Statistical Theory II: Chapter 10 - Hypothesis Testing

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Problem 10.38.

We know $H_0 : \mu \geq 64$. The rejection region for the test is given by:

$$z = \frac{\overline{y} - 64}{\sigma/\sqrt{n}} < -2.326$$

This is equivalent to,

$$\overline{y} < 64 - 2.36 \left(\frac{\sigma}{\sqrt{n}} \right) = 61.36$$

Given $\mu = 60$, we have

$$\begin{aligned} \beta = P(\overline{Y} > 61.36 | \mu = 60) &= P(Z > \frac{61.36 - 60}{8/\sqrt{50}} \\ &= P(Z > 1.2) = 0.1151 \end{aligned}$$

Problem 10.54.

a) We have $H_0 : p = 0.85$ and $H_a : p > 0.85$.

First we compute the test statistic:

$$\begin{aligned} z &= \frac{\hat{p} - p_0}{\sqrt{p_0(1 - p_0)/n}} \\ &= \frac{0.96 - 0.85}{\sqrt{0.85(0.15)/300}} \\ &\approx 5.34 \end{aligned}$$

We also note that for an upper-tail test with $\alpha = 0.01$, we check,

$$\begin{aligned} &z > z_\alpha \\ &z > z_{0.01} \\ &5.34 > 2.33 \end{aligned}$$

Since this inequality is true, we can reject the null hypothesis and conclude that the proportion of right-handed executives at large corporations is greater than 0.85.

b) The p-value for the test is given by (calculated in R using the function `pnorm()`):

$$p = P(Z > 5.34) = 4.647329\text{e}{-08}$$

This indicates that we have very strong evidence against H_0.

Problem 10.70.

a) We have $H_0 = \mu_1 - \mu_2 = 0$ and $H_a = \mu_1 - \mu_2 > 0$ where μ_1 is the mean for juveniles and μ_2 is the mean for nestlings. In this case, $D_0 = 0$.

Now we compute the test statistic,

$$\begin{aligned} T &= \frac{\overline{Y_1} - \overline{Y_2}}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \\ &= \frac{0.041 - 0.026}{0.0120178\sqrt{\frac{1}{10} + \frac{1}{13}}} \\ &\approx 2.967 \end{aligned}$$

We have degrees of freedom $v = 10 + 13 - 2 = 21$.

The rejection region is given for $\alpha = 0.05$ is given by,

$$\begin{aligned} &T > t_\alpha \\ &T > t_{0.05} \\ &2.967 > 1.721 \end{aligned}$$

Since this inequality is satisfied, we can reject H_0 and conclude that juveniles have a larger mean than nestlings.

b) We have $H_0 = \mu_1 - \mu_2 = 0.01$ and $H_a = \mu_1 - \mu_2 > 0.01$ where μ_1 is the mean for juveniles and μ_2 is the mean for nestlings. In this case, $D_0 = 0.01$.

ow we compute the test statistic,

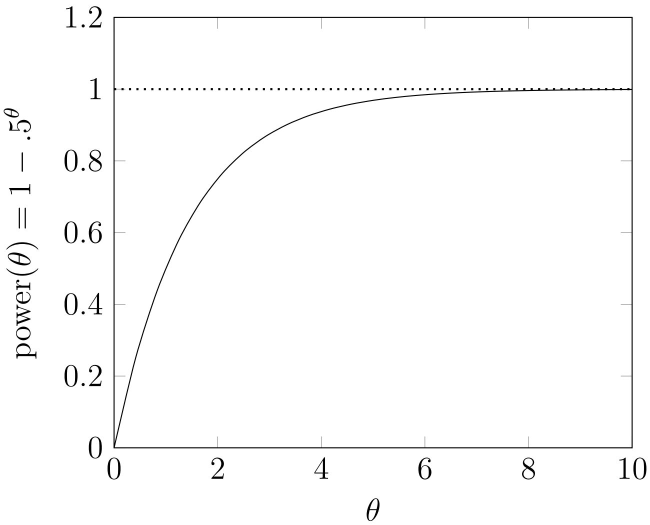
$$\begin{aligned} T &= \frac{\overline{Y_1} - \overline{Y_2} - D_0}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \\ &= \frac{0.041 - 0.026 - 0.01}{0.0120178\sqrt{\frac{1}{10} + \frac{1}{13}}} \\ &\approx 0.989 \end{aligned}$$

We have degrees of freedom $v = 10 + 13 - 2 = 21$.

Using the R function `pt()` and the test statistic and degrees of freedom from above, we yield a p-value of 0.1669612.

Problem 10.96.

a) Graph of power function of the test with rejection region $Y > 0.5$:



b) We have $H_0 : \theta = 1$ and $H_a : \theta = \theta_a$, $\theta_a < 1$.

The likelihood ratio test in this case is,

$$\begin{aligned} &\frac{L(\theta_0)}{L(\theta_a)} < k \\ \implies &\frac{L(1)}{L(\theta_a)} < k \\ \implies &\frac{1}{\theta_a y^{\theta_a - 1}} < k \\ \implies &y > \left(\frac{1}{\theta_a k}\right)^{\frac{1}{\theta_a - 1}} = c \end{aligned}$$

where c is chosen such that the test score is of size α. We can find this by,

$$\begin{aligned} &P(Y \geq c | \theta = 1) = \int_c^1 dy = 1 - c = \alpha \\ \implies &c = 1 - \alpha \end{aligned}$$

The rejection region does not depend on a specific value for θ_a, so this is a uniformly most powerful test.

Problem 10.98.

a) The uniformly most powerful test for testing $H_0 : \theta = \theta_0$ against $H_a : \theta > \theta_0$:

$$\frac{L(\theta_0)}{L(\theta_a)} = \left(\frac{\theta_a}{\theta_0}\right)^n exp\left[-\left(\frac{1}{\theta_0} - \frac{1}{\theta_a}\right)\sum_{i=1}^{n} y_i^m\right] < k$$

$$\Longrightarrow \sum_{i=1}^{n} y_i^m > -\left[\ln k + n\ln\left(\frac{\theta_0}{\theta_a}\right)\right] \text{X} \left[\frac{1}{\theta_0} - \frac{1}{\theta_a}\right]^{-1} = c$$

Thus, the rejection region has the form $T = \sum_{i=1}^{n} Y_i^m > c$ where c is chosen such that the rejection region is of size α.

The distribution of Y^m is exponential, so under H_0, we have

$$\frac{2T}{\theta_0} = \frac{2\sum_{i=1}^{n} Y_i^m}{\theta_0} > \frac{2c}{\theta_0}$$

This is a chi-square distribution with $2n$ degrees of freedom. This does not depend on the specific $\theta_a > \theta_0$, so this is the uniformly most powerful test.

b) We have,

$$\frac{2\sum_i^n Y_i^m}{\theta_a} \sim \chi^2_{2n}$$

which yields,

$$\frac{1}{4}\chi^2_{2n,0.05} = \chi^2_{2n,0.95}$$

We know that $\chi^2_{12,0.05} = 21.02$ and $\chi^2_{12,0.95} = 5.226$ from the table in the appendix. Thus the critical region will be,

$$\sum_{i=1}^{n} Y_i^m > \frac{\theta_0}{2}\chi^2_{12,0.05} = \frac{100}{2}(21.2) = 1051$$