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Lecture 3, Ex. B

1a. Let $u, u' \in V$, $\alpha \in F$

$$f_v(\alpha u) = \langle v, \alpha u \rangle \quad \text{(bilinear)}$$
$$= \alpha \langle v, u \rangle$$
$$= \alpha f_v(u)$$

$$f_v(u+u') = \langle v, u+u' \rangle \quad \text{(bilinear)}$$
$$= \langle v, u \rangle + \langle v, u' \rangle$$

b. Let $\beta \in F$, $v, v' \in V$

$$f_{\beta v}(u) = \langle \beta v, u \rangle$$
$$= \beta \langle v, u \rangle$$
$$= \beta f_v(u) \quad \checkmark$$

$$f_{v+v'}(u) = \langle v+v', u \rangle$$
$$= \langle v, u \rangle + \langle v', u \rangle$$
$$= f_v(u) + f_{v'}(u) \quad \checkmark$$

c. $f_v(u) = 0 \quad \forall u \in U$

$\Rightarrow \langle , \rangle$ is second coordinate degenerate

We know it is non-degenerate, so there cannot be a non-zero v.

$\Rightarrow v = 0$

$\Rightarrow \ker(f) = 0$

$\Rightarrow f$ injective

d. We have $\dim(V) = \dim(V^*) = n$

B be basis of V, $n = |B|$

Since f injective

$f(B) = \{f(b) \mid b \in B\}$ is linearly independent and size n

Thus $f(B)$ is basis of V^*. Hence

$V^* = F f(B) = f(FB) = f(V)$

$\Rightarrow f$ is also surjective

$\Rightarrow f$ isomorphism

2. $(1\ \ 1)\begin{pmatrix}1 & 2\\ 0 & -3\end{pmatrix}\begin{pmatrix}2\\ -1\end{pmatrix}$

$= (1\ \ -1)\begin{pmatrix}2\\ -1\end{pmatrix} = 3$

$(2\ \ -1)\begin{pmatrix}1 & 2\\ 0 & -3\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}$

$= (2\ \ 7)\begin{pmatrix}1\\ 1\end{pmatrix} = 9$

$(1,0)\begin{pmatrix}1 & 2\\ 0 & -3\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = (1\ \ 2)\begin{pmatrix}1\\ 0\end{pmatrix}$

$= 1$

$(1,0)\begin{pmatrix}1 & 2\\ 0 & -3\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = (1\ \ 2)\begin{pmatrix}0\\ 1\end{pmatrix}$

$= 2$

$(0,1)\begin{pmatrix}1 & 2\\ 0 & -3\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = (0\ \ -3)\begin{pmatrix}0\\ 1\end{pmatrix} = -3$

3 $\langle u,v \rangle_J = u^T J v$

$\langle v,u \rangle_{J^T} = v^T J^T u$

$= (u^T J v)^T$

Since $u^T J v$ is a 1 dimensional matrix (hence symmetric), we have

$u^T J v = v^T J^T u$

$\Longrightarrow \langle u,v \rangle_J = \langle v,u \rangle_{J^T}$

6 If J is symmetric (ie. $J = J^T$), then $\forall u,v \in V$,

$\langle u,v \rangle_J = \langle v,u \rangle_{J^T} = \langle v,u \rangle_J$

Hence $\langle , \rangle$ symmetric