Statistical Theory II: Chapter 7 - Sampling Distributions and the Central Limit Theorem

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Problem 7.10.

a

$$
\begin{aligned}
P(|\overline{Y} - \mu| \leq 0.3) &= P[-0.3 \leq (\overline{Y} - \mu) \leq 0.3] \\
&= P(-\frac{0.3}{\sigma/\sqrt{n}} \leq \frac{\overline{Y} - \mu}{\sigma/\sqrt{n}} \leq \frac{0.3}{\sigma/\sqrt{n}})
\end{aligned}
$$

Since $\frac{\overline{Y}-\mu}{\sigma/\sqrt{n}}$ has a standard normal distribution, we can write:

$$
\begin{aligned}
P(|\overline{Y} - \mu| \leq 0.3) &= P(-\frac{0.3}{2/\sqrt{9}} \leq Z \leq \frac{0.3}{2/\sqrt{9}}) \\
&= P(-0.45 \leq Z \leq 0.45)
\end{aligned}
$$

Using Table 4, Appendix 3 yields:

$$
\begin{aligned}
P(-0.45 \leq Z \leq 0.45) &= 1 - 2P(Z > 0.45) \\
&= 1 - 2(0.3264) = 0.3472
\end{aligned}
$$

Thus, the probability is 0.3472 that the sample mean will be within 0.3 ounces of the true mean.

b Using varying values of n

- $n = 25$

$$
\begin{aligned}
P(|\overline{Y} - \mu| \leq 0.3) &= P(-\frac{0.3}{2/\sqrt{25}} \leq Z \leq \frac{0.3}{2/\sqrt{25}}) \\
&= P(-0.75 \leq Z \leq 0.75)
\end{aligned}
$$

Using Table 4, Appendix 3 yields:

$$\begin{aligned} P(-0.75 \leq Z \leq 0.75) &= 1 - 2P(Z > 0.75) \\ &= 1 - 2(0.2266) = 0.5468 \end{aligned}$$

Thus, the probability is 0.5468 that the sample mean will be within 0.3 ounces of the true mean.

- $n = 36$

$$\begin{aligned} P(|\overline{Y} - \mu| \leq 0.3) &= P(-\frac{0.3}{2/\sqrt{36}} \leq Z \leq \frac{0.3}{2/\sqrt{36}}) \\ &= P(-0.9 \leq Z \leq 0.9) \end{aligned}$$

Using Table 4, Appendix 3 yields:

$$\begin{aligned} P(-0.9 \leq Z \leq 0.9) &= 1 - 2P(Z > 0.9) \\ &= 1 - 2(0.1841) = 0.6318 \end{aligned}$$

Thus, the probability is 0.6318 that the sample mean will be within 0.3 ounces of the true mean.

- $n = 49$

$$\begin{aligned} P(|\overline{Y} - \mu| \leq 0.3) &= P(-\frac{0.3}{2/\sqrt{49}} \leq Z \leq \frac{0.3}{2/\sqrt{49}}) \\ &= P(-1.05 \leq Z \leq 1.05) \end{aligned}$$

Using Table 4, Appendix 3 yields:

$$\begin{aligned} P(-1.05 \leq Z \leq 1.05) &= 1 - 2P(Z > 1.05) \\ &= 1 - 2(0.1469) = 0.7062 \end{aligned}$$

Thus, the probability is 0.7062 that the sample mean will be within 0.3 ounces of the true mean.

- $n = 64$

$$\begin{aligned} P(|\overline{Y} - \mu| \leq 0.3) &= P(-\frac{0.3}{2/\sqrt{64}} \leq Z \leq \frac{0.3}{2/\sqrt{64}}) \\ &= P(-1.2 \leq Z \leq 1.2) \end{aligned}$$

Using Table 4, Appendix 3 yields:

$$\begin{aligned} P(-1.2 \leq Z \leq 1.2) &= 1 - 2P(Z > 1.2) \\ &= 1 - 2(0.1151) = 0.7698 \end{aligned}$$

Thus, the probability is 0.7698 that the sample mean will be within 0.3 ounces of the true mean.

c As n increases, the probability that the sample mean is within 0.3 ounces of the true mean increases.

d When $\sigma = 1$, the probabilities are much higher. This occurs because there is less variability in the data and, as a result, there is less variability in the sample means.

Problem 7.48.

a

$$\begin{aligned} P(|\overline{Y} - \mu| \leq 1) &= P[-1 \leq (\overline{Y} - \mu) \leq 1] \\ &= P(-\frac{1}{\sigma/\sqrt{n}} \leq \frac{\overline{Y} - \mu}{\sigma/\sqrt{n}} \leq \frac{1}{\sigma/\sqrt{n}}) \end{aligned}$$

Since $\frac{\overline{Y}-\mu}{\sigma/\sqrt{n}}$ has a standard normal distribution, we can write:

$$\begin{aligned} P(|\overline{Y} - \mu| \leq 1) &= P(-\frac{1}{12/\sqrt{35}} \leq Z \leq \frac{1}{12/\sqrt{35}}) \\ &\approx P(-0.49 \leq Z \leq 0.49) \end{aligned}$$

Using Table 4, Appendix 3 yields:

$$\begin{aligned} P(-0.49 \leq Z \leq 0.49) &= 1 - 2P(Z > 0.49) \\ &= 1 - 2(0.3121) = 0.3758 \end{aligned}$$

Thus, the probability is about 0.3758 that the sample mean (26%) is within 1% of the mean of the population of estimates of all economists.

b No because each measurement is just an estimate of the percent tax saving.

Problem 7.62.

Let Y_i denote the service time for the ith customer. Then we have:
$P(\sum_{i=1}^{100} Y_i > 240) = P(\overline{Y} > \frac{240}{100}) = P(\overline{Y} > 2.40)$

Since the sample size is large, the central limit theorem tells us that $\overline{Y}$ is approximately normally distributed with mean $\mu_{\overline{Y}} = \mu = 2.5$ and variance $\sigma^2_{\overline{Y}} = \sigma^2/n = 4.0/100$. Therefore, using Table 4, Appendix 3 yields:

$$\begin{aligned} P(\overline{Y} > 2.40) &= P(\frac{\overline{Y} - 2.50}{\frac{2.0}{\sqrt{100}}} > \frac{2.40 - 2.50}{\frac{2.0}{\sqrt{100}}}) \\ &\approx P(Z \leq (2.4 - 2.5)\frac{10}{2}) = P(Z > -\frac{1}{2}) \\ &= 1 - P(Z > 0.5) = 1 - 0.3085 = 0.6915 \end{aligned}$$

Thus, the probability is about 0.6915 that it will take more than 4 hours to process the orders of 100 people.

Problem 7.92.

$$\begin{aligned} P(|\overline{X} - \overline{Y}| > 0.6) &= P(|Z| \leq \frac{0.6}{\sqrt{\frac{6.4^2 + 7.2^2}{64}}}) \\ &\approx P(|Z| \leq 0.5) = 0.6170 \end{aligned}$$

Problem 7.96.

Each Y_i has a beta distribution with $\alpha = 3$ and $\beta = 1$. By Theorem 4.11, $\mu = \frac{\alpha}{\alpha+\beta} = \frac{3}{4}$ and $\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{3}{80}$.
Thus,

$$\begin{aligned} P(\overline{Y} > 0.7) &\approx P(Z > \frac{0.7 - 0.75}{\sqrt{\frac{0.0375}{40}}}) \\ &= P(Z > -1.63) = 1 - P(Z > 1.63) = 1 - 0.0516 = 0.9484 \end{aligned}$$

Thus, the probability that the ore will be rejected is about 0.9484.