Real Analysis I: Assignment 5

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Problem 1.

Suppose $f : \mathbb{R} \to \mathbb{R}$. If f is continuous at a point $x \in \mathbb{R}$, then $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.

Thus, for a given ϵ, the region of continuity around x is $(x - \delta, x + \delta)$.

For each $x \in \mathbb{R}$ where f is continuous, let $\epsilon = \frac{1}{n}$ $\forall n \in \mathbb{N}$. Then the δ neighborhoods around each x will be shrinking in size as well.

Let x_i be the ith continuous point of f and let $O_n = \bigcup_{i=1}^{\infty} (x_i - \delta, x_i + \delta)$ for the δ corresponding to $\epsilon = \frac{1}{n}$. Then, each O_n is a union of open sets and thus is open as well.

It is clear from the construction of each δ neighborhood that, as $n \to \infty$, we have $\epsilon, \delta \to 0$.

Thus, if we take a specific point of continuity x_k, we have that for some δ, $x_k \in (x_k - \delta, x_k + \delta)$ for all choices of ϵ by definition. However, we can make δ arbitrarily small through our choice of ϵ, so for any $x_k \pm c$, we can choose an *epsilon* such that $x_k \pm c \notin (x_k - \delta, x_k + \delta)$. Hence, we have that x_k is in every O_n, but no other point contained in all of its delta neighborhoods is.

As a result, we have that $A = \bigcap_{i=1}^{\infty} O_i$ is the set of all points at which f is continuous.

Now, from the above, we have defined each O_n as a countable union of open sets, so each O_n is open. In addition, there are a countable number of sets O_n. Thus, A is a countable intersection of open sets and, hence, A is G_δ.

Problem 2.

We have that $m^*A = \inf_{A \subset \cup I_n} \sum l(I_n)$.

Let $B = \{I_n : n \in \mathbb{N}\}$ be the open cover consisting of open intervals that satisfies $\inf_{A \subset \cup I_n} \sum l(I_n)$.

Since B is a collection of open sets, it is an open cover for itself. Moreover, for an open set $I_k \in B$, let I_{k_n} be an open cover for I_k composed of open intervals. By the triangle inequality, we have that $l(I_k) \leq \sum l(I_{k_n})$.

Now let $O = \cup I_n$ such that $I_n \in B$. Since B is an open cover for A, we have that $A \subset O$. In addition, we have,

$$\begin{aligned} m^*O &= \sum l(I_n), \; I_n \in B \\ &= m^*A \end{aligned}$$

Hence, we can clearly see that $m^*O \leq m^*A + \epsilon$ for any choice of $\epsilon > 0$.

Now let $G = \cap \mathscr{C}$, where $\mathscr{C}$ is the collection of all open covers for A consisting solely of open intervals. Then G is the smallest such open cover for A and, as a result, $A \subset G$. In addition, since each set contained in $\mathscr{C}$ is open, we have that G is a G_δ set.

Since G is the smallest open cover consisting of intervals for A, and G is an open set consisting of open intervals, we have that $m^*G = l(G)$ and,

$$\begin{aligned} m^*A &= \inf_{A \subset \cup I_n} \sum l(I_n) \\ &= l(G) \\ &= m^*G \end{aligned}$$

Problem 3.

a) Let A be measurable and let $y \in \mathbb{R}$. Assume $A + y$ is also measurable. Then,

$$\begin{aligned} m^*A &= \inf_{A \subset \cup I_n} \sum l(I_n) \\ &= \inf \sum l(a_n, b_n) \\ &= \inf \sum (b_n - a_n) \end{aligned}$$

If we take $A + y$, we can also shift each I_n by y, preserving the status of $\cup I_n$ as an open cover for $A + y$. This is true because, if $x \in A$, x is in some $I_k = (a_k, b_k) \subset \cup I_n$. Hence, $a_k < x < b_k$. This implies that $a_k + y < x + y < b_k + y$ and thus $x \in (a_k + y, b_k + y)$. As a result, we have,

$$\begin{aligned} m^*(A + y) &= \inf_{A \subset \cup I_n + y} \sum l(I_n + y) \\ &= \inf \sum l(a_{n_k} + y, b_{n_k} + y) \\ &= \inf \sum (b_{n_k} + y - a_{n_k} - y) \\ &= \inf \sum (b_{n_k} - a_{n_k}) \end{aligned}$$

Thus, outer measure is translation invariant.

b) Let E be a measurable set, let $y \in \mathbb{R}$, and let A be any set. Since E is measurable, we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap \tilde{E})$$

In addition, since m^* is translation invariant, this yields,

$$\begin{aligned} m^*(A) = m^*(A - y) &= m^*((A - y) \cap E) + m^*((A - y) \cap \tilde{E}) \\ &= m^*(((A - y) \cap E) + y) + m^*(((A - y) \cap \tilde{E}) + y) \\ &= m^*(A \cap (E + y)) + m^*(A \cap (\tilde{E} + y)) \\ &= m^*(A \cap (E + y)) + m^*(A \cap (\widetilde{E + y})) \end{aligned}$$

Thus, $E + y$ satisfies the definition of measurability and hence is measurable.

Problem 4.

Suppose $m^*A = 0$ and B is measurable. Let I_{n_A} be an open cover consisting of open intervals for A and I_{n_B} be the same for B. Then,

$$\begin{aligned} m^*(A \cup B) &= \inf \sum l(I_n) \\ &= \inf \sum l((I_{n_A} \cup I_{n_B})) \\ &= \inf \sum [l(I_{n_B}) + l(I_{n_A} \setminus I_{n_B})] \\ &= m^*(B) - m^*(A \setminus B) \end{aligned}$$

Removing elements from A will only introduce the possibility of smaller open covers for the new set $A \setminus B$. Thus, $m^*(A \setminus B) \leq m^*(A) = 0$. Furthermore, we know that outer measure is non-negative, so $m^*(A \setminus B) = 0$.

Thus, we have,

$$\begin{aligned} m^*(A \cup B) &= m^*(B) - m^*(A \setminus B) \\ &= m^*(B) - 0 = m^*(B) \end{aligned}$$

Thus, if $m^*A = 0$, then $m^*(A \cup B) = m^*B$.