Real Analysis I: Assignment 4

Chris Hayduk

October 3, 2019

Problem 1.

Let $< F_n >$ be a sequence of non-empty, closed sets of real numbers with the property that $F_{n+1} \subset F_n$ for all $n \in \mathbb{N}$. Furthermore, assume there exists an m such that F_m is bounded.

Since F_m is bounded we know that $\exists c \in \mathbb{R}$ such that $\forall x \in F_m, |x| \leq c$. Now examine F_{m+1}. We know that $F_{m+1} \neq \emptyset$ and $F_{m+1} \subset F_m$. That is, $x \in F_{m+1} \implies x \in F_m$. As a result, every $x \in F_{m+1}$ is bounded by the same c.

This same line of reasoning applies for any $n \geq m$ since we have that $F_m \supset F_{m+1} \supset \cdots$ and $F_n \neq \emptyset \ \forall n$. Thus, we now have a subsequence of the original sequence which consists solely of closed, bounded, nonempty subsets.

We know that $\forall k \geq m, F_k$ is a non-empty, closed, and bounded set. As a result each F_k must contain its infimum, which we can denote by x_k.

We can see that $x_k \leq c$ and $x_k \leq x_{k+1}$. Thus, $< x_k >$ is a monotonically increasing, bounded sequence. By the monotone convergence theorem, this sequence converges to a limit point x.

Since the sets are nested, for any k, $x_j \in F_k \ \forall j \geq k$. Furthermore, F_k is closed, so we have $x \in F_k$. This applies for any $k \geq m$, so $x \in \bigcap_{i=m}^{\infty} F_i$.

Now that we have considered the subsequence of non-empty, closed, and bounded sets let us incorporate the finite collection of unbounded sets $F_1, \cdots, F_{m-1}$.

We know that $F_m \subset F_{m-1} \subset \cdots F_1$. Thus, $F_m \subset F_n \ \forall n < m$.

As a result, we clearly have $\bigcap_{i=0}^{m-1} F_i = F_m$.

Now if we combine the two intersections, we have

$$\begin{aligned}
&\bigcap_{i=0}^{\infty} F_i = \bigcap_{j=0}^{m-1} F_j \ \cap \ \bigcap_{k=m}^{\infty} F_k \\
\implies &(F_1 \cap F_2 \cap \cdots \cap F_{m-1}) \cap (F_m \cap F_{m+1} \cap \cdots) \\
\implies &F_m \cap (F_m \cap F_{m+1} \cap \cdots) \\
\implies &(F_m \cap F_m) \cap F_{m+1} \cap \cdots \\
\implies &F_m \cap F_{m+1} \cap \cdots = \bigcap_{k=m}^{\infty} F_k
\end{aligned}$$

Since we know that $x \in \bigcap_{i=m}^{\infty} F_i$, the derivation above implies that $x \in \bigcap_{i=0}^{\infty} F_i \neq \emptyset$ as well. Thus, the intersection of a nested sequence of non-empty, closed sets of real numbers is non-empty if one of the sets is bounded.

Now let $F_n = \{x \in \mathbb{R} : x \geq n\} = [n, \infty)$. Clearly each $F_n \subset \mathbb{R}$ is unbounded and closed.

Furthermore, we have that $F_{n+1} = \{x \in \mathbb{R} : x \geq n+1\} = [n+1, \infty)$. Then we can see that $F_{n+1} \subset F_n$ since $[n+1, \infty) \subset [n, \infty)$.

Now assume $\cap F_n \neq \emptyset$ and let $x \in \cap F_n$. Then $x \in F_n$ $\forall n$. Since $F_n \subset \mathbb{R}$, we have that $x \in \mathbb{R}$. Thus, by the Axiom of Archimedes, $\exists k \in \mathbb{N}$ such that $k > x$.

Hence, by the definition of F_n, we have that $x \notin F_k$. However, this contradicts the assumption that $x \in \cap F_n$ and, by extension, that $\cap F_n \neq \emptyset$.

As a result, if there is not at least one bounded F_n, we have that $\cap F_n = \emptyset$.

Problem 2.

Let $A = \{x \in [a, b] : f(x) \leq \gamma\}$. We have that $a \in A$ and A is bounded above by b. As a result, the supremum of A exists. Let $c = \sup A$.

We claim that $f(c) = \gamma$.

We know that f is continuous, so choose some $\epsilon > 0$. Then $\exists \delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$.

This yields,

$$f(x) - \epsilon < f(c) < f(x) + \epsilon$$

Since c is the supremum of the set, there exists $c' \in (c - \delta, c]$ such that

$$\begin{aligned}
&f(c) + \epsilon < f(c') \\
\implies &f(c) < f(c') + \epsilon \leq \gamma + \epsilon
\end{aligned}$$

In addition, there exists $c'' \in (c, c+\delta)$ such that

$$\begin{aligned} & f(c'') < f(c) + \epsilon \\ \implies & f(c) > f(c'') - \epsilon \geq \gamma - \epsilon \end{aligned}$$

Combining these two inequalities yields,

$$\gamma - \epsilon \leq f(c) \leq \gamma + \epsilon$$

Since $\epsilon > 0$ can be arbitrarily small, $f(c) = \gamma$.

Problem 3.

Firstly, let $g(x) = f(x)\ \forall x \in F$.

Since F is closed, we know that $\tilde{F}$ is open. By Proposition 8, $\tilde{F}$ is the union of a countable collection of disjoint open intervals. Define g on $\tilde{F}$ to be linear on these open intervals. Then g is defined on all of $\mathbb{R}$.

We already know that g is continuous on F, so it suffices to show that g is continuous on $\tilde{F}$.

Let (x_1, x_2) be one of the open intervals whose union equals $\tilde{F}$. Since g is linear, we have that $x \in (x_1, x_2) \implies g(x) = ax + b$ for some constants $a, b \in \mathbb{R}$. Assume $a \neq 0$.

Let $\epsilon = \delta|a|$. Then there is a $\delta > 0$ such that for $x, y \in (x_1, x_2)$,

$$\begin{aligned} |x - y| < \delta &\implies |g(x) - g(y)| < \epsilon \\ &\implies |ax + b - (ay + b)| < \epsilon \\ &\implies |a(x - y)| < \epsilon \\ &\implies |x - y| < \frac{\epsilon}{|a|} = \frac{\delta|a|}{|a|} = \delta \end{aligned}$$

Thus, $g(x)$ is continuous on (x_1, x_2) when $a \neq 0$.

Moreover, if $a = 0$, then $g(x) = b\ \forall x \in (x_1, x_2)$. Thus, $|g(x) - g(y)| = 0 < \epsilon\ \forall x, y \in (x_1, x_2)$ and $\forall \epsilon > 0$. Hence, $g(x)$ is continuous on (x_1, x_2) for any choice of $a, b \in \mathbb{R}$.

Since $g(x)$ is continuous for any choice of $a, b \in \mathbb{R}$, we can choose a, b such that $g(x_1) = f(x_1)$ and $g(x_2) = f(x_2)$. (We know that $x_1, x_2 \in F$ by the disjoint property of the open intervals). This yields $a = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$ and $b = f(x_1) - x_1 \frac{f(x_1) - f(x_2)}{x_1 - x_2}$

By this construction, $g(x)$ is continuous on $[x_1, x_2]$ and $g(x_1) = f(x_1), g(x_2) = f(x_2)$.

Since (x_1, x_2) is an arbitrary open interval from the collection of countable open intervals whose union equals $\tilde{F}$, we have that $g(x)$ is continuous on all such intervals with the property that $g(x_1) = f(x_1), g(x_2) = f(x_2)$ for the endpoints of said intervals.

We know that $g(x)$ is continuous at all interior points of F and $\tilde{F}$. Thus, we must show that $g(x)$ is continuous at the boundary points of F, which are precisely the boundary points of $\tilde{F}$. Namely, the endpoints of each open interval.

Choose a boundary point x and an $\epsilon > 0$. Since $g(x)$ is continuous in F and $x \in F$, there exists a $\delta_1 > 0$ such that $|x - y| < \delta \implies |g(x) - g(y)| < \epsilon \ \forall y \in F$. Furthermore, by the construction above of $g(x)$ on $\tilde{F}$, there exists a $\delta_2 > 0$ such that $|x - y| < \delta_2 \implies |g(x) - g(y)| < \epsilon \ \forall y \in \tilde{F}$. Take $\delta = \min\{\delta_1, \delta_2\}$, and we can clearly see that $g(x)$ is continuous at x. Since x is an arbitrary boundary point of F, $g(x)$ is continuous at all such boundary points (ie. the endpoints of each open interval comprising $\tilde{F}$).

Hence, we have that $g(x)$ is continuous at all points in F and $\tilde{F}$. Thus, g is continuous on $F \cup \tilde{F} = \mathbb{R}$.

Problem 4.

Since f is continuous on $[a, b]$ (a closed and bounded subset of $\mathbb{R}$), we have that f is uniformly continuous.

Let $\epsilon > 0$. Then there exists a $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{2}$ for all $x, y \in [a, b]$.

Choose $N \in \mathbb{N}$ such that $\frac{b-a}{N} < \delta$ and let $x_i = a + i\frac{b-a}{N}$. Furthermore, construct φ similarly to how we constructed g in the above problem: let φ be linear on each $[x_i, x_{i+1}]$ with the property that $\varphi(x_i) = f(x_i) \ \forall i$.

Now let $x \in [a, b]$. Then $\exists i$ such that $x \in [x_i, x_{i+1}]$. In addition, assume $f(x_i) \leq f(x_{i+1})$, which implies that $\varphi(x_i) \leq \varphi(x) \leq \varphi(x_{i+1})$ by properties of linear functions. Then,

$$\begin{aligned}|\varphi(x) - f(x)| &\leq |\varphi(x) - \varphi(x_i)| + |\varphi(x_i) - f(x)| \\ &\leq |\varphi(x_{i+1}) - \varphi(x_i)| + |f(x_i) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

If $f(x_i) > f(x_{i+1})$, we have that $\varphi(x_i) > \varphi(x) > \varphi(x_{i+1})$, yielding,

$$\begin{aligned}|\varphi(x) - f(x)| &< |\varphi(x) - \varphi(x_{i+1})| + |\varphi(x_{i+1}) - f(x)| \\ &< |\varphi(x_i) - \varphi(x_{i+1})| + |f(x_{i+1}) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

Thus, the inequality holds regardless of the inequality relationship between endpoints.

Since $x \in [a, b]$ was arbitrary, we have that $\varphi : [a, b] \to \mathbb{R}$ is a polygonal function with the property that $|f(x) - \varphi(x)| < \epsilon \ \forall x \in [a, b]$.