Mathematics A1200: Stochastic Calculus
Fall Semester 2019

Directions. *All work must be your individual effort.* In additional to mathematical symbols, you are to provide appropriate discussion which justifies your answer.

1. Let X and Y be independent normal random variables with means μ_X and μ_Y, respectively, and variances σ_X^2 and σ_Y^2, repectively. Prove that $X + Y$ is a normal random variable with mean $\mu_X + \mu_Y$ and variance $\sigma_X^2 + \sigma_Y^2$.

2. Let X and Y be independent standard uniform random variables. Compute the probability density function of $X/(Y + 1)$.

3. Compute the moment generating function of a standard normal random variable.

4. If $g(x)$ is a monotone increasing function, compute the probability density function of the random variable $g(X)$ given the probability density function f_X of X.

5. If $\phi(t)$ is the moment generating function of a random variable X, compute the moment generating function of $aX+b$ for constants $a \neq 0$ and b.

6. Let X and Y be independent random variables, each with finite mean and finite variance. Determine the probability density function $f_Z(z)$ of the sum $Z = X + Y$ given the density functions f_X and f_Y of X and Y, respectively. From this, compute the mean and variance of Z in terms of the means and variances of X and Y.

7. State, in precise mathematical terms, the (classical) Central Limit Theorem. Describe why one example of the theorem is the well-known normal approximation to the binomial random variable.

8. Let X and Y be independent random variables, each with finite mean and finite variance. Determine the probability density function $f_Z(z)$

of the product $Z = XY$ given the density functions f_X and f_Y of X and Y, respectively. From this, compute the mean and variance of Z in terms of the means and variances of X and Y.

9. Let $p > 1$ be any real number, and $f(x)$ a bounded function on the interval $[a, b]$. For any integer N sufficiently large, let $t_j = (j/N)(b - a) + a$, for $j = 0$ to N, determine a partition of $[a, b]$. Prove that

$$\lim_{N \to 0} \sum_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p = 0.$$

10. Assume that notation of problem 9. In addition, for each interval $[t_{j-1}, t_j]$ let $\Delta_{j-1}^{j} W$ denote a random variable which is normal, has mean zero and variance $t_j - t_{j-1}$. Furthermore, if $j \neq k$, assume that $\Delta_{j-1}^{j} W$ and $\Delta_{k-1}^{k} W$ are independent. Prove that

$$\lim_{N \to 0} \sum_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2 = \int_a^b f(t)dt.$$

11. Assume that notation of problems 9 and 10. Prove that for any pair of non-negative integers p and q such that the pair (p, q) is different from $(0, 0)$, $(1, 0)$, $(0, 1)$ or $(0, 2)$, one has that

$$\lim_{N \to 0} \sum_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p (\Delta_{j-1}^{j} W)^q = 0$$

12. Let $\{f_k\}$ and $\{g_k\}$ be two sequences of real numbers. Prove that

$$\sum_{k=m}^{n} f_k(g_{k+1} - g_k) = (f_n g_{n+1} - f_m g_m) - \sum_{k=m+1}^{n} g_k(f_k - f_{k-1}),$$

which can be viewed as a *summation by parts* formula.

13. From first principles (meaning, without Itô's lemma), prove that

$$\int_0^T W_t dW_t = \frac{1}{2}W_T^2 - \frac{1}{2}T.$$

14. Let $f(x)$ be any bounded and continuous function. Prove, from first principles, that the random variable

$$Y_t = \int_0^T f(t) dW_t$$

is normally distributed. What is its mean and variance?

15. Let

$$Z_t = \int_0^t W_s ds.$$

Prove, from first principles, that

$$Z_t = \int_0^t (t-s) dW_s.$$

16. Determine the conditions on the constants a and b such that the process

$$X_t = \int_0^t \left(a + \frac{bu}{t}\right) dW_u$$

has variance one.

17. Let X_t be a process which is defined by

$$dX_t = rX_t dt + \sigma X_t dW_t$$

and X_0 is a constant (with probability one). Compute the probability density function of $\log X_t$.

18. Using the notation of problem 17, let K be a constant greater than zero, let

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du.$$

Compute the expected value of $\max\{X_t - K, 0\}$, and express your answer in terms of $\Phi(x)$.

19. Use Itô's lemma to express the following increments in the form $b_t dt + \sigma_t dW_t$.

a. $d(W_t e^{W_t})$;

b. $d(e^{t+W_t^2})$;

c. $d\left(\frac{1}{t^\alpha} \int_0^t e^{W_s} ds\right)$.

20. Use Itô's lemma to express the following increments in the form $b_t dt + \sigma_t dW_t$.

a. $d(t\cos(W_t))$;

b. $d(e^t W_t^2)$;

c. $d(\sin(t) W_t^2)$.

21. If $W_t^{(1)}$ and $W_t^{(2)}$ are independent Brownian motion, discuss the rationale behind the (formal) algebraic expression $dW_t^{(1)} \cdot dW_t^{(2)} = 0$. (Hint: Use Problem 8.)

22. Let $f(u,v)$ be a "nice" function of two variables (meaning continuously differentiable to all orders, bounded, etc.) and set $F_t = f(W_t^{(1)}, W_t^{(2)})$. Prove that

$$\begin{aligned} dF_t &= \frac{\partial f}{\partial u}\Big|_{(u,v)=(W_t^{(1)},W_t^{(2)})} dW_t^{(1)} + \frac{\partial f}{\partial v}\Big|_{(u,v)=(W_t^{(1)},W_t^{(2)})} dW_t^{(2)} \\ &\quad + \frac{1}{2}\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right)\Big|_{(u,v)=(W_t^{(1)},W_t^{(2)})} dt. \end{aligned}$$

23. Verify the following stochastic integration formulas.

a. $\displaystyle\int_0^t e^{s/2}\sin(W_s)dW_s = 1 - e^{t/2}\cos(W_t);$

b. $\displaystyle\int_0^t \cos(W_s)dW_s = \sin(W_t) + \frac{1}{2}\sin(W_t)dt;$

c. All other formulas from pages 163 and 164 of the text by Calin.

24. Describe, in as much detail as possible, the probability distribution of X_t when $X_0 = 1$ is a constant (with probability one) and

$$dX_t = \frac{t}{1+t^2}dt + t^{3/2}dW_t.$$

25. Parts (a) and (b) of Exercise 8.2.2 of the text by Calin.

26. Prove that the stochastic differential equation

$$dX_t = (2tW_t^3 + 3t^2(1+W_t))dt + (3t^2W_t^2 + 1)dW_t,$$

with $X_0 = 0$, is exact. With this, solve for X_t.

27. Exercise 8.3.6 of the text by Calin.

28. Exercise 8.3.7 of the text by Calin.

28. Determine the solution of the Ornstein-Uhlenbeck mean-reverting process, which is given by

$$dX_t = (m - X_t)dt + \alpha dW_t$$

where X_0 is constant. Compute the mean and variance of X_t.

29. For a and b in $\mathbf{R}$, solve the stochastic differential equation

$$dX_t = \frac{b - X_t}{1-}dt + \sigma dW_t$$

for any $\sigma > 0$ which is constant, and $t \in [0,1]$. This process is known as the **Brownian bridge** since $X(0) = a$ and $X(1) = b$ almost surely.

30. Exercise 8.8.2 on pae 188 of Calin.