Real Analysis I: Assignment 9

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Problem 1.

Let f be measurable and B be a Borel set. Since B is Borel, let $\langle E_i \rangle$ be a sequence of open sets such that $B = \cup E_i$.

Since f is measurable, we know that the inverse image of each open set under f must be measurable. Thus, we know that $f^{-1}(E_i)$ is measurable for every i. As a result, we have

$$\begin{aligned} f^{-1}(B) &= f^{-1}(\cup E_i) \\ &= \cup f^{-1}(E_i) \end{aligned}$$

Thus, $f^{-1}(B)$ is measurable.

Problem 2.

Let f be a measurable real-valued function and let g be a continuous function defined on $(-\infty, \infty)$.

We know continuous functions are measurable, hence g is also measurable.

Now fix $c \in \mathbb{R}$. We have,

$$\begin{aligned} \{x : (g \circ f)(x) > c\} &= (g \circ f)^{-1}[(c, \infty)] \\ &= f^{-1}[g^{-1}[(c, \infty)]] \end{aligned}$$

We know that the inverse image of an open set under a continuous function is open. Thus, $O = g^{-1}[(c, \infty)]$ is an open set. In addition, from the previous problem, we know that the inverse image of an open set under a measurable function is measurable. Hence,

$$f^{-1}[g^{-1}[(c, \infty)]] = f^{-1}(O)$$

is measurable. As a result, $\{x : (g \circ f)(x) > c\}$, and thus $g \circ f$ satisfies the definition of a measurable function since c was arbitrary.

Problem 3.

Let $E = \mathbb{R}$. If $f_n = \chi_{[n,\infty)}$, then $f_n(x) \to 0$. Now let $\delta = 1$ and $\epsilon = 1$.

Choose $A \subset E$ with $mA < 1$ and let $N \in \mathbb{Z}$. For every $x \geq N$ such that $x \notin A$, we have that

$$|f_N(x) - f(x)| = |f_N(x) - 0| = |f_N(x)| \geq 1$$

Thus, there does not exist a measurable set $A \subset E$ with $mA < \delta$ such that for all $x \notin A$ and all $n \geq N$, $|f_n(x) - f(x)| < \epsilon$ when E has infinite measure.

Problem 4.

Let $\langle f_n \rangle$ be a sequence of measurable functions that converges to a real-valued function f a.e. on a measurable set E with $mE < \infty$. Let $\eta > 0$ be given.