Stochastic Differential Equations: Final Project

Chris Hayduk

December 15, 2019

NOTE: Still need to complete Problems 2, 7, 8, 15

Problem 1.

We know that the probability density of the sum of two independent random variables can be computed by the convolutions of f_X and f_Y. Thus, the probability density of $X + Y$, denoted by f_Z, is

$$f_Z(z) = (f_Y * f_X) = \int_{-\infty}^{\infty} f_Y(z - x) f_X(x) dx \tag{1}$$

We know that the normal density function is given by,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Plugging this density function into equation (1) and letting $\sigma_Z = \sqrt{\sigma_X^2 + \sigma_Y^2}$ yields,

$$
\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} \left[\frac{1}{\sigma_Y\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{(z-x)-\mu_Y}{\sigma_Y}\right)^2} \frac{1}{\sigma_X\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2} \right] dx \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{\sigma_Y\sigma_X(2\pi)} e^{-\frac{1}{2}\left[\left(\frac{z-x-\mu_Y}{\sigma_Y}\right)^2 + \left(\frac{x-\mu_X}{\sigma_X}\right)^2\right]} \right] dx \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{\sigma_Y\sigma_X(2\pi)} e^{-\frac{\sigma_X^2(z-x-\mu_Y)^2+\sigma_Y^2(x-\mu_X)^2}{2\sigma_Y^2\sigma_X^2}} \right] dx \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{(2\pi)\sigma_X\sigma_Y} e^{-\frac{x^2(\sigma_X^2+\sigma_Y^2)-2x(\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X)+\sigma_X^2(z^2+\mu_Y^2-2z\mu_Y)+\sigma_Y^2\mu_X^2}{2\sigma_Y^2\sigma_X^2}} \right] dx \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}\sigma_Z} e^{-\frac{\sigma_Z^2\left(\sigma_X^2(z-\mu_Y)^2+\sigma_Y^2\mu_X^2\right)-\left(\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X\right)^2}{2\sigma_Z^2(\sigma_X\sigma_Y)^2}} \frac{1}{\sqrt{2\pi}\frac{\sigma_X\sigma_Y}{\sigma_Z}} e^{-\frac{\left(x-\frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2}\right)^2}{2\left(\frac{\sigma_X\sigma_Y}{\sigma_Z}\right)^2}} \right] dx \\
&= \frac{1}{\sqrt{2\pi}\sigma_Z} e^{-\frac{(z-(\mu_X+\mu_Y))^2}{2\sigma_Z^2}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}\frac{\sigma_X\sigma_Y}{\sigma_Z}} e^{-\frac{\left(x-\frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2}\right)^2}{2\left(\frac{\sigma_X\sigma_Y}{\sigma_Z}\right)^2}} \right] dx
\end{aligned}
$$

The equation inside the integral symbol represents a valid normal density function for x, so we know it integrates to 1. Thus, the probability density function for $X+Y$ is given by

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma_Z} e^{-\frac{(z-(\mu_X+\mu_Y))^2}{2\sigma_Z^2}}$$

This is precisely a normal density function with mean $\mu_X + \mu_Y$ and variance $\sigma_Z^2 = (\sqrt{\sigma_X^2+\sigma_Y^2})^2 = \sigma_X^2 + \sigma_Y^2$.

Hence, we have shown that, given $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$, $X+Y$ is distributed as $N(\mu_X+\mu_Y, \sigma_X^2+\sigma_Y^2)$.

Problem 2.

Let $Z = Y + 1$. We'll begin by finding the probability density function for Z. Since Y is a standard uniform random variable, we know that

$$f(y) = \begin{cases} 1 & 0 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Thus, the distribution function approach yields,

$$\begin{aligned} F_Z(z) = P(Z \le z) = P(Y + 1 \le z) &= P(Y \le z - 1) \\ &= \int_{-\infty}^{z-1} f(y)dy \\ &= \int_{0}^{z-1} 1\, dy \\ &= z - 1 \end{aligned}$$

So, since as Y ranges from 0 to 1, Z ranges from 1 to 2, we have

$$F_Z(z) = \begin{cases} 0 & z < 1 \\ z - 1 & 1 \le z \le 2 \\ 1 & \text{elsewhere} \end{cases}$$

And the density function for Z is

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \begin{cases} 1 & 1 \le z \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

Now we can simplify the original function to be $U = X/Z$, where we now know the probability density function for both random variables ($X \sim \text{Unif}(0, 1)$ and $Z \sim \text{Unif}(1, 2)$).

So the distribution function is given by,

$$\begin{aligned} F_U(u) = P(U \le u) = P(X/Z \le u) &= P(X \le uZ) \\ &= \int_{-\infty}^{uz} f(x)dx \\ &= \int_{0}^{uz} 1\, dx \\ &= uz \end{aligned}$$

FINISH THIS LATER.

Problem 3.

Let Y be a standard normal variable. Then the moment-generating function of Y is given by,

$$\begin{aligned} m(t) &= E(e^{tY}) \\ &= \int_{-\infty}^{\infty} e^{ty} f(y)\, dy \end{aligned}$$

Since Y is a standard normal variable, we know that $\mu = 0$ and $\sigma = 1$. Thus, we have

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$

Plugging this into the above equation yields,

$$
\begin{aligned}
m(t) &= \int_{-\infty}^{\infty} e^{ty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}+ty}\, dy \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-t)^2} e^{\frac{1}{2}t^2}\, dy \\
&= e^{\frac{1}{2}t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-t)^2}\, dy
\end{aligned}
$$

We can see that the integral is precisely the integral for a normal random variable with $\mu = y - t$ and $\sigma = 1$. Thus, it integrates to 1, yielding,

$$m(t) = e^{\frac{1}{2}t^2}$$

Problem 4.

Suppose $g(x)$ is a monotone increasing function and X is a random variable with the probability density function f_X.

Let $U = g(X)$ where X has the above density function. Since $g(x)$ is an increasing function of x, then $g^{-1}(u)$ is an increasing function of u. Thus,

$$
\begin{aligned}
P(U \le u) &= P[g(X) \le u] \\
&= P\{g^{-1}[g(X)] \le g^{-1}(u)\} \\
&= P[X \le g^{-1}(u)]
\end{aligned}
$$

The above sequence of equalities implies that,

$$F_U(u) = F_X[g^{-1}(u)]$$

When we differentiate with respect to u, we get,

$$f_U(u) = \frac{dF_X[g^{-1}(u)]}{du} = f_X(g^{-1}(u)) \frac{d[g^{-1}(u)]}{du}$$

Problem 5.

We know that the moment-generating function of X is $\phi(t)$. Thus,

$$
\begin{aligned}
\phi(t) &= E(e^{tX}) \\
&= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx \\
&= \int_{-\infty}^{\infty} \left(1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots\right) f(x)\, dx \\
&= \int_{-\infty}^{\infty} f(x)dx + t\int_{-\infty}^{\infty} xf(x)dx + \frac{t^2}{2!}\int_{-\infty}^{\infty} x^2 f(x)dx + \cdots
\end{aligned}
$$

Now plugging in $aX + b$ to the above equation yields,

$$\begin{aligned} E(e^{t(aX+b)}) &= E(e^{taX+tb}) \\ &= E[e^{taX}(e^{tb})] \end{aligned}$$

Since expected value is a linear operator and e^{tb} is a constant, we can pull it out of the expectation operator,

$$\begin{aligned} E[e^{taX}(e^{tb})] &= e^{tb}E(e^{taX}) \\ &= e^{tb}\int_{-\infty}^{\infty} e^{tax} f(ax)\, dx \end{aligned}$$

We can see from the above equation that this is equal to $e^{tb}\phi(ta)$.

Thus, the moment-generating function for $aX + b$ with constants $a \neq 0$ and b is $e^{tb}\phi(ta)$.

Problem 6.

We know by Theorem 2.11.1 that the distribution of the sum of two random variables X and Y is given by the convolution of their densities, f_X and f_Y.

Thus, with $Z = X + Y$, we have,

$$f_Z(z) = (f * g)(z) = \int_{-\infty}^{\infty} f_X(z-y)f_Y(y)dy$$

The mean of this distribution is,

$$E(Z) = \int_{-\infty}^{\infty} zf_X(z-y)f_Y(y)dy$$

where $z = x + y$. Thus, we have,

$$\begin{aligned} E(Z) &= \int_{-\infty}^{\infty} zf_Z(z)dz \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [(x+y)f_X((x+y)-y)f_Y(y)]\, dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [(x+y)f_X(x)f_Y(y)]\, dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [(xf_X(x) + yf_X(x))f_Y(y)]\, dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [xf_X(x)f_Y(y) + yf_X(x)f_Y(y)]\, dxdy \\ &= \int_{-\infty}^{\infty} [\mu_X f_Y(y) + yf_Y(y)]\, dy \\ &= \mu_X + \mu_Y \end{aligned}$$

In the above equations, we are using the properties that, for any random variable X with density function $f(x)$ and mean μ, we have that $\int_{-\infty}^{\infty} f(x) = 1$ and $\int_{-\infty}^{\infty} xf(x) = \mu$.

Now for the variance, we know that for a given random variable X, $\mathrm{Var}(X) = E[X^2] - E[X]^2$. Plugging in Z to this formula yields,

$$\begin{aligned}
\mathrm{Var}(Z) = \sigma_Z^2 &= E[Z^2] - E[Z]^2 \\
&= \int_{-\infty}^{\infty} z^2 f_Z(z)dz + (\mu_X + \mu_Y)^2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[(x+y)^2 f_X((x+y)-y)f_Y(y)\right] dxdy - (\mu_X+\mu_Y)^2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[(x+y)^2 f_X(x)f_Y(y)\right] dxdy - (\mu_X+\mu_Y)^2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[(x^2+2xy+y^2) f_X(x)f_Y(y)\right] dxdy - (\mu_X+\mu_Y)^2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[x^2 f_X(x)f_Y(y) + 2xyf_X(x)f_Y(y) + y^2 f_X(x)f_Y(y)\right] dxdy - (\mu_X+\mu_Y)^2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f_X(x)f_Y(y)dxdy - \mu_X + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2xyf_X(x)f_Y(y)dxdy + \\
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f_X(x)f_Y(y)dxdy - \mu_Y \\
&= \left[\int_{-\infty}^{\infty} x^2 f_X(x)dx - \mu_X\right] + 2\mu_X\mu_Y + \left[\int_{-\infty}^{\infty} y^2 f_Y(y)dy - \mu_Y\right] \\
&= \sigma_X^2 + 2\mu_X\mu_Y + \sigma_Y^2
\end{aligned}$$

Thus, we have $\mathrm{Var}(Z) = \sigma_X^2 + 2\mu_X\mu_Y + \sigma_Y^2$.

Problem 7.

The definition of the Central Limit Theorem is as follows:

Let $Y_1, Y_2, ..., Y_n$ be independent and identically distributed random variables with $E(Y_i) = \mu$ and $V(Y_i) = \sigma^2 < \infty$. Define

$$U_n = \frac{\Sigma_{i=1}^n Y_i - n\mu}{\sigma\sqrt{n}} = \frac{\overline{Y} - \mu}{\sigma/\sqrt{n}}$$

where $\overline{Y} = \frac{1}{n}\Sigma_{i=1}^n Y_i$.

Then the distribution function of U_n converges to the standard normal distribution function as $n \to \infty$. That is,

$$\lim_{n\to\infty} P(U_n \le u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}dt$$

for all u.

Let $f_n(x)$ be the probability mass function for a binomial random variable with n trials and success probability p.

FINISH THIS LATER.

Problem 8.

Let X and Y be two independent, continuous random variables described by probability density functions f_X and f_Y. Also let $Z = XY$. We'll begin by finding the cumulative distribution function for Z. This yields,

$$\begin{aligned}
F_Z(z) &= P(Z \leq z) \\
&= P(XY \leq z) \\
&= P(XY \leq z, X \geq 0) + P(XY \leq z, X \leq 0) \\
&= P(Y \leq z/X, X \geq 0) + P(Y \geq z/X, X \leq 0) \\
&= \int_0^\infty f_X(x) \int_{-\infty}^{z/x} f_Y(y)dydx + \int_{-\infty}^0 f_X(x) \int_{z/X}^\infty f_Y(y)dydx
\end{aligned}$$

Now in order to find the probability density function for Z, we need to differentiate with respect to z on both sides of the above equation.

$$\begin{aligned}
f_Z(z) &= \frac{d}{dz} F_z(z) \\
&= \frac{d}{dz}\left[\int_0^\infty f_X(x) \int_{-\infty}^{z/x} f_Y(y)dydx + \int_{-\infty}^0 f_X(x) \int_{z/X}^\infty f_Y(y)dydx\right] \\
&= \int_0^\infty f_X(x)\left[f_Y(z/x)\left(\frac{1}{x}\right) - f(-\infty)\right]dx + \int_{-\infty}^0 f_X(x)\left[f_Y(\infty) - f_Y(z/x)\left(\frac{1}{x}\right)\right]dx
\end{aligned}$$

We obtained the above equation using the fundamental theorem of calculus and the chain rule. Now, we will use the fact that if $f(x)$ is a distribution function, as $x \to \infty$, $f(x) \to 0$. The same holds true as $x \to -\infty$. This yields,

$$\begin{aligned}
f_Z(z) &= \int_0^\infty f_X(x)\left[f_Y(z/x)\left(\frac{1}{x}\right) - f(-\infty)\right]dx + \int_{-\infty}^0 f_X(x)\left[f_Y(\infty) - f_Y(z/x)\left(\frac{1}{x}\right)\right]dx \\
&= \int_0^\infty f_X(x)f_Y(z/x)\frac{1}{x}dx + \int_{-\infty}^0 f_X(x)(-f_Y(z/x))\frac{1}{x}dx \\
&= \int_0^\infty f_X(x)f_Y(z/x)\frac{1}{x}dx - \int_{-\infty}^0 f_X(x)f_Y(z/x)\frac{1}{x}dx \\
&= \int_0^\infty f_X(x)f_Y(z/x)\frac{1}{|x|}dx + \int_{-\infty}^0 f_X(x)f_Y(z/x)\frac{1}{|x|}dx \\
&= \int_{-\infty}^\infty f_X(x)f_Y(z/x)\frac{1}{|x|}dx
\end{aligned}$$

Now we need to find the mean of Z where $Z = XY$,

$$
\begin{aligned}
E[Z] &= \int_{-\infty}^{\infty} z f_Z(z) dz \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (xy) f_X(x) f_Y(y) \frac{1}{|x|} dx dy \\
&= \mu_Y \int_{-\infty}^{\infty} \frac{x}{|x|} f_X(x) dx \\
&= \mu_Y \left[\int_{-\infty}^{0} -f_X(x) dx + \int_{0}^{\infty} f_X(x) dx \right] \\
&= \mu_Y \left[-\int_{-\infty}^{0} f_X(x) dx + \int_{0}^{\infty} f_X(x) dx \right]
\end{aligned}
$$

FINISH THIS LATER.

Problem 9.

We have that,

$$
\begin{aligned}
t_j - t_{j-1} &= (j/N)(b-a) + a - [(j-1/N)(b-a) + a] \\
&= (jb - ja)/N + a - (jb - ja - b + a)/N - a \\
&= (jb - ja - jb + ja + b - a)/N \\
&= (b-a)/N
\end{aligned}
$$

In addition, since the function f is bounded on $[a, b]$, we know that $\exists M \in \mathbb{N}$ such that $M \geq |f|$. These two facts yield the following,

$$
\begin{aligned}
\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p &= \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})((b-a)/N)^p \\
&\leq \lim_{N\to\infty} \Sigma_{j=1}^{N} M((b-a)/N)^p \\
&= \lim_{N\to\infty} MN((b-a)/N)^p \\
&= M \lim_{N\to\infty} N((b-a)/N)^p \\
&= M(b-a)^p \lim_{N\to\infty} N/N^p \\
&= M(b-a)^p \lim_{N\to\infty} N^{1-p}
\end{aligned}
$$

Since $1 - p < 0$, we have that $\lim_{N\to\infty} N^{1-p} = 0$. Thus, this gives us,

$$
\begin{aligned}
\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p &\leq M(b-a)^p \lim_{N\to\infty} N^{1-p} \\
&= [M(b-a)]0 = 0
\end{aligned}
$$

Now, if we use $-M$ for the lower bound, we get,

$$
\begin{aligned}
\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p &\geq -M(b-a)^p \lim_{N\to\infty} N^{1-p} \\
&= [-M(b-a)]0 = 0
\end{aligned}
$$

Since we have shown that

$$0 \leq \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p \leq 0$$

we have that $\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p = 0$.

Problem 10.

We know, by Proposition 4.9.1, that for a given stochastic process X_t, if $E[X_t] \to k$ a constant and $\text{Var}(X_t) \to 0$ as $t \to \infty$, then ms-$\lim_{t\to\infty} X_t = k$.

Thus, we will need to check the mean and variance of the given function. We will begin with the mean. Using the fact that the function f is a constant in terms of the expected value function, we get

$$E(\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2) = \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1}) E[(\Delta_{j-1}^{j} W)^2]$$

We know that the expected value of the square of a random variable is its variance, which is $t_j - t_{j-1}$ in this case. Hence,

$$\begin{aligned} E(\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2) &= \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1}) E[(\Delta_{j-1}^{j} W)^2] \\ &= \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1}) \end{aligned}$$

This is the equation for the Riemann integral of f over $[a, b]$. Since we already know f is bounded and defined on a closed interval, as long as f is continuous almost everywhere on $[a, b]$ we assert that the above limit exists and is equal to the integral of f. If that is the case, then we have

$$\begin{aligned} E(\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2) &= \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1}) \\ &= \int_a^b f(t)dt \end{aligned}$$

Now we need to show that the variance of the function is 0. Using Problem 8 to show that $\text{Var}[(\Delta_{j-1}^{j} W)^2] = 2(t_j - t_{j-1})^2$, we have

$$\begin{aligned} \text{Var}(\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2) &= \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 \text{Var}[(\Delta_{j-1}^{j} W)^2] \\ &= \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 \cdot 2(t_j - t_{j-1})^2 \\ &= 2 \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 (t_j - t_{j-1})^2 \end{aligned}$$

Now observe that if M is an upper bound for $|f|$ on $[a, b]$ (as in Problem 9), then M^2 is an upper bound for $|f|^2$ on $[a, b]$. This is true because increasing functions preserve inequalities, squaring is an increasing function for $x \geq 0$, and $|f| \geq 0$ for every x. In addition, observe that $|f|^2 = f^2$. Thus, we have that M^2 is an upper bound for f^2 on $[a, b]$.

We can now the above and the result from Problem 9 in order to obtain the following,

$$\begin{aligned} \mathrm{Var}(\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2) &= 2 \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 (t_j - t_{j-1})^2 \\ &\le 2M^2 \lim_{N\to\infty} \Sigma_{j=1}^{N} (t_j - t_{j-1})^2 \\ &= 2M^2(0) = 0 \end{aligned}$$

In addition, since we know that variance is always ≥ 0, we have that,

$$\mathrm{Var}(\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2) = 0$$

Thus, we have shown that this function satisfies Proposition 4.9.1 and can state that ms-$\lim_{t\to\infty}[\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2] = \int_a^b f(t)dt$.

Problem 11.

We need to show that the mean and variance of this function are 0. We'll begin with the mean,

$$E[\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p (\Delta_{j-1}^{j} W)^q] = \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p E[(\Delta_{j-1}^{j} W)^q]$$

If q is odd, then $E[(\Delta_{j-1}^{j} W)^q] = 0$. If q is even, then $E[(\Delta_{j-1}^{j} W)^q] = \sigma^q (q-1)!! = (\sqrt{t_j - t_{j-1}})^p (p-1)!!$ where !! is the double factorial. Since the expected value of the whole function is trivially 0 if q is odd, we will only consider the situation where q is even,

$$\begin{aligned} \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p E[(\Delta_{j-1}^{j} W)^q] &= \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p (\sqrt{t_j - t_{j-1}})^q (q-1)!! \\ &= \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p (t_j - t_{j-1})^{q/2} (q-1)!! \\ &= (q-1)!! \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^{p+q/2} \end{aligned}$$

By the constraints in the problem, we know that $p + q/2 > 1$. Hence, we can apply the result of Problem 9. Hence, the limit is equal to 0,

$$\begin{aligned} E[\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p (\Delta_{j-1}^{j} W)^q] &= (q-1)!! \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^{p+q/2} \\ &= (q-1)!!(0) \\ &= 0 \end{aligned}$$

Now we must show that the variance is equal to 0,

$$\begin{aligned} \mathrm{Var}[\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p (\Delta_{j-1}^{j} W)^q] &= \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 (t_j - t_{j-1})^{2p} \cdot \mathrm{Var}[(\Delta_{j-1}^{j} W)^q] \\ &= \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 (t_j - t_{j-1})^{2p} \cdot E[(\Delta_{j-1}^{j} W)^{2q}] \\ &= \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 (t_j - t_{j-1})^{2p} \cdot E[(\Delta_{j-1}^{j} W)^{2q}] \\ &= \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 (t_j - t_{j-1})^{2p} \cdot (t_j - t_{j-1})^q (2q-1) \\ &= (2q-1) \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 (t_j - t_{j-1})^{2p+q} \end{aligned}$$

By the same reasoning as in Problem 10, this limit is equal to 0. Thus,

$$\begin{aligned}\mathrm{Var}[\lim_{N\to\infty}\Sigma_{j=1}^{N} f(t_{j-1})(t_j-t_{j-1})^p(\Delta_{j-1}^{j}W)^q] &= (2q-1)\lim_{N\to\infty}\Sigma_{j=1}^{N}(f(t_{j-1}))^2(t_j-t_{j-1})^{2p+q}\\ &= 0\end{aligned}$$

Since the mean is equal to a constant (0 in this case) and the variance is equal to 0, we can say that the mean-squared limit of this function is 0.

Problem 12.

We have that,

$$\begin{aligned}\Sigma_{k=m}^{n} f_k(g_{k+1}-g_k) &= f_m(g_{m+1}-g_m)+f_{m+1}(g_{m+2}-g_{m+1})+f_{m+2}(g_{m+3}-g_{m+2})+\cdots\\ &= f_m(g_m+1)-f_m(g_m)+f_{m+1}(g_{m+2})-f_{m+1}(g_{m+1})+\cdots\\ &= -f_m(g_m)+g_{m+1}(f_m-f_{m+1})+g_{m+2}(f_{m+1}-f_{m+2})+\cdots+f_n(g_{n+1})\\ &= f_n(g_{n+1})-f_m(g_m)-[g_{m+1}(f_m-f_{m+1})+g_{m+2}(f_{m+1}-f_{m+2})+\cdots+g_n(f_{n-1}-f_n)]\\ &= f_n(g_{n+1})-f_m(g_m)-\Sigma_{k=m+1}^{n} g_k(f_{k-1}-f_k)\end{aligned}$$

Problem 13.

We have $F_t = W_t$ with the interval of interest being $[0,T]$. The partial sums of the integral $\int_0^T W_t dW_t$ are given by

$$\begin{aligned}S_n &= \Sigma_{i=0}^{n-1} F_{t_i}(W_{t_{i+1}}-W_{t_i})\\ &= \Sigma_{i=0}^{n-1} W_{t_i}(W_{t_{i+1}}-W_{t_i})\end{aligned}$$

We also have that

$$W_{t_i}(W_{t_{i+1}}-W_{t_i}) = \frac{1}{2}W_{t_{i+1}}^2-\frac{1}{2}W_{t_i}^2-\frac{1}{2}(W_{t_{i+1}}-W_{t_i})^2$$

Plugging this into the partial sums formula yields,

$$\begin{aligned}S_n &= \frac{1}{2}\Sigma_{i=0}^{n-1}W_{t_{i+1}}^2-\frac{1}{2}\Sigma_{i=0}^{n-1}W_{t_i}^2-\frac{1}{2}\Sigma_{i=0}^{n-1}(W_{t_{i+1}}-W_{t_i})^2\\ &= \frac{1}{2}W_{t_n}^2-\frac{1}{2}\Sigma_{i=0}^{n-1}(W_{t_{i+1}}-W_{t_i})^2\end{aligned}$$

If we let $t_n = T$, we get

$$S_n = \frac{1}{2}W_T^2-\frac{1}{2}\Sigma_{i=0}^{n-1}(W_{t_{i+1}}-W_{t_i})^2$$

Using Proposition 4.11.6 and the fact that $\frac{1}{2}W_T^2$ does not depend on n, we have

$$\begin{aligned}\text{ms-}\lim_{n\to\infty} S_n &= \frac{1}{2}W_T^2-\text{ms-}\lim_{n\to\infty}\frac{1}{2}\Sigma_{i=0}^{n-1}(W_{t_{i+1}}-W_{t_i})^2\\ &= \frac{1}{2}W_T^2-\frac{1}{2}T\end{aligned}$$

Since the Ito integral is defined as the mean-squared limit of the partial sums S_n, we have that

$$\begin{aligned}\int_0^T W_t dW_t &= \text{ms-}\lim_{n\to\infty} S_n \\ &= \frac{1}{2}W_T^2 - \frac{1}{2}T\end{aligned}$$

as required.

Problem 14.

We are given the Ito integral,

$$\int_0^T f(t)dW_t$$

where $f(t)$ is an arbitrary bounded and continuous function. Each increment is given by,

$$f(t_i)(W_{t_{i+1}} - W_{t_i})$$

So the partial sums are given by

$$S_n = \Sigma_{i=0}^{n-1} f(t_i)(W_{t_{i+1}} - W_{t_i})$$

We need to find the mean-squared limit of these partial sums. We'll start by finding the expected value,

$$\begin{aligned}E[\Sigma_{i=0}^{n-1} f(t_i)(W_{t_{i+1}} - W_{t_i})] &= \Sigma_{i=0}^{n-1} f(t_i)E[(W_{t_{i+1}} - W_{t_i})] \\ &= \Sigma_{i=0}^{n-1} f(t_i)(0) = 0\end{aligned}$$

Thus, the mean is 0. Now to find the variance,

$$\begin{aligned}\text{Var}[\Sigma_{i=0}^{n-1} f(t_i)(W_{t_{i+1}} - W_{t_i})] &= \Sigma_{i=0}^{n-1} f(t_i)^2 \text{Var}[(W_{t_{i+1}} - W_{t_i})] \\ &= \Sigma_{i=0}^{n-1} f(t_i)^2 (t_{i+1} - t_i)\end{aligned}$$

Now when we take the limit as $n \to \infty$ of the variance, we get,

$$\lim_{n\to\infty} \Sigma_{i=0}^{n-1} f(t_i)^2 (t_{i+1} - t_i) = \int_0^T f(t)^2$$

Thus, if $Y_t = \int_0^T f(t)dW_t$, then $Y_t \sim N(0, \int_0^T f(t)^2)$.

Problem 15.

Let $Z_t = \int_0^t W_s ds$. We will use integration by parts to solve this integral.

Let $u = W_s$ and $dv = ds$. This implies that $du = dW_s$ and $v = t - s$. Then using the formula,

$$\int udv = uv - \int vdu$$

we get,

$$\int_0^t W_s ds = W_s(t-s) - \int (t-s)dW_s$$

FINISH THIS LATER.

Problem 16.

We have $X_t = \int_0^t (a + \frac{bu}{t}) dW_u$. Since the non-infinitesimal part of the integral does not depend on a random variable, this is a Wiener integral.

By Proposition 5.6.1 , we know that Wiener integrals are normal random variables with mean 0 and variance,

$$\begin{aligned}\int_0^t (a + \frac{bu}{t})^2 du &= \int_0^t (a^2 + a\frac{bu}{t} + \frac{b^2u^2}{t^2}) du \\ &= a^2 t + a\frac{bt^2}{2t} + \frac{b^2t^3}{3t^2} \\ &= a^2 t + a\frac{bt}{2} + \frac{b^2 t}{3} \\ &= (a^2 + \frac{ab}{2} + \frac{b^2}{3})t\end{aligned}$$

So, in order for the variance to equal 1, we require,

$$\begin{aligned}(a^2 + \frac{ab}{2} + \frac{b^2}{3})t = 1 \\ \implies a^2 + \frac{ab}{2} + \frac{b^2}{3} = \frac{1}{t}\end{aligned}$$

Problem 17.