Real Analysis I: Assignment 11

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Problem 1.

Suppose f is measurable, non-negative, and $\int f = 0.$. Let $E = \{x : f(x) > 0\}$. Thus, if we define $E_n = \{x : f(x) \geq \frac{1}{n}\}$, we have that $\cup E_n = E$.

We know from Proposition 5 that $\int_{E_n} f = 0 \geq \left(\frac{1}{n}\right) mE_n$ for every n.

Since measure is non-negative and $\frac{1}{n}$ is always positive, $0 \geq \left(\frac{1}{n}\right) mE_n \implies mE_n = 0$ for every n.

As a result, we have that $mE = 0$. That is, the set of all x where $f(x) = 0$ has measure 0. Hence, $f = 0$ almost everywhere.

Problem 2.

Problem 3.

Problem 4.

Let $f_n = \chi_{[n,n+1]}$. It is clear that for each n, f_n equals 1 on $[n, n+1]$ and equals 0 everywhere else.

Since for any given x, $f_n(x) = 1$ for at most two values of n (if $x = n$ for some endpoint n), we have that

$$\lim_{n\to\infty} f_n(x) = 0 \ \forall x$$

Thus, we have that

$$\int \lim f_n = \int 0 = 0$$

Now fix $n \in \mathbb{N}$. Observe that

$$\int f_n = 1 \times m([n, n+1]) = 1(n + 1 - n) = 1$$

Since this holds for every $n \in \mathbb{N}$, we have that $\underline{\lim} \int f_n = 1$. Thus, we can see that,

$$\int \lim f_n = 0 < \underline{\lim} \int f_n = 1$$