Numerical Analysis: Homework #4

Chris Hayduk

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Problem 1.

Suppose that there are two polynomials $P_n(x)$ and $Q_n(x)$ of degree $\leq n$ which interpolate our $n+1$ data points $(x_0, y_0), (x_1, y_1), ..., (x_n, y_n)$. Define:

$$R(x) = P_n(x) - Q_n(x)$$

Since P_n and Q_n both have degree $\leq n$, we also have $\deg(R) \leq n$. Furthermore, observe that, by the properties of interpolation, we have:

$$R(x_i) = P_n(x_i) - Q_n(x_i) = y_i - y_i = 0$$

for $i = 0, ..., n$. Thus, $R(x)$ has n+1 distinct roots, but its degree is $\leq n$. By the fundamental theorem of algebra, this is not possible unless $R(x) \equiv 0$. But this implies that:

$$P_n(x) = Q_n(x)$$

Thus, there is only one polynomial of degree $\leq n$ that satisfies

$$P_n(x_i) = y_i, i = 0, ..., n$$

Problem 2.

a)

$$\begin{aligned} P_1(x) &= f(x_0) + (x - x_0)f[x_0, x_1] \\ &= 7 + (x - 0)(\frac{11 - 7}{2 - 0}) \\ &= 7 + 2x \end{aligned}$$

b)

$$\begin{aligned} P_2(x) &= P_1(x) + (x - x_0)(x - x_1)f[x_0, x_1, x_2] \\ &= 7 + 2x + (x - 0)(x - 2)(\frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}) \\ &= 7 + 2x + x(x - 2)(\frac{17 - 2}{3 - 0}) \\ &= 7 + 2x + (x^2 - 2x)5 \\ &= 7 + 2x + 5x^2 - 10x = 5x^2 - 8x + 7 \end{aligned}$$

c)

$$
\begin{aligned}
P_3(x) &= P_2(x) + (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3] \\
&= 5x^2 - 8x + 7 + (x - 0)(x - 2)(x - 3)(\frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}) \\
&= 5x^2 - 8x + 7 + x(x - 2)(x - 3)(\frac{(\frac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1}) - 5}{4 - 0}) \\
&= 5x^2 - 8x + 7 + x(x - 2)(x - 3)(\frac{(\frac{\frac{63-28}{4-3}-17}{4-2}) - 5}{4}) \\
&= 5x^2 - 8x + 7 + x(x - 2)(x - 3)(\frac{9 - 5}{4}) \\
&= 5x^2 - 8x + 7 + x(x - 2)(x - 3) \\
&= 5x^2 - 8x + 7 + (x^2 - 2x)(x - 3) \\
&= 5x^2 - 8x + 7 + x^3 - 3x^2 - 2x^2 + 6x = x^3 - 2x + 7
\end{aligned}
$$

Problem 3.

Since $n = 5$ and $f(x) = sin(x)$, $f^{n+1}(c) = f^6(c) = -sin(c)$. Furthermore, we know $(n+1)! = 6! = 720$. Thus,

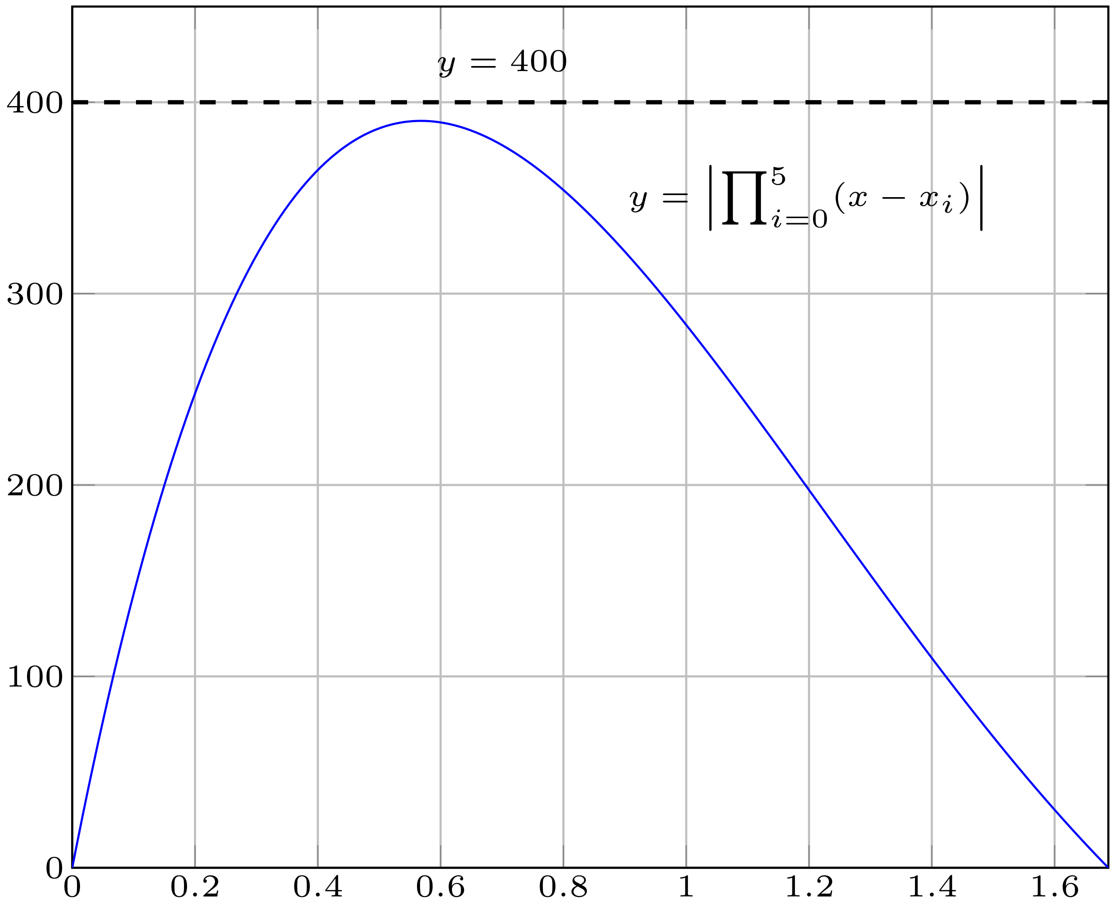
$$|\frac{f^6(c)}{6!}| \leq \frac{1}{720} \approx 0.001388889$$

Now we just need to bound the term $|\prod_{i=0}^{5}(x - x_i)|$.

Let $h = \frac{1.6875}{5}$ and $x_i = ih$. Then,

$$
\begin{aligned}
|\prod_{i=0}^{5}(x - x_i)| &= |(x)(x - 1(1.6875))(x - 2(1.6875))(x - 3(1.6875))(x - 4(1.6875))(x - 5(1.6875))| \\
&= |x^6 - 25.3125x^5 + 242.051x^4 - 1081.22x^3 + 2221.91x^2 - 1642.1x|
\end{aligned}
$$

Let's graph this function to try to find its upper bound on [0, 1.6875] now.



As evident in the plot,

$$y = |x^6 - 25.3125x^5 + 242.051x^4 - 1081.22x^3 + 2221.91x^2 - 1642.1x| \leq 400$$

Thus,

$$\begin{aligned} |f(x) - p(x)| &= \left|\frac{f^6(c)}{6!}\prod_{i=0}^{5}(x - x_i)\right| \\ &\leq \left|\left(\frac{1}{720}\right)400\right| = \frac{5}{9} \approx 0.555556 \end{aligned}$$

is an upper bound for the error $|f(x) - p(x)|$ on $[0, 1.6875]$.

Problem 4.

From (4.64) in the textbook,

$$\begin{aligned} \frac{1}{6}M_1 + \frac{2}{3}M_2 + \frac{1}{6}M_3 = 3 \\ \frac{1}{6}M_2 + \frac{2}{3}M_3 + \frac{1}{6}M_4 = 0 \\ \frac{1}{6}M_3 + \frac{2}{3}M_4 + \frac{1}{6}M_5 = -2 \end{aligned}$$

We know from (4.65) that $M_1 = M_5 = 0$. Combining this with the above system of equations yields:

$$M_2 = \tfrac{129}{28},\ M_3 = -\tfrac{3}{7},\ M_4 = -\tfrac{81}{28}$$

Thus, substituting into (4.63), we obtain:

$$s(x) = \begin{cases} \frac{43x^3}{56} - \frac{129x^2}{56} - \frac{13x}{28} + 5 & 1 \le x \le 2 \\ \frac{-47x^3}{56} + \frac{411x^2}{56} - \frac{79x}{4} + \frac{125}{7} & 2 \le x \le 3 \\ \frac{-23x^3}{56} + \frac{195x^2}{56} - \frac{229x}{28} + \frac{44}{7} & 3 \le x \le 4 \\ \frac{27x^3}{56} - \frac{81x^2}{14} + \frac{1213x}{56} - \frac{1201}{56} & 4 \le x \le 5 \end{cases}$$

Problem 5.

a) Each a_{ij} is comes from one of the following coefficients in (4.64):

$$\tfrac{x_j - x_{j-1}}{6},\ \tfrac{x_{j+1} - x_{j-1}}{3},\ \tfrac{x_{j+1} - x_j}{6}$$

We know $x_1 < x_2 < ... < x_n$. Thus, in the equations above, we have:

$$x_j > x_{j-1},\ x_{j+1} > x_{j-1},\ x_{j+1} > x_j$$

Thus, each numerator term is positive for any value $2 \le j \le n-1$. Since all the denominators are positive, each a_{ij} is positive.

Now we need to check that $a_{ii} = 2(a_{ii-1} + a_{ii+1})$ where $a_{10} = a_{nn+1} = 0$.

Let's examine the base case,

$$\begin{aligned} a_{11} &= 2\left(a_{10} + a_{12}\right) \\ &= 2\left(0 + \frac{x_3 - x_2}{6}\right) \\ &= \frac{x_3 - x_2}{3} \end{aligned}$$

However, we know that,

$$a_{11} = \frac{x_3 - x_1}{3} \neq \frac{x_3 - x_2}{3}$$

since $x_2 \neq x_1$. Thus, the matrix does not satisfy this condition for row 1. It can be shown that row n also does not satisfy the condition.

Let's try checking the interior rows $(2, ..., n-1)$ to see if they satisfy the condition.

$$\begin{aligned} a_{ii} &= 2\left(a_{ii-1} + a_{ii+1}\right) \\ &= 2\left(\frac{x_{i+1} - x_i}{6} + \frac{x_{i+2} - x_{i+1}}{6}\right) \\ &= \frac{x_{i+2} - x_i}{3} \end{aligned}$$

Since the indeces for $\vec{x}$ correspond to the indices for M offset by 1 (because we have $M_1 = M_n = 0$), we have $j = i + 1$ where j is the subscript for x_j in the formulas in (a) and i is the index of the row in the matrix. Plugging in that relationship yields,

$$\begin{aligned} a_{ii} &= \frac{x_{i+2} - x_i}{3} \\ &= \frac{x_{j+1} - x_j}{3} \end{aligned}$$

This is precisely the formula desired for each a_{ii}. Thus, the condition $a_{ii} = 2(a_{ii-1} + a_{ii+1})$ holds for rows $2, ..., n-1$.

b) Suppose A is invertible. Then we can write:

$$\begin{aligned} & A\vec{x} = \vec{b} \\ \Longrightarrow\ & A^{-1}(A\vec{x}) = A^{-1}(\vec{b}) \\ \Longrightarrow\ & \vec{x} = A^{-1}\vec{b} \end{aligned}$$

Now let's show that A^{-1} is unique. Suppose there are two matrices that are inverses of A called B and C. Then,

$$\begin{aligned} & AB = BA = AC = CA = I \\ \Longrightarrow\ & B = B(I) = B(AC) = (BA)C = (I)C = C \end{aligned}$$

Thus, $B = C$ and the inverse of A is unique.

Since matrix multiplication is a deterministic algorithm and both A^{-1} and $\vec{b}$ are unique, then there can only be one solution $\vec{x} = A^{-1}\vec{b}$.

c) For rows $2, ..., n-1$, we have

$$\begin{aligned} & a_{ii-1}x_{i-1} + a_{ii}x_i + a_{ii+1}x_{i+1} = 0 \\ \Longrightarrow\ & a_{ii-1}x_{i-1} + 2x_i(a_{ii-1} + a_{ii+1}) + a_{ii+1}x_{i+1} = 0 \end{aligned}$$

d) We have,

$$a_{11}x_1 + a_{12}x_2 = 0$$