

Numerical Analysis: Homework #4

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Problem 1.

Suppose that there are two polynomials $P_n(x)$ and $Q_n(x)$ of degree $\leq n$ which interpolate our $n + 1$ data points $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$. Define:

$$R(x) = P_n(x) - Q_n(x)$$

Since P_n and Q_n both have degree $\leq n$, we also have $\deg(R) \leq n$. Furthermore, observe that, by the properties of interpolation, we have:

$$R(x_i) = P_n(x_i) - Q_n(x_i) = y_i - y_i = 0$$

for $i = 0, \dots, n$. Thus, $R(x)$ has $n+1$ distinct roots, but its degree is $\leq n$. By the fundamental theorem of algebra, this is not possible unless $R(x) \equiv 0$. But this implies that:

$$P_n(x) = Q_n(x)$$

Thus, there is only one polynomial of degree $\leq n$ that satisfies

$$P_n(x_i) = y_i, i = 0, \dots, n$$

Problem 2.

1.

$$\begin{aligned} P_1(x) &= f(x_0) + (x - x_0)f[x_0, x_1] \\ &= 7 + (x - 0)\left(\frac{11 - 7}{2 - 0}\right) \\ &= 7 + 2x \end{aligned}$$

2.

$$\begin{aligned} P_2(x) &= P_1(x) + (x - x_0)(x - x_1)f[x_0, x_1, x_2] \\ &= 7 + 2x + (x - 0)(x - 2)\left(\frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}\right) \\ &= 7 + 2x + x(x - 2)\left(\frac{17 - 2}{3 - 0}\right) \\ &= 7 + 2x + (x^2 - 2x)5 \\ &= 7 + 2x + 5x^2 - 10x = 5x^2 - 8x + 7 \end{aligned}$$

3.

$$\begin{aligned}
P_3(x) &= P_2(x) + (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3] \\
&= 5x^2 - 8x + 7 + (x - 0)(x - 2)(x - 3)\left(\frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}\right) \\
&= 5x^2 - 8x + 7 + x(x - 2)(x - 3)\left(\frac{\left(\frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}\right) - 5}{4 - 0}\right) \\
&= 5x^2 - 8x + 7 + x(x - 2)(x - 3)\left(\frac{\left(\frac{\frac{63-28}{4-3} - 17}{4-2}\right) - 5}{4}\right) \\
&= 5x^2 - 8x + 7 + x(x - 2)(x - 3)\left(\frac{9 - 5}{4}\right) \\
&= 5x^2 - 8x + 7 + x(x - 2)(x - 3) \\
&= 5x^2 - 8x + 7 + (x^2 - 2x)(x - 3) \\
&= 5x^2 - 8x + 7 + x^3 - 3x^2 - 2x^2 + 6x = x^3 - 2x + 7
\end{aligned}$$

Problem 3.

Since $n = 5$ and $f(x) = \sin(x)$, $f^{n+1}(c) = f^6(c) = -\sin(c)$. Furthermore, we know $(n + 1)! = 6! = 720$. Thus,

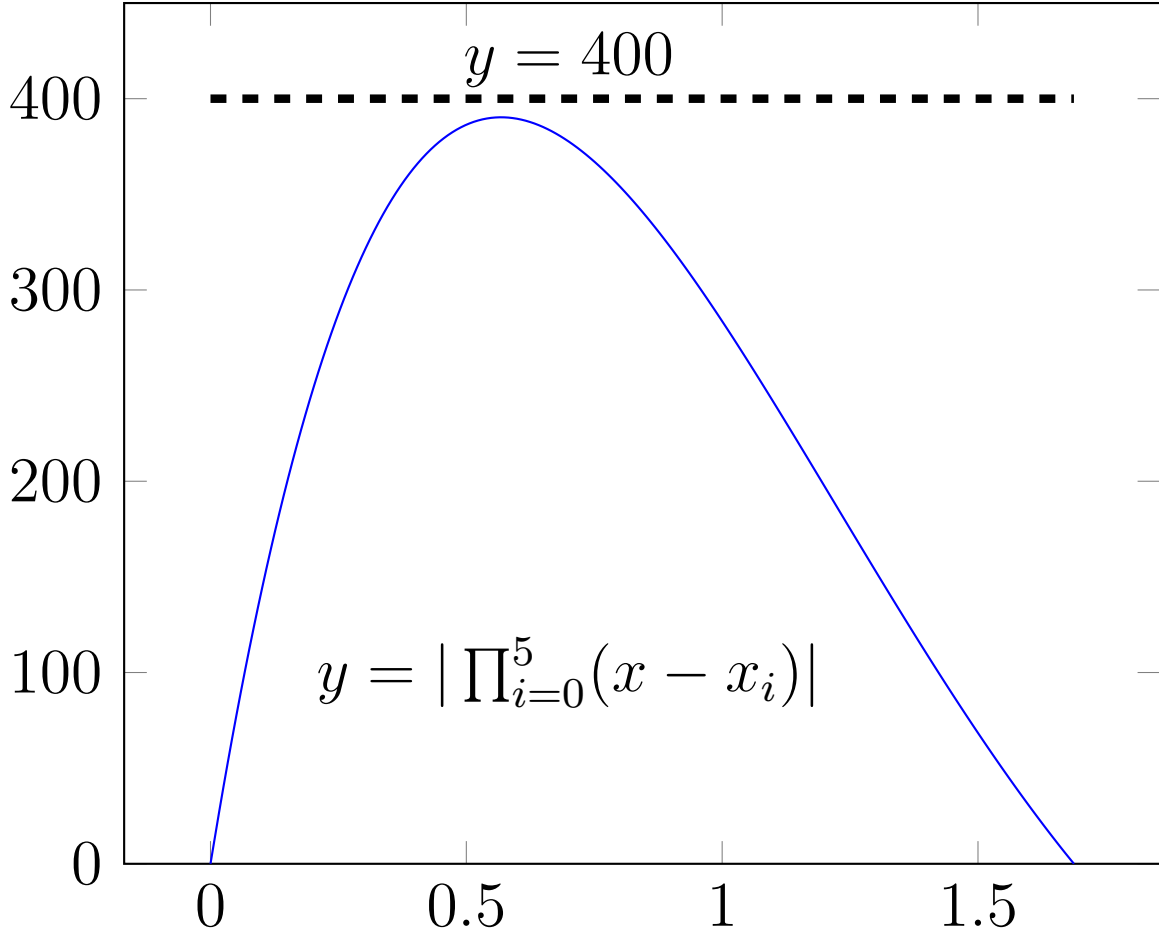
$$\left|\frac{f^6(c)}{6!}\right| \leq \frac{1}{720} \approx 0.001388889$$

Now we just need to bound the term $|\prod_{i=0}^5(x - x_i)|$.

Let $h = \frac{1.6875}{5}$ and $x_i = ih$. Then,

$$\begin{aligned}
\left|\prod_{i=0}^5(x - x_i)\right| &= |(x - 0(1.6875))(x - 1(1.6875))(x - 2(1.6875))(x - 3(1.6875))(x - 4(1.6875))(x - 5(1.6875))| \\
&= |x^6 - 25.3125x^5 + 242.051x^4 - 1081.22x^3 + 2221.91x^2 - 1642.1x|
\end{aligned}$$

Let's graph this function to try to find its upper bound on $[0, 1.6875]$ now.



As evident in the plot,

$$y = |x^6 - 25.3125x^5 + 242.051x^4 - 1081.22x^3 + 2221.91x^2 - 1642.1x| \leq 400$$

on $[0, 1.6875]$. Thus,

$$\begin{aligned} |f(x) - p(x)| &= \left| \frac{f^6(c)}{6!} \prod_{i=0}^5 (x - x_i) \right| \\ &= \left| \frac{f^6(c)}{6!} \right| \left| \prod_{i=0}^5 (x - x_i) \right| \\ &\leq \left(\frac{1}{720} \right) |400| = \frac{5}{9} \approx 0.555556 \end{aligned}$$

is the upper bound for the error $|f(x) - p(x)|$ on $[0, 1.6875]$.

Problem 4.

From (4.64) in the textbook,

$$\begin{aligned}\frac{1}{6}M_1 + 23M_2 + \frac{1}{6}M_3 &= 3 \\ \frac{1}{6}M_2 + 23M_3 + \frac{1}{6}M_4 &= 0 \\ \frac{1}{6}M_3 + 23M_4 + \frac{1}{6}M_5 &= -2\end{aligned}$$

We know from (4.65) that $M_1 = M_5 = 0$. Combining this with the above system of equations yields:

$$M_2 = \frac{129}{28}, M_3 = -\frac{3}{7}, M_4 = -\frac{81}{28}$$

Thus, substituting into (4.63), we obtain:

$$s(x) = \begin{cases} \frac{43x^3}{56} - \frac{129x^2}{56} - \frac{13x}{28} + 5 & 1 \leq x \leq 2 \\ \frac{-47x^3}{56} + \frac{411x^2}{56} - \frac{79x}{4} + \frac{125}{7} & 2 \leq x \leq 3 \\ \frac{-23x^3}{56} + \frac{195x^2}{56} - \frac{229x}{28} + \frac{44}{7} & 3 \leq x \leq 4 \\ \frac{27x^3}{56} - \frac{81x^2}{14} + \frac{1213x}{56} - \frac{1201}{56} & 4 \leq x \leq 5 \end{cases}$$

Problem 5.

a) Each a_{ij} comes from one of the following coefficients in (4.64):

$$\frac{x_j - x_{j-1}}{6}, \frac{x_{j+1} - x_{j-1}}{3}, \frac{x_{j+1} - x_j}{6}$$

We know $x_1 < x_2 < \dots < x_n$. Thus, in the equations above, we have:

$$x_j > x_{j-1}, x_{j+1} > x_{j-1}, x_{j+1} > x_j$$

Thus, each numerator term is positive for any value $2 \leq j \leq n-1$. Since all the denominators are also positive, each a_{ij} is positive.