Real Analysis I: Final Exam Review

Chris Hayduk

December 17, 2019

1 Definitions

- σ-algebra:

  A collection $\mathscr{A}$ of subsets of X is called an **algebra** of sets or a **Boolean algebra** if (i) $A \cup B$ is in $\mathscr{A}$ whenever A and B are, and (ii) $\tilde{A}$ is in $\mathscr{A}$ whenever A is.

  $\mathscr{A}$ is called a **σ-algebra**, or a **Borel field**, if it has the above properties *and* every union of a countable collection of sets in $\mathscr{A}$ is again in $\mathscr{A}$

- Uniform convergence of a sequence of functions:

  A sequence $\langle f_n \rangle$ of functions defined on a set E is said to converge **uniformly** on E if given $\epsilon > 0$, there is an N such that for all $x \in E$ and all $n \geq N$, we have $|f(x) - f_n(x)| < \epsilon$.

- Borel sets:

  The collection $\mathscr{B}$ of Borel sets is the smallest σ-algebra which contains all of the open sets.

- F_σ set:

  An F_σ set is a countable union of closed sets.

- G_δ set:

  A G_δ set is a countable intersection of open sets.

- Outer measure:

  $m^*A = \inf_{A \subset \cup I_n} \Sigma \ell(I_n)$, where $\{I_n\}$ represents a countable collections of open intervals that cover A.

- Measurable set:

  A set E is said to be **measurable** if for each set A we have $m^*A = m^*(A \cap E) + m^*(A \cap \tilde{E})$

- Measurable function:

  Let f be an extended real-valued function whose domain is measurable. Then the following statements are equivalent:

  1. For each real number α, the set $\{x : f(x) > \alpha\}$ is measurable.
  2. For each real number α, the set $\{x : f(x) \geq \alpha\}$ is measurable.
  3. For each real number α, the set $\{x : f(x) < \alpha\}$ is measurable.
  4. For each real number α, the set $\{x : f(x) \leq \alpha\}$ is measurable.

  These statements imply that, for each extended real number α, the set $\{x : f(x) = \alpha\}$ is measurable.

  An extended real-valued function f is said to be (Lebesgue) measurable if its domain is measurable and if it satisfies one of the first four statements above.

- Almost everywhere:

  A property is said to hold **almost everywhere** (abbreviated a.e.) if the set of points where it fails to hold is a set of measure zero.

- Lebesgue integral of simple functions: