Real Analysis I: Assignment 6

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Problem 1.

Let $A = \mathbb{Q} \cap [0,1]$ and let I_n be a finite collection of open intervals that cover A.

Fix $x \in [0,1]$ and suppose $x \notin \cup I_n$. Since I_n is a cover for $A = \mathbb{Q} \cap [0,1]$, we have that $x \in \mathbb{I} \cap [0,1]$.

We know that between any two rational numbers, there exists an irrational number. Thus, there are intervals contained in I_n which have elements arbitrarily close to x. Thus, the only way for x to not be included in $\cup I_n$ is for it to be the endpoint of one or more of the intervals. That is, the intervals would be of the structure (a, x) or (x, b) for some $a, b \in \mathbb{R}$.

Since there are only finitely many open intervals in the collection I_n, there must be finitely many numbers x. Let us construct a collection x_n where each $x \in x_n$ is a singleton set containing one such number x.

Now let $B = \left(\bigcup_{n=1}^{N} I_n\right) \cup \left(\bigcup_{k=1}^{M} x_k\right)$ where N represents the number of open intervals in I_n and M represents the number of x values such that $x \in \mathbb{I} \cap [0,1]$ and $x \notin \cup I_n$. We can see that B covers the entire interval $[0,1]$. That is, $[0,1] \subset B$. Thus, $mB \geq 1$.

We can see that each set x_k is disjoint from all other x_n, as well as from each interval in I_n. Thus,

$$\begin{aligned}
mB &= m\left(\bigcup_{n=1}^{N} I_n \cup \bigcup_{k=1}^{M} x_k\right) \\
&= m\left(\bigcup_{n=1}^{N} I_n\right) + m\left(\bigcup_{k=1}^{M} x_k\right) \\
&= m\left(\bigcup_{n=1}^{N} I_n\right) + m(x_1) + m(x_2) + \cdots + m(x_M) \\
&= m\left(\bigcup_{n=1}^{N} I_n\right) + 0 + 0 + \cdots + 0 \\
&= m\left(\bigcup_{n=1}^{N} I_n\right) \geq 1
\end{aligned}$$

Thus, by Proposition 2 in Chapter 3 of the textbook, we have that,

$$1 \leq m\left(\bigcup_{n=1}^{N} I_n\right) \leq \sum m(I_n)$$

Since $m(I) = \ell(I)$ for any interval I, we can see that,

$$\sum m(I_n) = \sum \ell(I_n) \geq 1$$

Problem 2.

We can see that $E_1 \cup E_2 = E_1 \cup (E_2 \setminus E_1)$, with E_1 and $E_2 \setminus E_1$ disjoint. Thus,

$$\begin{aligned} m(E_1 \cup E_2) + m(E_1 \cap E_2) &= m(E_1 \cup (E_2 \setminus E_1)) + m(E_1 \cap E_2) \\ &= m(E_1) + m(E_2 \setminus E_1) + m(E_1 \cap E_2) \end{aligned}$$

In addition, we have that $E_2 \setminus E_1$ and $E_1 \cap E_2$ are disjoint sets. Thus, $m((E_2 \setminus E_1) \cup (E_1 \cap E_2)) = m(E_2 \setminus E_1) + m(E_1 \cap E_2)$. Hence,

$$\begin{aligned} m(E_1) + m(E_2 \setminus E_1) + m(E_1 \cap E_2) &= m(E_1) + m((E_2 \setminus E_1) \cup (E_1 \cap E_2)) \\ &= m(E_1) + m((E_2 \setminus E_1) \cup (E_2 \setminus \tilde{E}_1)) \\ &= m(E_1) + m(E_2) \end{aligned}$$

Problem 3.

Let $E_n = [n, \infty)$. We know from Lemma 11 in Chapter 3 of the textbook that E_n is measurable for each n. In addition, we have that the sequence $< E_n >$ is a decreasing sequence. That is, $E_{n+1} \subset E_n$. This clearly follows from the definition of each E_n. Writing out this definition, we can easily see that $[n+1, \infty) \subset [n, \infty)$. Thus, $< E_n >$ is a decreasing sequence of measurable sets.

Now assume $\cap E_n \neq \emptyset$ and let $x \in \cap E_n$. This implies that $x \in [n, \infty)$ for every $n \in \mathbb{N}$. However, by the Axiom of Archimedes, $\exists n_1 \in \mathbb{N}$ such that $x < n_1$. Thus, $x \notin E_{n_1}$. This contradicts the assumption that $\cap E_n$ is non-empty, so we have that $\cap E_n = \emptyset$.

Now we need to show that $mE_n = \infty$ for every n. Fix $n \in \mathbb{N}$. We can separate E_n into a countable union of pairwise disjoint intervals. That is, $E_n = [n, \infty) = [n, n+1) \cup [n+1, n+2) \cdots$.

By Proposition 13, we have that $m([n, n+1) \cup [n+1, n+2) \cdots) = m([n, n+1)) + m([n+1, n+2)) + \cdots$. Thus,

$$\begin{aligned} mE_n = m([n, \infty)) &= m([n, n+1) \cup [n+1, n+2) \cdots) \\ &= m\left(\bigcup_{k=0}^{\infty} [n+k, n+k+1)\right) \\ &= \sum_{k=0}^{\infty} m\left([n+k, n+k+1)\right) \\ &= \sum_{k=0}^{\infty} 1 = \infty \end{aligned}$$

Since n was chosen to be arbitrary, this holds for each E_n.

Hence, we have that $< E_n >$ is a decreasing sequence of measurable sets with $\cap E_n = \emptyset$ and $mE_n = \infty$ for every n.