Real Analysis I: Assignment 2

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Problem 1.

Let the relation R be $\leq$ on the set of real numbers $\mathbb{R}$. Now let $A = \mathbb{R} \cup c$ for some c that cannot be compared to any $x \in \mathbb{R}$ and extend R to this set A.

Since c cannot be compared to any $x \in \mathbb{R}$, we have that $c \not\leq x$ and $x \not\leq c \; \forall x \in \mathbb{R}$. Thus, c satisfies the definition of a minimal element in A, but does not satisfy the definition of the smallest element.

Now, let's fix a $y \in \mathbb{R}$ and assume it is a minimal element. Since $\mathbb{R}$ is closed under addition and we have $-1, y \in \mathbb{R}$, this yields $y + (-1) \in \mathbb{R}$. However, we know that $y + (-1) \leq y$. Thus, we have a contradiction and $\mathbb{R}$ has no minimal element under the relation $\leq$. Since the smallest element must also be minimal, $\mathbb{R}$ has no smallest element as well.

As a result, A has a unique minimal element c but has no smallest element.

Problem 2.

Suppose that $\inf E < \sup E$ and that E has only one element b. Then $\inf E \leq b \leq \sup E$.

From the definitions of infimum and supremum and the fact that E contains only one element, we have that $\inf E + \epsilon > b$ and $\sup E - \epsilon < b \; \forall \epsilon > 0$. Since ϵ can be made arbitrarily close to 0, we have $\inf E = b = \sup E$.

This is a contradiction, and so if $\inf E < \sup E$, then E must contain at least two elements.

Now suppose that E has at least two elements.

Choose two elements $a, b \in E$. Since $\mathbb{R}$ is an ordered field and $E \subset \mathbb{R}$, we have that $a \leq b$ or $b \leq a$. Assume $a \leq b$ without loss of generality.

Since every element of a set is distinct, $a \neq b$ and we have $a < b$.

By definition of $\inf E$, we have that $\inf E \leq E \implies \inf E \leq a$. Similarly, we have $E \leq \sup E \implies b \leq \sup E$.

These two statements yield,

$$\inf E \leq a < b \leq \sup E$$

Thus, if E has at least two elements, $\inf E < \sup E$.

Problem 3.

Suppose that a sequence $\langle x_n \rangle$ has two distinct limits, l_1 and l_2.

Then $\forall \epsilon > 0, \exists N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \implies |x_n - l_1| < \frac{\epsilon}{2}$ and $n \geq N_2 \implies |x_n - l_2| < \frac{\epsilon}{2}$.

Now, let $N = max\{N_1, N_2\}$ and fix $n \geq N$. Then, by the triangle inequality, we have

$$|l_1 - l_2| \leq |x_n - l_1| + |x_n - l_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since ϵ can be made arbitrarily close to 0, the distance between l_1 and l_2 is arbitrarily close to 0 as well and hence, $l_1 = l_2$.

This is a contradiction to our supposition that l_1 and l_2 are distinct. As a result, a sequence can have at most one limit.

Problem 4.

Suppose that a sequence $\langle x_n \rangle$ is Cauchy.

That is, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that whenever $n, m \geq N, |x_n - x_m| < \epsilon$.

Now take $\epsilon = 1$. Then we have (taking $n \geq N$),

$$x_n = |(x_n - x_N) + x_N| \leq |x_n - x_N| + x_N \leq \epsilon + x_N = 1 + x_N$$

for all $n \geq N$.

Let $M = 1 + max\{|x_1|, |x_2|, ..., |x_N|\}$. Then it is clear that $x_k \in [-M, M]\ \forall k$. Thus, $\langle x_n \rangle$ is a bounded sequence.

Define a "peak" as x_n such that $x_n \geq x_m\ \forall m \geq n$. If there are finitely many peaks in $\langle x_n \rangle$, then it can be shown $\langle x_n \rangle$ has a monotone increasing subsequence. If there are infinitely many peaks, $\langle x_n \rangle$ has a monotone decreasing subsequence. In both cases, $\exists$ a monotone subsequence $\langle x_{n_k} \rangle$. Since $\langle x_n \rangle$ constitutes a bounded set of real numbers, we know that it has an inf and sup. Thus, the monotone sequence $\langle x_{n_k} \rangle$ must converge to either the inf or sup of the set depending upon whether it is decreasing or increasing, respectively. This sketches a proof for the Bolzano-Weierstrass theorem, which states that every bounded sequence contains a convergent subsequence. Since we know that our sequence $\langle x_{n_k} \rangle$ is bounded, we can apply this theorem.

Define a subsequence $\langle x_{n_k} \rangle$ such that the sequence converges to some limit l. Fix $\epsilon > 0$ and define N_1 such that whenever $n_k \geq N_1, |x_{n_k} - l| < \frac{\epsilon}{2}$. Now, define an N_2 for the parent sequence $\langle x_n \rangle$ such that whenever $n, m \geq N_2, |x_n - x_m| < \frac{\epsilon}{2}$.

Now take $N = max\{N_1, N_2\}$. It is clear then that we have the following,

$$|x_n - l| = |x_n - x_{n_k} + x_{n_k} - l| \leq |x_n - x_{n_k}| + |x_{n_k} - l| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

whenever $n, n_k \geq N$.

This satisfies the definition of a limit and so $\langle x_n \rangle \to l$.