Real Analysis I: Assignment 8

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Problem 1.

Let $\langle E_i \rangle = \langle P_i \rangle$ where $\langle P_i \rangle$ is the sequence of sets that yielded the non-measurable set P from class. We have that,

$$[0, 1) = \sqcup_{i=0}^{\infty} E_i$$
$$\implies m^*[0, 1) = m^*(\sqcup_{i=0}^{\infty} E_i) = 1$$

By the countable subadditivity of outer measure, we have that,

$$m^*(\sqcup_{i=0}^{\infty} E_i) = 1 \leq \sum_{i=0}^{\infty} m^* E_i$$

We also know that $m^* E_k = m^* E_j$ for every $k, j \in \mathbb{N}$ since outer measure is modulo addition translation invariant (by our in class proof). So $m^* E_i = c$ for some $c \geq 0 \in \mathbb{R}$ and for every $i \in \mathbb{N}$. However, observe that the above inequality does not hold if $c = 0$. Thus, we have that $c > 0$.

Since $c > 0$, we have that $\sum_{i=0}^{\infty} m^* E_i = \sum_{i=0}^{\infty} c = \infty$. Hence, $\langle E_i \rangle$ is a sequence of disjoint sets with,

$$m^*(\sqcup E_i) = 1 < \sum m^*(E_i) = \infty$$

Problem 2.

Define $f(x)$ as,

$$f(x) = \begin{cases} -e^x & x \notin P \\ e^x & x \in P \end{cases}$$

where P is the non-measurable set that we defined in class. By this definition, we can see that for every $x \notin P$, $f(x) < 0$. In addition, for every $x \in P$, $x > 0$.

It is clear that $-e^{x_1} = -e^{x_2} \implies x_1 = x_2$ and $e^{x_1} = e^{x_2} \implies x_1 = x_2$ by taking the logarithm of both sides of the equality. So to check that $f(x)$ assumes each value at most once, we need to check that $-e^{x_1} \neq e^{x_2}$ for any x_1, x_2 in the domain. However, this again is clear because $e^x > 0 \; \forall x$ and $-e^x < 0 \; \forall x$. Thus, it will never be the case that $-e^{x_1} \neq e^{x_2}$.

Since $f(x)$ is one-to-one, the pre-image of $f(x) = \alpha$ for any α is a singleton, which we know is measurable with measure 0. Thus, statement (v) from Proposition 18 is satisfied. However, we have that $\{x : f(x) > 0\} = P$, which we know is a non-measurable set. Thus, statement (i) from Proposition 18 has been violated. We proved in class that (i) and (iv) are equivalent, so (iv) does not hold as well.

Problem 3.

Suppose the restrictions of f to D and E are measurable. We know that f is measurable on $D \cup E$ if $D \cup E$ is measurable and if f it satisfies one of the statements in Proposition 18.

Since the measurable sets are a σ-algebra and both D and E are measurable, $D \cup E$ is also measurable. Thus, we just need to show that f satisfies one of the statements in Proposition 18. We will use statement (i) in this proof.

Hence, in order for f to satisfy statement (i), we need that for each real number α, the set $\{x \in D \cup E : f(x) > \alpha\}$ is measurable.

We see that we can rewrite the above statement as: for each real number α, the set $\{x \in D : f(x) > \alpha\} \cup \{x \in E : f(x) > \alpha\}$ is measurable. Since f is measurable when restricted to either D or E, both of these sets are measurable. Furthermore, since the measurable sets are a σ-algebra, their union is also measurable. Thus, we have

$$\{x \in D : f(x) > \alpha\} \cup \{x \in E : f(x) > \alpha\} = \{x \in D \cup E : f(x) > \alpha\}$$

is measurable for each real number α. As a result, f is measurable on $D \cup E$.

Now suppose that f is measurable on $D \cup E$ and suppose both D and E are measurable. We need to show that it is measurable when restricted to D and E.

Again observe that $\{x \in D \cup E : f(x) > \alpha\} = \{x \in D : f(x) > \alpha\} \cup \{x \in E : f(x) > \alpha\}$. Since the measurable sets are a σ-algebra, let us perform operations on the set $\{x \in D \cup E : f(x) > \alpha\}$ in such a way that yields $\{x \in D : f(x) > \alpha\}$ and $\{x \in E : f(x) > \alpha\}$. If we are able to do this, then we will have that f is measurable when restricted to D and E.

Now for the derivation:

$$\begin{aligned}
\{x \in D \cup E : f(x) > \alpha\} &= (\{x \in D \cup E : f(x) > \alpha\} \cap D) \cup (\{x \in D \cup E : f(x) > \alpha\} \cap E) \\
&= \{x \in D : f(x) > \alpha\} \cup \{x \in E : f(x) > \alpha\}
\end{aligned}$$

Since D and E are both measurable, the intersections of each of these sets with the original set are measurable. Thus, both $\{x \in D : f(x) > \alpha\}$ and $\{x \in E : f(x) > \alpha\}$ are measurable sets, and we have that f is measurable when restricted to either D or E.

Problem 4.

Let ϕ and ψ be simple functions defined on some measurable set E. Then we have,

$$\phi = \sum_{k=1}^{n_1} c_k \cdot \chi_{E_{ck}}$$

where $E_{ck} = \{x : \phi(x) = c_k\}$ and

$$\psi = \sum_{k=1}^{n_2} d_k \cdot \chi_{E_{dk}}$$

where $E_{dk} = \{x : \phi(x) = d_k\}$.

If $n_1 < n_2$, let $c_{n_1+1}, \cdots, c_{n_2} = 0$. If $n_2 < n_1$, let $d_{n_2+1}, \cdots, d_{n_1} = 0$. Also let $n = \max\{n_1, n_2\}$. Then we have that,

$$\phi = \sum_{k=1}^{n_1} c_k \cdot \chi_{E_{ck}} = \sum_{k=1}^{n} c_k \cdot \chi_{E_{ck}}$$

and

$$\psi = \sum_{k=1}^{n_2} d_k \cdot \chi_{E_{dk}} = \sum_{k=1}^{n} d_k \cdot \chi_{E_{dk}}$$

Thus, $\phi + \psi$ yields,

$$\begin{aligned}
\phi + \psi &= \sum_{k=1}^{n_1} c_k \cdot \chi_{E_{ck}} + \sum_{k=1}^{n_2} d_k \cdot \chi_{E_{dk}} \\
&= \sum_{k=1}^{n} c_k \cdot \chi_{E_{ck}} + \sum_{k=1}^{n} d_k \cdot \chi_{E_{dk}} \\
&= \sum_{k=1}^{n} (c_k \cdot \chi_{E_{ck}} + d_k \cdot \chi_{E_{dk}})
\end{aligned}$$

We know that $\phi + \psi$ is measurable because simple functions are measurable and, by Proposition 19, the sum of any two measurable functions is measurable.

Furthermore, since there are finitely many terms c and d, there are only finitely many combinations $c + d$. In particular, there are n values of c and n values of d. We know that $(\phi + \psi)(x)$ is a linear combination of these c and d values for each x. There are 2^n ways to take linear combinations of c and 2^n ways to take linear combinations of d, so $\phi + \psi$ takes on $2^n + 2^n = 2^{n+1}$ possible values. Thus, $\phi + \psi$ takes on finitely many values and, hence, $\phi + \psi$ is a simple function.

Now, $\phi \cdot \psi$ gives,

$$
\begin{aligned}
\phi \cdot \psi &= \sum_{k=1}^{n_1} c_k \cdot \chi_{E_{ck}} \cdot \sum_{k=1}^{n_2} d_k \cdot \chi_{E_{dk}} \\
&= \sum_{k=1}^{n} c_k \cdot \chi_{E_{ck}} \cdot \sum_{k=1}^{n} d_k \cdot \chi_{E_{dk}}
\end{aligned}
$$

Once again, we have that $\phi \cdot \psi$ is measurable because simple functions are measurable and, by Proposition 19, multiplying any two measurable functions yields a measurable function.

And again, we have that there are 2^n possible linear combinations of the c values and 2^n possible linear combinations of the d values. Thus, there are $2^n \cdot 2^n = 2^{2n} = 4^n$ possible values for $\phi \cdot \psi$. As a result, we have that $\phi \cdot \psi$ is measurable and takes on finitely many values, so it is a simple function.