Real Analysis I: Assignment 2

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Problem 1.

Let the relation R be $\leq$ on the set of real numbers $\mathbb{R}$. Now let $A = \mathbb{R} \cup c$ for some c that cannot be compared to any $x \in \mathbb{R}$ and extend R to this set A.

Since c cannot be compared to any $x \in \mathbb{R}$, we have that $c \not\leq x$ and $x \not\leq c \; \forall x \in \mathbb{R}$. Thus, c satisfies the definition of a minimal element in A, but does not satisfy the definition of the smallest element.

Now, let's fix a $y \in \mathbb{R}$ and assume it is a minimal element. Since $\mathbb{R}$ is closed under addition and we have $-1, y \in \mathbb{R}$, this yields $y + (-1) \in \mathbb{R}$. However, we know that $y + (-1) \leq y$. Thus, we have a contradiction and $\mathbb{R}$ has no minimal element under the relation $\leq$. Since the smallest element must also be minimal, $\mathbb{R}$ has no smallest element as well.

As a result, A has a unique minimal element c but has no smallest element.

Problem 2.

Suppose that $\inf E < \sup E$ and that E has only one element b. Then $\inf E \leq b \leq \sup E$.

From the definitions of infimum and supremum and the fact that E contains only one element, we have that $\inf E + \epsilon > b$ and $\sup E - \epsilon < b \; \forall \epsilon > 0$. As a result, $\inf E = b = \sup E$.

This is a contradiction, and so if $\inf E < \sup E$, then E must contain at least two elements.

Now suppose that E has at least two elements.

Choose two elements $a, b \in E$. Since $\mathbb{R}$ is an ordered field and $E \subset \mathbb{R}$, we have that $a \leq b$ or $b \leq a$. Assume $a \leq b$ without loss of generality.

Since every element of a set is distinct, $a \neq b$ and we have $a < b$.

By definition of $\inf E$, we have that $\inf E \leq E \implies \inf E \leq a$. Similarly, we have $E \leq \sup E \implies b \leq \sup E$.

These two statements yield,

$$\inf E \leq a < b \leq \sup E$$

Thus, if E has at least two elements, $\inf E < \sup E$.