There must be only finitely many of these simple submodules (why?), so M is a finite direct sum of simple modules. Now let us assume that M is semisimple and a finite direct sum of simple modules and work in reverse. □

4. (a) Let R and S be rings such that $M_m(R) \cong M_n(S)$ for some $m, n \in \mathbb{Z}_{\geq 1}$. Does this imply that $m = n$ and $R \cong S$? If so, why? If not, give a counter-example.

Proof. This is not true. Let us take $R = M_k(\mathbb{C})$ and $S = M_n(\mathbb{C})$ with $n \neq k$ and $n, k \geq 1$. Then $S \ncong R$ since the matrices are of different dimension, but we have that,

$$M_n(R) = M_n(M_k(\mathbb{C})) \cong M_{nk}(\mathbb{C})$$

and

$$M_k(S) = M_k(M_n(\mathbb{C})) \cong M_{nk}(\mathbb{C})$$

Hence, since $M_n(R) \cong M_{nk}(\mathbb{C})$ and $M_k(S) \cong M_{nk}(\mathbb{C})$, we must have that $M_n(R) \cong M_k(S)$. □

(b) We call A a *full matrix ring* if $A \cong M_n(R)$ for some ring R and some $n \in \mathbb{Z}_{\geq 1}$. Is the homomorphic image of a matrix ring necessarily itself a matrix ring? If so, prove it. If not, give a counterexample.

Proof. Let B be the homomorphic image of A. By The First Isomorphism Theorem from Dummit and Foote, we have that $B \cong A/\ker \phi$. In addition, from Theroem 7(2) in Dummit and Foote, we have that $\ker \phi \cong I$ where I is some ideal of A. Hence, $B \cong A/I \cong M_n(R)/I$. Moreover, we know from Lam Theorem 3.1 that the ideals of $M_n(R)$ are in bijection with the ideals of R. That is, for some ideal I_R of R, we have,

$$\begin{aligned} B &\cong M_n(R)/I \\ &\cong M_n(R)/M_n(I_R) \\ &\cong M_n(R/I_R) \end{aligned}$$

Hence, B is a matrix ring as well. □

To receive credit for this assignment, include the following in your solutions [edited appropriately]:

Academic integrity statement: I *did not violate* the CUNY Academic Integrity Policy in completing this assignment. *Chris Hayduk*
