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Lec. 8, Ex. B

1. Let $U, V$ be $A$-modules

Consider $U \oplus V$

We have,

$$(u, v) + (u', v')$$

$$= (u + u', v + v')$$

$U,V$ abelian $= (u' + u, v + v')$

$$= (u', v') + (u, v)$$

So $U \oplus V$ abelian

Now fix $a, b \in A$

$$(a+b) \cdot (u, v) = ((a+b)u, (a+b)v)$$

$$= (au + bu, av + bv)$$

$$= a(u,v) + b(u,v) \checkmark$$

$$(ab) \cdot (u, v) = ((ab)u, (ab)v)$$

$$= (a(bu), a(bv))$$

$$= a(b(u,v)) \checkmark$$

$$a((u,v)+(u',v'))$$
$$= a(u+u', v+v')$$
$$= (a(u+u'), a(v+v'))$$
$$= (au+au', av+av')$$
$$= (au, av) + (au', av')$$
$$= a(u,v) + a(u',v') \checkmark$$

So $U \oplus V$ is an $A$-module

2. $X, Y \in M_m(F) \oplus M_n(F)$, $\alpha \in F$

$$X = \begin{pmatrix} \alpha_{1,1}^{(x)} & \cdots & \alpha_{1,m}^{(x)} & & 0 & \\ \vdots & & & & & \\ \alpha_{m,1}^{(x)} & \cdots & \alpha_{m,m}^{(x)} & & & \\ & & & \beta_{1,1}^{(x)} & \cdots & \beta_{1,n}^{(x)} \\ & 0 & & \vdots & & \\ & & & \beta_{n,1}^{(x)} & \cdots & \beta_{n,n}^{(x)} \end{pmatrix}$$

$$Y = \begin{pmatrix} \alpha^{(y)}_{1,1} & \cdots & \alpha^{(y)}_{1,m} & & & \\ \vdots & & \vdots & & 0 & \\ \alpha^{(y)}_{m,1} & \cdots & \alpha^{(y)}_{m,m} & & & \\ & & & \beta^{(y)}_{1,1} & \cdots & \beta^{(y)}_{1,n} \\ & 0 & & \vdots & & \vdots \\ & & & \beta^{(y)}_{n,1} & \cdots & \beta^{(y)}_{n,n} \end{pmatrix}$$

Observe for $X+Y$ we are adding corresponding components.

$$\alpha^{(x)}_{i_1,j_1} + \alpha^{(y)}_{i_2,j_2} \in F \quad \forall i_1, j_1, i_2, j_2$$

and

$$\beta^{(y)}_{i_1,j_1} + \beta^{(y)}_{i_1,j_1} \in F \quad \forall i_1, j_1, i_2, j_2$$

So $X+Y \in M_m(F) \oplus M_n(F)$

Note $\alpha \cdot \alpha^{(x)}_{i_1,j_1}, \ \alpha \cdot \alpha^{(y)}_{i_2,j_2},$

$\alpha \cdot \beta^{(x)}_{i_1,j_1}, \ \alpha \cdot \beta^{(y)}_{i_2,j_2} \in F \quad \forall i_1, j_1, i_2, j_2$

So $\alpha X \in M_m(F) \oplus M_n(F)$

The entries of $XY$ will also be in $F$ because any linear combination of elements of $F$ is in $F$. Also, the $0$ blocks will not have their position changed

$$XY \in M_m(F) \oplus M_n(F)$$

3a. $C_x(\lambda) = \det(\lambda I - \rho(x))$

$$= \det\left( \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right)$$

$$= \det\left( \begin{pmatrix} \lambda & 0 & -1 \\ -1 & \lambda & 0 \\ 0 & -1 & \lambda \end{pmatrix} \right)$$

$$= \lambda(\lambda^2 - 0) - 0 \cdots + -1(1-0)$$

$$= \lambda^3 - 1$$

$\lambda^3 - 1 = 0 \Rightarrow \lambda_1 = 1 \in \mathbb{R}\ (\in \mathbb{C})$

$\lambda_2 = -\sqrt[3]{-1} \in \mathbb{C}$

$\lambda_3 = (-1)^{2/3} \in \mathbb{C}$

b. $\rho \cong \lambda_1 \oplus \lambda_2 \oplus \lambda_3$

$\lambda_1 : x \mapsto 1, \quad \lambda_2 : x \mapsto -\sqrt[3]{-1}, \quad \lambda_3 : x \mapsto (-1)^{2/3}$

$$V_1 = \ker(\rho(x) - 1)$$

$$= \ker \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= \mathbb{C} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$V_2 = \ker(\rho(x) + \sqrt[3]{-1})$$

$$= \ker \begin{pmatrix} \sqrt[3]{-1} & 0 & 1 \\ 1 & \sqrt[3]{-1} & 0 \\ 0 & 1 & \sqrt[3]{-1} \end{pmatrix}$$

$$= \mathbb{C} \begin{pmatrix} (-1)^{2/3} \\ -\sqrt[3]{-1} \\ 1 \end{pmatrix}$$

$$V_3 = \ker\left(\rho(x) - (-1)^{2/3}\right)$$

$$= \ker\begin{pmatrix} -(-1)^{2/3} & 0 & 1 \\ 1 & -(-1)^{2/3} & 0 \\ 0 & 1 & -(-1)^{2/3} \end{pmatrix}$$

$$= \mathbb{C}\begin{pmatrix} -\sqrt[3]{-1} \\ (-1)^{2/3} \\ 1 \end{pmatrix}$$

So,

$$P = \begin{pmatrix} 1 & (-1)^{2/3} & -\sqrt[3]{-1} \\ 1 & -\sqrt[3]{-1} & (-1)^{2/3} \\ 1 & 1 & 1 \end{pmatrix}$$

$$\lambda_1 \oplus \lambda_2 \oplus \lambda_3 = P^{-1}\rho(a)P \quad \forall a \in \mathbb{C}\mathbb{Z}_3$$

c.(i) $\rho(x)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$= \lambda\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(ii) $x \cdot v_1 = v_2 - v_3 = w_2 \in W$

$x \cdot w_2 = v_3 - v_1 = -w_1 - w_2 \in W$

(iii) $\psi(x) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$

(iv) $C_{\psi(x)}(\lambda) = \det(\lambda I - \psi(x))$

$= \det\begin{pmatrix} \lambda & 1 \\ -1 & \lambda+1 \end{pmatrix}$

$= \lambda(\lambda+1) - 1(-1)$

$= \lambda^2 + \lambda + 1$

$\uparrow$ doesn't factor over $\mathbb{R}$

(v)