Real Analysis I: Assignment 4

Chris Hayduk

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Problem 1.

Let $< F_n >$ be a sequence of non-empty, closed sets of real numbers with the property that $F_{n+1} \subset F_n$ for all $n \in \mathbb{N}$. Furthermore, assume there exists an m such that F_m is bounded.

Since F_m is bounded we know that $\exists c \in \mathbb{R}$ such that $\forall x \in F_m, |x| \leq c$. Now examine F_{m+1}. We know that $F_{m+1} \neq \emptyset$ and $F_{m+1} \subset F_m$. That is, $x \in F_{m+1} \implies x \in F_m$. As a result, every $x \in F_{m+1}$ is bounded by the same c.

This same line of reasoning applies for any $n \geq m$ since we have that $F_m \supset F_{m+1} \supset \cdots$ and $F_n \neq \emptyset \; \forall n$. Thus, we now have a subsequence of the original sequence which consists solely of closed, bounded, nonempty subsets.

We know that $\forall k \geq m, F_k$ is a non-empty, closed, and bounded set. As a result each F_k must contain its infimum, which we can denote by x_k.

We can see that $x_k \leq c$ and $x_k \leq x_{k+1}$. Thus, $< x_k >$ is a monotonically increasing, bounded sequence. By the monotone convergence theorem, this sequence converges to a limit point x.

Since the sets are nested, for any k, $x_j \in F_k \; \forall j \geq k$. Furthermore, F_k is closed, so we have $x \in F_k$. This applies for any $k \geq m$, so $x \in \bigcap_{i=m}^{\infty} F_i$.

Now that we have considered the subsequence of non-empty, closed, and bounded sets let us incorporate the finite collection of unbounded sets $F_1, \cdots, F_{m-1}$.

We know that $F_m \subset F_{m-1} \subset \cdots F_1$. Thus, $F_m \subset F_n \; \forall n < m$.

As a result, we clearly have $\bigcap_{i=0}^{m-1} F_i = F_m$.

Now if we combine the two intersections, we have

$$\begin{aligned}
&\bigcap_{i=0}^{\infty} F_i = \bigcap_{j=0}^{m-1} F_j \ \cap \ \bigcap_{k=m}^{\infty} F_k \\
\implies &(F_1 \cap F_2 \cap \cdots \cap F_{m-1}) \cap (F_m \cap F_{m+1} \cap \cdots) \\
\implies &F_m \cap (F_m \cap F_{m+1} \cap \cdots) \\
\implies &(F_m \cap F_m) \cap F_{m+1} \cap \cdots \\
\implies &F_m \cap F_{m+1} \cap \cdots = \bigcap_{k=m}^{\infty} F_k
\end{aligned}$$

Since we know that $x \in \bigcap_{i=m}^{\infty} F_i$, the derivation above implies that $x \in \bigcap_{i=0}^{\infty} F_i \neq \emptyset$ as well. Thus, the intersection of a nested sequence of non-empty, closed sets of real numbers is non-empty if one of the sets is bounded.

Now let $F_n = \{x \in \mathbb{R} : x \geq n\} = [n, \infty)$. Clearly each $F_n \subset \mathbb{R}$ is unbounded and closed.

Furthermore, we have that $F_{n+1} = \{x \in \mathbb{R} : x \geq n+1\} = [n+1, \infty)$. Then we can see that $F_{n+1} \subset F_n$ since $[n+1, \infty) \subset [n, \infty)$.

Now assume $\cap F_n \neq \emptyset$ and let $x \in \cap F_n$. Then $x \in F_n \ \forall n$. Since $F_n \subset \mathbb{R}$, we have that $x \in \mathbb{R}$. Thus, by the Axiom of Archimedes, $\exists k \in \mathbb{N}$ such that $k > x$.

Hence, by the definition of F_n, we have that $x \notin F_k$. However, this contradicts the assumption that $x \in \cap F_n$ and, by extension, that $\cap F_n \neq \emptyset$.

As a result, if there is not at least one bounded F_n, we have that $\cap F_n = \emptyset$.

Problem 2.

Let $A = \{x \in [a, b] : f(x) \leq \gamma\}$. We have that $a \in A$ and A is bounded above by b. As a result, the supremum of A exists. Let $c = \sup A$.

We claim that $f(c) = \gamma$.

We know that f is continuous, so choose some $\epsilon > 0$. Then $\exists \delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$.

This yields,

$$f(x) - \epsilon < f(c) < f(x) + \epsilon$$

Since c is the supremum of the set, there exists $c' \in (c - \delta, c]$ such that

$$\begin{aligned}
&f(c) + \epsilon < f(c') \\
\implies &f(c) < f(c') + \epsilon \leq \gamma + \epsilon
\end{aligned}$$

In addition, there exists $c'' \in (c, c+\delta)$ such that

$$\begin{aligned} & f(c'') < f(c) + \epsilon \\ \implies & f(c) > f(c'') - \epsilon \geq \gamma - \epsilon \end{aligned}$$

Combining these two inequalities yields,

$$\gamma - \epsilon \leq f(c) \leq \gamma + \epsilon$$

Since $\epsilon > 0$ is arbitrary, $f(c) = \gamma$.

Problem 3.

Let $g(x) = f(x)\ \forall x \in F$.

Problem 4.