Real Analysis I: Assignment 7

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Problem 1.

Let $C = [0,1] \sim [\cup O_n]$, where O_n is defined as in the problem. Then $C = [0,1] \cap \widetilde{[\cup O_n]}$. Since each O_n is an open interval, and we know that a countable union of open sets is open, then $\cup O_n$ is open as well and, hence, $\widetilde{[\cup O_n]}$ is closed. Since $[0,1]$ is closed, we then have that C is closed.

In addition, by Theorem 12, we know that each closed set is measurable, so C is measurable.

Now take $C_k = [0,1] \sim [\bigcup_{n=1}^{k} O_n]$. From the construction of C, we know that C_k is the disjoint union of 2^k closed intervals, each with length $\frac{1}{3^k}$.

Since $C = C_k \sim [\bigcup_{n=k+1}^{\infty} O_n]$, we have that $C \subset C_k$ for every $k \in \mathbb{N}$. Hence, $m(C) \leq m(C_k)$ for every k.

Thus, by the countable additivity of Lebesgue measure and the property that, for any interval $I, m(I) = \ell(I)$, we have

$$\begin{aligned} m(C) \leq m(C_k) &= \sum_{n=1}^{2^k} \frac{1}{3^k} \\ &= 2^k \left(\frac{1}{3^k}\right) = \left(\frac{2}{3}\right)^k \end{aligned}$$

Since $m(C) \leq \left(\frac{2}{3}\right)^k$ for any choice of k and $\left(\frac{2}{3}\right)^k \to 0$ as $k \to \infty$, we have that $m(C) = 0$.

Problem 2.

Fix $\epsilon > 0$ and assume there is a finite union U of open intervals such that $m^*(U \Delta E) < \epsilon/3$. That is, $m^*(U \sim E) + m^*(E \sim U) < \epsilon/3$.

Moreover, there is an open set V such that $E \sim U \subset V$ and $m^*V \leq m^*(E \sim U) + \epsilon/3$.

Hence, we have that $E \subset U \cup V = O$ and,

$$\begin{aligned}
m^*(O \sim E) &= m^*((U \cup V) \sim E) \\
&= m^*((U \sim E) \cup (V \sim E)) \\
&\leq m^*((U \sim E) \cup (E \sim U) \cup (V \sim (E \sim U))) \\
&\leq m^*(U \sim E) + m^*(E \sim U) + m^*(V \sim (E \sim U)) \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon
\end{aligned}$$

Problem 3.

We have, by countable subadditivity, that,

$$\begin{aligned}
m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) &= m^*\left(\bigcup_{i=1}^{\infty} A \cap E_i\right) \\
&\leq \sum_{i=1}^{\infty} m^*(A \cap E_i)
\end{aligned}$$

In addition, we have for every $n \in \mathbb{N}$,

$$\begin{aligned}
\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) &\supset \left(\bigcup_{i=1}^{n} A \cap E_i\right) \\
\implies m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) &\geq m^*\left(\bigcup_{i=1}^{n} A \cap E_i\right) \\
\implies m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) &\geq \sum_{i=1}^{n} m^*(A \cap E_i)
\end{aligned}$$

Since the left side of this inequality does not depend on the choice of n, we have that,

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i)$$

Thus,

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(A \cap E_i)$$

Problem 4.

Suppose E is measurable and $E \subset P$. Define $E_i = E \overset{\circ}{+} r_i$, where r_i is defined as in the proof from our notes. Then $E_i \subset P_i$ for every i.

In addition, since $< P_i >$ is a disjoint sequence of measurable sets and $E_i \subset P_i$, we have that $< E_i >$ is a disjoint sequence of measurable sets as well. Moreover, $\cup E_i \subset \cup P_i \subset [0, 1)$.

From the above statements, we have that,

$$m(\cup E_i) = \sum mE_i \leq m([0,1)) = 1$$

Now, measure is modulo addition invariant, so $mE = mE_i$ for every i. Suppose $mE > 0$. Then,

$$\sum mE = \sum E_i = m(\cup E_i) \to \infty$$

Thus, in order for $m(\cup E_i) < m([0,1)) = 1$ to hold, $mE_i = mE = 0$.