Real Analysis I: Assignment 12

Chris Hayduk

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Problem 1.

- $D^+f(0)$:

$$\begin{aligned} D^+f(0) &= \overline{\lim}_{h\to 0^+} \frac{f(0+h)-f(0)}{h} \\ &= \overline{\lim}_{h\to 0^+} \frac{f(h)-f(0)}{h} \\ &= \overline{\lim}_{h\to 0^+} \frac{f(h)-0}{h} \\ &= \overline{\lim}_{h\to 0^+} \frac{h\sin(\frac{1}{h})}{h} \\ &= \overline{\lim}_{h\to 0^+} \sin\left(\frac{1}{h}\right) \end{aligned}$$

  Since $\sin\left(\frac{1}{x}\right)$ oscillates between -1 and 1 infinitely often as we approach 0, we have that,

$$D^+f(0) = \overline{\lim}_{h\to 0^+} \sin\left(\frac{1}{h}\right) = 1$$

- $D^-f(0)$:

$$\begin{aligned} D^-f(0) &= \overline{\lim}_{h\to 0^+} \frac{f(0)-f(0-h)}{h} \\ &= \overline{\lim}_{h\to 0^+} \frac{f(0)-f(-h)}{h} \\ &= \overline{\lim}_{h\to 0^+} \frac{0-f(-h)}{h} \\ &= \overline{\lim}_{h\to 0^+} \frac{-(-h)\sin(\frac{1}{-h})}{h} \\ &= \overline{\lim}_{h\to 0^+} \sin\left(\frac{1}{-h}\right) \end{aligned}$$

Since we know $\sin\left(\frac{1}{x}\right)$ oscillates between -1 and 1 infinitely often as we approach 0, we have that $\sin\left(\frac{1}{-x}\right)$ oscillates between -1 and 1 infinitely as well. Hence,

$$D^- f(0) = \overline{\lim}_{h\to 0^+} \sin\left(\frac{1}{-h}\right) = 1$$

By the same reasoning from the two above,

- $D_+ f(0)$:

$$\begin{aligned}
D_+ f(0) &= \underline{\lim}_{h\to 0^+} \frac{f(0+h) - f(0)}{h} \\
&= \underline{\lim}_{h\to 0^+} \frac{f(h) - f(0)}{h} \\
&= \underline{\lim}_{h\to 0^+} \frac{f(h) - 0}{h} \\
&= \underline{\lim}_{h\to 0^+} \frac{h \sin(\frac{1}{h})}{h} \\
&= \underline{\lim}_{h\to 0^+} \sin\left(\frac{1}{h}\right) \\
&= -1
\end{aligned}$$

- $D_- f(0)$:

$$\begin{aligned}
D_- f(0) &= \underline{\lim}_{h\to 0^+} \frac{f(0) - f(0-h)}{h} \\
&= \underline{\lim}_{h\to 0^+} \frac{f(0) - f(-h)}{h} \\
&= \underline{\lim}_{h\to 0^+} \frac{0 - f(-h)}{h} \\
&= \underline{\lim}_{h\to 0^+} \frac{-(-h) \sin(\frac{1}{-h})}{h} \\
&= \underline{\lim}_{h\to 0^+} \sin\left(\frac{1}{-h}\right) \\
&= -1
\end{aligned}$$

Problem 2.

Since we know that f assumes a local minimum at c, we know that $f(c) \leq f(c+h)$ for $h \in \mathbb{R}$ sufficiently small. As a result, we have that $f(c)-f(c-h) \leq 0$ and $f(c+h)-f(c) \geq 0$.

This result implies that,

$$\frac{f(c+h)-f(c)}{h} \geq 0$$

and

$$\frac{f(c)-f(c-h)}{h} \leq 0$$

for $h > 0$ and h sufficiently close to 0. Thus, if we take limits with h approaching 0 from above, these inequalities are preserved. Hence,

$$\begin{aligned} D_{-}f(c) &= \underline{\lim}_{h\to 0^+}\frac{f(c)-f(c-h)}{h} \leq 0 \\ D^{-}f(c) &= \overline{\lim}_{h\to 0^+}\frac{f(c)-f(c-h)}{h} \leq 0 \\ D_{+}f(c) &= \underline{\lim}_{h\to 0^+}\frac{f(c+h)-f(c)}{h} \geq 0 \\ D^{+}f(c) &= \overline{\lim}_{h\to 0^+}\frac{f(c+h)-f(c)}{h} \geq 0 \end{aligned}$$

Now, by the definitions of $\liminf$ and $\limsup$, we also have that

$$D_{-}f(c) \leq D^{-}f(c)$$

and

$$D_{+}f(c) \leq D^{+}f(c)$$

Putting it all together, we have

$$D_{-}f(c) \leq D^{-}f(c) \leq 0 \leq D_{+}f(c) \leq D^{+}f(c)$$

when f attains a local minimum at c.

Problem 3.

Let g be a function on $[a,b]$ with $D^+g(x) \geq \epsilon$ for some $\epsilon > 0$ and for all values of $x \in (a,b)$. Then we have,

$$D^+g(x) = \overline{\lim}_{h\to 0^+} \frac{g(x+h)-g(x)}{h} \geq \epsilon$$

Suppose that g is not non-decreasing. That is, there exist $x, y \in [a,b]$ with $x < y$ such that $g(x) > g(y)$.

By similar reasoning to question 2, we know that $D^+f(c) \leq 0$ when some function f attains a maximum at some point c. Since $D^+g(x) \geq \epsilon > 0$ for all $x \in (a,b)$, g does not attain a local maximum on (a,b). Hence, g is decreasing on $(a,y]$ and $D^+g(c) \leq 0$ for all $c \in (a,y)$, a contradiction. Thus, g must be non-decreasing on $[a,b]$.

Now let $f(x)$ be a continuous function on $[a,b]$ and assume $D^+f \geq 0$ on (a,b).

For any $\epsilon > 0$, $D^+(f(x)+\epsilon x) \geq \epsilon$ on (a,b). Thus, $f(x)+\epsilon x$ is non-decreasing on $[a,b]$.

Now let $x < y$. f non-decreasing implies that $f(x)+\epsilon x \leq f(y)+\epsilon y$.

Suppose for contradiction that $f(x) > f(y)$. Then, by the above inequality, we have

$$0 < f(x) - f(y) \leq \epsilon(x-y)$$

Now choose $\epsilon = (f(x)-f(y))/(2(y-x))$. This yields,

$$f(x) - f(y) \leq (f(x)-f(y))/2$$

This is a contradiction. Thus, when $x < y$, we have that $f(x) \leq f(y)$ and so f is non-decreasing on $[a,b]$.

Problem 4.

For any x, we have that,

$$\begin{aligned}
D^+(f+g)(x) &= \overline{\lim}_{h\to 0^+} \frac{(f+g)(x+h)-(f+g)(x)}{h} \\
&= \overline{\lim}_{h\to 0^+} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} \\
&= \overline{\lim}_{h\to 0^+} \frac{f(x+h)-f(x)+g(x+h)-g(x)}{h} \\
&= \overline{\lim}_{h\to 0^+} \left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right] \\
&\leq \overline{\lim}_{h\to 0^+} \frac{f(x+h)-f(x)}{h} + \overline{\lim}_{h\to 0^+} \frac{g(x+h)-g(x)}{h} \\
&= D^+f + D^+g
\end{aligned}$$