

Statement: Let F be a field and V be a vector space over F . Fix $\varphi \in \text{End}(V)$. For $\lambda \in F$, prove that the weight space V_λ and the generalized weight space V^λ are both subspaces of V .

Problem: **1C**No. stars: **2**

Proof. Let F be a field and V be a vector space over F . Fix $\lambda \in F$. By definition, every $v \in V_\lambda$ is also an element of V . Hence, we have $V_\lambda \subset V$. Now observe that $\lambda 0 = 0$ for any $\lambda \in F$. Hence, $0 \in V_\lambda$ and thus V_λ is non-empty. Now, by the submodule criterion, we just need to show that $x + ry \in V_\lambda$ for all $r \in F$ and for all $x, y \in V_\lambda$. Let us start by applying φ to this element and using the properties of the linearity of φ ,

$$\begin{aligned}\varphi(x + ry) &= \varphi(x) + r\varphi(y) \\ &= \lambda x + r\lambda y \\ &= \lambda(x + ry)\end{aligned}$$

Hence, $x + ry \in V_\lambda$ and so V_λ is a subspace of V .

By definition, every $v \in V^\lambda$ is also an element of V . Hence, we have $V^\lambda \subset V$. Now observe that, for any $\lambda \in F$,

$$\begin{aligned}\varphi(v) &= \lambda 0 = 0v = 0 \\ \iff (\varphi - \lambda \cdot \text{id})(0) &= 0\end{aligned}$$

Hence, $0 \in V^\lambda$ and thus V^λ is non-empty. Again, by the submodule criterion, we just need to show that $x + ry \in V^\lambda$ for all $r \in F$ and for all $x, y \in V^\lambda$. Since, $x, y \in V^\lambda$, we have that

$$\begin{aligned}(\varphi - \lambda \cdot \text{id})^\ell(x) &= 0 \\ (\varphi - \lambda \cdot \text{id})^m(y) &= 0\end{aligned}$$

Let $k = \max\{\ell, m\}$. Then by the fact that if $(\varphi - \lambda \cdot \text{id})^m(v) = 0$, then $(\varphi - \lambda \cdot \text{id})^n v = 0$ for all integers $n \geq m$ and by the fact that linear combinations and compositions of linear functions are linear, we have that,

$$\begin{aligned}(\varphi - \lambda \cdot \text{id})^k(x + ry) &= (\varphi - \lambda \cdot \text{id})^k(x) + r(\varphi - \lambda \cdot \text{id})^k(y) \\ &= 0 + r0 \\ &= 0\end{aligned}$$

Hence, $x + ry \in V^\lambda$ and so V^λ is also a subspace of V . □

	Points Possible					
complete	0	1	2	3	4	5
mathematically valid	0	1	2	3	4	5
readable/fluent	0	1	2	3	4	5
Total:	(out of 15)					

Statement: Define $\langle, \rangle : M_n(F) \times M_n(F) \rightarrow F$ by $\langle A, B \rangle = \text{tr}(AB)$. Show that \langle, \rangle is symmetric and nondegenerate.

Problem:	2A.II
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No. stars:	2
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Proof. Fix $A, B \in M_n(F)$. Then we have,

$$\begin{aligned}
 \langle A, B \rangle &= \text{tr}(AB) \\
 &= \sum_{i=1}^n AB_{ii} \\
 &= \sum_{i=1}^n \sum_{k=1}^n a_{ik} b_{ki} \\
 &= \sum_{i=1}^n a_{i1} b_{1i} + a_{i2} b_{2i} + \cdots + a_{in} b_{ni} \\
 &= a_{11} b_{11} + a_{12} b_{21} + \cdots + a_{1n} b_{n1} + a_{21} b_{12} + a_{22} b_{22} + \cdots + a_{2n} b_{n2} + \cdots + a_{n1} b_{1n} + a_{n2} b_{2n} + \cdots + a_{nn} b_{nn}
 \end{aligned}$$

and,

$$\begin{aligned}
 \langle B, A \rangle &= \text{tr}(BA) \\
 &= \sum_{i=1}^n BA_{ii} \\
 &= \sum_{i=1}^n \sum_{k=1}^n b_{ik} a_{ki} \\
 &= \sum_{i=1}^n b_{i1} a_{1i} + b_{i2} a_{2i} + \cdots + b_{in} a_{ni} \\
 &= b_{11} a_{11} + b_{12} a_{21} + \cdots + b_{1n} a_{n1} + b_{21} a_{12} + b_{22} a_{22} + \cdots + b_{2n} a_{n2} + \cdots + b_{n1} a_{1n} + b_{n2} a_{2n} + \cdots + b_{nn} a_{nn}
 \end{aligned}$$

Note from the above two expressions, we can see that each term in $\text{tr}(BA)$ appears in $\text{tr}(AB)$, just with the order of a and b switched. Since F is a field, we have that multiplication is commutative and so, since the terms are the same in each expanded addition, we must have that

$$\text{tr}(AB) = \text{tr}(BA)$$

Hence, trace is symmetric.

Now we want to show that trace is a bilinear form. Let us fix $\alpha \in F$. We have that,

$$\begin{aligned}
 \langle \alpha A, B \rangle &= \text{tr}(\alpha AB) \\
 &= \sum_{i=1}^n \sum_{k=1}^n (\alpha a_{ik}) b_{ki} \\
 &= \alpha \sum_{i=1}^n \sum_{k=1}^n a_{ik} b_{ki} \\
 &= \alpha \text{tr}(AB) \\
 &= \alpha \langle A, B \rangle
 \end{aligned}$$

Moreover, we have that,

$$\begin{aligned}
 \langle \alpha A, B \rangle &= \text{tr}(\alpha AB) \\
 &= \sum_{i=1}^n \sum_{k=1}^n (\alpha a_{ik}) b_{ki} \\
 &= \sum_{i=1}^n \sum_{k=1}^n a_{ik} (\alpha b_{ki}) \\
 &= \text{tr}(A(\alpha B)) \\
 &= \langle A, \alpha B \rangle
 \end{aligned}$$

Now in addition to A and B , let us fix $A', B' \in M_n(F)$. Then we have,

$$\begin{aligned}
 \langle A + A', B \rangle &= \text{tr}((A + A')B) \\
 &= \text{tr}(AB + A'B) \\
 &= \sum_{i=1}^n \sum_{k=1}^n (a_{ik}b_{ki} + a'_{ik}b_{ki}) \\
 &= \sum_{i=1}^n \sum_{k=1}^n a_{ik}b_{ki} + \sum_{i=1}^n \sum_{k=1}^n a'_{ik}b_{ki} \\
 &= \text{tr}(AB) + \text{tr}(A'B) \\
 &= \langle A, B \rangle + \langle A', B \rangle
 \end{aligned}$$

In addition,

$$\begin{aligned}
 \langle A, B + B' \rangle &= \text{tr}(A(B + B')) \\
 &= \text{tr}(AB + AB') \\
 &= \sum_{i=1}^n \sum_{k=1}^n (a_{ik}b_{ki} + a_{ik}b'_{ki}) \\
 &= \sum_{i=1}^n \sum_{k=1}^n a_{ik}b_{ki} + \sum_{i=1}^n \sum_{k=1}^n a_{ik}b'_{ki} \\
 &= \text{tr}(AB) + \text{tr}(AB') \\
 &= \langle A, B \rangle + \langle A, B' \rangle
 \end{aligned}$$

Thus, we have shown that trace is a symmetric bilinear form. Now we must show that it is nondegenerate. Fix $A \in M_n(F)$ and suppose

$$\langle A, B \rangle = 0$$

for all $B \in M_n(F)$. Then we have,

$$\begin{aligned}
 \langle A, B \rangle &= \text{tr}(AB) \\
 &= \sum_{i=1}^n AB_{ii} \\
 &= \sum_{i=1}^n \sum_{k=1}^n a_{ik} b_{ki} \\
 &= \sum_{i=1}^n a_{i1} b_{1i} + a_{i2} b_{2i} + \cdots + a_{in} b_{ni} \\
 &= a_{11} b_{11} + a_{12} b_{21} + \cdots + a_{1n} b_{n1} + a_{21} b_{12} + a_{22} b_{22} + \cdots + a_{2n} b_{n2} + \cdots + a_{n1} b_{1n} + a_{n2} b_{2n} + \cdots + a_{nn} b_{nn} \\
 &= 0
 \end{aligned}$$

The above expression implies that every term of the form $a_{ij} b_{jk} = 0$. Since this holds for all $B \in M_n(F)$, we can assume that our matrix B has every entry greater than 0. Then, the fact that $a_{ij} b_{jk} = 0$ for every $1 \leq i, j, k \leq n$ implies that $a_{ij} = 0$ for all i, j . Hence, $\langle A, B \rangle = 0$ for all $B \in M_n(F)$ if and only if $A = \mathbf{0}$

Similarly, fix $B \in M_n(F)$ and suppose

$$\langle A, B \rangle = 0$$

for all $A \in M_n(F)$. Then we have,

$$\begin{aligned}
 \langle A, B \rangle &= \text{tr}(AB) \\
 &= \sum_{i=1}^n AB_{ii} \\
 &= \sum_{i=1}^n \sum_{k=1}^n a_{ik} b_{ki} \\
 &= \sum_{i=1}^n a_{i1} b_{1i} + a_{i2} b_{2i} + \cdots + a_{in} b_{ni} \\
 &= a_{11} b_{11} + a_{12} b_{21} + \cdots + a_{1n} b_{n1} + a_{21} b_{12} + a_{22} b_{22} + \cdots + a_{2n} b_{n2} + \cdots + a_{n1} b_{1n} + a_{n2} b_{2n} + \cdots + a_{nn} b_{nn} \\
 &= 0
 \end{aligned}$$

The above expression implies that every term of the form $a_{ij} b_{jk} = 0$. Since this holds for all $A \in M_n(F)$, we can assume that our matrix A has every entry greater than 0. Then, the fact that $a_{ij} b_{jk} = 0$ for every $1 \leq i, j, k \leq n$ implies that $b_{jk} = 0$ for all i, j . Hence, $\langle A, B \rangle = 0$ for all $A \in M_n(F)$ if and only if $B = \mathbf{0}$

As a result, we have now shown that trace is a symmetric, nondegenerate bilinear form. \square

	Points Possible					
complete	0	1	2	3	4	5
mathematically valid	0	1	2	3	4	5
readable/fluent	0	1	2	3	4	5
Total:	(out of 15)					

Statement: Use the definition of determinant to verify that $\det(I_n) = 1$, where I_n is the identity matrix in $M_n(F)$.

Problem:	2B
No. stars:	1

Proof. Recall that the determinant function $\det : M_n(F) \rightarrow F$ is defined by

$$\det((\alpha_{i,j})) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \alpha_{1,\sigma(1)} \alpha_{2,\sigma(2)} \cdots \alpha_{n,\sigma(n)}.$$

Note that for I_n , we have that $\alpha_{ii} = 1$ for all $1 \leq i \leq n$ and $\alpha_{ij} = 0$ for all $i \neq j$. Hence, in the above definition of the determinant, we must have that the only non-zero term in the summation is the one corresponding to the identity $\sigma = 1$. This gives us,

$$\begin{aligned} \det(I_n) &= \operatorname{sgn}(1) \alpha_{1,1} \alpha_{2,2} \cdots \alpha_{n,n} \\ &= 1 \cdot 1 \cdot 1 \cdots 1 \\ &= 1 \end{aligned}$$

□

	Points Possible					
complete	0	1	2	3	4	5
mathematically valid	0	1	2	3	4	5
readable/fluent	0	1	2	3	4	5
Total:	(out of 15)					

Statement: Prove that determinant is invariant under change of basis. [The details required in this proof are outlined in Homework 2; be sure to hit all the beats highlighted in that problem statement.]

Problem:	2C
No. stars:	2

Proof. Let $A \in \text{GL}_n(F)$. Then A is invertible with inverse A^{-1} . But A^{-1} is also invertible with inverse A , so $A^{-1} \in \text{GL}_n(F)$. Hence, by fact (2) we have $\det(A^{-1}), \det(A)^{-1} \neq 0$. Furthermore, since $\text{GL}_n(F) \subset M_n(F)$, we have that $A, A^{-1} \in M_n(F)$ and so we can apply fact (3). Thus, we have that,

$$\begin{aligned}\det(AA^{-1}) &= \det(I_n) \\ &= 1 \\ &= \det(A) \det(A^{-1})\end{aligned}$$

Since $\det(A) \det(A^{-1}) = 1$, we have that $\det(A^{-1}) = \det(A)^{-1}$.

Now let $B \in \text{GL}_n(F)$. Consider $\det(ABA^{-1})$ and using fact (3) along with the associativity of matrix multiplication, we get,

$$\begin{aligned}\det((AB)A^{-1}) &= \det(AB) \det(A^{-1}) \\ &= \det(A) \det(B) \det(A^{-1})\end{aligned}$$

By our initial derivation, we have that $\det(A^{-1}) = \det(A)^{-1}$ and so, by the fact that F is a field and hence commutative, we have,

$$\begin{aligned}\det((AB)A^{-1}) &= \det(A) \det(B) \det(A^{-1}) \\ &= \det(A) \det(B) \det(A)^{-1} \\ &= \det(A) \det(A)^{-1} \det(B) \\ &= 1 \cdot \det(B) \\ &= \det(B)\end{aligned}$$

FINISH THIS PROOF

□

	Points Possible					
complete	0	1	2	3	4	5
mathematically valid	0	1	2	3	4	5
readable/fluent	0	1	2	3	4	5
Total:	(out of 15)					

Statement: Prove that determinant is invariant under change of basis. [The details required in this proof are outlined in Homework 2; be sure to hit all the beats highlighted in that problem statement.]

Problem:	3B
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No. stars:	1
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Proof.

□

	Points Possible					
complete	0	1	2	3	4	5
mathematically valid	0	1	2	3	4	5
readable/fluent	0	1	2	3	4	5
Total:	(out of 15)					