Real Analysis I: Assignment 12

Chris Hayduk

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Problem 1.

- $D^+f(0)$:

$$\begin{aligned}
D^+f(0) &= \overline{\lim}_{h\to 0^+}\frac{f(0+h)-f(0)}{h} \\
&= \overline{\lim}_{h\to 0^+}\frac{f(h)-f(0)}{h} \\
&= \overline{\lim}_{h\to 0^+}\frac{f(h)-0}{h} \\
&= \overline{\lim}_{h\to 0^+}\frac{h\sin(\frac{1}{h})}{h} \\
&= \overline{\lim}_{h\to 0^+}\sin\left(\frac{1}{h}\right)
\end{aligned}$$

  Since $\sin\left(\frac{1}{x}\right)$ oscillates between -1 and 1 infinitely often as we approach 0, we have that,

$$D^+f(0) = \overline{\lim}_{h\to 0^+}\sin\left(\frac{1}{h}\right) = 1$$

- $D^-f(0)$:

$$\begin{aligned}
D^-f(0) &= \overline{\lim}_{h\to 0^+}\frac{f(0)-f(0-h)}{h} \\
&= \overline{\lim}_{h\to 0^+}\frac{f(0)-f(-h)}{h} \\
&= \overline{\lim}_{h\to 0^+}\frac{0-f(-h)}{h} \\
&= \overline{\lim}_{h\to 0^+}\frac{-(-h)\sin(\frac{1}{-h})}{h} \\
&= \overline{\lim}_{h\to 0^+}\sin\left(\frac{1}{-h}\right)
\end{aligned}$$

Since we know $\sin\left(\frac{1}{x}\right)$ oscillates between -1 and 1 infinitely often as we approach 0, we have that $\sin\left(\frac{1}{-x}\right)$ oscillates between -1 and 1 infinitely as well. Hence,

$$D^- f(0) = \overline{\lim}_{h\to 0^+} \sin\left(\frac{1}{-h}\right) = 1$$

By the same reasoning from the two above,

- $D_+ f(0)$:

$$\begin{aligned} D_+ f(0) &= \underline{\lim}_{h\to 0^+} \frac{f(0+h) - f(0)}{h} \\ &= \underline{\lim}_{h\to 0^+} \frac{f(h) - f(0)}{h} \\ &= \underline{\lim}_{h\to 0^+} \frac{f(h) - 0}{h} \\ &= \underline{\lim}_{h\to 0^+} \frac{h\sin(\frac{1}{h})}{h} \\ &= \underline{\lim}_{h\to 0^+} \sin\left(\frac{1}{h}\right) \\ &= -1 \end{aligned}$$

- $D_- f(0)$:

$$\begin{aligned} D_- f(0) &= \underline{\lim}_{h\to 0^+} \frac{f(0) - f(0-h)}{h} \\ &= \underline{\lim}_{h\to 0^+} \frac{f(0) - f(-h)}{h} \\ &= \underline{\lim}_{h\to 0^+} \frac{0 - f(-h)}{h} \\ &= \underline{\lim}_{h\to 0^+} \frac{-(-h)\sin(\frac{1}{-h})}{h} \\ &= \underline{\lim}_{h\to 0^+} \sin\left(\frac{1}{-h}\right) \\ &= -1 \end{aligned}$$

Problem 2.

Since we know that f assumes a local minimum at c, we know that $f(c) \leq f(c+h)$ for $h \in \mathbb{R}$ sufficiently small. As a result, we have that $f(c) - f(c-h) \leq 0$ and $f(c+h) - f(c) \geq 0$.

This result implies that,

$$\frac{f(c+h) - f(c)}{h} \geq 0$$

and

$$\frac{f(c) - f(c-h)}{h} \leq 0$$

for $h > 0$ and h sufficiently close to 0. Thus, if we take limits with h approaching 0 from above, these inequalities are preserved. Hence,

$$\begin{aligned}
D_{-}f(c) &= \underline{\lim}_{h \to 0^+} \frac{f(c) - f(c-h)}{h} \leq 0 \\
D^{-}f(c) &= \overline{\lim}_{h \to 0^+} \frac{f(c) - f(c-h)}{h} \leq 0 \\
D_{+}f(c) &= \underline{\lim}_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \geq 0 \\
D^{+}f(c) &= \overline{\lim}_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \geq 0
\end{aligned}$$

Now, by the definitions of lim inf and lim sup, we also have that

$$D_{-}f(c) \leq D^{-}f(c)$$

and

$$D_{+}f(c) \leq D^{+}f(c)$$

Putting it all together, we have

$$D_{-}f(c) \leq D^{-}f(c) \leq 0 \leq D_{+}f(c) \leq D^{+}f(c)$$

when f attains a local minimum at c.

If f attains a local minimum at a, we know from the above that $0 \leq D_{+}f(a) \leq D^{+}f(a)$ since $\frac{f(a+h)-f(a)}{h} \geq 0$ for small, positive values of h. However, we cannot make a statement about $D^{-}f(a)$ or $D_{-}f(a)$ because we do not know how the function behaves for values of $x < a$.

Similarly, if f attains a local minimum at b, we know from the above that $D_{-}f(b) \leq D^{-}f(b) \leq 0$ since $\frac{f(b)-f(b-h)}{h} \leq 0$ for small, positive values of h. However, we cannot make a statement about $D^{+}f(b)$ or $D_{+}f(b)$ because we do not know how the function behaves for values of $x > b$.

Problem 3.

Let g be a function on $[a, b]$ with $D^+g(x) \geq \epsilon$ for some $\epsilon > 0$ and for all values of $x \in (a, b)$. Then we have,

$$D^+g(x) = \overline{\lim}_{h \to 0^+} \frac{g(x+h) - g(x)}{h} \geq \epsilon$$

Suppose that g is not non-decreasing. That is, there exist $x, y \in [a, b]$ with $x < y$ such that $g(x) > g(y)$.

By similar reasoning to question 2, we know that $D^+f(c) \leq 0$ when some function f attains a maximum at some point c. Since $D^+g(x) \geq \epsilon > 0$ for all $x \in (a, b)$, g does not attain a local maximum on (a, b). Hence, g is decreasing on $(a, y]$ and $D^+g(c) \leq 0$ for all $c \in (a, y)$, a contradiction. Thus, g must be non-decreasing on $[a, b]$.

Now let $f(x)$ be a continuous function on $[a, b]$ and assume $D^+f \geq 0$ on (a, b).

For any $\epsilon > 0$, $D^+(f(x) + \epsilon x) \geq \epsilon$ on (a, b). Thus, $f(x) + \epsilon x$ is non-decreasing on $[a, b]$.

Now let $x < y$. f non-decreasing implies that $f(x) + \epsilon x \leq f(y) + \epsilon y$.

Suppose for contradiction that $f(x) > f(y)$. Then, by the above inequality, we have

$$0 < f(x) - f(y) \leq \epsilon(x - y)$$

Now choose $\epsilon = (f(x) - f(y))/(2(y - x))$. This yields,

$$f(x) - f(y) \leq (f(x) - f(y))/2$$

This is a contradiction. Thus, when $x < y$, we have that $f(x) \leq f(y)$ and so f is non-decreasing on $[a, b]$.

Problem 4.

For any x, we have that,

$$\begin{aligned}
D^+(f+g)(x) &= \overline{\lim}_{h \to 0^+} \frac{(f+g)(x+h) - (f+g)(x)}{h} \\
&= \overline{\lim}_{h \to 0^+} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} \\
&= \overline{\lim}_{h \to 0^+} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h} \\
&= \overline{\lim}_{h \to 0^+} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right] \\
&\leq \overline{\lim}_{h \to 0^+} \frac{f(x+h) - f(x)}{h} + \overline{\lim}_{h \to 0^+} \frac{g(x+h) - g(x)}{h} \\
&= D^+f + D^+g
\end{aligned}$$