HOMEWORK 1
MATH B4900
DUE: 2/12/2021

1. **Class sums.** Let R be a commutative ring with 1, and let G be a finite group. Before starting, review our work considering groups acting on themselves by conjugation (Section 4.3 in D&F). In particular, the conjugacy classes of group elements partition the group. For example, in S_n, the conjugacy classes are in bijection with cycle type; in S_3 in particular, the classes are
$$\{1\}, \quad \{(12),(13),(23)\}, \quad \text{and} \quad \{(123),(132)\}.$$

   (a) In RG, a *class sum* corresponding to a conjugacy class
   $$\mathcal{K}_g = \{h \in G \mid h = aga^{-1} \text{ for some } a \in G\} \qquad \text{is} \qquad \kappa_g = \sum_{h \in \mathcal{K}_g} h.$$
   For example, the class sums in RS_3 are
   $$\kappa_1 = 1, \quad \kappa_{(12)} = (12) + (13) + (23), \quad \text{and} \quad \kappa_{(123)} = (123) + (132).$$
   Compute the class sums in RD_8 and RA_4.

   (b) For each of the class sums κ in RD_8, compute $r\kappa r^{-1}$ and $s\kappa s^{-1}$. Use your results to argue that $g\kappa = \kappa g$ for all $g \in D_8$.

   (c) **Claim:** the center of the group algebra RG is the R-span of the class sums of G,
   $$Z(RG) = R\{\kappa_g \mid g \in G\} = \{r_1\kappa_1 + \cdots r_\ell \kappa_\ell \mid r_i \in R\},$$
   where $\kappa_1, \dots, \kappa_\ell$ denote the ℓ class sums of G.

   Let's prove it:

   (i) For each $g \in G$, show that for all $h \in G$, we have $h\kappa_g h^{-1} = \kappa_g$. Conclude that $a\kappa_g = \kappa_g a$ for all $a \in RG$ (showing that $\kappa_g \in Z(RG)$).

   (ii) Use the previous part to show that $r_1\kappa_1 + \cdots r_\ell\kappa_\ell \in Z(RG)$ for all $r_i \in R$ (showing that $R\{\kappa_i \mid i = 1, \dots \ell\} \subseteq Z(RG)$).

   (iii) Conversely, show that for $a = \sum_{g \in G} s_g g \in RG$, if $hah^-1 = a$ for all $h \in G$, then $s_g = s_{g'}$ whenever g is conjugate to g (i.e. the coefficients are constant across conjugacy classes). [Hint: Start one at a time: if $hah^-1 = a$, then compare both sides to get $s_g = s_{h^{-1}gh}$. Try on your examples in part (b) to get started if you need help.]

   (iv) Let $a \in RG$. Show that if $ha = ah$ for all $h \in G$, then $ba = ab$ for all $b \in RG$.

   (d) Let F be a field with $n! \neq 0$ in F.[1] Show that
   $$e_+ = \sum_{\sigma \in S_n} \sigma \qquad \text{and} \qquad e_- = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\sigma$$
   are essential idempotents in FS_n and are central, and compute the corresponding (pure) idempotents. [*Hint:* Do e_+ first, using the fact that any group acts transitively on itself by left multiplication. For e_-, do some small examples first, and modify your proof for e_+ appropriately.]

[1] As usual, as an element of F, $n!$ means $1 + 1 + \cdots 1$ ($n!$ terms).

2. **Vector spaces.** U, V, and W denote vector spaces over a common field F; φ and ψ denote linear transformations; $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ denote bases; A, B, and C denote matrices in $M_n(F)$.

   (a) Let $\varphi : V \to V$ be a linear map. An element $v \in V$ satisfying $\varphi(v) = \lambda v$ for some $\lambda \in F$ is called a *weight vector* of *weight* λ (otherwise known as an *eigenvector* of *eigenvalue* λ). More restrictively, element $\lambda \in F$ is called a *weight* or *eigenvalue* of φ if it is the weight of some non-zero weight vector of φ. Given a weight of φ, the *weight space* of V associated to λ is
   $$V_\lambda = \{v \in V \mid \varphi(v) = \lambda v\}$$
   (the set of weight vectors in V of weight λ).

   Show that V_λ is a subspace of V.

   (b) Check briefly that $\varphi(v) = \lambda v$ is equivalent to $(\varphi - \lambda \cdot \mathrm{id})(v) = 0$.

   (c) Given a weight λ of φ, the *generalized weight space* associated to λ is
   $$V^\lambda = \{v \in V \mid (\varphi - \lambda \cdot \mathrm{id})^m(v) = 0 \text{ for some } m \in \mathbb{Z}_{>0}\}.^2$$

      (i) Let
      $$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$
      Check that $v \in V^2$ but $v \notin V_2$.

      (ii) Briefly argue that $V_\lambda \subseteq V^\lambda$.

      (iii) Show that V^λ is a subspace of V.
      [*Hint:* If $(\varphi - \lambda \cdot \mathrm{id})^m(v) = 0$, then $(\varphi - \lambda \cdot \mathrm{id})^n v = 0$ for all integers $n \geq m$.]

To receive credit for this assignment, include the following in your solutions [edited appropriately]:

Academic integrity statement: I *[violated/did not violate]* the CUNY Academic Integrity Policy in completing this assignment. *[enter your full name as a digital signature here]*

For example:
Academic integrity statement: I *did not violate* the CUNY Academic Integrity Policy in completing this assignment. *Zajj B. Daugherty*

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[2] Here, ψ^m means $\psi \circ \psi \circ \cdots \circ \psi$ (m terms).