Numerical Analysis: Homework #5

Chris Hayduk

March 10, 2019

Problem 1.

a) To minimize the size of the error, we need to minimize the absolute value of,

$$\omega(x) = (x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5)$$

on $-1 \leq x \leq 1$.

We can see that $\omega(x)$ is a monic polynomial of degree 6. Thus, from Theorem 4.5.3, we have,

$$\omega(x) = \frac{T_6(x)}{2^5} = \frac{T_6(x)}{32}$$

Let $n = 6$. Then,

$$T_6(x) = 2xT_5(x) - T_4(x)$$

and,

$$T_5(x) = 2xT_4(x) - T_3(x)$$

From (4.87) and Example 4.5.2, this yields,

$$\begin{aligned} T_5(x) &= 2x(8x^4 - 8x^2 + 1) - (4x^3 - 3x) \\ &= 16x^5 - 16x^3 + 2x - 4x^3 + 3x \\ &= 16x^5 - 20x^3 + 5x \end{aligned}$$

And thus we get,

$$\begin{aligned} T_6(x) &= 2x(16x^5 - 20x^3 + 5x) - (8x^4 - 8x^2 + 1) \\ &= 32x^6 - 40x^4 + 10x^2 - 8x^4 + 8x^2 - 1 \\ &= 32x^6 - 48x^4 + 18x^2 - 1 \end{aligned}$$

Thus, we need to minimize,

$$\omega(x) = \frac{32x^6 - 48x^4 + 18x^2 - 1}{32}$$

The node points are the zeros of $\omega(x)$ and from (4.96), they must be the zeros of $T_6(x)$.

From definition (4.84) and (4.85), we know that,

$$T_6(x) = cos(6\theta), \quad x = cos(\theta)$$

This expression is zero when,

$$\begin{aligned}
&6\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, ... \\
&\theta = \pm\frac{\pi}{12}, \pm\frac{3\pi}{12}, \pm\frac{5\pi}{12}, \pm 7\pi 12, ... \\
&x = \cos(\frac{\pi}{12}), \cos(\frac{3\pi}{12}), \cos(\frac{5\pi}{8}), \cos(\frac{7\pi}{8}), \cos(\frac{9\pi}{8}), ...
\end{aligned}$$

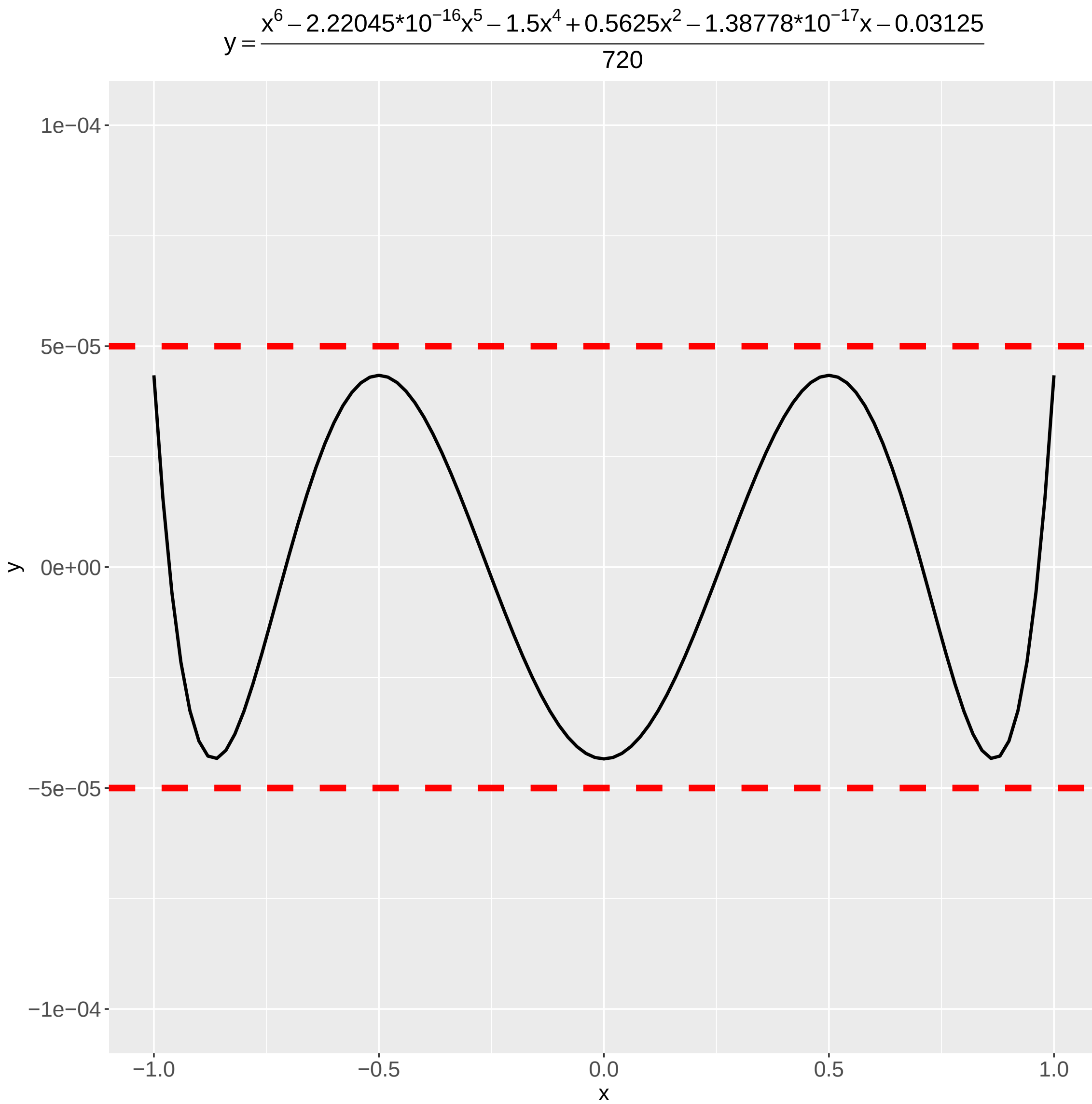
The first six values of x are distinct, but the successive values repeat the first four values. Thus, when we evaluate (4.97), the nodes are approximately,

$$\pm 0.258819, \pm 0.707107, \pm 0.965926$$

Thus, we now have,

$$\begin{aligned}
f(x) - P_5(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{6!} f^{(6)}(c_x) \\
&= \frac{x^6 - 2.22045 * 10^{-16}x^5 - 1.5x^4 + 0.5625x^2 - 1.38778 * 10^{-17}x - 0.03125}{720} f^{(6)}(c_x)
\end{aligned}$$

Let's graph the first part of this function,



As can be seen from the graph,

$$\left|\frac{x^6 - 2.22045 * 10^{-16}x^5 - 1.5x^4 + 0.5625x^2 - 1.38778 * 10^{-17}x - 0.03125}{720}\right| \leq 5 * 10^{-5}$$

when $-1 \leq x \leq 1$.

Let's now find a bound for the sixth derivative of $f(x) = e^{2x}$ on $x \in [-1, 1]$.

$$\begin{aligned}
&f'(x) = 2e^{2x} \\
&f''(x) = 2(2e^{2x}) = 4e^{2x} \\
&f'''(x) = 4(2e^{2x}) = 8e^{2x} \\
&\dots \\
&f^{(6)}(x) = 2^6 e^{2x} = 64e^{2x}
\end{aligned}$$

It is clear that e^{2x} is an increasing function of x. Thus, this equation takes on its maximum value when $c_x = 1$. Plugging into the equation yields,

$$f^{(6)}(1) = 64e^2 \approx 472.89959 \leq 473$$

Now that we have an upper bound for both terms in the error equation on $x \in [-1, 1]$, we can find an upper bound for the error of $P_5(x)$,

$$\begin{aligned}
|f(x) - P_5(x)| &= \left| \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{6!} f^{(6)}(c_x) \right| \\
&\leq \left| 5 * 10^{-5} * 473 \right| = 0.02365
\end{aligned}$$

Thus, we now have 0.02365 as an upper bound for the error $|f(x) - P_5(x)|$ on $x \in [-1, 1]$.

b) If we use a minimax approximation of $f(x)$, then we have,

$$\begin{aligned}
\rho(f) &\leq \frac{[(1-(-1))/2]^{n+1}}{(n+1)!2^n} \max_{-1 \leq x \leq 1} \left| f^{(n+1)}(x) \right| \\
&= \frac{1}{(n+1)!2^n} \left| 2^{n+1} e^2 \right| \\
&= \frac{2e^2}{(n+1)!}
\end{aligned}$$

Thus, we need to find n such that,

$$\begin{aligned}
&\frac{2e^2}{(n+1)!} \leq 10^{-6} \\
\implies &(n+1)! \geq \frac{2e^2}{10^{-6}} \approx 14,778,000
\end{aligned}$$

From this, we can see that $n = 10$ results in an error less than or equal to 10^{-6} on $[-1, 1]$.

Problem 2.

a)

$$\begin{aligned}\widetilde{T}_n(x) &= \frac{1}{2^{n-1}} T_n(x) \\ &= \frac{1}{2^{n-1}} \cos(n \cos^{-1} x)\end{aligned}$$

We know that we need,

$$\begin{aligned}&\left|\cos(m \cos^{-1} x)\right| = 1 \\ \implies &(m \cos^{-1} x) = 0, \quad (m \cos^{-1} x) = \pi \\ \implies &\cos^{-1} x = 0, \quad \cos^{-1} x = \frac{\pi}{m} \\ \implies &x = 1, \quad x = \cos\left(\frac{\pi}{m}\right)\end{aligned}$$

Thus, $x = \cos(\frac{\pi}{m})$ for every $m = 1, 2, 3, ..., n$ and $x = 1$ results in,

$$\begin{aligned}\left|\widetilde{T}_n(x)\right| &= \left|\frac{1}{2^{n-1}} T_n(x)\right| \\ &= 1\end{aligned}$$

Hence, there are $n+1$ such points on $[-1, 1]$.

b)

c) Since both $\widetilde{T}$ and $p(x)$ are both degree n monic polynomials, we know that,

$$\begin{aligned}r(x) &= \widetilde{T} - p(x) \\ &= (x^n) + \text{lower-degree terms from}\, \widetilde{T} - ((x^n) + \text{lower-degree terms from}\, p(x) \\ &= \text{lower-degree terms from}\, \widetilde{T} - \text{lower-degree terms from}\, p(x)\end{aligned}$$

Thus, we can see that $r(x)$ has degree $\leq n-1$. Hence, r cannot have n distinct roots and $r(x) = 0$. As a result, $\widetilde{T}(x) = p(x)$.

Problem 3.

a)

$$\begin{aligned}dist(\vec{v}, \vec{v}) &= \|\vec{v} - \vec{v}\| \\ &= \left\|\vec{0}\right\| = 0\end{aligned}$$

b)

$$\begin{aligned} dist(\vec{v},\vec{w}) &= \|\vec{v}-\vec{w}\| \\ &= \|(v_1-w_1, v_2-w_2, v_3-w_3, ..., v_n-w_n)\| \\ &= \|(-w_1+v_1, -w_2+v_2, -w_3+v_3, ..., -w_n+v_n)\| \\ &= \|-1(w_1-v_1, w_2-v_2, w_3-v_3, ..., w_n-v_n)\| \\ &= \|-1(\vec{w}-\vec{v})\| \\ &= |-1|\,\|\vec{w}-\vec{v}\| \\ &= \|\vec{w}-\vec{v}\| \end{aligned}$$

c)

$$\begin{aligned} dist(\vec{v},\vec{w}) &= \|\vec{v}-\vec{w}\| \\ &= \|\vec{v}+(-\vec{w})\| \\ &= \|\vec{v}+(-\vec{x}+\vec{x})+(-\vec{w})\| \\ &= \|\vec{v}-\vec{x}+\vec{x}-\vec{w}\| \\ &\le \|\vec{v}-\vec{x}\|+\|\vec{x}-\vec{w}\| \\ &= dist(\vec{v},\vec{x})+dist(\vec{x},\vec{w}) \end{aligned}$$

Problem 4.

Problem 5.

a)

$$\begin{aligned} \|A\|_\infty &= \max |a_{ij}| \\ &= 3 \end{aligned}$$

b)

$$\begin{aligned} \|A\|_1 &= \sum |a_{ij}| \\ &= |1|+|2|+|1|+|2|+|2|+|3|+|-1|+|-3|+|0| \\ &= 15 \end{aligned}$$

c)

$$\begin{aligned} \|A\|_2 &= \sqrt{\sum a_{ij}^2} \\ &= \sqrt{1^2+2^2+1^2+2^2+2^2+3^2+(-1)^2+(-3)^2+0^2} \\ &= \sqrt{33} \approx 5.74456 \end{aligned}$$

d)

$$\begin{aligned}
\|A\|_h &= \max_{1\leq i\leq 3}(\sum_{j=1}^{3}|a_{ij}|) \\
&= \max[(|1|+|2|+|1|),(|2|+|2|+|3|),(|-1|,|-3|,|0|)] \\
&= \max[4,7,4] \\
&= 7
\end{aligned}$$