# Numerical Analysis: Homework #9

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April 23, 2019

**Problem 1.**

a) 0 is an eigenvalue of $A$ if $A\vec{v} = 0\vec{v}$ for some $\vec{v} \neq \vec{0}$.

   We can simplify this as,

$$A\vec{v} = 0\vec{v} = \vec{0}$$

   Since we know $A$ is invertible, the equation $A\vec{x} = \vec{0}$ only has the trivial solution $\vec{x} = \vec{0}$.

   This implies that $\vec{v} = 0$. However, we assumed that $\vec{v} \neq 0$, a contradiction. Thus, 0 is not an eigenvalue of $A$ when $A$ is invertible.

b) Suppose $\lambda$ is a non-zero eigenvalue of $A$. Then, $\exists \vec{v} \neq \vec{0}$ such that $A\vec{v} = \lambda\vec{v}$.

   Now let's multiply $A^{-1}$ on both sides,

$$\begin{aligned} A^{-1}(A\vec{v}) &= A^{-1}(\lambda\vec{v}) \\ \vec{v} &= A^{-1}\lambda\vec{v} \\ A^{-1}\vec{v} &= \lambda^{-1}\vec{v} \end{aligned}$$

   This satisfies the definition of an eigenvalue. Thus, $\lambda^{-1}$ is an eigenvalue of $A^{-1}$ if $\lambda$ is an eigenvalue of $A$.

**Problem 2.**

After coding a power method program in Python with the following initial vector:

$$\vec{z}^{(0)} = \begin{bmatrix} 1.0 \\ 0.30683515 \\ 0.50975501 \end{bmatrix}$$

I obtained the following output:

$\lambda_1^{(m)} = 5.031308512575385$

$$\vec{z}^{(m)} = \begin{bmatrix} 0.87931865 \\ 1.0 \\ -0.07802744 \end{bmatrix}$$

**Problem 3.**

We have,

$$\int_a^b \sqrt{1 + (3x^2)^2} dx$$
$$\int_a^b \sqrt{1 + 9x^4} dx$$

Thus,

$$f(x) = \sqrt{1 + 9x^4}$$

From (5.22), we have,

$$\begin{aligned} S_6(f) &= \frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)] \\ &= \frac{b-a}{18}[\sqrt{1+9a^4} + 4\sqrt{1+9(a+h)^4} + 2\sqrt{1+9(a+2h)^4} + 4\sqrt{1+9(a+3h)^4} + \\ & 2\sqrt{1+9(a+4h)^4} + 4\sqrt{1+9(a+5h)^4} + \sqrt{1+9(a+6h)^4}] \end{aligned}$$

**Problem 4.**

From the error formula (5.35), we have,

$$\begin{aligned} E_n(f) &= -\frac{h^4(b-a)}{180} f^{(4)}(c_n) \\ &= -\frac{\frac{1-0}{n}^4 (1-0)}{180}[4e^{-c_n^2}(4c_n^4 - 12c_n^2 + 3)] \\ &= -\frac{\frac{1}{n}^4}{180}[4e^{-c_n^2}(4c_n^4 - 12c_n^2 + 3)] \end{aligned}$$

with $c_n \in [0, 1]$.

This expression is maximized at $c_n = 0$. This yields,

$$
\begin{aligned}
|E_n(f)| &\leq 12\frac{\frac{1}{n}^4}{180} \\
&= \frac{\frac{1}{n}^4}{15} \\
&= \frac{1}{15n^4}
\end{aligned}
$$

Thus we have,

$$
\begin{aligned}
|E_n(f)| \leq 10^{-8} &\implies \frac{1}{15n^4} \leq 10^{-8} \\
&\implies 15n^4 \geq 10^8 \\
&\implies n^4 \geq \frac{10^8}{15} \\
&\implies n \geq 50.81327
\end{aligned}
$$

Thus, we must choose $n \geq 51$ to ensure that the absolute value of the error is less than $10^{-8}$.

After coding a Python function for Simpson's Rule, I approximated the integral using $n = 51$. This yielded an approximation of

$$\int_0^1 e^{-x^2} dx \approx S_{51}(f) = 0.7443725443100901$$