Real Analysis I: Assignment 1

Chris Hayduk

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Problem 1.

Suppose $f : X \to Y$ is onto.

Let $B \subset Y$ be nonempty. Thus, $\exists b \in B$.

Suppose $f^{-1}(B) = \emptyset$. This implies that there is no $x \in X$ such that $f(x) = b$.

However, by the definition of onto, $\forall y \in Y$, $\exists x \in X$ such that $f(x) = y$. Since $b \in B \subset Y$ and f is assumed to be onto, this definition applies.

Thus, we have a contradiction and $f^{-1}(B) \neq \emptyset$.

Problem 2.

Let $\mathscr{A}$ be a collection of sets and assume properties (ii) and (iii) from the question.

Let $A, B \in \mathscr{A}$. By property (ii), $\tilde{A}, \tilde{B} \in \mathscr{A}$ as well.

Thus, by property (iii),

$$\tilde{A} \cap \tilde{B} \in \mathscr{A}$$

Then, by (ii) and DeMorgan's Laws, we have,

$$\begin{aligned}
&\widetilde{\tilde{A} \cap \tilde{B}} \in \mathscr{A} \\
\implies &\tilde{\tilde{A}} \cup \tilde{\tilde{B}} \in \mathscr{A} \\
\implies &A \cup B \in \mathscr{A}
\end{aligned}$$

Thus, by properties (ii) and (iii), whenever $A, B \in \mathscr{A}$, $A \cup B \in \mathscr{A}$ as well.

Problem 3.

Let $\mathscr{F}$ be the family of all σ-algebras that contain the collection $\mathscr{C}$ of subsets of X.

We know that $P(X)$ is an algebra containing $\mathscr{C}$. Let there be a sequence of sets (X_i) in $P(X)$. Since $P(X)$ contains only possible subsets of X, then $x \in X_k \implies x \in X$ for every $X_k \in (X_i)$. Hence, every element of each set in the sequence (X_i) is contained in X.

Thus, if we take the union of all sets in the sequence (X_i), every element of $\bigcup_{i=1}^{\infty} X_i$ is an element of X. In other words,

$$\bigcup_{i=1}^{\infty} X_i \subset X$$

Since $P(X)$ contains all possible subsets of X,

$$\bigcup_{i=1}^{\infty} X_i \subset P(X)$$

Since this holds for an arbitrary sequence (X_i), we know that $P(X)$ is a σ-algebra. Thus, $P(X) \in \mathscr{F}$ and $\mathscr{F}$ is nonempty.

Let $\mathscr{A} = \cap\{\mathscr{B} : \mathscr{B} \in \mathscr{F}\}$

Since each $\mathscr{B} \in \mathscr{F}$ contains $\mathscr{C}$, then $\mathscr{C}$ is a subcollection of $\mathscr{A}$.

If $A, B \in \mathscr{A}$, then $\forall \mathscr{B} \in \mathscr{F}, A, B \in \mathscr{B}$.

Since each $\mathscr{B}$ is a σ-algebra, then $A \cap B, A \cup B \in \mathscr{B}\ \forall \mathscr{B} \in \mathscr{F}$. So we see that $A \cap B$ and $A \cup B \in \mathscr{A}$.

Similarly, given an $A \in \mathscr{A}$, $A \in \mathscr{B}\ \forall \mathscr{B} \in \mathscr{F}$.

Since each $\mathscr{B}$ is a σ-algebra, then $X \setminus A \in \mathscr{B}$ and because this holds $\forall \mathscr{B} \in \mathscr{F}$, then $X \setminus A \in \mathscr{A}$ as well.

Therefore, $\mathscr{A}$ is an algebra.

Now let there be a sequence of sets $(A_i) \in \mathscr{A}$. We have that $\forall \mathscr{B} \in \mathscr{F}, (A_i) \in \mathscr{B}$.

Since every $\mathscr{B} \in \mathscr{F}$ is a σ-algebra, $\bigcup_{i=1}^{\infty} A_i \subset \mathscr{B}\ \forall \mathscr{B} \in \mathscr{F}$. Thus, we see that $\bigcup_{i=1}^{\infty} A_i \subset \mathscr{A}$.

Hence, $\mathscr{A}$ is a σ-algebra.

Now, let $\mathscr{B}$ be a σ-algebra containing $\mathscr{C}$. Then $\mathscr{B} \in \mathscr{F}$.

Since $\mathscr{A} \subset \bigcap_{\mathscr{B} \in \mathscr{F}} \mathscr{B}$, then $\mathscr{A} \subset \mathscr{B}$, so $\mathscr{A}$ is the smallest such σ-algebra.