Real Analysis I: Assignment 3

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Problem 1.

Assume $\ell = \inf_n \sup_{k \geq n} x_k$.

Define $y_n = \sup_{k \geq n} x_k = \sup\{x_n, x_{n+1}, ...\}$

It is clear that $\forall n, y_n \geq y_{n+1}$. This holds because x_n is included in the set of points for y_n, but not for y_{n+1}. Hence $y_n \to \inf y_n = \ell$ as $n \to \infty$.

Thus, given $\epsilon > 0$, we can find an n_1 such that $y_{n_1} < \ell + \epsilon$. That is, $x_k < \ell + \epsilon \; \forall k \geq n_1$, satisfying condition (i).

Now let ϵ and n be given. Since $y_n \to \inf y_n = \ell$ as $n \to \infty$, then $y_n > \ell - \epsilon$. So $\exists k \geq n$ such that $x_k > \ell - \epsilon$, satisfying condition (ii).

Now suppose conditions (i) and (ii) hold.

Condition (i) states that $\forall \epsilon > 0, \exists n \in \mathbb{N}$ such that whenever $k \geq n$, $x_k < \ell + \epsilon$. This implies that for every $\epsilon > 0$, there exists an n such that $\sup_{k \geq n} x_k \leq \ell + \epsilon$.

Condition (ii) states that $\forall \epsilon > 0$ and every $n \in \mathbb{N}$, there exists $k \geq n$ for which $x_k > \ell - \epsilon$. Thus, for any ϵ and n, $\sup_{k \geq n} x_k \geq \ell - \epsilon$.

Since $\ell - \epsilon \leq \sup_{k \geq n} x_k \leq \ell + \epsilon$ given an n of sufficient size, we see that $\ell = \inf_n \sup_{k \geq n} x_k$.

Problem 2.

i) Let $\ell_1 = \limsup x_n$ and $\ell_2 = \liminf x_n$. Suppose $\ell_2 > \ell_1$.

   By the definition of $\limsup$, $\forall \epsilon > 0 \; \exists n_1 \in N$ such that whenever $k \geq n_1$, $x_k < \ell_1 + \epsilon$.

   Similarly for $\liminf$, we have $\exists n_2 \in N$ such that whenever $k \geq n_2$, $x_k > \ell_2 - \epsilon$.

Let $0 < \epsilon < \frac{\ell_2 - \ell_1}{2}$. Thus, we have

$$\ell_2 - \epsilon > \ell_1 + \epsilon$$

Now, choose $n_1 \in N$ such that whenever $k \geq n_1$, $x_k < \ell_1 + \epsilon$ and $n_2 \in N$ such that whenever $k \geq n_2$, $x_k > \ell_2 - \epsilon$. Let $n = \max\{n_1, n_2\}$.

This yields, $\forall k \geq n$, $\ell_2 - \epsilon < x_k < \ell_1 + \epsilon$.

However, we know that $\ell_2 - \epsilon > \ell_1 + \epsilon$, a contradiction. Hence, $\ell_2 \leq \ell_1$.

ii) Suppose $\limsup x_n = \liminf x_n = \ell$.

Then $\forall \epsilon > 0, \exists n_1 \in N$ such that whenever $k \geq n_1$, $x_k < \ell + \epsilon$ and $n_2 \in N$ such that whenever $k \geq n_2$, $x_k > \ell - \epsilon$.

Let $n = \max\{n_1, n_2\}$. Then, $\forall k \geq n$,

$$\begin{aligned} &\ell - \epsilon < x_k < \ell + \epsilon \\ \implies &|x_k - \ell| < \epsilon \end{aligned}$$

This is precisely the definition of a limit, and so $\lim x_n = \ell$.

Now start by supposing $\lim x_n = \ell$.

This implies that $\forall \epsilon > 0, \exists n \in \mathbb{N}$ such that $k \geq n \implies |x_k - \ell| < \epsilon$.

Thus, we have that

$$\begin{aligned} &-\epsilon < x_k - \ell < \epsilon \\ \implies &\ell - \epsilon < x_k < \ell + \epsilon \end{aligned}$$

As a result, we can see that for every $\epsilon > 0, \exists n$ such that whenever $k \geq n, x_k < \ell + \epsilon$, satisfying the definition $\ell = \limsup x_k$. Similarly, for every $\epsilon > 0, \exists n$ such that whenever $k \geq n, \ell - \epsilon < x_k$, satisfying the definition $\ell = \liminf x_k$

Problem 3.

The set of rational numbers $\mathbb{Q}$ is neither open nor closed.

Any open interval of any size centered around an $x \in \mathbb{Q}$ will contain a $y \in \mathbb{I}$. Since $\mathbb{Q} \cap \mathbb{I} = \emptyset$, there is no $\delta > 0$ such that $(x - \delta, x + \delta) \subset \mathbb{Q}$. Thus, $\mathbb{Q}$ is not open.

Now take $\langle x_n \rangle = (1+\frac{1}{n})^n$. This sequence is known to converge to e, which is an irrational number. However, since every term in $\langle x_n \rangle$ consists of the multiplication of rational numbers and $\mathbb{Q}$ is closed under multiplication, we have a sequence of rational numbers that converges to an irrational number.

As a result, e is a point of closure of $\mathbb{Q}$ but is not contained in $\mathbb{Q}$. Thus, $\mathbb{Q} \neq \overline{\mathbb{Q}}$ and $\mathbb{Q}$ is not closed.

Problem 4.

Let $A = (1, 2)$ and $B = (2, 3)$. It is clear from these two intervals that we have $A \cap B = \emptyset$.

We can see that A and B share a point of closure (namely 2). However, both of these sets do not contain this point of closure. Since they do not contain that point and are disjoint intervals, their intersection is the empty set.

The closure of a set X is the collection of points of closure of X. We know that $x \in X$ is a point of closure of X if $\forall \delta > 0, (x - \delta, x + \delta) \cap X \neq \emptyset$.

It is quite clear from this definition that all points contained in A and B are points of closure for each set. Moreover, we can see that 1 and 2 are points of closure for A, while 2 and 3 are points of closure for B.

To show that 2 is a point of closure for A, fix $\delta > 0$. Then we need to check that $(2-\delta, 2+\delta) \cap A \neq \emptyset$. We can see that $A = \{a : 1 < a < 2\}$. It is clear that $2-\delta < 2 < 2+\delta$.

If we assume that $\delta \geq 1$, then $A \cap (2-\delta, 2+\delta) = A$. If $0 < \delta < 1$, $A \cap (2-\delta, 2+\delta) = (2-\delta, 2)$ Thus, this interval will intersect A for every $\delta > 0$, satisfying the definition for a point of closure.

A similar argument holds for 2 being a point of closure of B.

Since 2 is a point of closure for both A and B, we can see that $2 \in \overline{A}, \overline{B}$.

Thus, we have that $\overline{A} \cap \overline{B} = \{2\}$.

Problem 5.

Let x be an irrational number. Then x can be expressed as a non-terminal, non-repeating decimal.

Now construct a sequence $\langle x_n \rangle$ from this decimal expansion for x, where x_1 contains the first term in the decimal expansion of x, x_2 contains the first two terms, and so on. That is, each x_k is a natural number containing k terms from the decimal expansion of the original

x.

Furthermore, let m be the absolute value of the exponent that 10 must be raised to in order to put the terms in correspondence with the original decimal expansion. For example, for a decimal 0.0003, we would have $0.0003 = 3 \cdot 10^{-4}$, which yields $m = |-4| = 4$.

Then we see can that $\langle x_n \rangle = \frac{x_n}{10^{m+(n-1)}}$.

By construction, $\langle x_n \rangle \to x$. This convergence means that, $\forall \epsilon > 0, \exists N$ such that if $n \geq N$, then $|x - x_n| < \epsilon$.

Now take $\delta > 0$ and construct an open interval around x, $(x - \delta, x + \delta)$. In addition, let $X = \{x_k : x_k \in \langle x_n \rangle\}$.

Let $\epsilon = \delta$. Then $\exists N$ such that if $n \geq N$, then

$$\begin{aligned}
&|x - x_n| < \epsilon = \delta \\
\implies &-\delta < x - x_n < \delta \\
\implies &-x - \delta < -x_n < -x + \delta \\
\implies &x - \delta < x_n < x + \delta
\end{aligned}$$

Thus, when $n \geq N$, $x_n \in (x - \delta, x + \delta)$. As a result, x is point of closure for the set X, which is a set of rational numbers. Hence, x is a point of closure for $\mathbb{Q}$. Since x was defined as an arbitrary irrational number, this holds for any irrational.

Now, let y be an arbitrary rational number and let $\delta > 0$. Construct an open interval around y, $(y - \delta, y + \delta)$.

It is clear that $y - \delta < y < y < y + \delta$. Hence, $y \in (y - \delta, y + \delta)$. Thus, $\mathbb{Q} \cap (y - \delta, y + \delta) \neq \emptyset$.

As a result, y is a point of closure for $\mathbb{Q}$ and, by extension, every rational is a point of closure for $\mathbb{Q}$.

Since both the rationals and irrationals are points of closure for $\mathbb{Q}$, we can see that $\overline{\mathbb{Q}} = \mathbb{Q} \cup \mathbb{I}$. This is precisely the definition of $\mathbb{R}$, and so $\overline{\mathbb{Q}} = \mathbb{R}$.