Real Analysis I: Final Exam Review

Chris Hayduk

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1 Definitions

- σ-algebra:

  A collection $\mathscr{A}$ of subsets of X is called an **algebra** of sets or a **Boolean algebra** if (i) $A \cup B$ is in $\mathscr{A}$ whenever A and B are, and (ii) $\tilde{A}$ is in $\mathscr{A}$ whenever A is.

  $\mathscr{A}$ is called a **σ-algebra**, or a **Borel field**, if it has the above properties *and* every union of a countable collection of sets in $\mathscr{A}$ is again in $\mathscr{A}$

- Uniform convergence of a sequence of functions:

  A sequence $\langle f_n \rangle$ of functions defined on a set E is said to converge **uniformly** on E if given $\epsilon > 0$, there is an N such that for all $x \in E$ and all $n \geq N$, we have $|f(x) - f_n(x)| < \epsilon$.

- Borel sets:

  The collection $\mathscr{B}$ of Borel sets is the smallest σ-algebra which contains all of the open sets.

- F_σ set:

  An F_σ set is a countable union of closed sets.

- G_δ set:

  A G_δ set is a countable intersection of open sets.

- Outer measure:

  $m^*A = \inf_{A \subset \cup I_n} \Sigma \ell(I_n)$, where $\{I_n\}$ represents a countable collections of open intervals that cover A.

- Measurable set:

  A set E is said to be **measurable** if for each set A we have $m^*A = m^*(A \cap E) + m^*(A \cap \tilde{E})$

- Measurable function:

  Let f be an extended real-valued function whose domain is measurable. Then the following statements are equivalent:

  1. For each real number α, the set $\{x : f(x) > \alpha\}$ is measurable.
  2. For each real number α, the set $\{x : f(x) \geq \alpha\}$ is measurable.
  3. For each real number α, the set $\{x : f(x) < \alpha\}$ is measurable.
  4. For each real number α, the set $\{x : f(x) \leq \alpha\}$ is measurable.

  These statements imply that, for each extended real number α, the set $\{x : f(x) = \alpha\}$ is measurable.

  An extended real-valued function f is said to be (Lebesgue) measurable if its domain is measurable and if it satisfies one of the first four statements above.

- Almost everywhere:

  A property is said to hold **almost everywhere** (abbreviated a.e.) if the set of points where it fails to hold is a set of measure zero.

- Lebesgue integral of simple functions:

  If ϕ is a simple function and $\{a_1, ..., a_n\}$ the set of non-zero values of ϕ, then

  $$\phi = \Sigma a_i \chi_{A_i}$$

  where $A_i = \{x : \phi(x) = a_i\}$. This representation for ϕ is called the canonical representation, and it is characterized by the fact that the A_i are disjoint and the a_i distinct and non-zero.

  If ϕ vanishes outside a set of finite measure, we define the integral of ϕ by

  $$\int \phi(x)dx = \Sigma_{i=1}^{n} a_i m A_i$$

- Lebesgue integral of bounded measurable functions that vanish outside of a set of finite measure:

  If f is a bounded measurable function defined on a measurable set E with mE finite, we define the (Lebesgue) integral of f over E by

  $$\int_E f(x)dx = \inf \int_E \psi(x)dx$$

  for all simple functions $\psi \geq f$.

- Lebesgue integral of non-negative measurable functions:

  If f is a non-negative measurable function defined on a measurable set E, we define,

  $$\int_E f = \sup_{h \leq f} \int_E h$$

  where h is a bounded measurable function such that $m\{x : h(x) \neq 0\}$ is finite.

- Lebesgue integral of a general measurable function:

  A measurable function f is said to be integrable over E if f^+ and f^- are both integrable over E (that is, $\int_E f^+ < \infty$ and $\int_E f^- < \infty$). In this case, we define,

  $$\int_E f = \int_E f^+ - \int_E f^-$$

  where $f^+ = \max\{f(x), 0\}$ and $f^- = \max\{-f(x), 0\}$

- Convergence in measure:

  A sequence $\langle f_n \rangle$ of measurable functions is said to converge to f in measure if, given $\epsilon > 0$, there is an N such that for all $n \geq N$ we have

  $$m\{x : |f(x) - f_n(x)| \geq \epsilon\} < \epsilon$$

- Vitali cover:

  Let g be a collection of intervals. Then we say that g covers a set E in the sense of Vitali if, for each $\epsilon > 0$ and any $x \in E$, there is an interval $I \in g$ such that $x \in I$ and $\ell(I) < \epsilon$. The intervals may be open, closed, or half-open, but we do not allow degenerate intervals consisting of only one point.

- Total variation:

  Let f be a real-valued function defined on the interval $[a, b]$, and let $a = x_0 < x_1 < \cdots < x_k = b$ be any subdivision of $[a, b]$. Define

$$
\begin{aligned}
p &= \Sigma_{i=1}^{k}[f(x_i) - f(x_{i-1})]^+ \\
n &= \Sigma_{i=1}^{k}[f(x_i) - f(x_{i-1})]^- \\
t = n + p &= \Sigma_{i=1}^{k}|f(x_i) - f(x_{i-1})|
\end{aligned}
$$

  where we use r^+ to denote r if $r \geq 0$ and 0 if $r \leq 0$, and set $r^- = |r| - r^+$. We have $f(b) - f(a) = p - n$. Set,

$$
\begin{aligned}
P &= \sup p \\
N &= \sup n \\
T &= \sup t
\end{aligned}
$$

  where we take the supremum over all possible subdivisions of $[a, b]$.

  We clearly have $P \leq T \leq P + N$. We call P, N, T the positive, negative, and total variations of f over $[a, b]$. We sometimes write T_a^b, $T_a^b(f)$, etc. to denote the dependence on the interval $[a, b]$ or on the function f.

  If $T < \infty$, we say that f is of bounded variation over $[a, b]$.

2 Theorems

- Heine-Borel Theorem:

  Let F be a closed and bounded set of real numbers. Then each open covering of F has a finite subcovering.

  That is, if a collection $\mathscr{C}$ is a collection of open sets such that $F \subset \cup\{O : O \in \mathscr{C}\}$, then there is a collection $\{O_1, O_2, ..., O_n\}$ of sets in $\mathscr{C}$ such that,

  $$F \subset \cup_{i=1}^{n} O_i$$

- Egoroff's Theorem:

  If $\langle f_n \rangle$ is a sequence of measurable functions that converge to a real-valued function f a.e. on a measurable set E of finite measure, then given $\eta > 0$, there is a subset $A \subset E$ with $mA < \eta$ such that f_n converges to f uniformly on $E \sim A$

- Fatou's Lemma:

  If $\langle f_n \rangle$ is a sequence of non-negative measurable functions and $f_n(x) \to f(x)$ almost everywhere on a set E, then

  $$\int_E f \leq \underline{\lim} \int_E f_n$$

- Monotone Convergence Theorem:

  Let $\langle f_n \rangle$ be an increasing sequence of non-negative measurable functions, and let $f = \lim f_n$ a.e. Then,

  $$\int f = \lim \int f_n$$

- Lebesgue Convergence Theorem:

  Let g be integrable over E and let $\langle f_n \rangle$ be a sequence of measurable functions such that $|f_n| \leq g$ on E and for almost all x in E we have $f(x) = \lim f_n(x)$. Then,

  $$\int_E f = \lim \int_E f_n$$

- Vitali Covering Lemma:

  Let E be a set of finite outer measure and g a collection of intervals that covers E in the sense of Vitali. Then, given $\epsilon > 0$, there is a finite disjoint collection $\{I_1, ..., I_n\}$ of intervals in g such that,

$$m^* \left[E \sim \cup_{n=1}^{N} I_n \right] < \epsilon$$

3 Proofs

- If f is measurable and $f = g$ a.e., then g is also measurable:

  Let E be the set $\{x : f(x) \neq g(x)\}$. By hypothesis, $mE = 0$. Now,

  $$\{x : g(x) > \alpha\} = [\{x : f(x) > \alpha\} \cup \{x \in E : g(x) > \alpha\}] \sim \{x \in E : g(x) \leq \alpha\}$$

  The first set on the right is measurable since f is a measurable function. The last two sets on the right are measurable since they are subsets of E and $mE = 0$. Thus, $\{x : g(x) > \alpha\}$ is measurable for each α and so g is measurable.

- Bounded Convergence Theorem:

  Theorem statement: Let $\langle f_n \rangle$ be a sequence of measurable functions defined on a set E of finite measure, and suppose that there is a real number M such that $|f_n(x)| \leq M$ for all n and all x. If $f(x) = \lim f_n(x)$ for each x in E, then

  $$\int_E f = \lim \int_E f_n$$

  Proof:

  By Proposition 3.23, we have that, given $\epsilon > 0$, there is an N and a measurable set $A \subset E$ with $mA < \frac{\epsilon}{4M}$ such that for $n \geq N$ and $x \in E \sim A$, we have $|f_n(x) - f(x)| < \frac{\epsilon}{2mE}$. Then,

  $$\begin{aligned} \left| \int_E f_n - \int_E f \right| &= \left| \int_E f_n - f \right| \\ &\leq \int_E |f_n - f| \\ &= \int_{E \sim A} |f_n - f| + \int_A |f_n - f| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

  Hence,

  $$\int_E f_n \to \int_E f$$

- Proposition 14 from Ch. 4:

  Proposition statement: Let f be a non-negative function which is integrable over a set E. Then given $\epsilon > 0$, there is a $\delta > 0$ such that for every set $A \subset E$ with $mA < \delta$, we have

  $$\int_A f < \epsilon$$

Proof:

The proposition would be trivial if f were bounded, so assume f is unbounded. Set $f_n(x) = f(x)$ if $f(x) \leq n$ and $f_n(x) = n$ otherwise. Then each $f_n(x)$ is bounded and f_n converges to f at each point.

By the Monotone Convergence Theorem, there is an N such that $\int_E f_N > \int_E f - \epsilon/2$ and $\int_E f - f_N < \epsilon/2$.

Choose $\delta < \frac{\epsilon}{2N}$. If $mA < \delta$, we have

$$\begin{aligned}
\int_A f &= \int_A (f - f_N) + \int_A f_N \\
&< \int_E (f - f_N) + NmA \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

as required.

- A function is of bounded variation iff it is the difference of two monotone real-valued functions:

Let f be of bounded variation and set $g(x) = P_a^x$ and $h(x) = N_a^x$. Then g and h are monotone increasing functions which are real valued, since $0 \leq P_a^x \leq T_a^x \leq T_a^b < \infty$ and $0 \leq N_a^x \leq T_a^x \leq T_a^b < \infty$.

But $f(x) = g(x) - h(x) + f(a) = g(x) - [h(x) - f(a)]$ by Lemma 4. Since $h - f(a)$ is a monotone function, we have f expressed as the difference of two monotone functions.

On the other hand, if $f = g - h$ on $[a, b]$ with g and h increasing, then for any subdivision we have

$$\begin{aligned}
t = \Sigma|f(x_i) - f(x_{i-1})| &\leq \Sigma[g(x_i) - g(x_{i-1})] + \Sigma[h(x_i) - h(x_{i-1})] \\
&= g(b) - g(a) + h(b) - h(a) < \infty
\end{aligned}$$

Since this holds for any subdivision of $[a, b]$, we have that $t < \infty$ for all such subdivisions. Hence, $\sup t = T < \infty$ and so f is of bounded variation.