Stochastic Differential Equations: Final Project

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Problem 1.

We know that the probability density of the sum of two independent random variables can be computed by the convolutions of f_X and f_Y. Thus, the probability density of $X+Y$, denoted by f_Z, is

$$f_Z(z) = (f_Y * f_X) = \int_{-\infty}^{\infty} f_Y(z-x)f_X(x)dx \tag{1}$$

We know that the normal density function is given by,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Plugging this density function into equation (1) and letting $\sigma_Z = \sqrt{\sigma_X^2 + \sigma_Y^2}$ yields,

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty}\left[\frac{1}{\sigma_Y\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{(z-x)-\mu_Y}{\sigma_Y}\right)^2}\frac{1}{\sigma_X\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}\right]dx \\
&= \int_{-\infty}^{\infty}\left[\frac{1}{\sigma_Y\sigma_X(2\pi)}e^{-\frac{1}{2}\left[\left(\frac{z-x-\mu_Y}{\sigma_Y}\right)^2+\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right]}\right]dx \\
&= \int_{-\infty}^{\infty}\left[\frac{1}{\sigma_Y\sigma_X(2\pi)}e^{-\frac{\sigma_X^2(z-x-\mu_Y)^2+\sigma_Y^2(x-\mu_X)^2}{2\sigma_Y^2\sigma_X^2}}\right]dx \\
&= \int_{-\infty}^{\infty}\left[\frac{1}{(2\pi)\sigma_X\sigma_Y}e^{-\frac{x^2(\sigma_X^2+\sigma_Y^2)-2x(\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X)+\sigma_X^2(z^2+\mu_Y^2-2z\mu_Y)+\sigma_Y^2\mu_X^2}{2\sigma_Y^2\sigma_X^2}}\right]dx \\
&= \int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}\sigma_Z}e^{-\frac{\sigma_Z^2\left(\sigma_X^2(z-\mu_Y)^2+\sigma_Y^2\mu_X^2\right)-\left(\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X\right)^2}{2\sigma_Z^2(\sigma_X\sigma_Y)^2}}\frac{1}{\sqrt{2\pi}\frac{\sigma_X\sigma_Y}{\sigma_Z}}e^{-\frac{\left(x-\frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2}\right)^2}{2\left(\frac{\sigma_X\sigma_Y}{\sigma_Z}\right)^2}}\right]dx \\
&= \frac{1}{\sqrt{2\pi}\sigma_Z}e^{-\frac{(z-(\mu_X+\mu_Y))^2}{2\sigma_Z^2}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}\frac{\sigma_X\sigma_Y}{\sigma_Z}}e^{-\frac{\left(x-\frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2}\right)^2}{2\left(\frac{\sigma_X\sigma_Y}{\sigma_Z}\right)^2}}\right]dx
\end{aligned}$$

The equation inside the integral symbol represents a valid normal density function for x, so we know it integrates to 1. Thus, the probability density function for $X+Y$ is given by

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma_Z} e^{-\frac{(z-(\mu_X+\mu_Y))^2}{2\sigma_Z^2}}$$

This is precisely a normal density function with mean $\mu_X + \mu_Y$ and variance $\sigma_Z^2 = (\sqrt{\sigma_X^2 + \sigma_Y^2})^2 = \sigma_X^2 + \sigma_Y^2$.

Hence, we have shown that, given $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$, $X+Y$ is distributed as $N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$.

Problem 2.

Let $Z = Y + 1$. We'll begin by finding the probability density function for Z. Since Y is a standard uniform random variable, we know that

$$f(y) = \begin{cases} 1 & 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Thus, the distribution function approach yields,

$$\begin{aligned} F_Z(z) = P(Z \le z) = P(Y+1 \le z) &= P(Y \le z-1) \\ &= \int_{-\infty}^{z-1} f(y)dy \\ &= \int_0^{z-1} 1\, dy \\ &= z - 1 \end{aligned}$$

So, since as Y ranges from 0 to 1, Z ranges from 1 to 2, we have

$$F_Z(z) = \begin{cases} 0 & z < 1 \\ z - 1 & 1 \le z \le 2 \\ 1 & \text{elsewhere} \end{cases}$$

And the density function for Z is

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \begin{cases} 1 & 1 \le z \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

Now we can simplify the original function to be $U = X/Z$, where we now know the probability density function for both random variables ($X \sim \text{Unif}(0,1)$ and $Z \sim \text{Unif}(1,2)$).

So the distribution function is given by,

$$\begin{aligned}F_U(u) = P(U \leq u) = P(X/Z \leq u) &= P(X \leq uZ) \\ &= \int_{-\infty}^{uz} f(x)dx \\ &= \int_0^{uz} 1\,dx \\ &= uz\end{aligned}$$

And the density function for U is

$$f_U(u) = \frac{dF_U(u)}{du} = z = y + 1$$

Problem 3.

Let Y be a standard normal variable. Then the moment-generating function of Y is given by,

$$\begin{aligned}m(t) &= E(e^{tY}) \\ &= \int_{-\infty}^{\infty} e^{ty} f(y)\,dy\end{aligned}$$

Since Y is a standard normal variable, we know that $\mu = 0$ and $\sigma = 1$. Thus, we have

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$

Plugging this into the above equation yields,

$$\begin{aligned}m(t) &= \int_{-\infty}^{\infty} e^{ty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\,dy \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}+ty}\,dy \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-t)^2} e^{\frac{1}{2}t^2}\,dy \\ &= e^{\frac{1}{2}t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-t)^2}\,dy\end{aligned}$$

We can see that the integral is precisely the integral for a normal random variable with $\mu = y - t$ and $\sigma = 1$. Thus, it integrates to 1, yielding,

$$m(t) = e^{\frac{1}{2}t^2}$$

Problem 4.

Suppose $g(x)$ is a monotone increasing function and X is a random variable with the probability density function f_X.

Let $U = g(X)$ where X has the above density function. Since $g(x)$ is an increasing function of x, then $g^{-1}(u)$ is an increasing function of u. Thus,

$$\begin{aligned} P(U \leq u) &= P[g(X) \leq u] \\ &= P\{g^{-1}[g(X)] \leq g^{-1}(u)\} \\ &= P[X \leq g^{-1}(u)] \end{aligned}$$

The above sequence of equalities implies that,

$$F_U(u) = F_X[g^{-1}(u)]$$

When we differentiate with respect to u, we get,

$$f_U(u) = \frac{dF_X[g^{-1}(u)]}{du} = f_X(g^{-1}(u))\frac{d[g^{-1}(u)]}{du}$$

Problem 5.

We know that the moment-generating function of X is $\phi(t)$. Thus,

$$\begin{aligned} \phi(t) &= E(e^{tX}) \\ &= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx \\ &= \int_{-\infty}^{\infty} \left(1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots\right) f(x)\, dx \\ &= \int_{-\infty}^{\infty} f(x)dx + t\int_{-\infty}^{\infty} xf(x)dx + \frac{t^2}{2!}\int_{-\infty}^{\infty} x^2 f(x)dx + \cdots \end{aligned}$$

Now plugging in $aX + b$ to the above equation yields,

$$\begin{aligned} E(e^{t(aX+b)}) &= E(e^{taX+tb}) \\ &= E[e^{taX}(e^{tb})] \end{aligned}$$

Since expected value is a linear operator and e^{tb} is a constant, we can pull it out of the expectation operator,

$$\begin{aligned} E[e^{taX}(e^{tb})] &= e^{tb}E(e^{taX}) \\ &= e^{tb}\int_{-\infty}^{\infty} e^{tax} f(ax)\, dx \end{aligned}$$

We can see from the above equation that this is equal to $e^{tb}\phi(ta)$.

Thus, the moment-generating function for $aX + b$ with constants $a \neq 0$ and b is $e^{tb}\phi(ta)$.

Problem 6.

We know by Theorem 2.11.1 that the distribution of the sum of two random variables X and Y is given by the convolution of their densities, f_X and f_Y.

Thus, with $Z = X + Y$, we have,

$$f_Z(z) = (f * g)(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy$$

The mean of this distribution is,

$$E(Z) = \int_{-\infty}^{\infty} z f_X(z-y) f_Y(y) dy$$

where $z = x + y$. Thus, we have,

$$\begin{aligned}
E(Z) &= \int_{-\infty}^{\infty} z f_Z(z) dz \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[(x+y) f_X((x+y)-y) f_Y(y)\right] dxdy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[(x+y) f_X(x) f_Y(y)\right] dxdy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[(x f_X(x) + y f_X(x)) f_Y(y)\right] dxdy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[x f_X(x) f_Y(y) + y f_X(x) f_Y(y)\right] dxdy \\
&= \int_{-\infty}^{\infty} \left[\mu_X f_Y(y) + y f_Y(y)\right] dy \\
&= \mu_X + \mu_Y
\end{aligned}$$

In the above equations, we are using the properties that, for any random variable X with density function $f(x)$ and mean μ, we have that $\int_{-\infty}^{\infty} f(x) = 1$ and $\int_{-\infty}^{\infty} x f(x) = \mu$.

Now for the variance, we know that for a given random variable X, $\text{Var}(X) = E[X^2] - E[X]^2$. Plugging in Z to this formula yields,

$$\begin{aligned}
\text{Var}(Z) = \sigma_Z^2 &= E[Z^2] - E[Z]^2 \\
&= \int_{-\infty}^{\infty} z^2 f_Z(z) dz + (\mu_X + \mu_Y)^2 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[(x+y)^2 f_X((x+y)-y) f_Y(y)\right] dxdy - (\mu_X + \mu_Y)^2 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[(x+y)^2 f_X(x) f_Y(y)\right] dxdy - (\mu_X + \mu_Y)^2 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[(x^2 + 2xy + y^2) f_X(x) f_Y(y)\right] dxdy - (\mu_X + \mu_Y)^2 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[x^2 f_X(x) f_Y(y) + 2xy f_X(x) f_Y(y) + y^2 f_X(x) f_Y(y)\right] dxdy - (\mu_X + \mu_Y)^2
\end{aligned}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f_X(x) f_Y(y) dx dy - \mu_X + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2xy f_X(x) f_Y(y) dx dy +$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f_X(x) f_Y(y) dx dy - \mu_Y$$
$$= \left[\int_{-\infty}^{\infty} x^2 f_X(x) dx - \mu_X\right] + 2\mu_X\mu_Y + \left[\int_{-\infty}^{\infty} y^2 f_Y(y) dy - \mu_Y\right]$$
$$= \sigma_X^2 + 2\mu_X\mu_Y + \sigma_Y^2$$

Thus, we have $\text{Var}(Z) = \sigma_X^2 + 2\mu_X\mu_Y + \sigma_Y^2$.

Problem 7.

The definition of the Central Limit Theorem is as follows:

Let $Y_1, Y_2, ..., Y_n$ be independent and identically distributed random variables with $E(Y_i) = \mu$ and $V(Y_i) = \sigma^2 < \infty$. Define

$$U_n = \frac{\Sigma_{i=1}^n Y_i - n\mu}{\sigma\sqrt{n}} = \frac{\overline{Y} - \mu}{\sigma/\sqrt{n}}$$

where $\overline{Y} = \frac{1}{n}\Sigma_{i=1}^n Y_i$.

Then the distribution function of U_n converges to the standard normal distribution function as $n \to \infty$. That is,

$$\lim_{n\to\infty} P(U_n \leq u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

for all u.

Let $f_n(x)$ be the probability mass function for a binomial random variable with n trials and success probability p and failure probability $q = 1 - p$ We know that the mean for n trials is given by np and variance by npg..

Now let X_i be distributed by this probability function and define $S_n = X_1 + X_2 + \cdots + X_n$. Thus, S_n is the number of successes in n trials.

Now define the standardized sum of S_n to be,

$$S_n^* = \frac{S_n - np}{\sqrt{npq}}$$

Thus, we can see that our S_n^* are in the same form as the U_n defined above. Hence, as we let $n \to \infty$, these standardized sums of binomial random variables will converge to the standard normal distribution.

Problem 8.

Let X and Y be two independent, continuous random variables described by probability density functions f_X and f_Y. Also let $Z = XY$. We'll begin by finding the cumulative distribution function for Z. This yields,

$$\begin{aligned}
F_Z(z) &= P(Z \leq z) \\
&= P(XY \leq z) \\
&= P(XY \leq z, X \geq 0) + P(XY \leq z, X \leq 0) \\
&= P(Y \leq z/X, X \geq 0) + P(Y \geq z/X, X \leq 0) \\
&= \int_0^\infty f_X(x) \int_{-\infty}^{z/x} f_Y(y)dydx + \int_{-\infty}^0 f_X(x) \int_{z/X}^\infty f_Y(y)dydx
\end{aligned}$$

Now in order to find the probability density function for Z, we need to differentiate with respect to z on both sides of the above equation.

$$\begin{aligned}
f_Z(z) &= \frac{d}{dz}F_z(z) \\
&= \frac{d}{dz}\left[\int_0^\infty f_X(x) \int_{-\infty}^{z/x} f_Y(y)dydx + \int_{-\infty}^0 f_X(x) \int_{z/X}^\infty f_Y(y)dydx\right] \\
&= \int_0^\infty f_X(x)\left[f_Y(z/x)\left(\frac{1}{x}\right) - f(-\infty)\right]dx + \int_{-\infty}^0 f_X(x)\left[f_Y(\infty) - f_Y(z/x)\left(\frac{1}{x}\right)\right]dx
\end{aligned}$$

We obtained the above equation using the fundamental theorem of calculus and the chain rule. Now, we will use the fact that if $f(x)$ is a distribution function, as $x \to \infty$, $f(x) \to 0$. The same holds true as $x \to -\infty$. This yields,

$$\begin{aligned}
f_Z(z) &= \int_0^\infty f_X(x)\left[f_Y(z/x)\left(\frac{1}{x}\right) - f(-\infty)\right]dx + \int_{-\infty}^0 f_X(x)\left[f_Y(\infty) - f_Y(z/x)\left(\frac{1}{x}\right)\right]dx \\
&= \int_0^\infty f_X(x)f_Y(z/x)\frac{1}{x}dx + \int_{-\infty}^0 f_X(x)(-f_Y(z/x))\frac{1}{x}dx \\
&= \int_0^\infty f_X(x)f_Y(z/x)\frac{1}{x}dx - \int_{-\infty}^0 f_X(x)f_Y(z/x)\frac{1}{x}dx \\
&= \int_0^\infty f_X(x)f_Y(z/x)\frac{1}{|x|}dx + \int_{-\infty}^0 f_X(x)f_Y(z/x)\frac{1}{|x|}dx \\
&= \int_{-\infty}^\infty f_X(x)f_Y(z/x)\frac{1}{|x|}dx
\end{aligned}$$

Now we need to find the mean of Z where $Z = XY$,

$$\begin{aligned}
E[Z] &= \int_{-\infty}^\infty z f_Z(z)dz \\
&= \int_{-\infty}^\infty \int_{-\infty}^\infty (xy)f_X(x)f_Y(y)\frac{1}{|x|}dxdy \\
&= \mu_Y \int_{-\infty}^\infty \frac{x}{|x|}f_X(x)dx
\end{aligned}$$

$$
\begin{aligned}
&= \mu_Y \left[\int_{-\infty}^{0} -f_X(x)dx + \int_{0}^{\infty} f_X(x)dx \right] \\
&= \mu_Y \left[-\int_{-\infty}^{0} f_X(x)dx + \int_{0}^{\infty} f_X(x)dx \right] \\
&= \mu_X \mu_Y
\end{aligned}
$$

For the variance of Z, we have,

$$
\begin{aligned}
\text{Var}(Z) &= E[Z^2] - E[Z]^2 \\
&= E[X^2 Y^2] - [\mu_X \mu_Y]^2 \\
&= (\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) - \mu_X^2 \mu_Y^2
\end{aligned}
$$

Problem 9.

We have that,

$$
\begin{aligned}
t_j - t_{j-1} &= (j/N)(b-a) + a - [(j - 1/N)(b-a) + a] \\
&= (jb - ja)/N + a - (jb - ja - b + a)/N - a \\
&= (jb - ja - jb + ja + b - a)/N \\
&= (b-a)/N
\end{aligned}
$$

In addition, since the function f is bounded on $[a, b]$, we know that $\exists M \in \mathbb{N}$ such that $M \geq |f|$. These two facts yield the following,

$$
\begin{aligned}
\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p &= \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})((b-a)/N)^p \\
&\leq \lim_{N\to\infty} \Sigma_{j=1}^{N} M((b-a)/N)^p \\
&= \lim_{N\to\infty} MN((b-a)/N)^p \\
&= M \lim_{N\to\infty} N((b-a)/N)^p \\
&= M(b-a)^p \lim_{N\to\infty} N/N^p \\
&= M(b-a)^p \lim_{N\to\infty} N^{1-p}
\end{aligned}
$$

Since $1 - p < 0$, we have that $\lim_{N\to\infty} N^{1-p} = 0$. Thus, this gives us,

$$
\begin{aligned}
\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p &\leq M(b-a)^p \lim_{N\to\infty} N^{1-p} \\
&= [M(b-a)]0 = 0
\end{aligned}
$$

Now, if we use $-M$ for the lower bound, we get,

$$
\begin{aligned}
\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p &\geq -M(b-a)^p \lim_{N\to\infty} N^{1-p} \\
&= [-M(b-a)]0 = 0
\end{aligned}
$$

Since we have shown that

$$
0 \leq \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p \leq 0
$$

we have that $\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p = 0$.

Problem 10.

We know, by Proposition 4.9.1, that for a given stochastic process X_t, if $E[X_t] \to k$ a constant and $\text{Var}(X_t) \to 0$ as $t \to \infty$, then $\text{ms-}\lim_{t\to\infty} X_t = k$.

Thus, we will need to check the mean and variance of the given function. We will begin with the mean. Using the fact that the function f is a constant in terms of the expected value function, we get

$$E(\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2) = \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1}) E[(\Delta_{j-1}^{j} W)^2]$$

We know that the expected value of the square of a random variable is its variance, which is $t_j - t_{j-1}$ in this case. Hence,

$$\begin{aligned} E(\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2) &= \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1}) E[(\Delta_{j-1}^{j} W)^2] \\ &= \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1}) \end{aligned}$$

This is the equation for the Riemann integral of f over $[a, b]$. Since we already know f is bounded and defined on a closed interval, as long as f is continuous almost everywhere on $[a, b]$ we assert that the above limit exists and is equal to the integral of f. If that is the case, then we have

$$\begin{aligned} E(\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2) &= \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1}) \\ &= \int_a^b f(t)dt \end{aligned}$$

Now we need to show that the variance of the function is 0. Using Problem 8 to show that $\text{Var}[(\Delta_{j-1}^{j} W)^2] = 2(t_j - t_{j-1})^2$, we have

$$\begin{aligned} \text{Var}(\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2) &= \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 \text{Var}[(\Delta_{j-1}^{j} W)^2] \\ &= \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 \cdot 2(t_j - t_{j-1})^2 \\ &= 2 \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 (t_j - t_{j-1})^2 \end{aligned}$$

Now observe that if M is an upper bound for $|f|$ on $[a, b]$ (as in Problem 9), then M^2 is an upper bound for $|f|^2$ on $[a, b]$. This is true because increasing functions preserve inequalities, squaring is an increasing function for $x \geq 0$, and $|f| \geq 0$ for every x. In addition, observe that $|f|^2 = f^2$. Thus, we have that M^2 is an upper bound for f^2 on $[a, b]$.

We can now the above and the result from Problem 9 in order to obtain the following,

$$\begin{aligned} \text{Var}(\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2) &= 2 \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 (t_j - t_{j-1})^2 \\ &\leq 2M^2 \lim_{N\to\infty} \Sigma_{j=1}^{N} (t_j - t_{j-1})^2 \\ &= 2M^2(0) = 0 \end{aligned}$$

In addition, since we know that variance is always ≥ 0, we have that,

$$\mathrm{Var}(\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2) = 0$$

Thus, we have shown that this function satisfies Proposition 4.9.1 and can state that ms- $\lim_{t\to\infty}[\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2] = \int_a^b f(t)dt$.

Problem 11.

We need to show that the mean and variance of this function are 0. We'll begin with the mean,

$$E[\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p (\Delta_{j-1}^{j} W)^q] = \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p E[(\Delta_{j-1}^{j} W)^q]$$

If q is odd, then $E[(\Delta_{j-1}^{j} W)^q] = 0$. If q is even, then $E[(\Delta_{j-1}^{j} W)^q] = \sigma^q (q-1)!! = (\sqrt{t_j - t_{j-1}})^p (p-1)!!$ where !! is the double factorial. Since the expected value of the whole function is trivially 0 if q is odd, we will only consider the situation where q is even,

$$\begin{aligned}
\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p E[(\Delta_{j-1}^{j} W)^q] &= \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p (\sqrt{t_j - t_{j-1}})^q (q-1)!! \\
&= \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p (t_j - t_{j-1})^{q/2} (q-1)!! \\
&= (q-1)!! \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^{p+q/2}
\end{aligned}$$

By the constraints in the problem, we know that $p + q/2 > 1$. Hence, we can apply the result of Problem 9. Hence, the limit is equal to 0,

$$\begin{aligned}
E[\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p (\Delta_{j-1}^{j} W)^q] &= (q-1)!! \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^{p+q/2} \\
&= (q-1)!!(0) \\
&= 0
\end{aligned}$$

Now we must show that the variance is equal to 0,

$$\begin{aligned}
\mathrm{Var}[\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p (\Delta_{j-1}^{j} W)^q] &= \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 (t_j - t_{j-1})^{2p} \cdot \mathrm{Var}[(\Delta_{j-1}^{j} W)^q] \\
&= \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 (t_j - t_{j-1})^{2p} \cdot E[(\Delta_{j-1}^{j} W)^{2q}] \\
&= \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 (t_j - t_{j-1})^{2p} \cdot E[(\Delta_{j-1}^{j} W)^{2q}] \\
&= \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 (t_j - t_{j-1})^{2p} \cdot (t_j - t_{j-1})^q (2q-1) \\
&= (2q-1) \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 (t_j - t_{j-1})^{2p+q}
\end{aligned}$$

By the same reasoning as in Problem 10, this limit is equal to 0. Thus,

$$\begin{aligned}
\mathrm{Var}[\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p (\Delta_{j-1}^{j} W)^q] &= (2q-1) \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 (t_j - t_{j-1})^{2p+q} \\
&= 0
\end{aligned}$$

Since the mean is equal to a constant (0 in this case) and the variance is equal to 0, we can say that the mean-squared limit of this function is 0.

Problem 12.

We have that,

$$\begin{aligned}
\Sigma_{k=m}^{n} f_k(g_{k+1}-g_k) &= f_m(g_{m+1}-g_m) + f_{m+1}(g_{m+2}-g_{m+1}) + f_{m+2}(g_{m+3}-g_{m+2}) + \cdots \\
&= f_m(g_m+1) - f_m(g_m) + f_{m+1}(g_{m+2}) - f_{m+1}(g_{m+1}) + \cdots \\
&= -f_m(g_m) + g_{m+1}(f_m - f_{m+1}) + g_{m+2}(f_{m+1} - f_{m+2}) + \cdots + f_n(g_{n+1}) \\
&= f_n(g_{n+1}) - f_m(g_m) - [g_{m+1}(f_m - f_{m+1}) + g_{m+2}(f_{m+1} - f_{m+2}) + \cdots + g_n(f_{n-1} - f_n)] \\
&= f_n(g_{n+1}) - f_m(g_m) - \Sigma_{k=m+1}^{n} g_k(f_{k-1} - f_k)
\end{aligned}$$

Problem 13.

We have $F_t = W_t$ with the interval of interest being $[0, T]$. The partial sums of the integral $\int_0^T W_t dW_t$ are given by

$$\begin{aligned}
S_n &= \Sigma_{i=0}^{n-1} F_{t_i}(W_{t_{i+1}} - W_{t_i}) \\
&= \Sigma_{i=0}^{n-1} W_{t_i}(W_{t_{i+1}} - W_{t_i})
\end{aligned}$$

We also have that

$$W_{t_i}(W_{t_{i+1}} - W_{t_i}) = \frac{1}{2}W_{t_{i+1}}^2 - \frac{1}{2}W_{t_i}^2 - \frac{1}{2}(W_{t_{i+1}} - W_{t_i})^2$$

Plugging this into the partial sums formula yields,

$$\begin{aligned}
S_n &= \frac{1}{2}\Sigma_{i=0}^{n-1} W_{t_{i+1}}^2 - \frac{1}{2}\Sigma_{i=0}^{n-1} W_{t_i}^2 - \frac{1}{2}\Sigma_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2 \\
&= \frac{1}{2}W_{t_n}^2 - \frac{1}{2}\Sigma_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2
\end{aligned}$$

If we let $t_n = T$, we get

$$S_n = \frac{1}{2}W_T^2 - \frac{1}{2}\Sigma_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2$$

Using Proposition 4.11.6 and the fact that $\frac{1}{2}W_T^2$ does not depend on n, we have

$$\begin{aligned}
\text{ms-}\lim_{n\to\infty} S_n &= \frac{1}{2}W_T^2 - \text{ms-}\lim_{n\to\infty} \frac{1}{2}\Sigma_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2 \\
&= \frac{1}{2}W_T^2 - \frac{1}{2}T
\end{aligned}$$

Since the Ito integral is defined as the mean-squared limit of the partial sums S_n, we have that

$$\begin{aligned}
\int_0^T W_t dW_t &= \text{ms-}\lim_{n\to\infty} S_n \\
&= \frac{1}{2}W_T^2 - \frac{1}{2}T
\end{aligned}$$

as required.

Problem 14.

We are given the Ito integral,

$$\int_0^T f(t)dW_t$$

where $f(t)$ is an arbitrary bounded and continuous function. Each increment is given by,

$$f(t_i)(W_{t_{i+1}} - W_{t_i})$$

So the partial sums are given by

$$S_n = \Sigma_{i=0}^{n-1} f(t_i)(W_{t_{i+1}} - W_{t_i})$$

We need to find the mean-squared limit of these partial sums. We'll start by finding the expected value,

$$\begin{aligned} E[\Sigma_{i=0}^{n-1} f(t_i)(W_{t_{i+1}} - W_{t_i})] &= \Sigma_{i=0}^{n-1} f(t_i)E[(W_{t_{i+1}} - W_{t_i})] \\ &= \Sigma_{i=0}^{n-1} f(t_i)(0) = 0 \end{aligned}$$

Thus, the mean is 0. Now to find the variance,

$$\begin{aligned} \mathrm{Var}[\Sigma_{i=0}^{n-1} f(t_i)(W_{t_{i+1}} - W_{t_i})] &= \Sigma_{i=0}^{n-1} f(t_i)^2 \mathrm{Var}[(W_{t_{i+1}} - W_{t_i})] \\ &= \Sigma_{i=0}^{n-1} f(t_i)^2 (t_{i+1} - t_i) \end{aligned}$$

Now when we take the limit as $n \to \infty$ of the variance, we get,

$$\lim_{n\to\infty} \Sigma_{i=0}^{n-1} f(t_i)^2 (t_{i+1} - t_i) = \int_0^T f(t)^2$$

Thus, if $Y_t = \int_0^T f(t)dW_t$, then $Y_t \sim N(0, \int_0^T f(t)^2)$.

Problem 15.

We have,

$$\begin{aligned} \int_0^t W_s\, ds &= \int_0^t \int_0^t \mathbf{1}_{[0,s]}(u)\, dW_u\, ds \\ &= \int_0^t \int_0^t \mathbf{1}_{[0,s]}(u)\, ds\, dW_u \\ &= \int_0^t (t-u)\, dW_u \end{aligned}$$

where $\mathbf{1}_{[0,s]}(u)$ is the indicator function. That is, $\mathbf{1}_{[0,s]}(u) = 1$ if $u \in [0, s]$ and 0 otherwise.

Problem 16.

We have $X_t = \int_0^t (a + \frac{bu}{t}) dW_u$. Since the non-infinitesimal part of the integral does not depend on a random variable, this is a Wiener integral.

By Proposition 5.6.1 , we know that Wiener integrals are normal random variables with mean 0 and variance,

$$\begin{aligned}\int_0^t (a + \frac{bu}{t})^2 du &= \int_0^t (a^2 + a\frac{bu}{t} + \frac{b^2u^2}{t^2}) du \\ &= a^2 t + a\frac{bt^2}{2t} + \frac{b^2t^3}{3t^2} \\ &= a^2 t + a\frac{bt}{2} + \frac{b^2 t}{3} \\ &= (a^2 + \frac{ab}{2} + \frac{b^2}{3})t\end{aligned}$$

So, in order for the variance to equal 1, we require,

$$\begin{aligned}(a^2 + \frac{ab}{2} + \frac{b^2}{3})t = 1 \\ \implies a^2 + \frac{ab}{2} + \frac{b^2}{3} = \frac{1}{t}\end{aligned}$$

Problem 17.

We have that,

$$dX_t = rX_t dt + \sigma X_t dW_t$$

Dividing by X_t and integrating yields,

$$\int \frac{dX_t}{X_t} = \int r dt + \int \sigma dW_t \tag{2}$$

which gives us,

$$\log X_t = rt + \sigma W_t + c$$

where c is an integration constant. Exponentiation on both sides yields,

$$X_t = e^{rt + \sigma W_t + c}$$

Now we assume the constant c is replaced by a function $c(t)$. We then apply Ito's formula, which gives

$$dX_t = X_t(r + c'(t) + \frac{\sigma^2}{2})dt + \sigma X_t dW_t$$

When we subtract the initial equation, we get

$$(c'(t) + \frac{\sigma^2}{2})dt = 0$$

which implies that $c'(t) = -\frac{\sigma^2}{2}$ Thus, $c(t) = -\frac{\sigma^2}{2}t + k$ where k is a constant.

Substituting back in gives us,

$$\begin{aligned} X_t &= e^{rt+\sigma W_t - \frac{\sigma^2}{2}t+k} \\ &= e^{(r-\frac{\sigma^2}{2})t+\sigma W_t+k} \\ &= X_0 e^{(r-\frac{\sigma^2}{2})t+\sigma W_t} \end{aligned}$$

Now we have,

$$\begin{aligned} \log X_t &= \log X_0 + \log\left(e^{(r-\frac{\sigma^2}{2})t+\sigma W_t}\right) \\ &= \log X_0 + (r - \frac{\sigma^2}{2})t + \sigma W_t \end{aligned}$$

Problem 18.

Problem 19.

Ito's formula states that, if X_t is a stochastic process satisfying $dX_t = b_t dt + \sigma_t dW_t$ with b_t and σ_t measurable, and if $F_t = f(X_t)$ with f twice continuously differentiable, we have

$$dF_t = [b_t f'(X_t) + \frac{\sigma_t^2}{2} f''(X_t)]dt + \sigma_t f'(X_t)dW_t$$

a) In the case that $X_t = W_t$, Corollary 6.2.3, we have the following when we let $F_t = f(W_t)$,

$$dF_t = \frac{1}{2} f''(W_t)dt + f'(W_t)dW_t$$

We will apply this formula to the increment below.

We have $d(W_t e^{W_t})$, so let $f(x) = xe^x$. Then $f'(x) = xe^x + e^x$ and $f''(x) = xe^x + e^x + e^x$. Let $F_t = f(W_t)$ and we have,

$$\begin{aligned} dF_t &= \frac{1}{2}(W_t e^{W_t})'' dt + (W_t e^{W_t})' dW_t \\ &= \frac{1}{2}[W_t e^{W_t} + e^{W_t} + e^{W_t}]dt + [W_t e^{W_t} + e^{W_t}]dW_t \\ &= (\frac{1}{2}W_t e^{W_t} + e^{W_t})dt + e^{W_t}(W_t + 1)dW_t \\ &= e^{W_t}(\frac{1}{2}W_t + 1)dt + e^{W_t}(W_t + 1)dW_t \end{aligned}$$

b) We have $d(e^{t+W_t^2})$, so let $X_t = t+W_t^2$. Then $dX_t = dt+(2W_tdW_t+dt) = 2dt+2W_tdW_t$

Thus, we have that $b_t = 2$ and $\sigma_t = 2W_t$ satisfying the processes from Ito's lemma. Letting $f(x) = e^x$, we have $f'(x) = f''(x) = e^x$. Now let $F_t = f(X_t)$ and applying Ito's formula yields,

$$\begin{aligned} dF_t &= [2f'(X_t) + \frac{4W_t^2}{2}f''(X_t)]dt + 2W_tf'(X_t)dW_t \\ &= [2e^{t+W_t^2} + (2W_t^2)e^{t+W_t^2}]dt + 2W_te^{t+W_t^2}dW_t \\ &= 2e^{t+W_t^2}[1+W_t^2]dt + 2W_te^{t+W_t^2}dW_tn \end{aligned}$$

c) We have $d\left(\frac{1}{t^\alpha}\int_0^t e^{W_s}d_s\right)$. Then,

$$\begin{aligned} d\left(\frac{1}{t^\alpha}\int_0^t e^{W_s}d_s\right) &= \left(\frac{1}{t^\alpha}\right)'\int_0^t e^{W_s}d_s + \frac{1}{t^\alpha}\left(\int_0^t e^{W_s}d_s\right)' \\ &= \frac{-\alpha}{t^{\alpha+1}}dt\int_0^t e^{W_s}d_s + \frac{1}{t^\alpha}e^{W_t}dt \\ &= \frac{1}{t^\alpha}\left[e^{W_t} - \frac{\alpha}{t}\int_0^t e^{W_s}d_s\right]dt \end{aligned}$$

Problem 20.

a) We are given $d(t\cos(W_t))$. Then we have that $f(t,x) = t\cos(x)$ and $X_t = W_t$. In addition, we have that $\partial_t f = \cos(x)$, $\partial_x f = -t\sin(x)$, and $\partial_x^2 f = -t\cos(x)$.

Applying these identities to equation 6.2.7 yields,

$$\begin{aligned} df(t,X_t) = d(t\cos(W_t)) &= \cos(W_t)dt - t\sin(W_t)dW_t - \frac{1}{2}t\cos(W_t)(dW_t)^2 \\ &= \cos(W_t)dt - t\sin(W_t)dW_t - \frac{1}{2}t\cos(W_t)dt \\ &= \cos(W_t)\left[1-\frac{1}{2}t\right]dt - t\sin(W_t)dW_t \end{aligned}$$

b) We are given $d(e^tW_t^2)$. Then we have that $f(t,x) = e^tx^2$ and $X_t = W_t$. In addition, we have that $\partial_t f = e^tx^2$, $\partial_x f = 2e^tx$, and $\partial_x^2 f = 2e^t$.

Applying these identities to equation 6.2.7 yields,

$$\begin{aligned} df(t,X_t) = d(e^tW_t^2) &= e^tW_t^2dt + 2e^tW_tdW_t + \frac{1}{2}(2e^t)(dW_t)^2 \\ &= e^tW_t^2dt + 2e^tW_tdW_t + e^tdt \\ &= e^t\left[W_t^2+1\right]dt + 2e^tW_tdW_t \end{aligned}$$

c) We are given $d(\sin(t)W_t^2)$. Then we have that $f(t,x) = \sin(t)x^2$ and $X_t = W_t$. In addition, we have that $\partial_t f = \cos(t)x^2$, $\partial_x f = 2\sin(t)x$, and $\partial_x^2 f = 2\sin(t)$.

Applying these identities to equation 6.2.7 yields,

$$\begin{aligned} df(t,X_t) = d(\sin(t)W_t^2) &= \cos(t)W_t^2 dt + 2\sin(t)W_t dW_t + \frac{1}{2}(2\sin(t))(dW_t)^2 \\ &= \cos(t)W_t^2 dt + 2\sin(t)W_t dW_t + \sin(t)dt \\ &= \Big[\cos(t)W_t^2 + \sin(t)\Big]\, dt + 2\sin(t)W_t dW_t \end{aligned}$$

Problem 21.

From Problem 8, we know that for any two independent random variables X and Y, we have $E(XY) = E(X)E(Y)$ and $\mathrm{Var}(XY) = \mathrm{Var}(X)\mathrm{Var}(Y) + \mathrm{Var}(X)\,(E(Y))^2 + \mathrm{Var}(Y)\,(E(X))^2$.

Now consider two independent Brownian motions $W_t^{(1)}$ and $W_t^{(2)}$. Furthermore, consider the differences for each Brownian motion: $W_{t_1+h}^{(1)} - W_{t_1}^{(1)}$ and $W_{t_2+g}^{(2} - W_{t_2}^{(2)}$ with $h > 0$ and $g > 0$.

We know that these differences both have mean 0 and variances $|(t+h) - t| = |h| = h$ and $|(t+g) - t| = |g| = g$, respectively.

Now we will take the limit of each difference as $h \to 0^+$ and $g \to 0^+$. In this case, we get that,

$$W_{t_1+h}^{(1)} - W_{t_1}^{(1)} \to dW_t^{(1)}$$

as $h \to 0$ and,

$$\mathrm{Var}(W_{t_1+h}^{(1)} - W_{t-1}^{(1)}) = h \to dt_1$$

as $h \to 0$.

The same holds for the differences of $W_t^{(2)}$.

Thus, $dW_t^{(1)} \sim N(0, dt_1)$ and $dW_t^{(2)} \sim N(0, dt_2)$. From Problem 8, we have that,

$$E(dW_t^{(1)} \cdot dW_t^{(2)}) = E(dW_t^{(1)}) \cdot E(dW_t^{(2)}) = 0$$

and,

$$\begin{aligned} \mathrm{Var}(dW_t^{(1)} \cdot dW_t^{(2)}) &= \mathrm{Var}(dW_t^{(1)})\mathrm{Var}(dW_t^{(2)}) + \mathrm{Var}(dW_t^{(1)}) \left(E(dW_t^{(2)})\right)^2 + \mathrm{Var}(dW_t^{(2)}) \left(E(dW_t^{(1)})\right)^2 \\ &= dt_1 dt_2 + dt_1(0)^2 + dt_2(0)^2 \\ &= dt_1 dt_2 \end{aligned}$$

Since $dt_1 dt_2$ can be made arbitrarily close to 0, we can say that $\mathrm{Var}(dW_t^{(1)} \cdot dW_t^{(2)})$.

Since the mean of $dW_t^{(1)} \cdot dW_t^{(2)}$ is a constant (0 in this case) and the variance approaches 0, we can say that the mean-squared limit of this quantity is 0. Thus, we have that,

$$dW_t^{(1)} \cdot dW_t^{(2)} = 0$$

Problem 22.

Problem 23.

a) We will begin by showing that the derivative of the answer is equal to the integrand.

Let $f(t,x) = 1 - e^{t/2}\cos(x)$. Then $\partial_t f = -\frac{1}{2}e^{t/2}\cos(x)$, $\partial_x f = e^{t/2}\sin(x)$, and $\partial_x^2 f = e^{t/2}\cos(x)$.

Applying these identities to equation 6.2.7 yields,

$$\begin{aligned}
df(t, X_t) = d(1 - e^{t/2}\cos(W_t)) &= -\frac{1}{2}e^{t/2}\cos(W_t)dt + e^{t/2}\sin(W_t)dW_t + \frac{1}{2}e^{t/2}\cos(W_t)(dW_t)^2 \\
&= -\frac{1}{2}e^{t/2}\cos(W_t)dt + e^{t/2}\sin(W_t)dW_t + \frac{1}{2}e^{t/2}\cos(W_t)dt \\
&= \frac{1}{2}\left[e^{t/2}\cos(W_t) - e^{t/2}\cos(W_t)\right]dt + e^{t/2}\sin(W_t)dW_t \\
&= \frac{1}{2}[0]dt + e^{t/2}\sin(W_t)dW_t \\
&= e^{t/2}\sin(W_t)dW_t
\end{aligned}$$

So from the problem statement and the above derivation we have that,

$$\begin{aligned}
\int_0^t e^{s/2}\sin(W_s)dW_s &= \int_0^t df(s, W_s) \\
&= f(t, W_t) \\
&= 1 - e^{t/2}\cos(W_t)
\end{aligned}$$

b) Let us take the derivative of the integrated function. By the sum rule of the derivative, we have,

$$df(W_t) = d\left(\sin(W_t) + \frac{1}{2}\int_0^T \sin(W_t)dt\right) = d\left[\sin(W_t)\right] + \frac{1}{2}\left[d\left(\int_0^T \sin(W_t)dt\right)\right]$$

A direct result of Corollary 6.2.3 is that $d(\sin(W_t)) = \cos(W_t)dW_t - \frac{1}{2}\sin(W_t)dt$. In addition we know that the derivative of the integral over the whole domain (in this case 0 to T) is precisely the integrand. Thus,

$$df(W_t) = d\left[\sin(W_t)\right] + \frac{1}{2}\left[d\left(\int_0^T \sin(W_t)dt\right)\right] = \cos(W_t)dW_t - \frac{1}{2}\sin(W_t)dt + \frac{1}{2}\sin(W_t)dt$$

$$= \cos(W_t)dW_t$$

So we have,

$$\int_0^T df(W_t)dW_t = f(W_t)$$
$$\implies \int_0^T \cos(W_t)dW_t = \sin(W_t) + \frac{1}{2}\int_0^T \sin(W_t)dt$$

c) The below formulas are from pages 163 and 164 in the textbook:

Problem 24.

By Proposition 8.2.1, we know that if both the drift and volatility of a stochastic differential equation are just functions of time t, then the solution is Gaussian distributed with the mean $X_0 + \int_0^t a(s)ds$ and variance $\int_0^t b^2(s)ds$.

In the given stochastic differential equation, we see that

$$a(t) = \frac{t}{1+t^2}$$

and,

$$b(t) = t^{3/2}$$

Thus, we can apply Proposition 8.2.1 in this case. As a result, the mean is given by,

$$X_0 + \int_0^t a(s)ds = 1 + \int_0^t \frac{s}{1+s^2}ds$$
$$= 1 + \frac{1}{2}\ln(t^2+1)$$

and the variance is given by,

$$\int_0^t b^2(s)ds = \int_0^t \left(s^{3/2}\right)^2 ds$$
$$= \int_0^t s^3 ds$$
$$= \frac{t^4}{4}$$

Thus, the distribution of the solution is given by $X_t \sim N(1+\frac{1}{2}\ln(t^2+1), \frac{t^4}{4})$.

Moreover, the formula for the solution X_t, given by 8.1.2, is

$$X_t = X_0 + \int_0^t a(s)ds + \int_0^t b(s)dW_s$$
$$= 1 + \int_0^t \frac{s}{1+s^2}ds + \int_0^t t^{3/2}dW_s$$
$$= 1 + \frac{1}{2}\ln(t^2+1) + \int_0^t t^{3/2}dW_s$$

Problem 25.

a) We are given $dX_t = \cos t dt - \sin t dW_t$, $X_0 = 1$

As a result, we have $a(t) = \cos t$ and $b(t) = -\sin t$ and can use Proposition 8.2.1 as above.

Hence, the mean is given by,

$$\begin{aligned} X_0 + \int_0^t a(s)ds &= 1 + \int_0^t \cos(s)ds \\ &= 1 + [\sin(t) - \sin(0)] = 1 + \sin(t) \end{aligned}$$

and the variance is given by,

$$\begin{aligned} \int_0^t b^2(s)ds &= \int_0^t (-\sin s)^2 \, ds \\ &= \int_0^t \sin^2(s)ds \\ &= \frac{t}{2} - \frac{1}{4}\sin(2t) \\ &= \frac{1}{2}\left[t - \sin(t)\cos(t)\right] \end{aligned}$$

Thus, $X_t \sim N\left(1 + \sin(t), \frac{1}{2}\left[t - \sin(t)\cos(t)\right]\right)$ and the solution is given by,

$$\begin{aligned} X_t &= X_0 + \int_0^t a(s)ds + \int_0^t b(s)\,dW_s \\ &= 1 + \int_0^t \cos(s)ds + \int_0^t -\sin s\,dW_s \\ &= 1 + \sin(t) - \int_0^t \sin s\,dW_s \end{aligned}$$

b) We are given $dX_t = e^t dt + \sqrt{t} dW_t$, $X_0 = 0$

As a result, we have $a(t) = e^t$ and $b(t) = \sqrt{t}$ and can use Proposition 8.2.1 as above.

Hence, the mean is given by,

$$\begin{aligned} X_0 + \int_0^t a(s)ds &= 0 + \int_0^t e^s ds \\ &= e^t - e^0 \\ &= e^t - 1 \end{aligned}$$

and the variance is given by,

$$\begin{aligned}\int_0^t b^2(s)ds &= \int_0^t \left(\sqrt{s}\right)^2 ds \\ &= \int_0^t sds \\ &= \frac{t^2}{2}\end{aligned}$$

Thus, $X_t \sim N\left(e^t - 1, \frac{t^2}{2}\right)$ and the solution is given by,

$$\begin{aligned}X_t &= X_0 + \int_0^t a(s)ds + \int_0^t b(s)\, dW_s \\ &= 0 + \int_0^t e^s ds + \int_0^t \sqrt{s}\, dW_s \\ &= e^t - 1 + \int_0^t \sqrt{s}\, dW_s\end{aligned}$$

Problem 26.

We have,

$$a(t,x) = 2tx^3 + 3t^2(1+x)$$

and

$$b(t,x) = 3t^2x^2 + 1$$

So the associated system is,

$$\begin{aligned}2tx^3 + 3t^2(1+x) &= \partial_t f(t,x) + \frac{1}{2}\partial_x^2 f(t,x) \\ 3t^2x^2 + 1 &= \partial_x f(t,x)\end{aligned}$$

Integrating partially in x in the second equation yields,

$$f(t,x) = \int (3t^2x^2+1)dx = t^2x^3 + x + T(t)$$

Thus, $\partial_t f = 2tx^3 + T'(t)$. Using the first equation, we have

$$2tx^3 + 3t^2(1+x) = 2tx^3 + T'(t) + 3t^2x$$

This implies that $T'(t) = 3t^2$. As a result, $T(t) = t^3 + c$. Hence,

$$f(t,x) = t^2x^3 + x + t^3 + c$$

And we have,

$$X_t = f(t, W_t) = t^2(W_t)^3 + W_t + t^3 + c$$

Since $X_0 = 0$ is given, we have,

$$X_0 = f(0, W_0) = 0^2(W_0)^3 + W_0 + 0^3 + c = 0$$

Thus, $c = 0$. The solution is then,

$$X_t = t^2(W_t)^3 + W_t + t^3$$

Problem 27.

a) We have $a(t, x) = e^t$ and $b(t, x) = x^2 - t$.

So the associated system is,

$$\begin{aligned} e^t &= \partial_t f(t, x) + \frac{1}{2}\partial_x^2 f(t, x) \\ x^2 - t &= \partial_x f(t, x) \end{aligned}$$

Integrating partially in x in the second equation yields,

$$f(t, x) = \int (x^2 - t)dx = \frac{x^3}{3} - tx + T(t)$$

Thus, $\partial_t f = -x + T'(t)$. Using the first equation, we have

$$e^t = -x + T'(t) + x = T'(t)$$

This implies that $T'(t) = e^t$. As a result, $T(t) = e^t + c$. Hence,

$$f(t, x) = \frac{x^3}{3} - tx + e^t + c$$

Since $X_0 = 1$ is given, we have,

$$X_0 = f(0, W_0) = \frac{W_0^3}{3} - 0W_0 + e^0 + c = 1 + c = 1$$

Thus, $c = 0$. The solution is then,

$$X_t = \frac{W_t^3}{3} - t(W_t) + e^t$$

b) We have $a(t, x) = \sin t$ and $b(t, x) = x^2 - t$.

So the associated system is,

$$\sin t = \partial_t f(t, x) + \frac{1}{2}\partial_x^2 f(t, x)$$

$$x^2 - t = \partial_x f(t,x)$$

Integrating partially in x in the second equation yields,

$$f(t,x) = \int (x^2 - t)dx = \frac{x^3}{3} - tx + T(t)$$

Thus, $\partial_t f = -x + T'(t)$. Using the first equation, we have

$$\sin t = -x + T'(t) + x = T'(t)$$

This implies that $T'(t) = \sin t$. As a result, $T(t) = -\cos t + c$. Hence,

$$f(t,x) = \frac{x^3}{3} - tx - \cos t + c$$

Since $X_0 = -1$ is given, we have,

$$X_0 = f(0, W_0) = \frac{W_0^3}{3} - 0W_0 - \cos(0) + c = -1 + c = -1$$

Thus, $c = 0$. The solution is then,

$$X_t = \frac{W_t^3}{3} - t(W_t) - \cos t$$

c) We have $a(t,x) = t^2$ and $b(t,x) = e^{x - \frac{t}{2}}$

So the associated system is,

$$t^2 = \partial_t f(t,x) + \frac{1}{2}\partial_x^2 f(t,x)$$
$$e^{x-\frac{t}{2}} = \partial_x f(t,x)$$

Integrating partially in x in the second equation yields,

$$f(t,x) = \int e^{x-\frac{t}{2}}dx = e^{x-\frac{t}{2}} + T(t)$$

Thus, $\partial_t f = -\frac{1}{2}e^{x-\frac{t}{2}} + T'(t)$. Using the first equation, we have

$$t^2 = -\frac{1}{2}e^{x-\frac{t}{2}} + T'(t) + \frac{1}{2}e^{x-\frac{t}{2}} = T'(t)$$

This implies that $T'(t) = t^2$. As a result, $T(t) = \frac{t^3}{3} + c$. Hence,

$$f(t,x) = e^{x-\frac{t}{2}} + \frac{t^3}{3} + c$$

Since $X_0 = 0$ is given, we have,

$$\begin{aligned} X_0 = f(0, W_0) &= e^{W_0 - \frac{0}{2}} + \frac{0^3}{3} + c \\ &= e^0 + 0 + c \\ &= 1 + c = 0 \end{aligned}$$

Thus, $c = -1$. The solution is then,

$$X_t = e^{W_t - \frac{t}{2}} + \frac{t^3}{3} - 1$$

d) We have $a(t, x) = t$ and $b(t, x) = e^{t/2}(\cos x)$

So the associated system is,

$$\begin{aligned} t &= \partial_t f(t, x) + \frac{1}{2}\partial_x^2 f(t, x) \\ e^{t/2}(\cos x) &= \partial_x f(t, x) \end{aligned}$$

Integrating partially in x in the second equation yields,

$$f(t, x) = \int e^{t/2}(\cos x)dx = e^{t/2}(\sin x) + T(t)$$

Thus, $\partial_t f = \frac{1}{2}e^{t/2}(\sin x) + T'(t)$. Using the first equation, we have

$$t = \frac{1}{2}e^{t/2}(\sin x) + T'(t) - \frac{1}{2}e^{t/2}(\sin x) = T'(t)$$

This implies that $T'(t) = t$. As a result, $T(t) = \frac{t^2}{2} + c$. Hence,

$$f(t, x) = e^{t/2}(\sin x) + \frac{t^2}{2} + c$$

Since $X_0 = 1$ is given, we have,

$$\begin{aligned} X_0 = f(0, W_0) &= e^{0/2}(\sin W_0) + \frac{0^2}{2} + c \\ &= e^0(\sin 0) + 0 + c \\ &= 0(0) + 0 + c = c = 1 \end{aligned}$$

Thus, $c = 1$. The solution is then,

$$X_t = e^{t/2}(\sin W_t) + \frac{t^2}{2} + 1$$

Problem 28.

a) We have $a(t,x) = x + \frac{3}{2}x^2$ and $b(t,x) = t + x^3$

We will first verify the closeness condition:

$$\partial_x a = 1 + 3x$$

and,

$$\begin{aligned}\partial_t b + \frac{1}{2}\partial_x^2 b &= 1 + \frac{1}{2}(6x)\\ &= 1 + 3x\end{aligned}$$

Thus, we have $\partial_x a = \partial_t b + \frac{1}{2}\partial_x^2 b$ and the closeness condition is satisfied.

So the associated system is,

$$\begin{aligned}x + \frac{3}{2}x^2 &= \partial_t f(t,x) + \frac{1}{2}\partial_x^2 f(t,x)\\ t + x^3 &= \partial_x f(t,x)\end{aligned}$$

Integrating partially in x in the second equation yields,

$$f(t,x) = \int t + x^3 dx = tx + \frac{x^4}{4} + T(t)$$

Thus, $\partial_t f = x + T'(t)$. Using the first equation, we have

$$\begin{aligned}x + \frac{3}{2}x^2 &= x + T'(t) + \frac{1}{2}3x^2\\ \implies T'(t) &= 0\end{aligned}$$

This implies that $T'(t) = 0$. As a result, $T(t) = c$. Hence,

$$f(t,x) = tx + \frac{x^4}{4} + c$$

Since $X_0 = 0$ is given, we have,

$$\begin{aligned}X_0 = f(0, W_0) &= 0(W_0) + \frac{W_0^4}{4} + c\\ &= 0 + 0 + c\\ &= c = 0\end{aligned}$$

Thus, $c = 0$. The solution is then,

$$X_t = t(W_t) + \frac{W_t^4}{4}$$

b) We have $a(t,x) = 2tx$ and $b(t,x) = t^2 + x$

We will first verify the closeness condition:

$$\partial_x a = 2t$$

and,

$$\begin{aligned}\partial_t b + \frac{1}{2}\partial_x^2 b &= 2t + \frac{1}{2}0 \\ &= 2t\end{aligned}$$

Thus, we have $\partial_x a = \partial_t b + \frac{1}{2}\partial_x^2 b$ and the closeness condition is satisfied.

So the associated system is,

$$\begin{aligned}2tx &= \partial_t f(t,x) + \frac{1}{2}\partial_x^2 f(t,x) \\ t^2 + x &= \partial_x f(t,x)\end{aligned}$$

Integrating partially in x in the second equation yields,

$$f(t,x) = \int (t^2 + x)dx = t^2 x + \frac{x^2}{2} + T(t)$$

Thus, $\partial_t f = 2tx + T'(t)$. Using the first equation, we have

$$\begin{aligned}2tx &= 2tx + T'(t) + \frac{1}{2}1 \\ \implies T'(t) &= -\frac{1}{2}\end{aligned}$$

This implies that $T'(t) = -\frac{1}{2}$. As a result, $T(t) = -\frac{1}{2}t + c$. Hence,

$$f(t,x) = t^2 x + \frac{x^2}{2} - \frac{1}{2}t + c$$

Since $X_0 = 0$ is given, we have,

$$\begin{aligned}X_0 = f(0, W_0) &= 0^2(W_0) + \frac{W_0^2}{2} - \frac{1}{2}0 + c \\ &= 0 + 0 + 0 + c \\ &= c = 0\end{aligned}$$

Thus, $c = 0$. The solution is then,

$$X_t = t^2(W_t) + \frac{W_t^2}{2} - \frac{1}{2}t$$

c) We have $a(t,x) = e^t x + \frac{1}{2}\cos x$ and $b(t,x) = e^t + \sin x$

We will first verify the closeness condition:

$$\partial_x a = e^t - \frac{1}{2}\sin x$$

and,

$$\partial_t b + \frac{1}{2}\partial_x^2 b = e^t - \frac{1}{2}\sin x$$

Thus, we have $\partial_x a = \partial_t b + \frac{1}{2}\partial_x^2 b$ and the closeness condition is satisfied.

So the associated system is,

$$\begin{aligned} e^t x + \frac{1}{2}\cos x &= \partial_t f(t,x) + \frac{1}{2}\partial_x^2 f(t,x) \\ e^t + \sin x &= \partial_x f(t,x) \end{aligned}$$

Integrating partially in x in the second equation yields,

$$f(t,x) = \int (e^t + \sin x)dx = e^t x - \cos x + T(t)$$

Thus, $\partial_t f = e^t x + T'(t)$. Using the first equation, we have

$$\begin{aligned} e^t x + \frac{1}{2}\cos x &= e^t x + T'(t) + \frac{1}{2}\cos x \\ \implies T'(t) &= -\frac{1}{2} \end{aligned}$$

This implies that $T'(t) = 0$. As a result, $T(t) = c$. Hence,

$$f(t,x) = e^t x - \cos x + c$$

Since $X_0 = 0$ is given, we have,

$$\begin{aligned} X_0 = f(0, W_0) &= e^0(W_0) - \cos(W_0) + c \\ &= 1(0) - 1 + c \\ &= -1 + c = 0 \end{aligned}$$

Thus, $c = 1$. The solution is then,

$$X_t = e^t(W_t) - \cos(W_t) + 1$$

d) We have $a(t,x) = e^x(1+\frac{t}{2})$ and $b(t,x) = te^x$

We will first verify the closeness condition:

$$\partial_x a = e^x(1+\frac{t}{2})$$

and,

$$\begin{aligned}\partial_t b + \frac{1}{2}\partial_x^2 b &= e^x + \frac{1}{2}te^x \\ &= e^x(1+\frac{t}{2})\end{aligned}$$

Thus, we have $\partial_x a = \partial_t b + \frac{1}{2}\partial_x^2 b$ and the closeness condition is satisfied.

So the associated system is,

$$\begin{aligned}e^x(1+\frac{t}{2}) &= \partial_t f(t,x) + \frac{1}{2}\partial_x^2 f(t,x) \\ te^x &= \partial_x f(t,x)\end{aligned}$$

Integrating partially in x in the second equation yields,

$$f(t,x) = \int (te^x)dx = te^x + T(t)$$

Thus, $\partial_t f = e^x + T'(t)$. Using the first equation, we have

$$\begin{aligned}e^x(1+\frac{t}{2}) &= e^x + T'(t) + \frac{1}{2}te^x \\ \implies e^x(1+\frac{t}{2}) &= e^x(1+\frac{t}{2}) + T'(t) \\ \implies T'(t) &= 0\end{aligned}$$

This implies that $T'(t) = 0$. As a result, $T(t) = c$. Hence,

$$f(t,x) = te^x + c$$

Since $X_0 = 2$ is given, we have,

$$\begin{aligned}X_0 = f(0,W_0) &= 0e^{W_0} + c \\ &= 0(1) + c \\ &= c = 2\end{aligned}$$

Thus, $c = 2$. The solution is then,

$$X_t = te^{W_t} + 2$$

Problem 29.

We have $dX_t = (m - X_t)dt + \alpha dW_t$ where X_0 is constant.

If we add $X_t dt$ to both sides and multiply by the integrating factor e^t, we get,

$$d(e^t X + t) = me^t dt + \alpha e^t dW_t$$

Integrating both sides yields,

$$\begin{aligned}
e^t X_t &= \int_0^t [me^s ds + \alpha e^s dW_s] \\
&= \int_0^t me^s ds + \int_0^t \alpha e^s dW_s \\
&= m\int_0^t e^s ds + \alpha \int_0^t e^s dW_s \\
&= m(e^t - e^0) + \alpha \int_0^t e^s dW_s \\
&= m(e^t - 1) + \alpha \int_0^t e^s dW_s
\end{aligned}$$

Dividing by e^t on both sides give us,

$$\begin{aligned}
X_t &= X_0 e^{-t} + m - e^{-t} + \alpha e^{-t} \int_0^t e^s dW_s \\
&= m + (X_0 - m)e^{-t} + \alpha \int_0^t e^{s-t} dWs
\end{aligned}$$

We know that $\alpha \int_0^t e^{s-t} dWs$ is a Wiener integral. Thus, by Proposition 5.6.1, X_t is Gaussian with,

$$\begin{aligned}
E[X_t] &= m + (X_0 - m)e^{-t} + E\left[\alpha \int_0^t e^{s-t} dWs\right] = m + (X_0 - m)e^{-t} \\
\text{Var}(X_t) &= \text{Var}\left[\alpha \int_0^t e^{s-t} dW_s\right] = \alpha^2 e^{-2t} \int_0^t e^{2s} ds \\
&= \alpha^2 e^{-2t} \frac{e^{2t} - 1}{2} = \frac{1}{2}\alpha^2 (1 - e^{-2t})
\end{aligned}$$

Problem 30.

Problem 31.

a) We have $dX_t = \alpha X_t dW_t$ with α a constant. The integrating factor is given by,

$$\begin{aligned}
\rho_t &= e^{-\int_0^t g(s)dW_s + \frac{1}{2}\int_0^t g^2(s)ds} \\
&= e^{-\int_0^t \alpha dW_s + \frac{1}{2}\int_0^t \alpha^2 ds} \\
&= e^{\frac{1}{2}\alpha^2 t - \alpha W_t}
\end{aligned}$$

By Ito's formula, we have that,

$$d\rho_t = \rho_t(\alpha^2 dt - \alpha dW_t)$$

Using $dt^2 = dtdW_t = 0$ and $(dW_t)^2 = dt$, we obtain,

$$\begin{aligned} dX_t d\rho_t &= \alpha X_t dW_t[\rho_t(\alpha^2 dt - \alpha dW_t)] \\ &= \alpha X_t dW_t[\rho_t \alpha^2 dt - \rho_t \alpha dW_t] \\ &= \alpha X_t[\rho_t \alpha^2 dW_t dt - \rho_t \alpha (dW_t)^2] \\ &= \alpha X_t[-\rho_t \alpha dt] \\ &= -\alpha^2 \rho_t X_t dt \end{aligned}$$

Multiplying by ρ_t, the initial equation becomes,

$$\rho_t dX_t - \alpha \rho_t X_t dW_t = 0$$

and adding and subtracting $\alpha^2 \rho_t X_t dt$ from the left side yields,

$$\rho_t dX_t - \alpha \rho_t X_t dW_t + \alpha^2 \rho_t X_t dt - \alpha^2 \rho_t X_t dt = 0$$

This can be written as,

$$\rho_t dX_t + X_t d\rho_t + d\rho_t dX_t = 0$$

which, by virtue of the product rule, becomes

$$d(\rho_t X_t) = 0$$

Integrating yields,

$$\begin{aligned} \rho_t X_t = \rho_0 X_0 + \int_0^t 0 ds &= \rho_0 X_0 \\ &= (e^{\frac{1}{2}\alpha^2(0) - \alpha W_0}) X_0 \\ &= (e^0) X_0 = X_0 \end{aligned}$$

And hence the solution is,

$$\begin{aligned} X_t &= \frac{X_0}{\rho_t} \\ &= \frac{X_0}{e^{\frac{1}{2}\alpha^2 t - \alpha W_t}} \end{aligned}$$

b) We have $dX_t = X_t dt + \alpha X_t dW_t$ with α a constant. The integrating factor is given by,

$$\begin{aligned}\rho_t &= e^{-\int_0^t g(s)dW_s + \frac{1}{2}\int_0^t g^2(s)ds} \\ &= e^{-\int_0^t \alpha dW_s + \frac{1}{2}\int_0^t \alpha^2 ds} \\ &= e^{\frac{1}{2}\alpha^2 t - \alpha W_t}\end{aligned}$$

By Ito's formula, we have that,

$$d\rho_t = \rho_t(\alpha^2 dt - \alpha dW_t)$$

Using $dt^2 = dtdW_t = 0$ and $(dW_t)^2 = dt$, we obtain,

$$\begin{aligned}dX_t d\rho_t &= [X_t dt + \alpha X_t dW_t][\rho_t(\alpha^2 dt - \alpha dW_t)] \\ &= [X_t dt + \alpha X_t dW_t][\rho_t \alpha^2 dt - \rho_t \alpha dW_t] \\ &= \rho_t X_t \alpha^2 (dt)^2 - \rho_t X_t \alpha dt dW_t + \rho_t X_t \alpha^3 dt dW_t - \rho_t X_t \alpha^2 (dW_t)^2 \\ &= -\alpha^2 \rho_t X_t dt\end{aligned}$$

Multiplying by ρ_t, the initial equation becomes,

$$\rho_t dX_t - \alpha \rho_t X_t dW_t = \rho_t X_t dt$$

and adding and subtracting $\alpha^2 \rho_t X_t dt$ from the left side yields,

$$\rho_t dX_t - \alpha \rho_t X_t dW_t + \alpha^2 \rho_t X_t dt - \alpha^2 \rho_t X_t dt = \rho_t X_t dt$$

which, by virtue of the product rule, becomes

$$d(\rho_t X_t) = \rho_t X_t dt$$

So, with $Y_t = \rho_t X_t$, we have

$$\begin{aligned}& d(Y_t) = Y_t dt \\ \Longrightarrow\ & Y_t = Y_0 e^t \\ \Longrightarrow\ & \rho_t X_t = X_0 e^t \\ \Longrightarrow\ & X_t = X_0 e^{(1-\frac{\alpha^2}{2})t + \alpha W_t}\end{aligned}$$

c) We have $dX_t = \frac{1}{X_t}dt + \alpha X_t dW_t$ with α a constant and $X_0 > 0$. The integrating factor is given by,

$$\begin{aligned}\rho_t &= e^{-\int_0^t g(s)dW_s + \frac{1}{2}\int_0^t g^2(s)ds} \\ &= e^{-\int_0^t \alpha dW_s + \frac{1}{2}\int_0^t \alpha^2 ds} \\ &= e^{\frac{1}{2}\alpha^2 t - \alpha W_t}\end{aligned}$$

By Ito's formula, we have that,

$$d\rho_t = \rho_t(\alpha^2 dt - \alpha dW_t)$$

Using $dt^2 = dtdW_t = 0$ and $(dW_t)^2 = dt$, we obtain,

$$\begin{aligned} dX_t d\rho_t &= [\frac{1}{X_t}dt + \alpha X_t dW_t][\rho_t(\alpha^2 dt - \alpha dW_t)] \\ &= [\frac{1}{X_t}dt + \alpha X_t dW_t][\rho_t \alpha^2 dt - \rho_t \alpha dW_t] \\ &= \rho_t \frac{1}{X_t}\alpha^2 (dt)^2 - \rho_t \frac{1}{X_t}\alpha dt dW_t + \rho_t X_t \alpha^3 dt dW_t - \rho_t X_t \alpha^2 (dW_t)^2 \\ &= -\alpha^2 \rho_t X_t dt \end{aligned}$$

Multiplying by ρ_t, the initial equation becomes,

$$\rho_t dX_t - \alpha \rho_t X_t dW_t = \rho_t \frac{1}{X_t} dt$$

and adding and subtracting $\alpha^2 \rho_t X_t dt$ from the left side yields,

$$\rho_t dX_t - \alpha \rho_t X_t dW_t + \alpha^2 \rho_t X_t dt - \alpha^2 \rho_t X_t dt = \rho_t \frac{1}{X_t} dt$$

which, by virtue of the product rule, becomes

$$d(\rho_t X_t) = \rho_t \frac{1}{X_t} dt$$

Integrating yields,

$$\begin{aligned} \rho_t X_t &= \rho_0 X_0 + \int_0^t \rho_s \frac{1}{X_s} ds \\ &= (e^{\frac{1}{2}\alpha^2(0) - \alpha W_0}) X_0 + \int_0^t \rho_s \frac{1}{X_s} ds \\ &= (e^0) X_0 + \int_0^t \rho_s \frac{1}{X_s} ds \\ &= X_0 + \int_0^t \rho_s \frac{1}{X_s} ds \end{aligned}$$

And hence the solution is,

$$\begin{aligned} X_t &= \frac{1}{\rho_t}\left[X_0 + \int_0^t \rho_s \frac{1}{X_s} ds\right] \\ &= \frac{1}{e^{\frac{1}{2}\alpha^2 t - \alpha W_t}}\left[X_0 + \int_0^t e^{\frac{1}{2}\alpha^2 s - \alpha W_s} \frac{1}{X_s} ds\right] \end{aligned}$$