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Math B4900
Homework 6
3/26/2021

Let A be a ring with 1, and let M be an A-module.

1. Investigate whether or not the left regular module for $\mathbb{F}_2 S_2$ is decomposable. If so, give the decomposition. If not, why not? What simple submodules does it contain, and what do the corresponding quotients look like?
   [Recall that you decomposed $\mathbb{C}S_2$ in Homework 5. Also note that $\mathbb{F}_2 S_2$ is small—it has only 4 elements.]

   Proof. Observe that A is an $F-$algebra. □

2. Prove that M is simple if and only if $Am = M$ for any non-zero $m \in M$.

3. Let M be a free module with basis $\mathcal{B}$ and let I be an ideal of A. We showed on Homework 4 that IM is a submodule of M, and it follows similarly that Im is a submodule of M for any $m \in M$.[1]

   Prove that as A-modules, we have
   $$M/IM \cong \bigoplus_{b \in \mathcal{B}} Ab/Ib.$$
   [*Hint:* Since $M \cong \mathcal{F}\mathrm{r}(\mathcal{B})$ is free, we know it decomposes similarly to the right-hand side of this expression. So what's the most natural homomorphism from M to $\bigoplus_{b\in\mathcal{B}} Ab/Ib$? What's its kernel? (You may assume the isomorphism theorems, as found in Lang §III.1 and D & F §10.2.) See also, D& F, Exercises 10.2.11 and 10.2.12.]

4. **Ranks of free modules.**
   (a) **Rank is well-defined for commutative rings.**[2] Show that if A is a commutative ring with 1, that $A^m \cong A^n$ if and only if $n = m$.
   [Hint: Let I be a maximal ideal of A (what kind of ring does that make A/I?). Now consider A^m/IA^m, as in Problem 3. Recall that we showed earlier in the semester that two finite-dimensional *vector spaces* were isomorphic if and only if they had the same dimension.]

   (b) **Rank is not well-defined in general.**[3] Let M be the $\mathbb{Z}$-module
   $$M = \mathbb{Z} \times \mathbb{Z} \times \cdots = \{(n_1, n_2, \dots) \mid n_i \in \mathbb{Z}\}.$$
   Note the difference here between this and $N = \bigoplus_{i\in\mathbb{Z}_{>0}} \mathbb{Z}$, which is the submodule of M where all but finitely many n_i are 0. In particular, N is free, but M is not (see D&F Exercise 10.3.24).

   Let
   $$A = \mathrm{End}_{\mathbb{Z}}(M) = \{\ \mathbb{Z}\text{-module homomorphisms } \varphi : M \to M\ \},$$
   where, as usual, addition is defined point-wise (i.e. $(\varphi + \psi)(a) = \varphi(a) + \psi(a)$) and multiplication is defined by function composition (i.e. $\varphi\psi = \varphi \circ \psi$). See also D&F Exercise 7.1.30.

[1] This follows considering $M' = Am$, which is a submodule of M. Then since $1 \in A$, we have $IA = A$, so that $IM' = (IA)m = Im$ is also a submodule. Or, you know, just check it directly.

[2] D&F Exercise 10.3.2.

[3] D&F Exercise 10.3.27.

Define $\varphi_1, \varphi_2 \in R$ by

$$\varphi_1(n_1, n_2, n_3, \dots) = (n_1, n_3, n_5, \dots) \quad \text{and} \quad \varphi_2(n_1, n_2, n_3, \dots) = (n_2, n_4, n_6, \dots).$$

(i) **Show that $\{\varphi_1, \varphi_2\}$ is a free basis of the the left regular module of A.** [*Hnit:* Define $\psi_1, \psi_2 \in R$ by

$$\psi_1(n_1, n_2, n_3, \dots) = (n_1, 0, n_2, 0, n_3, \dots) \quad \text{and} \quad \psi_2(n_1, n_2, n_3, \dots) = (0, n_1, 0, n_2, 0, \dots).$$

Verify that

$$\varphi_i\psi_i = 1, \quad \varphi_1\psi_2 = 0 = \varphi_2\psi_1, \quad \text{and} \quad \psi_1\varphi_1 + \psi_2\varphi_2 = 1.$$

Use these relations to prove that φ_1, φ_2 are independent and generate A as a left A-module.]

(ii) Use the previous part to **prove that $A \cong A^2$ as A-modules**. Deduce that $A \cong A^n$ as A-modules for all $n \in \mathbb{Z}_{>0}$.

5. In class, we showed that for any A-module X, we have $\operatorname{Hom}_A(A, X) \cong A$ as groups (or as A-modules if A is commutative). It is *not* necessarily true that $\operatorname{Hom}_A(X, A) \cong A$.

Now suppose A is commutative and let $\mathcal{B}$ be a finite set of size n.

Prove that $\operatorname{Hom}_A(\mathcal{F}\mathrm{r}(\mathcal{B}), A) \cong \mathcal{F}\mathrm{r}(\mathcal{B})$ as A-modules.

(Namely, just like any finite-dimensional vector space is isomorphic its dual, we have any free module of finite rank is also isomorphic to its dual.) We say $\mathcal{F}\mathrm{r}(\mathcal{B})$ is *self-dual* (up to isomorphism). [*Hint:* Reasonable tactics include either showing that $\operatorname{Hom}_A(\mathcal{F}\mathrm{r}(\mathcal{B}), A)$ is free of rank n and using 4(a); or using the last proposition from Lecture 10.]

To receive credit for this assignment, include the following in your solutions [edited appropriately]:

Academic integrity statement: I *did not violate* the CUNY Academic Integrity Policy in completing this assignment. *Christopher Hayduk*
