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Math B4900
Homework 9
4/30/2021

Let A be a ring with 1.

1. Let M be a completely reducible A-module. Show that for any submodule $N \subseteq M$, we have M/N is completely reducible as well. Moreover, if

$$M \cong \bigoplus_{\lambda \in \Lambda} M_\lambda, \quad \text{then} \quad M/N \cong \bigoplus_{\lambda \in \Gamma} M_\lambda,$$

for some $\Gamma \subseteq \Lambda$.

[*Hint:* if $M \cong \bigoplus_{\lambda \in \Lambda} M_\lambda$ (with M_λ simple), then, more simply, $M = \sum_{\lambda \in \Lambda} M_\lambda$ (identifying M_λ with $\hat{M}_\lambda$). Show that $M/N = \sum_\lambda (M_\lambda + N)/N$ (write out the cosets!), and then use the second isomorphism theorem on each piece. Finally, check that, for all $\mu \in \Lambda$, we have

$$(M_\mu + N)/N \cap \sum_{\lambda \neq \mu} (M_\lambda + N)/N = 0.]$$

Proof. Since M is completely reducible, we have that $M \cong \bigoplus_{\lambda \in \Lambda} M_\lambda$ where M_λ is simple. This is equivalent to $M = \sum_{\lambda \in \Lambda} M_\lambda$ with $\left(\sum_{\lambda \in \Lambda - \mu} M_\lambda\right) \cap M_\mu = 0$. Hence, for any $m \in M$, we have that $m = \sum_{\lambda \in \Lambda} m_\lambda$ uniquely.

Let $N \subset M$. Then,

$$\begin{aligned}
M/N &= \left(\sum_{\lambda \in \Lambda} M_\lambda\right)/N \\
&= \{m + N \mid m \in \sum_{\lambda \in \Lambda;\text{ finite}} M_\lambda\}
\end{aligned}$$

Thus, for $m + N \in M/N$, we have,

$$\begin{aligned}
m + N &= \sum_{\lambda;\text{ finite}} m_\lambda + N \\
&= \sum_{\lambda;\text{ finite}} m_\lambda + \sum_{\lambda; m_\lambda \neq 0} N \\
&= \{\sum_{\lambda;\ m_\lambda \neq 0} n_\lambda \mid n_\lambda \in N\} \\
&= N \\
&= \sum_{\lambda;\text{ finite}} (m_\lambda + N) \in (M_\lambda + N)/N
\end{aligned}$$

The above derivation thus gives us that $M/N = \sum_\lambda (M_\lambda + N)/N$. Now, applying the second isomorphism theorem for modules, we have that,

$$\begin{aligned} M/N &= \sum_\lambda (M_\lambda + N)/N \\ &= \sum_\lambda M_\lambda/(M_\lambda \cap N) \end{aligned}$$

Observe that, since each M_λ is simple, we have that $M_\lambda \cap N = 0$ if $M_\lambda \not\subset N$ and $M_\lambda \cap N = M_\lambda$ if $M_\lambda \subset N$. These are the only two possible values for $M_\lambda \cap N$. Thus, for a fixed M_λ, we have either that,

$$\begin{aligned} M_\lambda/(M_\lambda \cap N) &= M_\lambda/0 \\ &= \{m_\lambda + 0 \mid m_\lambda \in M_\lambda\} \\ &= M_\lambda \end{aligned}$$

or,

$$\begin{aligned} M_\lambda/(M_\lambda \cap N) &= M_\lambda/M_\lambda \\ &= 0 \end{aligned}$$

Thus, we have that $\sum_\lambda M_\lambda/(M_\lambda \cap N)$ corresponds to some subset $\Gamma \subset \Lambda$, since the terms are either M_λ for some λ or 0. This gives us that,

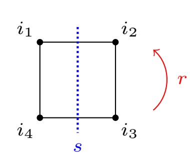
$$\begin{aligned} M/N &= \sum_\lambda (M_\lambda + N)/N \\ &= \sum_\lambda M_\lambda/(M_\lambda \cap N) \\ &= \sum_{\lambda \in \Gamma} M_\lambda \\ &\cong \bigoplus_{\lambda \in \Gamma} M_\lambda \end{aligned}$$

as required. $\square$

2. Let $\{z_\lambda \mid \lambda \in \Lambda\}$ be the centrally primitive idempotents in a semisimple ring A, and let U_λ be the simple A-module corresponding to $\lambda \in \Lambda$. Let M be an A-module (not necessarily the left-regular module. Use Artin-Wedderburn to show that $z_\lambda M \cong \bigoplus_{i \in \mathcal{I}} U_\lambda$ (i.e. z_λ projects onto a (not necessarily finite) direct sum of a bunch of copies of U_λ—called the λ-*isotypic component* of M).

3. Let $V = \mathbb{C}^2 = \mathbb{C}\{v_1, v_2\}$. Let $\mathbb{C}D_8$ act on $V^{\otimes 4} = V \otimes V \otimes V \otimes V$ by identifying the copies of V with the vertices of the square, and applying the corresponding factor permutation:

$$r \cdot (v_{i_1} \otimes v_{i_2} \otimes v_{i_3} \otimes v_{i_4}) = v_{i_2} \otimes v_{i_3} \otimes v_{i_4} \otimes v_{i_1}$$
and
$$s \cdot (v_{i_1} \otimes v_{i_2} \otimes v_{i_3} \otimes v_{i_4}) = v_{i_2} \otimes v_{i_1} \otimes v_{i_4} \otimes v_{i_3}$$



(where $i_1, i_2, i_3, i_4 \in \{1, 2\}$). For example, r fixes $v_1 \otimes v_1 \otimes v_1 \otimes v_1$, but $r \cdot v_1 \otimes v_2 \otimes v_1 \otimes v_1 = v_2 \otimes v_1 \otimes v_1 \otimes v_1$.

Use the primitive central idempotents of $\mathbb{C}D_8$ to decompose $V^{\otimes 4}$ into its isotypic components (you computed these idempotents in HW 5; you should also know which corresponds to which simple representations of $\mathbb{C}D_8$). Then make a dimension argument to classify the decomposition of $V^{\otimes 4}$ up to isomorphism—and make a complete decomposition if you can.

[*See p. 2 for some help.*]

Proof. We have that $M = V \otimes V \otimes V \otimes V = \mathbb{C}\{v_{i_1} \otimes v_{i_2} \otimes v_{i_3} \otimes v_{i_4} \mid v_{i_j} \text{ is a basis vector of } V\}$. We can think about each element of V as a length 4 word with alphabet $\{1, 2\}$. For example, we have $v_1 \otimes v_1 \otimes v_2 \otimes v_1 \mapsto 1121$. In addition, we have $\dim(M) = \dim(V)^4 = 16$. $\square$

4. Let $S_3 \leq S_4$ in the usual way, and let $\mathcal{W}$ be the reflection representation. Compute the action of $s_1 = (12)$, $s_2 = (23)$, and $s_3 = (34)$ on $\mathrm{Ind}_{\mathbb{C}S_3}^{\mathbb{C}S_4}(\mathcal{W})$. [*Hint:* Stay organized!]

Proof. We have that $S_3 = \{1, (12), (23), (13), (123), (132)\}$ and $S_4 = \{1, (ab), (abc), (abcd), (ab)(cd) \mid a, b, c, d \in \{1, 2, 3, 4\} \text{ distinct}\}$. By Lagrange's theorem, we have that $|S_4 : S_3| = |S_4|/|S_3| = 4!/3! = 4$. Thus, there should be 4 left costs of S_3 in S_4. We will compute these cosets using the transpositions: $1, (14), (24), (34)$. This gives us,

$$\begin{aligned}
1\{1, (12), (23), (13), (123), (132)\} &= \{1, (12), (23), (13), (123), (132)\} \\
(14)\{1, (12), (23), (13), (123), (132)\} &= \{(14), (124), (14)(23), (134), (1234), (1324)\} \\
(24)\{1, (12), (23), (13), (123), (132)\} &= \{(24), (142), (234), (24)(13), (1423), (1342)\} \\
(34)\{1, (12), (23), (13), (123), (132)\} &= \{(34), (34)(12), (243), (143), (1243), (1432)\}
\end{aligned}$$

So $a_1 = 1, a_2 = (14), a_3 = (24), a_4 = (34)$.

Observe that S_4 is generated by $\langle (1234), (12) \rangle$, and so for each a_i, we will compute $(1234)a_i = a_j\sigma$ and $(12)a_i = a_k\tau$ for some $j, k \in \{1, 2, 3, 4\}$ and some $\sigma, \tau \in S_3$:

$$\begin{aligned}
(1234)a_1 &= (1234) \cdot 1 = 1 \cdot (1234) = a_1 \cdot (1234) \\
(1234)a_2 &= (1234) \cdot (14) = (234) = a_3 \cdot (23) \\
(1234)a_3 &= (1234) \cdot (24) = (21)(34) = (34)(12) = a_4 \cdot (12) \\
(1234)a_4 &= (1234) \cdot (34) = (312) = (123) = a_1 \cdot (123) \\
(12)a_1 &= (12) \cdot 1 = 1 \cdot (12) = a_1 \cdot (12) \\
(12)a_2 &= (12) \cdot (14) = (142) = (24) \cdot (12) = a_3 \cdot (12) \\
(12)a_3 &= (12) \cdot (24) = (241) = (124) = (14) \cdot (12) = a_2 \cdot (12) \\
(12)a_4 &= (12) \cdot (34) = (12)(34) = (34)(12) = a_4 \cdot (12)
\end{aligned}$$

Now let us fix any $0 \neq \alpha \in \mathbb{C}$ as our basis. Then $\mathbb{C}S_4 \otimes_{\mathbb{C}S_3}$ has basis,

$$v_1 = a_1 \otimes \alpha, \quad v_2 = a_2 \otimes \alpha, \quad v_3 = a_3 \otimes \alpha, \quad v_4 = a_4 \otimes \alpha$$

By the previous computations, on this basis we have,

$$\begin{aligned}
(12)v_1 = (12)a_1 \otimes \alpha &= a_1 \otimes (12)\alpha \\
&= a_1 \otimes (-\alpha) = -a_1 \otimes \alpha = -v_1
\end{aligned}$$

$$\begin{aligned}(12)v_2 = (12)a_2 \otimes \alpha &= a_3 \otimes (12)\alpha \\ &= a_3 \otimes (-\alpha) = -a_1 \otimes \alpha = -v_3\end{aligned}$$

$$\begin{aligned}(12)v_3 = (12)a_3 \otimes \alpha &= a_2 \otimes (12)\alpha \\ &= a_2 \otimes (-\alpha) = -a_2 \otimes \alpha = -v_2\end{aligned}$$

$$\begin{aligned}(12)v_4 = (12)a_4 \otimes \alpha &= a_4 \otimes (12)\alpha \\ &= a_4 \otimes (-\alpha) = -a_4 \otimes \alpha = -v_4\end{aligned}$$

So $(12)v_1 = -v_1$, $(12)v_2 = -v_3$, $(12)v_3 = -v_2$, and $(12)v_4 = -v_4$. This yields,

$$\rho((12)) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Now we will compute the action of (1234) on $\square$

To receive credit for this assignment, include the following in your solutions [edited appropriately]:

Academic integrity statement: I *did not violate* the CUNY Academic Integrity Policy in completing this assignment. *Christopher Hayduk*

Help with #3: This is a big computational problem. But with a little bit of care, it won't be too bad. One tip is to encode a basis vector like $v_{i_1} \otimes v_{i_2} \otimes v_{i_3} \otimes v_{i_4}$ as $i_1 i_2 i_3 i_4$. For example, $v_1 \otimes v_2 \otimes v_1 \otimes v_1$ becomes 1211, and $r \cdot 1211 = 2111$. Another trick you have up your sleeve is action graphs; namely, the action of $\mathbb{C}D_8$ on $V^{\otimes 4}$ is a linear extension of the action of D_8 on $\{i_1 i_2 i_3 i_4 \mid i_\ell \in \{1,2\}\}$. For example, one part of your action graph will look like

$$\begin{tikzcd}
1112 \arrow[r, "r"] \arrow[r, bend right, leftrightarrow, "s"'] & 1121 \arrow[d, "r"] \\
2111 \arrow[u, "r"] & 1211 \arrow[l, "r"] \arrow[l, bend right, leftrightarrow, "s"']
\end{tikzcd}.$$

Next, your job is to compute $z_j V^{\otimes 4}$ for each $j = 1, \ldots, 5$. But since the simple tensors $\{i_1 i_2 i_3 i_4 \mid i_\ell \in \{1,2\}\}$ form a spanning set of $V^{\otimes 4}$; the action of z_j on this set, $\{z_j \cdot i_1 i_2 i_3 i_4 \mid i_\ell \in \{1,2\}\}$, sill form a spanning set of $z_j V^{\otimes 4}$. To compute $z_j V^{\otimes 4}$, you just need to compute $z_j \cdot i_1 i_2 i_3 i_4$ for each set of $i_\ell \in \{1,2\}$, and taking the span of the result.

Now, recall that the coefficients in z_1 correspond to setting $r = 1$ and $s = 1$; the coefficients in z_2 correspond to setting $r = -1$ and $s = 1$; and so on...; so the first four of these computations essentially amount to walking around the vertices of this graph, assigning ± 1 coefficients by what edge we walk along, and then summing up the result. So for example, the computation of z_2 acting on 1112 looks like (starting from the upper-left corner, corresponding to the action of 1, and moving out)

$$\begin{tikzcd}
\text{start} \arrow[r] & \overset{1}{\boxed{1112}}_{+} \arrow[r, "r"] \arrow[d, bend right, "s"'] & \overset{r}{\boxed{1121}}_{-} \arrow[r, "r"] & \overset{r^2}{\boxed{1211}}_{+} \arrow[r, "r"] & \overset{r^3}{\boxed{2111}}_{-} \\
& \underset{s}{\boxed{1121}}^{+} \arrow[r, "r"'] & \underset{rs}{\boxed{1211}}^{-} \arrow[r, "r"'] & \underset{r^2 s}{\boxed{2111}}^{+} \arrow[r, "r"'] & \underset{r^3 s}{\boxed{1112}}^{-}
\end{tikzcd}$$

so $z_2 \cdot 1112 = \frac{1}{8}\Big(\boxed{1112} - \boxed{1121} + \boxed{1211} - \boxed{2111} + \boxed{1121} - \boxed{1211} + \boxed{2111} - \boxed{1112}\Big) = 0.$)

Continue computing the actions of the z_j on the basis vectors, organize your computations by orbits. For example, setting

$$b_1 = 1112, \quad b_2 = 1121, \quad b_3 = 1211, \quad \text{and} \quad b_4 = 2111,$$

we have

$$\begin{aligned}
&z_1 b_i = \frac{1}{4}(b_1 + b_2 + b_3 + b_4) && \text{for } i = 1,2,3,4; \\
&z_2 b_i = 0 \quad \text{and} \quad z_4 b_i = 0 && \text{for } i = 1,2,3,4; \\
&z_3 b_1 = z_3 b_3 = -z_3 b_2 = -z_3 b_4 = \frac{1}{4}(b_1 - b_2 + b_3 - b_4); \\
&z_5 b_1 = -z_5 b_3 = \frac{1}{2}(b_1 - b_3); \quad \text{and} \quad z_5 b_2 = -z_5 b_4 = \frac{1}{2}(b_2 - b_4).
\end{aligned}$$

So

$$\begin{aligned}
&z_1 V^{\otimes 4} \text{ contains } b_1 + b_2 + b_3 + b_4; \\
&z_3 V^{\otimes 4} \text{ contains } b_1 - b_2 + b_3 - b_4; \quad \text{and} \\
&z_5 V^{\otimes 4} \text{ contains } b_1 - b_3 \text{ and } b_2 - b_4.
\end{aligned}$$

(We have accounted for 4 of 16 dim'ns in $V^{\otimes 4}$, so we're now 1/4 done with this computation!)