ID: **5514**
Math B4900
Proof portfolio 2
4/11/2021

Problem:	**4AI**
No. stars:	**1**

Statement: Let z be a central element of A. Show that zM is a submodule of M and that $\varphi : M \to M$ defined by $m \mapsto zm$ is an A-module endomorphism.

Proof. Suppose A is a ring with 1, z is a central element of A and M is an A module. We will first show that zM is a submodule of M using the submodule criterion. First, let us show that $zM \subset M$. Fix $x \in zM$. By the definition of zM, we must have that $x = zm$ for some $m \in M$. Since M is an A module, we have that it is closed under action by the ring elements, and so $zm \in M$. Thus, $zM \subset M$.

Second, let us show that zM is non-empty. Since M is an A module, it must be non-empty, and so there exists $m \in M$. Moreover, am is defined for all $a \in A$ and $m \in M$. In particular, zm is defined with our choice of z and m, and so $zm \in zM$ and so $zM \neq \varnothing$.

Lastly, fix $x, y \in zM$ and $a \in A$. We can write $x = zm_1, y = zm_2$ for some $m_1, m_2 \in M$. Hence, using the properties of M as a module of A and z as a central element of A, we have,

$$\begin{aligned} x + ay &= zm_1 + a(zm_2) \\ &= zm_1 + z(am_2) \\ &= z(m_1 + am_2) \in zM \end{aligned}$$

Thus, we have shown that zM is a submodule of M. Now consider $\varphi : M \to M$ defined by $m \mapsto zm$. As we have shown above, zm is defined for every $m \in M$ and $zm \in M$ for all $m \in M$. Hence, φ is well-defined and does indeed map from $M \to M$. Now we will show that it is an endomorphism on M. Let $x, y \in M$ and $a \in A$,

$$\begin{aligned} \varphi(ax + y) &= z(ax + y) \\ &= z(ax) + zy \\ &= a(zx) + zy \\ &= a\varphi(x) + \varphi(y) \end{aligned}$$

Hence, φ is an endomorphism on M as required. $\square$

	Points Possible
complete	0 1 2 3 4 5
mathematically valid	0 1 2 3 4 5
readable/fluent	0 1 2 3 4 5
Total:	(out of 15)

Problem:	**4C**
No. stars:	**1**

Statement: Let X and Y be submodules of M. Show that

$$0 \hookrightarrow X \cap Y \xrightarrow{f:x \mapsto (x,-x)} X \oplus Y \xrightarrow{g:(x,y) \mapsto x+y} X + Y \to 0$$

is a short exact sequence of A-modules.

Proof. We must verify that f is injective, g is surjective, and $\mathrm{img}(f) = \ker(g)$. First, suppose $f(x_1) = f(x_2)$. Then $(x_1, -x_1) = (x_2, -x_2)$, which implies that $x_1 = x_2$ and $x_2 = -x_2$. Hence, we have that $f(x_1) = f(x_2)$ implies $x_1 = x_2$, and so f is injective. Now fix $z \in X + Y$. Observe that by the definition of $X + Y$, $z = x_1 + y_1$ for some $x_1 \in X$ and $y_1 \in Y$. Since $X \oplus Y = \{(x, y) : x \in X, y \in Y\}$, we must have that $(x_1, y_1) \in X \oplus Y$ and hence $g(x_1, y_1) = x_1 + y_1 = z$, as required. Thus, g is surjective.

Now note that $\mathrm{img}(f) = \{(x, -x) : x \in X \cap Y\}$. We have that $\ker(g) = \{(x, y) : x \in X, y \in Y, y = -x\}$. Hence, every element in $\mathrm{img}(f)$ is of the form $(x, -x)$ and so $g(\mathrm{img}(f)) = 0$, which gives us that $\mathrm{img}(f) \subset \ker(g)$. In addition, since Y is a submodule of M and hence must be an abelian group under addition, if $y = -x$, then we must also have that $x \in Y$. Thus, every element $(x, y) \in \ker(g)$ as defined previously must be such that $x \in X \cap Y$ and $(x, y) = (x, -x)$. Thus, $\ker(g) \subset \mathrm{img}(f)$ and so $\mathrm{img}(f) = \ker(g)$. Hence, we have that this is a short exact sequence, as required. $\square$

Problem:	**5A**
No. stars:	**2**

Statement: Prove that $\mathrm{Hom}_A(*, M)$ is an exact functor. Namely, show that if $0 \hookrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is a split exact sequence of A-modules, then so is

$$0 \hookrightarrow \mathrm{Hom}_A(M,X) \xrightarrow{F} \mathrm{Hom}_A(M,Y) \xrightarrow{G} \mathrm{Hom}_A(M,Z) \to 0,$$

where $F(\varphi) = f \circ \varphi$ and $G(\varphi) = g \circ \varphi$. [You may use any other propositions or theorems from Lecture 10 or before.]

Proof. By Proposition 2.2 in Section 3 of Lang, we have that,

$$0 \hookrightarrow \mathrm{Hom}_A(M,X) \xrightarrow{F} \mathrm{Hom}_A(M,Y) \xrightarrow{G} \mathrm{Hom}_A(M,Z)$$

is exact. Moreover, we can assert that

$$0 \hookrightarrow \mathrm{Hom}_A(M,X) \xrightarrow{F} \mathrm{Hom}_A(M,Y) \xrightarrow{G} \mathrm{Hom}_A(M,Z) \hookrightarrow 0$$

is a short exact sequence because $\mathrm{Im}(G) = \mathrm{Hom}_A(M,Z)$ and $\ker(0) = \mathrm{Hom}_A(M,Z)$. Now define $\mu : \mathrm{Hom}_A(M,Z) \to \mathrm{Hom}_A(M,Y)$ by $\mu(\varphi) = \varphi^{-1} \circ g^{-1}$ and define $\lambda : \mathrm{Hom}_A(M,Y) \to \mathrm{Hom}_A(M,X)$ by $\lambda(\varphi) = \varphi^{-1} \circ f^{-1}$. Then

$$\begin{aligned} G\mu &= g \circ \varphi \circ \varphi^{-1} \circ g^{-1} \\ &= \mathrm{id} \end{aligned}$$

and

$$\begin{aligned} \lambda f &= \varphi^{-1} \circ f^{-1} \circ f \circ \varphi \\ &= \mathrm{id} \end{aligned}$$

Hence, by the Proposition from Lecture 10 part B, we have that our short exact sequence is also split, with μ and λ as the splitting homomorphisms. □

Problem:	**6A**
No. stars:	**2**

Statement: Let A be a ring with 1, and let M be an $A-$module. Prove that M is simple if and only if $Am = M$ for any non-zero $m \in M$.

Proof. Suppose M is simple. Then the only submodules of M are 0 and itself. By the footnote on Homework 6, we have that Am is a submodule of M. Since M is simple, we have that either $Am = 0$ and $Am = M$. However, we know that $m \neq 0$. Since A is a ring with 1, we have that $1m = m \in Am$ and so $Am \neq 0$. Thus, we must have that $Am = m$ for any non-zero $m \in M$.

Now suppose that $Am = M$ for any non-zero $m \in M$. Let $N \subset M$ be a submodule and suppose $N \neq 0$. Thus there is some $m \in N \backslash \{0\} \subset M \backslash \{0\}$. Now since $Am = M$ for every non-zero $m \in M$, we must have that $Am = M$ for this particular choice of m. Since N is a submodule and closed under the action of A on N, we must have that $N = M$. Thus, M is simple. $\square$

Problem:	**6C**
No. stars:	**2**

Statement: Show that if A is a commutative ring with 1, that $A^m \cong A^n$ if and only if $n = m$.

Proof. Suppose A is is a commutative ring with 1 and suppose that $A^m \cong A^n$. Let I be a maximal ideal of A. Since $A^m \cong A^n$, we have that $IA^m \cong IA^n$, and so $A^m/IA^m \cong A^n/IA^n$. Moreover, from Q3 on Homework 6, we have that,

$$A^m/IA^m \cong \oplus_{b\in\mathcal{B}} Ab/Ib$$
$$A^n/IA^n \cong \oplus_{c\in\mathcal{C}} Ac/Ic$$

Thus, we have that $\oplus_{b\in\mathcal{B}} Ab/Ib \cong \oplus_{c\in\mathcal{C}} Ac/Ic$.

Since I is a maximal ideal of A, by Proposition 12 in Section 7.4, we have that Ab/Ib and Ac/Ic are fields for every b, c.

Now suppose $n = m$. Then we must have $A^m = A^n$ and so $A^m \cong A^n$ by the identity map. $\square$