Stochastic Differential Equations: Final Project

Chris Hayduk

December 14, 2019

NOTE: Still need to complete Problems 2, 7, 8

Problem 1.

We know that the probability density of the sum of two independent random variables can be computed by the convolutions of f_X and f_Y. Thus, the probability density of $X+Y$, denoted by f_Z, is

$$f_Z(z) = (f_Y * f_X) = \int_{-\infty}^{\infty} f_Y(z-x)f_X(x)dx \tag{1}$$

We know that the normal density function is given by,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Plugging this density function into equation (1) and letting $\sigma_Z = \sqrt{\sigma_X^2 + \sigma_Y^2}$ yields,

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} \left[\frac{1}{\sigma_Y\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{(z-x)-\mu_Y}{\sigma_Y}\right)^2} \frac{1}{\sigma_X\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2} \right] dx \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{\sigma_Y\sigma_X(2\pi)} e^{-\frac{1}{2}\left[\left(\frac{z-x-\mu_Y}{\sigma_Y}\right)^2 + \left(\frac{x-\mu_X}{\sigma_X}\right)^2\right]} \right] dx \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{\sigma_Y\sigma_X(2\pi)} e^{-\frac{\sigma_X^2(z-x-\mu_Y)^2+\sigma_Y^2(x-\mu_X)^2}{2\sigma_Y^2\sigma_X^2}} \right] dx \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{(2\pi)\sigma_X\sigma_Y} e^{-\frac{x^2(\sigma_X^2+\sigma_Y^2)-2x(\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X)+\sigma_X^2(z^2+\mu_Y^2-2z\mu_Y)+\sigma_Y^2\mu_X^2}{2\sigma_Y^2\sigma_X^2}} \right] dx \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}\sigma_Z} e^{-\frac{\sigma_Z^2\left(\sigma_X^2(z-\mu_Y)^2+\sigma_Y^2\mu_X^2\right)-\left(\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X\right)^2}{2\sigma_Z^2(\sigma_X\sigma_Y)^2}} \frac{1}{\sqrt{2\pi}\frac{\sigma_X\sigma_Y}{\sigma_Z}} e^{-\frac{\left(x-\frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2}\right)^2}{2\left(\frac{\sigma_X\sigma_Y}{\sigma_Z}\right)^2}} \right] dx \\
&= \frac{1}{\sqrt{2\pi}\sigma_Z} e^{-\frac{(z-(\mu_X+\mu_Y))^2}{2\sigma_Z^2}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}\frac{\sigma_X\sigma_Y}{\sigma_Z}} e^{-\frac{\left(x-\frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2}\right)^2}{2\left(\frac{\sigma_X\sigma_Y}{\sigma_Z}\right)^2}} \right] dx
\end{aligned}$$

The equation inside the integral symbol represents a valid normal density function for x, so we know it integrates to 1. Thus, the probability density function for $X + Y$ is given by

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma_Z} e^{-\frac{(z-(\mu_X+\mu_Y))^2}{2\sigma_Z^2}}$$

This is precisely a normal density function with mean $\mu_X + \mu_Y$ and variance $\sigma_Z^2 = (\sqrt{\sigma_X^2 + \sigma_Y^2})^2 = \sigma_X^2 + \sigma_Y^2$.

Hence, we have shown that, given $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$, $X + Y$ is distributed as $N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$.

Problem 2.

Let $Z = Y + 1$. We'll begin by finding the probability density function for Z. Since Y is a standard uniform random variable, we know that

$$f(y) = \begin{cases} 1 & 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Thus, the distribution function approach yields,

$$\begin{aligned}
F_Z(z) = P(Z \le z) = P(Y+1 \le z) &= P(Y \le z-1) \\
&= \int_{-\infty}^{z-1} f(y)dy \\
&= \int_{0}^{z-1} 1\,dy \\
&= z-1
\end{aligned}$$

So, since as Y ranges from 0 to 1, Z ranges from 1 to 2, we have

$$F_Z(z) = \begin{cases} 0 & z < 1 \\ z-1 & 1 \le z \le 2 \\ 1 & \text{elsewhere} \end{cases}$$

And the density function for Z is

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \begin{cases} 1 & 1 \le z \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

Now we can simplify the original function to be $U = X/Z$, where we now know the probability density function for both random variables ($X \sim \text{Unif}(0,1)$ and $Z \sim \text{Unif}(1,2)$).

So the distribution function is given by,

$$\begin{aligned}
F_U(u) = P(U \le u) = P(X/Z \le u) &= P(X \le uZ) \\
&= \int_{-\infty}^{uz} f(x)dx \\
&= \int_{0}^{uz} 1\,dx \\
&= uz
\end{aligned}$$

FINISH THIS LATER.

Problem 3.

Let Y be a standard normal variable. Then the moment-generating function of Y is given by,

$$\begin{aligned}
m(t) &= E(e^{tY}) \\
&= \int_{-\infty}^{\infty} e^{ty} f(y)\,dy
\end{aligned}$$

Since Y is a standard normal variable, we know that $\mu = 0$ and $\sigma = 1$. Thus, we have

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$

Plugging this into the above equation yields,

$$\begin{aligned} m(t) &= \int_{-\infty}^{\infty} e^{ty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}+ty}\, dy \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-t)^2} e^{\frac{1}{2}t^2}\, dy \\ &= e^{\frac{1}{2}t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-t)^2}\, dy \end{aligned}$$

We can see that the integral is precisely the integral for a normal random variable with $\mu = y - t$ and $\sigma = 1$. Thus, it integrates to 1, yielding,

$$m(t) = e^{\frac{1}{2}t^2}$$

Problem 4.

Suppose $g(x)$ is a monotone increasing function and X is a random variable with the probability density function f_X.

Let $U = g(X)$ where X has the above density function. Since $g(x)$ is an increasing function of x, then $g^{-1}(u)$ is an increasing function of u. Thus,

$$\begin{aligned} P(U \le u) &= P[g(X) \le u] \\ &= P\{g^{-1}[g(X)] \le g^{-1}(u)\} \\ &= P[X \le g^{-1}(u)] \end{aligned}$$

The above sequence of equalities implies that,

$$F_U(u) = F_X[g^{-1}(u)]$$

When we differentiate with respect to u, we get,

$$f_U(u) = \frac{dF_X[g^{-1}(u)]}{du} = f_X(g^{-1}(u)) \frac{d[g^{-1}(u)]}{du}$$

Problem 5.

We know that the moment-generating function of X is $\phi(t)$. Thus,

$$\begin{aligned} \phi(t) &= E(e^{tX}) \\ &= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx \\ &= \int_{-\infty}^{\infty} \left(1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots \right) f(x)\, dx \\ &= \int_{-\infty}^{\infty} f(x)dx + t\int_{-\infty}^{\infty} xf(x)dx + \frac{t^2}{2!}\int_{-\infty}^{\infty} x^2 f(x)dx + \cdots \end{aligned}$$

Now plugging in $aX + b$ to the above equation yields,

$$\begin{aligned} E(e^{t(aX+b)}) &= E(e^{taX+tb}) \\ &= E[e^{taX}(e^{tb})] \end{aligned}$$

Since expected value is a linear operator and e^{tb} is a constant, we can pull it out of the expectation operator,

$$\begin{aligned} E[e^{taX}(e^{tb})] &= e^{tb}E(e^{taX}) \\ &= e^{tb}\int_{-\infty}^{\infty} e^{tax} f(ax)\, dx \end{aligned}$$

We can see from the above equation that this is equal to $e^{tb}\phi(ta)$.

Thus, the moment-generating function for $aX + b$ with constants $a \neq 0$ and b is $e^{tb}\phi(ta)$.

Problem 6.

We know by Theorem 2.11.1 that the distribution of the sum of two random variables X and Y is given by the convolution of their densities, f_X and f_Y.

Thus, with $Z = X + Y$, we have,

$$f_Z(z) = (f * g)(z) = \int_{-\infty}^{\infty} f_X(z-y)f_Y(y)dy$$

The mean of this distribution is,

$$E(Z) = \int_{-\infty}^{\infty} zf_X(z-y)f_Y(y)dy$$

where $z = x + y$. Thus, we have,

$$\begin{aligned} E(Z) &= \int_{-\infty}^{\infty} zf_Z(z)dz \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [(x+y)f_X((x+y)-y)f_Y(y)]\, dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [(x+y)f_X(x)f_Y(y)]\, dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [(xf_X(x) + yf_X(x))f_Y(y)]\, dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [xf_X(x)f_Y(y) + yf_X(x)f_Y(y)]\, dxdy \\ &= \int_{-\infty}^{\infty} [\mu_X f_Y(y) + yf_Y(y)]\, dy \\ &= \mu_X + \mu_Y \end{aligned}$$

In the above equations, we are using the properties that, for any random variable X with density function $f(x)$ and mean μ, we have that $\int_{-\infty}^{\infty} f(x) = 1$ and $\int_{-\infty}^{\infty} xf(x) = \mu$.

Now for the variance, we know that for a given random variable X, $\text{Var}(X) = E[X^2] - E[X]^2$. Plugging in Z to this formula yields,

$$
\begin{aligned}
\text{Var}(Z) = \sigma_Z^2 &= E[Z^2] - E[Z]^2 \\
&= \int_{-\infty}^{\infty} z^2 f_Z(z)dz + (\mu_X + \mu_Y)^2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[(x+y)^2 f_X((x+y)-y)f_Y(y)\right] dxdy - (\mu_X + \mu_Y)^2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[(x+y)^2 f_X(x)f_Y(y)\right] dxdy - (\mu_X + \mu_Y)^2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[(x^2+2xy+y^2) f_X(x)f_Y(y)\right] dxdy - (\mu_X + \mu_Y)^2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[x^2 f_X(x)f_Y(y) + 2xyf_X(x)f_Y(y) + y^2 f_X(x)f_Y(y)\right] dxdy - (\mu_X + \mu_Y)^2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f_X(x)f_Y(y)dxdy - \mu_X + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2xyf_X(x)f_Y(y)dxdy + \\
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f_X(x)f_Y(y)dxdy - \mu_Y \\
&= \left[\int_{-\infty}^{\infty} x^2 f_X(x)dx - \mu_X\right] + 2\mu_X\mu_Y + \left[\int_{-\infty}^{\infty} y^2 f_Y(y)dy - \mu_Y\right] \\
&= \sigma_X^2 + 2\mu_X\mu_Y + \sigma_Y^2
\end{aligned}
$$

Thus, we have $\text{Var}(Z) = \sigma_X^2 + 2\mu_X\mu_Y + \sigma_Y^2$.

Problem 7.

The definition of the Central Limit Theorem is as follows:

Let $Y_1, Y_2, ..., Y_n$ be independent and identically distributed random variables with $E(Y_i) = \mu$ and $V(Y_i) = \sigma^2 < \infty$. Define

$$U_n = \frac{\Sigma_{i=1}^n Y_i - n\mu}{\sigma\sqrt{n}} = \frac{\overline{Y} - \mu}{\sigma/\sqrt{n}}$$

where $\overline{Y} = \frac{1}{n}\Sigma_{i=1}^n Y_i$.

Then the distribution function of U_n converges to the standard normal distribution function as $n \to \infty$. That is,

$$\lim_{n\to\infty} P(U_n \le u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

for all u.

Let $f_n(x)$ be the probability mass function for a binomial random variable with n trials and success probability p.

FINISH THIS LATER.

Problem 8.

Let X and Y be two independent, continuous random variables described by probability density functions f_X and f_Y. Also let $Z = XY$. We'll begin by finding the cumulative distribution function for Z. This yields,

$$\begin{aligned} F_Z(z) &= P(Z \le z) \\ &= P(XY \le z) \\ &= P(XY \le z, X \ge 0) + P(XY \le z, X \le 0) \\ &= P(Y \le z/X, X \ge 0) + P(Y \ge z/X, X \le 0) \\ &= \int_0^\infty f_X(x) \int_{-\infty}^{z/x} f_Y(y) dy dx + \int_{-\infty}^0 f_X(x) \int_{z/X}^\infty f_Y(y) dy dx \end{aligned}$$

Now in order to find the probability density function for Z, we need to differentiate with respect to z on both sides of the above equation.

$$\begin{aligned} f_Z(z) &= \frac{d}{dz} F_z(z) \\ &= \frac{d}{dz}\left[\int_0^\infty f_X(x) \int_{-\infty}^{z/x} f_Y(y) dy dx + \int_{-\infty}^0 f_X(x) \int_{z/X}^\infty f_Y(y) dy dx\right] \\ &= \int_0^\infty f_X(x)\left[f_Y(z/x)\left(\frac{1}{x}\right) - f(-\infty)\right] dx + \int_{-\infty}^0 f_X(x)\left[f_Y(\infty) - f_Y(z/x)\left(\frac{1}{x}\right)\right] dx \end{aligned}$$

We obtained the above equation using the fundamental theorem of calculus and the chain rule. Now, we will use the fact that if $f(x)$ is a distribution function, as $x \to \infty$, $f(x) \to 0$. The same holds true as $x \to -\infty$. This yields,

$$\begin{aligned} f_Z(z) &= \int_0^\infty f_X(x)\left[f_Y(z/x)\left(\frac{1}{x}\right) - f(-\infty)\right] dx + \int_{-\infty}^0 f_X(x)\left[f_Y(\infty) - f_Y(z/x)\left(\frac{1}{x}\right)\right] dx \\ &= \int_0^\infty f_X(x) f_Y(z/x) \frac{1}{x} dx + \int_{-\infty}^0 f_X(x)(-f_Y(z/x)) \frac{1}{x} dx \\ &= \int_0^\infty f_X(x) f_Y(z/x) \frac{1}{x} dx - \int_{-\infty}^0 f_X(x) f_Y(z/x) \frac{1}{x} dx \\ &= \int_0^\infty f_X(x) f_Y(z/x) \frac{1}{|x|} dx + \int_{-\infty}^0 f_X(x) f_Y(z/x) \frac{1}{|x|} dx \\ &= \int_{-\infty}^\infty f_X(x) f_Y(z/x) \frac{1}{|x|} dx \end{aligned}$$

Now we need to find the mean of Z where $Z = XY$,

$$\begin{aligned}
E[Z] &= \int_{-\infty}^{\infty} z f_Z(z) dz \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (xy) f_X(x) f_Y(y) \frac{1}{|x|} dx dy \\
&= \mu_Y \int_{-\infty}^{\infty} \frac{x}{|x|} f_X(x) dx \\
&= \mu_Y \left[\int_{-\infty}^{0} -f_X(x) dx + \int_{0}^{\infty} f_X(x) dx \right] \\
&= \mu_Y \left[-\int_{-\infty}^{0} f_X(x) dx + \int_{0}^{\infty} f_X(x) dx \right]
\end{aligned}$$

FINISH THIS LATER.

Problem 9.

We have that,

$$\begin{aligned}
t_j - t_{j-1} &= (j/N)(b-a) + a - [(j-1/N)(b-a) + a] \\
&= (jb - ja)/N + a - (jb - ja - b + a)/N - a \\
&= (jb - ja - jb + ja + b - a)/N \\
&= (b-a)/N
\end{aligned}$$

In addition, since the function f is bounded on $[a, b]$, we know that $\exists M \in \mathbb{N}$ such that $M \geq |f|$. These two facts yield the following,

$$\begin{aligned}
\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p &= \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})((b-a)/N)^p \\
&\leq \lim_{N\to\infty} \Sigma_{j=1}^{N} M((b-a)/N)^p \\
&= \lim_{N\to\infty} MN((b-a)/N)^p \\
&= M \lim_{N\to\infty} N((b-a)/N)^p \\
&= M(b-a)^p \lim_{N\to\infty} N/N^p \\
&= M(b-a)^p \lim_{N\to\infty} N^{1-p}
\end{aligned}$$

Since $1 - p < 0$, we have that $\lim_{N\to\infty} N^{1-p} = 0$. Thus, this gives us,

$$\begin{aligned}
\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p &\leq M(b-a)^p \lim_{N\to\infty} N^{1-p} \\
&= [M(b-a)]0 = 0
\end{aligned}$$

Now, if we use $-M$ for the lower bound, we get,

$$\begin{aligned}
\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p &\geq -M(b-a)^p \lim_{N\to\infty} N^{1-p} \\
&= [-M(b-a)]0 = 0
\end{aligned}$$

Since we have shown that

$$0 \leq \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p \leq 0$$

we have that $\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p = 0$.

Problem 10.

We know, by Proposition 4.9.1, that for a given stochastic process X_t, if $E[X_t] \to k$ a constant and $\text{Var}(X_t) \to 0$ as $t \to \infty$, then ms-$\lim_{t\to\infty} X_t = k$.

Thus, we will need to check the mean and variance of the given function. We will begin with the mean. Using the fact that the function f is a constant in terms of the expected value function, we get

$$E(\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2) = \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1}) E[(\Delta_{j-1}^{j} W)^2]$$

We know that the expected value of the square of a random variable is its variance, which is $t_j - t_{j-1}$ in this case. Hence,

$$\begin{aligned} E(\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2) &= \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1}) E[(\Delta_{j-1}^{j} W)^2] \\ &= \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1}) \end{aligned}$$

This is the equation for the Riemann integral of f over $[a, b]$. Since we already know f is bounded and defined on a closed interval, as long as f is continuous almost everywhere on $[a, b]$ we assert that the above limit exists and is equal to the integral of f. If that is the case, then we have

$$\begin{aligned} E(\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2) &= \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1}) \\ &= \int_a^b f(t)dt \end{aligned}$$

Now we need to show that the variance of the function is 0. Using Problem 8 to show that $\text{Var}[(\Delta_{j-1}^{j} W)^2] = 2(t_j - t_{j-1})^2$, we have

$$\begin{aligned} \text{Var}(\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2) &= \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 \text{Var}[(\Delta_{j-1}^{j} W)^2] \\ &= \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 \cdot 2(t_j - t_{j-1})^2 \\ &= 2 \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 (t_j - t_{j-1})^2 \end{aligned}$$

Now observe that if M is an upper bound for $|f|$ on $[a, b]$ (as in Problem 9), then M^2 is an upper bound for $|f|^2$ on $[a, b]$. This is true because increasing functions preserve inequalities, squaring is an increasing function for $x \geq 0$, and $|f| \geq 0$ for every x. In addition, observe that $|f|^2 = f^2$. Thus, we have that M^2 is an upper bound for f^2 on $[a, b]$.

We can now the above and the result from Problem 9 in order to obtain the following,

$$\begin{aligned}\mathrm{Var}(\lim_{N\to\infty}\Sigma_{j=1}^{N}f(t_{j-1})(\Delta_{j-1}^{j}W)^2) &= 2\lim_{N\to\infty}\Sigma_{j=1}^{N}(f(t_{j-1}))^2(t_j-t_{j-1})^2\\ &\leq 2M^2\lim_{N\to\infty}\Sigma_{j=1}^{N}(t_j-t_{j-1})^2\\ &= 2M^2(0)=0\end{aligned}$$

In addition, since we know that variance is always ≥ 0, we have that,

$$\mathrm{Var}(\lim_{N\to\infty}\Sigma_{j=1}^{N}f(t_{j-1})(\Delta_{j-1}^{j}W)^2)=0$$

Thus, we have shown that this function satisfies Proposition 4.9.1 and can state that ms-$\lim_{t\to\infty}[\lim_{N\to\infty}\Sigma_{j=1}^{N}f(t_{j-1})(\Delta_{j-1}^{j}W)^2]=\int_a^b f(t)dt$.

Problem 11.

We need to show that the mean and variance of this function are 0. We'll begin with the mean,

$$E[\lim_{N\to\infty}\Sigma_{j=1}^{N}f(t_{j-1})(t_j-t_{j-1})^p(\Delta_{j-1}^{j}W)^q]=\lim_{N\to\infty}\Sigma_{j=1}^{N}f(t_{j-1})(t_j-t_{j-1})^pE[(\Delta_{j-1}^{j}W)^q]$$

If q is odd, then $E[(\Delta_{j-1}^{j}W)^q]=0$. If q is even, then $E[(\Delta_{j-1}^{j}W)^q]=\sigma^q(q-1)!!=(\sqrt{t_j-t_{j-1}})^p(p-1)!!$ where !! is the double factorial. Since the expected value of the whole function is trivially 0 if q is odd, we will only consider the situation where q is even,

$$\begin{aligned}\lim_{N\to\infty}\Sigma_{j=1}^{N}f(t_{j-1})(t_j-t_{j-1})^pE[(\Delta_{j-1}^{j}W)^q] &= \lim_{N\to\infty}\Sigma_{j=1}^{N}f(t_{j-1})(t_j-t_{j-1})^p(\sqrt{t_j-t_{j-1}})^q(q-1)!!\\ &= \lim_{N\to\infty}\Sigma_{j=1}^{N}f(t_{j-1})(t_j-t_{j-1})^p(t_j-t_{j-1})^{q/2}(q-1)!!\\ &= (q-1)!!\lim_{N\to\infty}\Sigma_{j=1}^{N}f(t_{j-1})(t_j-t_{j-1})^{p+q/2}\end{aligned}$$

By the constraints in the problem, we know that $p+q/2>1$. Hence, we can apply the result of Problem 9. Hence, the limit is equal to 0,

$$\begin{aligned}E[\lim_{N\to\infty}\Sigma_{j=1}^{N}f(t_{j-1})(t_j-t_{j-1})^p(\Delta_{j-1}^{j}W)^q] &= (q-1)!!\lim_{N\to\infty}\Sigma_{j=1}^{N}f(t_{j-1})(t_j-t_{j-1})^{p+q/2}\\ &= (q-1)!!(0)\\ &= 0\end{aligned}$$

Now we must show that the variance is equal to 0,

$$\begin{aligned}\mathrm{Var}[\lim_{N\to\infty}\Sigma_{j=1}^{N}f(t_{j-1})(t_j-t_{j-1})^p(\Delta_{j-1}^{j}W)^q] &= 2\lim_{N\to\infty}\Sigma_{j=1}^{N}(f(t_{j-1}))^2(t_j-t_{j-1})^{2p}\cdot\mathrm{Var}[(\Delta_{j-1}^{j}W)^q]\\ &= 2\lim_{N\to\infty}\Sigma_{j=1}^{N}(f(t_{j-1}))^2(t_j-t_{j-1})^{2p}\cdot E[(\Delta_{j-1}^{j}W)^{2q}]\\ &= 2\lim_{N\to\infty}\Sigma_{j=1}^{N}(f(t_{j-1}))^2(t_j-t_{j-1})^{2p}\cdot E[(\Delta_{j-1}^{j}W)^{2q}]\\ &= 2\lim_{N\to\infty}\Sigma_{j=1}^{N}(f(t_{j-1}))^2(t_j-t_{j-1})^{2p}\cdot(t_j-t_{j-1})^q(2q-1)\\ &= 2(2q-1)\lim_{N\to\infty}\Sigma_{j=1}^{N}(f(t_{j-1}))^2(t_j-t_{j-1})^{2p+q}\end{aligned}$$

By the same reasoning as in Problem 10, this limit is equal to 0. Thus,

$$\begin{aligned}\text{Var}[\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p(\Delta_{j-1}^j W)^q] &= 2(2q-1) \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2(t_j - t_{j-1})^{2p+q} \\ &= 0\end{aligned}$$

Since the mean is equal to a constant (0 in this case) and the variance is equal to 0, we can say that the mean-squared limit of this function is 0.

Problem 12.

We have that,

$$\begin{aligned}\Sigma_{k=m}^{n} f_k(g_{k+1} - g_k) &= f_m(g_{m+1} - g_m) + f_{m+1}(g_{m+2} - g_{m+1}) + f_{m+2}(g_{m+3} - g_{m+2}) + \cdots \\ &= f_m(g_m + 1) - f_m(g_m) + f_{m+1}(g_{m+2}) - f_{m+1}(g_{m+1}) + \cdots \\ &= -f_m(g_m) + g_{m+1}(f_m - f_{m+1}) + g_{m+2}(f_{m+1} - f_{m+2}) + \cdots + f_n(g_{n+1}) \\ &= f_n(g_{n+1}) - f_m(g_m) - [g_{m+1}(f_m - f_{m+1}) + g_{m+2}(f_{m+1} - f_{m+2}) + \cdots + g_n(f_{n-1} - f_n)] \\ &= f_n(g_{n+1}) - f_m(g_m) - \Sigma_{k=m+1}^{n} g_k(f_{k-1} - f_k)\end{aligned}$$

Problem 13.

We have $F_t = W_t$ with the interval of interest being $[0, T]$. The partial sums of the integral $\int_0^T W_t dW_t$ are given by

$$\begin{aligned}S_n &= \Sigma_{i=0}^{n-1} F_{t_i}(W_{t_{i+1}} - W_{t_i}) \\ &= \Sigma_{i=0}^{n-1} W_{t_i}(W_{t_{i+1}} - W_{t_i})\end{aligned}$$

We also have that

$$W_{t_i}(W_{t_{i+1}} - W_{t_i}) = \frac{1}{2}W_{t_{i+1}}^2 - \frac{1}{2}W_{t_i}^2 - \frac{1}{2}(W_{t_{i+1}} - W_{t_i})^2$$

Plugging this into the partial sums formula yields,

$$\begin{aligned}S_n &= \frac{1}{2}\Sigma_{i=0}^{n-1} W_{t_{i+1}}^2 - \frac{1}{2}\Sigma_{i=0}^{n-1} W_{t_i}^2 - \frac{1}{2}\Sigma_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2 \\ &= \frac{1}{2}W_{t_n}^2 - \frac{1}{2}\Sigma_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2\end{aligned}$$

If we let $t_n = T$, we get

$$S_n = \frac{1}{2}W_T^2 - \frac{1}{2}\Sigma_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2$$

Using Proposition 4.11.6 and the fact that $\frac{1}{2}W_T^2$ does not depend on n, we have

$$\begin{aligned}\text{ms-}\lim_{n\to\infty} S_n &= \frac{1}{2}W_T^2 - \text{ms-}\lim_{n\to\infty} \frac{1}{2}\Sigma_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2 \\ &= \frac{1}{2}W_T^2 - \frac{1}{2}T\end{aligned}$$

Since the Ito integral is defined as the mean-squared limit of the partial sums S_n, we have that

$$\begin{aligned}\int_0^T W_t dW_t &= \text{ms-}\lim_{n\to\infty} S_n \\ &= \frac{1}{2}W_T^2 - \frac{1}{2}T\end{aligned}$$

as required.

Problem 14.

We are given the Ito integral,

$$\int_0^T f(t)dW_t$$

where $f(t)$ is an arbitrary bounded and continuous function. The partial sums for this integral are given by,

$$S_n = \Sigma_{i=0}^{n-1} f(t_i)(W_{t_{i+1}} - W_{t_i})$$