Stochastic Differential Equations: Final Project

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NOTE: Still need to complete Problems 2, 7, 8, 15, 17, 18, 22, 23c, 29, 30, 31bc

Problem 1.

We know that the probability density of the sum of two independent random variables can be computed by the convolutions of f_X and f_Y. Thus, the probability density of $X+Y$, denoted by f_Z, is

$$f_Z(z) = (f_Y * f_X) = \int_{-\infty}^{\infty} f_Y(z-x) f_X(x) dx \tag{1}$$

We know that the normal density function is given by,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Plugging this density function into equation (1) and letting $\sigma_Z = \sqrt{\sigma_X^2 + \sigma_Y^2}$ yields,

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} \left[\frac{1}{\sigma_Y\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{(z-x)-\mu_Y}{\sigma_Y}\right)^2} \frac{1}{\sigma_X\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}\right] dx \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{\sigma_Y\sigma_X(2\pi)} e^{-\frac{1}{2}\left[\left(\frac{z-x-\mu_Y}{\sigma_Y}\right)^2 + \left(\frac{x-\mu_X}{\sigma_X}\right)^2\right]}\right] dx \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{\sigma_Y\sigma_X(2\pi)} e^{-\frac{\sigma_X^2(z-x-\mu_Y)^2 + \sigma_Y^2(x-\mu_X)^2}{2\sigma_Y^2\sigma_X^2}}\right] dx \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{(2\pi)\sigma_X\sigma_Y} e^{-\frac{x^2(\sigma_X^2+\sigma_Y^2) - 2x(\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X) + \sigma_X^2(z^2+\mu_Y^2-2z\mu_Y)+\sigma_Y^2\mu_X^2}{2\sigma_Y^2\sigma_X^2}}\right] dx \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}\sigma_Z} e^{-\frac{\sigma_Z^2\left(\sigma_X^2(z-\mu_Y)^2+\sigma_Y^2\mu_X^2\right) - \left(\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X\right)^2}{2\sigma_Z^2(\sigma_X\sigma_Y)^2}} \frac{1}{\sqrt{2\pi}\frac{\sigma_X\sigma_Y}{\sigma_Z}} e^{-\frac{\left(x-\frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2}\right)^2}{2\left(\frac{\sigma_X\sigma_Y}{\sigma_Z}\right)^2}}\right] dx
\end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_Z} e^{-\frac{(z-(\mu_X+\mu_Y))^2}{2\sigma_Z^2}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}\frac{\sigma_X \sigma_Y}{\sigma_Z}} e^{-\frac{\left(x - \frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2}\right)^2}{2\left(\frac{\sigma_X\sigma_Y}{\sigma_Z}\right)^2}} \right] dx$$

The equation inside the integral symbol represents a valid normal density function for x, so we know it integrates to 1. Thus, the probability density function for $X + Y$ is given by

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma_Z} e^{-\frac{(z-(\mu_X+\mu_Y))^2}{2\sigma_Z^2}}$$

This is precisely a normal density function with mean $\mu_X + \mu_Y$ and variance $\sigma_Z^2 = (\sqrt{\sigma_X^2 + \sigma_Y^2})^2 = \sigma_X^2 + \sigma_Y^2$.

Hence, we have shown that, given $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$, $X + Y$ is distributed as $N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$.

Problem 2.

Let $Z = Y + 1$. We'll begin by finding the probability density function for Z. Since Y is a standard uniform random variable, we know that

$$f(y) = \begin{cases} 1 & 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Thus, the distribution function approach yields,

$$\begin{aligned} F_Z(z) = P(Z \le z) = P(Y + 1 \le z) &= P(Y \le z - 1) \\ &= \int_{-\infty}^{z-1} f(y)dy \\ &= \int_{0}^{z-1} 1\, dy \\ &= z - 1 \end{aligned}$$

So, since as Y ranges from 0 to 1, Z ranges from 1 to 2, we have

$$F_Z(z) = \begin{cases} 0 & z < 1 \\ z - 1 & 1 \le z \le 2 \\ 1 & \text{elsewhere} \end{cases}$$

And the density function for Z is

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \begin{cases} 1 & 1 \le z \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

Now we can simplify the original function to be $U = X/Z$, where we now know the probability density function for both random variables ($X \sim \text{Unif}(0,1)$ and $Z \sim \text{Unif}(1,2)$).

So the distribution function is given by,

$$\begin{aligned} F_U(u) = P(U \leq u) = P(X/Z \leq u) &= P(X \leq uZ) \\ &= \int_{-\infty}^{uz} f(x)dx \\ &= \int_{0}^{uz} 1\, dx \\ &= uz \end{aligned}$$

FINISH THIS LATER.

Problem 3.

Let Y be a standard normal variable. Then the moment-generating function of Y is given by,

$$\begin{aligned} m(t) &= E(e^{tY}) \\ &= \int_{-\infty}^{\infty} e^{ty} f(y)\, dy \end{aligned}$$

Since Y is a standard normal variable, we know that $\mu = 0$ and $\sigma = 1$. Thus, we have

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$

Plugging this into the above equation yields,

$$\begin{aligned} m(t) &= \int_{-\infty}^{\infty} e^{ty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}+ty}\, dy \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-t)^2} e^{\frac{1}{2}t^2}\, dy \\ &= e^{\frac{1}{2}t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-t)^2}\, dy \end{aligned}$$

We can see that the integral is precisely the integral for a normal random variable with $\mu = y - t$ and $\sigma = 1$. Thus, it integrates to 1, yielding,

$$m(t) = e^{\frac{1}{2}t^2}$$

Problem 4.

Suppose $g(x)$ is a monotone increasing function and X is a random variable with the probability density function f_X.

Let $U = g(X)$ where X has the above density function. Since $g(x)$ is an increasing function of x, then $g^{-1}(u)$ is an increasing function of u. Thus,

$$\begin{aligned} P(U \le u) &= P[g(X) \le u] \\ &= P\{g^{-1}[g(X)] \le g^{-1}(u)\} \\ &= P[X \le g^{-1}(u)] \end{aligned}$$

The above sequence of equalities implies that,

$$F_U(u) = F_X[g^{-1}(u)]$$

When we differentiate with respect to u, we get,

$$f_U(u) = \frac{dF_X[g^{-1}(u)]}{du} = f_X(g^{-1}(u))\frac{d[g^{-1}(u)]}{du}$$

Problem 5.

We know that the moment-generating function of X is $\phi(t)$. Thus,

$$\begin{aligned} \phi(t) &= E(e^{tX}) \\ &= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \\ &= \int_{-\infty}^{\infty} \left(1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots\right) f(x)\,dx \\ &= \int_{-\infty}^{\infty} f(x)dx + t\int_{-\infty}^{\infty} xf(x)dx + \frac{t^2}{2!}\int_{-\infty}^{\infty} x^2f(x)dx + \cdots \end{aligned}$$

Now plugging in $aX + b$ to the above equation yields,

$$\begin{aligned} E(e^{t(aX+b)}) &= E(e^{taX+tb}) \\ &= E[e^{taX}(e^{tb})] \end{aligned}$$

Since expected value is a linear operator and e^{tb} is a constant, we can pull it out of the expectation operator,

$$\begin{aligned} E[e^{taX}(e^{tb})] &= e^{tb}E(e^{taX}) \\ &= e^{tb}\int_{-\infty}^{\infty} e^{tax} f(ax)\,dx \end{aligned}$$

We can see from the above equation that this is equal to $e^{tb}\phi(ta)$.

Thus, the moment-generating function for $aX + b$ with constants $a \neq 0$ and b is $e^{tb}\phi(ta)$.

Problem 6.

We know by Theorem 2.11.1 that the distribution of the sum of two random variables X and Y is given by the convolution of their densities, f_X and f_Y.

Thus, with $Z = X + Y$, we have,

$$f_Z(z) = (f * g)(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy$$

The mean of this distribution is,

$$E(Z) = \int_{-\infty}^{\infty} z f_X(z-y) f_Y(y) dy$$

where $z = x + y$. Thus, we have,

$$\begin{aligned}
E(Z) &= \int_{-\infty}^{\infty} z f_Z(z) dz \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[(x+y) f_X((x+y)-y) f_Y(y)\right] dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[(x+y) f_X(x) f_Y(y)\right] dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[(x f_X(x) + y f_X(x)) f_Y(y)\right] dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[x f_X(x) f_Y(y) + y f_X(x) f_Y(y)\right] dxdy \\
&= \int_{-\infty}^{\infty} \left[\mu_X f_Y(y) + y f_Y(y)\right] dy \\
&= \mu_X + \mu_Y
\end{aligned}$$

In the above equations, we are using the properties that, for any random variable X with density function $f(x)$ and mean μ, we have that $\int_{-\infty}^{\infty} f(x) = 1$ and $\int_{-\infty}^{\infty} x f(x) = \mu$.

Now for the variance, we know that for a given random variable X, $\text{Var}(X) = E[X^2] - E[X]^2$. Plugging in Z to this formula yields,

$$\begin{aligned}
\text{Var}(Z) = \sigma_Z^2 &= E[Z^2] - E[Z]^2 \\
&= \int_{-\infty}^{\infty} z^2 f_Z(z) dz + (\mu_X + \mu_Y)^2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[(x+y)^2 f_X((x+y)-y) f_Y(y)\right] dxdy - (\mu_X + \mu_Y)^2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[(x+y)^2 f_X(x) f_Y(y)\right] dxdy - (\mu_X + \mu_Y)^2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[(x^2 + 2xy + y^2) f_X(x) f_Y(y)\right] dxdy - (\mu_X + \mu_Y)^2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[x^2 f_X(x) f_Y(y) + 2xy f_X(x) f_Y(y) + y^2 f_X(x) f_Y(y)\right] dxdy - (\mu_X + \mu_Y)^2
\end{aligned}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f_X(x)f_Y(y)dxdy - \mu_X + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2xyf_X(x)f_Y(y)dxdy +$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f_X(x)f_Y(y)dxdy - \mu_Y$$
$$= \left[\int_{-\infty}^{\infty} x^2 f_X(x)dx - \mu_X\right] + 2\mu_X\mu_Y + \left[\int_{-\infty}^{\infty} y^2 f_Y(y)dy - \mu_Y\right]$$
$$= \sigma_X^2 + 2\mu_X\mu_Y + \sigma_Y^2$$

Thus, we have $\mathrm{Var}(Z) = \sigma_X^2 + 2\mu_X\mu_Y + \sigma_Y^2$.

Problem 7.

The definition of the Central Limit Theorem is as follows:

Let $Y_1, Y_2, ..., Y_n$ be independent and identically distributed random variables with $E(Y_i) = \mu$ and $V(Y_i) = \sigma^2 < \infty$. Define

$$U_n = \frac{\Sigma_{i=1}^n Y_i - n\mu}{\sigma\sqrt{n}} = \frac{\overline{Y} - \mu}{\sigma/\sqrt{n}}$$

where $\overline{Y} = \frac{1}{n}\Sigma_{i=1}^n Y_i$.

Then the distribution function of U_n converges to the standard normal distribution function as $n \to \infty$. That is,

$$\lim_{n\to\infty} P(U_n \leq u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

for all u.

Let $f_n(x)$ be the probability mass function for a binomial random variable with n trials and success probability p.

FINISH THIS LATER.

Problem 8.

Let X and Y be two independent, continuous random variables described by probability density functions f_X and f_Y. Also let $Z = XY$. We'll begin by finding the cumulative distribution function for Z. This yields,

$$\begin{aligned}
F_Z(z) &= P(Z \leq z) \\
&= P(XY \leq z) \\
&= P(XY \leq z, X \geq 0) + P(XY \leq z, X \leq 0) \\
&= P(Y \leq z/X, X \geq 0) + P(Y \geq z/X, X \leq 0) \\
&= \int_0^{\infty} f_X(x) \int_{-\infty}^{z/x} f_Y(y)dydx + \int_{-\infty}^{0} f_X(x) \int_{z/X}^{\infty} f_Y(y)dydx
\end{aligned}$$

Now in order to find the probability density function for Z, we need to differentiate with respect to z on both sides of the above equation.

$$\begin{aligned}
f_Z(z) &= \frac{d}{dz}F_z(z) \\
&= \frac{d}{dz}\left[\int_0^\infty f_X(x)\int_{-\infty}^{z/x} f_Y(y)dydx + \int_{-\infty}^0 f_X(x)\int_{z/X}^\infty f_Y(y)dydx\right] \\
&= \int_0^\infty f_X(x)\left[f_Y(z/x)\left(\frac{1}{x}\right) - f(-\infty)\right]dx + \int_{-\infty}^0 f_X(x)\left[f_Y(\infty) - f_Y(z/x)\left(\frac{1}{x}\right)\right]dx
\end{aligned}$$

We obtained the above equation using the fundamental theorem of calculus and the chain rule. Now, we will use the fact that if $f(x)$ is a distribution function, as $x \to \infty$, $f(x) \to 0$. The same holds true as $x \to -\infty$. This yields,

$$\begin{aligned}
f_Z(z) &= \int_0^\infty f_X(x)\left[f_Y(z/x)\left(\frac{1}{x}\right) - f(-\infty)\right]dx + \int_{-\infty}^0 f_X(x)\left[f_Y(\infty) - f_Y(z/x)\left(\frac{1}{x}\right)\right]dx \\
&= \int_0^\infty f_X(x)f_Y(z/x)\frac{1}{x}dx + \int_{-\infty}^0 f_X(x)(-f_Y(z/x))\frac{1}{x}dx \\
&= \int_0^\infty f_X(x)f_Y(z/x)\frac{1}{x}dx - \int_{-\infty}^0 f_X(x)f_Y(z/x)\frac{1}{x}dx \\
&= \int_0^\infty f_X(x)f_Y(z/x)\frac{1}{|x|}dx + \int_{-\infty}^0 f_X(x)f_Y(z/x)\frac{1}{|x|}dx \\
&= \int_{-\infty}^\infty f_X(x)f_Y(z/x)\frac{1}{|x|}dx
\end{aligned}$$

Now we need to find the mean of Z where $Z = XY$,

$$\begin{aligned}
E[Z] &= \int_{-\infty}^\infty zf_Z(z)dz \\
&= \int_{-\infty}^\infty\int_{-\infty}^\infty (xy)f_X(x)f_Y(y)\frac{1}{|x|}dxdy \\
&= \mu_Y\int_{-\infty}^\infty \frac{x}{|x|}f_X(x)dx \\
&= \mu_Y\left[\int_{-\infty}^0 -f_X(x)dx + \int_0^\infty f_X(x)dx\right] \\
&= \mu_Y\left[-\int_{-\infty}^0 f_X(x)dx + \int_0^\infty f_X(x)dx\right]
\end{aligned}$$

FINISH THIS LATER.

Problem 9.

We have that,

$$\begin{aligned}
t_j - t_{j-1} &= (j/N)(b-a) + a - [(j-1/N)(b-a) + a] \\
&= (jb - ja)/N + a - (jb - ja - b + a)/N - a \\
&= (jb - ja - jb + ja + b - a)/N
\end{aligned}$$

$$= (b-a)/N$$

In addition, since the function f is bounded on $[a,b]$, we know that $\exists M \in \mathbb{N}$ such that $M \geq |f|$. These two facts yield the following,

$$\begin{aligned}
\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p &= \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})((b-a)/N)^p \\
&\leq \lim_{N\to\infty} \Sigma_{j=1}^{N} M((b-a)/N)^p \\
&= \lim_{N\to\infty} MN((b-a)/N)^p \\
&= M \lim_{N\to\infty} N((b-a)/N)^p \\
&= M(b-a)^p \lim_{N\to\infty} N/N^p \\
&= M(b-a)^p \lim_{N\to\infty} N^{1-p}
\end{aligned}$$

Since $1-p<0$, we have that $\lim_{N\to\infty} N^{1-p} = 0$. Thus, this gives us,

$$\begin{aligned}
\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p &\leq M(b-a)^p \lim_{N\to\infty} N^{1-p} \\
&= [M(b-a)]0 = 0
\end{aligned}$$

Now, if we use $-M$ for the lower bound, we get,

$$\begin{aligned}
\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p &\geq -M(b-a)^p \lim_{N\to\infty} N^{1-p} \\
&= [-M(b-a)]0 = 0
\end{aligned}$$

Since we have shown that

$$0 \leq \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p \leq 0$$

we have that $\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p = 0$.

Problem 10.

We know, by Proposition 4.9.1, that for a given stochastic process X_t, if $E[X_t] \to k$ a constant and $\mathrm{Var}(X_t) \to 0$ as $t \to \infty$, then ms-$\lim_{t\to\infty} X_t = k$.

Thus, we will need to check the mean and variance of the given function. We will begin with the mean. Using the fact that the function f is a constant in terms of the expected value function, we get

$$E(\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2) = \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1}) E[(\Delta_{j-1}^{j} W)^2]$$

We know that the expected value of the square of a random variable is its variance, which is $t_j - t_{j-1}$ in this case. Hence,

$$E(\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2) = \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1}) E[(\Delta_{j-1}^{j} W)^2]$$

$$= \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})$$

This is the equation for the Riemann integral of f over $[a, b]$. Since we already know f is bounded and defined on a closed interval, as long as f is continuous almost everywhere on $[a, b]$ we assert that the above limit exists and is equal to the integral of f. If that is the case, then we have

$$\begin{aligned} E(\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2) &= \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1}) \\ &= \int_a^b f(t)dt \end{aligned}$$

Now we need to show that the variance of the function is 0. Using Problem 8 to show that $\text{Var}[(\Delta_{j-1}^{j} W)^2] = 2(t_j - t_{j-1})^2$, we have

$$\begin{aligned} \text{Var}(\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2) &= \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 \text{Var}[(\Delta_{j-1}^{j} W)^2] \\ &= \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 \cdot 2(t_j - t_{j-1})^2 \\ &= 2 \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 (t_j - t_{j-1})^2 \end{aligned}$$

Now observe that if M is an upper bound for $|f|$ on $[a, b]$ (as in Problem 9), then M^2 is an upper bound for $|f|^2$ on $[a, b]$. This is true because increasing functions preserve inequalities, squaring is an increasing function for $x \geq 0$, and $|f| \geq 0$ for every x. In addition, observe that $|f|^2 = f^2$. Thus, we have that M^2 is an upper bound for f^2 on $[a, b]$.

We can now the above and the result from Problem 9 in order to obtain the following,

$$\begin{aligned} \text{Var}(\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2) &= 2 \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 (t_j - t_{j-1})^2 \\ &\leq 2M^2 \lim_{N\to\infty} \Sigma_{j=1}^{N} (t_j - t_{j-1})^2 \\ &= 2M^2(0) = 0 \end{aligned}$$

In addition, since we know that variance is always ≥ 0, we have that,

$$\text{Var}(\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2) = 0$$

Thus, we have shown that this function satisfies Proposition 4.9.1 and can state that $\text{ms-}\lim_{t\to\infty} [\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(\Delta_{j-1}^{j} W)^2] = \int_a^b f(t)dt$.

Problem 11.

We need to show that the mean and variance of this function are 0. We'll begin with the mean,

$$E[\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p (\Delta_{j-1}^{j} W)^q] = \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p E[(\Delta_{j-1}^{j} W)^q]$$

If q is odd, then $E[(\Delta_{j-1}^{j} W)^q] = 0$. If q is even, then $E[(\Delta_{j-1}^{j} W)^q] = \sigma^q (q-1)!! = (\sqrt{t_j - t_{j-1}})^p (p-1)!!$ where !! is the double factorial. Since the expected value of the whole function is trivially 0 if q is odd, we will only consider the situation where q is even,

$$\begin{aligned}
\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p E[(\Delta_{j-1}^{j} W)^q] &= \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p (\sqrt{t_j - t_{j-1}})^q (q-1)!! \\
&= \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p (t_j - t_{j-1})^{q/2} (q-1)!! \\
&= (q-1)!! \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^{p+q/2}
\end{aligned}$$

By the constraints in the problem, we know that $p + q/2 > 1$. Hence, we can apply the result of Problem 9. Hence, the limit is equal to 0,

$$\begin{aligned}
E[\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p (\Delta_{j-1}^{j} W)^q] &= (q-1)!! \lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^{p+q/2} \\
&= (q-1)!!(0) \\
&= 0
\end{aligned}$$

Now we must show that the variance is equal to 0,

$$\begin{aligned}
\mathrm{Var}[\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p (\Delta_{j-1}^{j} W)^q] &= \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 (t_j - t_{j-1})^{2p} \cdot \mathrm{Var}[(\Delta_{j-1}^{j} W)^q] \\
&= \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 (t_j - t_{j-1})^{2p} \cdot E[(\Delta_{j-1}^{j} W)^{2q}] \\
&= \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 (t_j - t_{j-1})^{2p} \cdot E[(\Delta_{j-1}^{j} W)^{2q}] \\
&= \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 (t_j - t_{j-1})^{2p} \cdot (t_j - t_{j-1})^q (2q-1) \\
&= (2q-1) \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 (t_j - t_{j-1})^{2p+q}
\end{aligned}$$

By the same reasoning as in Problem 10, this limit is equal to 0. Thus,

$$\begin{aligned}
\mathrm{Var}[\lim_{N\to\infty} \Sigma_{j=1}^{N} f(t_{j-1})(t_j - t_{j-1})^p (\Delta_{j-1}^{j} W)^q] &= (2q-1) \lim_{N\to\infty} \Sigma_{j=1}^{N} (f(t_{j-1}))^2 (t_j - t_{j-1})^{2p+q} \\
&= 0
\end{aligned}$$

Since the mean is equal to a constant (0 in this case) and the variance is equal to 0, we can say that the mean-squared limit of this function is 0.

Problem 12.

We have that,

$$\begin{aligned}
\Sigma_{k=m}^{n} f_k (g_{k+1} - g_k) &= f_m (g_{m+1} - g_m) + f_{m+1}(g_{m+2} - g_{m+1}) + f_{m+2}(g_{m+3} - g_{m+2}) + \cdots \\
&= f_m (g_m + 1) - f_m (g_m) + f_{m+1}(g_{m+2}) - f_{m+1}(g_{m+1}) + \cdots \\
&= -f_m (g_m) + g_{m+1}(f_m - f_{m+1}) + g_{m+2}(f_{m+1} - f_{m+2}) + \cdots + f_n (g_{n+1}) \\
&= f_n (g_{n+1}) - f_m (g_m) - [g_{m+1}(f_m - f_{m+1}) + g_{m+2}(f_{m+1} - f_{m+2}) + \cdots + g_n (f_{n-1} - f_n)] \\
&= f_n (g_{n+1}) - f_m (g_m) - \Sigma_{k=m+1}^{n} g_k (f_{k-1} - f_k)
\end{aligned}$$

Problem 13.

We have $F_t = W_t$ with the interval of interest being $[0, T]$. The partial sums of the integral $\int_0^T W_t dW_t$ are given by

$$\begin{aligned} S_n &= \Sigma_{i=0}^{n-1} F_{t_i}(W_{t_{i+1}} - W_{t_i}) \\ &= \Sigma_{i=0}^{n-1} W_{t_i}(W_{t_{i+1}} - W_{t_i}) \end{aligned}$$

We also have that

$$W_{t_i}(W_{t_{i+1}} - W_{t_i}) = \frac{1}{2}W_{t_{i+1}}^2 - \frac{1}{2}W_{t_i}^2 - \frac{1}{2}(W_{t_{i+1}} - W_{t_i})^2$$

Plugging this into the partial sums formula yields,

$$\begin{aligned} S_n &= \frac{1}{2}\Sigma_{i=0}^{n-1} W_{t_{i+1}}^2 - \frac{1}{2}\Sigma_{i=0}^{n-1} W_{t_i}^2 - \frac{1}{2}\Sigma_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2 \\ &= \frac{1}{2}W_{t_n}^2 - \frac{1}{2}\Sigma_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2 \end{aligned}$$

If we let $t_n = T$, we get

$$S_n = \frac{1}{2}W_T^2 - \frac{1}{2}\Sigma_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2$$

Using Proposition 4.11.6 and the fact that $\frac{1}{2}W_T^2$ does not depend on n, we have

$$\begin{aligned} \text{ms-}\lim_{n\to\infty} S_n &= \frac{1}{2}W_T^2 - \text{ms-}\lim_{n\to\infty} \frac{1}{2}\Sigma_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2 \\ &= \frac{1}{2}W_T^2 - \frac{1}{2}T \end{aligned}$$

Since the Ito integral is defined as the mean-squared limit of the partial sums S_n, we have that

$$\begin{aligned} \int_0^T W_t dW_t &= \text{ms-}\lim_{n\to\infty} S_n \\ &= \frac{1}{2}W_T^2 - \frac{1}{2}T \end{aligned}$$

as required.

Problem 14.

We are given the Ito integral,

$$\int_0^T f(t) dW_t$$

where $f(t)$ is an arbitrary bounded and continuous function. Each increment is given by,

$$f(t_i)(W_{t_{i+1}} - W_{t_i})$$

So the partial sums are given by

$$S_n = \Sigma_{i=0}^{n-1} f(t_i)(W_{t_{i+1}} - W_{t_i})$$

We need to find the mean-squared limit of these partial sums. We'll start by finding the expected value,

$$\begin{aligned} E[\Sigma_{i=0}^{n-1} f(t_i)(W_{t_{i+1}} - W_{t_i})] &= \Sigma_{i=0}^{n-1} f(t_i) E[(W_{t_{i+1}} - W_{t_i})] \\ &= \Sigma_{i=0}^{n-1} f(t_i)(0) = 0 \end{aligned}$$

Thus, the mean is 0. Now to find the variance,

$$\begin{aligned} \mathrm{Var}[\Sigma_{i=0}^{n-1} f(t_i)(W_{t_{i+1}} - W_{t_i})] &= \Sigma_{i=0}^{n-1} f(t_i)^2 \mathrm{Var}[(W_{t_{i+1}} - W_{t_i})] \\ &= \Sigma_{i=0}^{n-1} f(t_i)^2 (t_{i+1} - t_i) \end{aligned}$$

Now when we take the limit as $n \to \infty$ of the variance, we get,

$$\lim_{n\to\infty} \Sigma_{i=0}^{n-1} f(t_i)^2 (t_{i+1} - t_i) = \int_0^T f(t)^2$$

Thus, if $Y_t = \int_0^T f(t) dW_t$, then $Y_t \sim N(0, \int_0^T f(t)^2)$.

Problem 15.

Let $Z_t = \int_0^t W_s ds$.

FINISH THIS LATER.

Problem 16.

We have $X_t = \int_0^t (a + \frac{bu}{t}) dW_u$. Since the non-infinitesimal part of the integral does not depend on a random variable, this is a Wiener integral.

By Proposition 5.6.1 , we know that Wiener integrals are normal random variables with mean 0 and variance,

$$\begin{aligned} \int_0^t (a + \frac{bu}{t})^2 du &= \int_0^t (a^2 + a\frac{bu}{t} + \frac{b^2u^2}{t^2}) du \\ &= a^2 t + a\frac{bt^2}{2t} + \frac{b^2t^3}{3t^2} \\ &= a^2 t + a\frac{bt}{2} + \frac{b^2 t}{3} \\ &= (a^2 + \frac{ab}{2} + \frac{b^2}{3})t \end{aligned}$$

So, in order for the variance to equal 1, we require,

$$\begin{aligned} (a^2 + \frac{ab}{2} + \frac{b^2}{3})t = 1 \\ \implies a^2 + \frac{ab}{2} + \frac{b^2}{3} = \frac{1}{t} \end{aligned}$$

Problem 17.

FINISH THIS LATER.

Problem 18.

FINISH THIS LATER.

Problem 19.

Ito's formula states that, if X_t is a stochastic process satisfying $dX_t = b_t dt + \sigma_t dW_t$ with b_t and σ_t measurable, and if $F_t = f(X_t)$ with f twice continuously differentiable, we have

$$dF_t = [b_t f'(X_t) + \frac{\sigma_t^2}{2} f''(X_t)]dt + \sigma_t f'(X_t) dW_t$$

a) In the case that $X_t = W_t$, Corollary 6.2.3, we have the following when we let $F_t = f(W_t)$,

$$dF_t = \frac{1}{2} f''(W_t)dt + f'(W_t)dW_t$$

We will apply this formula to the increment below.

We have $d(W_t e^{W_t})$, so let $f(x) = xe^x$. Then $f'(x) = xe^x + e^x$ and $f''(x) = xe^x + e^x + e^x$. Let $F_t = f(W_t)$ and we have,

$$\begin{aligned} dF_t &= \frac{1}{2}(W_t e^{W_t})'' dt + (W_t e^{W_t})' dW_t \\ &= \frac{1}{2}[W_t e^{W_t} + e^{W_t} + e^{W_t}]dt + [W_t e^{W_t} + e^{W_t}]dW_t \\ &= (\frac{1}{2} W_t e^{W_t} + e^{W_t})dt + e^{W_t}(W_t + 1)dW_t \\ &= e^{W_t}(\frac{1}{2} W_t + 1)dt + e^{W_t}(W_t + 1)dW_t \end{aligned}$$

b) We have $d(e^{t+W_t^2})$, so let $X_t = t + W_t^2$. Then $dX_t = dt + (2W_t dW_t + dt) = 2dt + 2W_t dW_t$

Thus, we have that $b_t = 2$ and $\sigma_t = 2W_t$ satisfying the processes from Ito's lemma. Letting $f(x) = e^x$, we have $f'(x) = f''(x) = e^x$. Now let $F_t = f(X_t)$ and applying Ito's formula yields,

$$\begin{aligned} dF_t &= [2f'(X_t) + \frac{4W_t^2}{2} f''(X_t)]dt + 2W_t f'(X_t)dW_t \\ &= [2e^{t+W_t^2} + (2W_t^2)e^{t+W_t^2}]dt + 2W_t e^{t+W_t^2} dW_t \\ &= 2e^{t+W_t^2}[1 + W_t^2]dt + 2W_t e^{t+W_t^2} dW_t n \end{aligned}$$

c) We have $d\left(\frac{1}{t^\alpha}\int_0^t e^{W_s} d_s\right)$. Then,

$$\begin{aligned}
d\left(\frac{1}{t^\alpha}\int_0^t e^{W_s} d_s\right) &= \left(\frac{1}{t^\alpha}\right)'\int_0^t e^{W_s} d_s + \frac{1}{t^\alpha}\left(\int_0^t e^{W_s} d_s\right)' \\
&= \frac{-\alpha}{t^{\alpha+1}} dt \int_0^t e^{W_s} d_s + \frac{1}{t^\alpha} e^{W_t} dt \\
&= \frac{1}{t^\alpha}\left[e^{W_t} - \frac{\alpha}{t}\int_0^t e^{W_s} d_s\right] dt
\end{aligned}$$

Problem 20.

a) We are given $d(t\cos(W_t))$. Then we have that $f(t,x) = t\cos(x)$ and $X_t = W_t$. In addition, we have that $\partial_t f = \cos(x)$, $\partial_x f = -t\sin(x)$, and $\partial_x^2 f = -t\cos(x)$.

Applying these identities to equation 6.2.7 yields,

$$\begin{aligned}
df(t,X_t) = d(t\cos(W_t)) &= \cos(W_t)dt - t\sin(W_t)dW_t - \frac{1}{2}t\cos(W_t)(dW_t)^2 \\
&= \cos(W_t)dt - t\sin(W_t)dW_t - \frac{1}{2}t\cos(W_t)dt \\
&= \cos(W_t)\left[1 - \frac{1}{2}t\right] dt - t\sin(W_t)dW_t
\end{aligned}$$

b) We are given $d(e^t W_t^2)$. Then we have that $f(t,x) = e^t x^2$ and $X_t = W_t$. In addition, we have that $\partial_t f = e^t x^2$, $\partial_x f = 2e^t x$, and $\partial_x^2 f = 2e^t$.

Applying these identities to equation 6.2.7 yields,

$$\begin{aligned}
df(t,X_t) = d(e^t W_t^2) &= e^t W_t^2 dt + 2e^t W_t dW_t + \frac{1}{2}(2e^t)(dW_t)^2 \\
&= e^t W_t^2 dt + 2e^t W_t dW_t + e^t dt \\
&= e^t\left[W_t^2 + 1\right] dt + 2e^t W_t dW_t
\end{aligned}$$

c) We are given $d(\sin(t)W_t^2)$. Then we have that $f(t,x) = \sin(t)x^2$ and $X_t = W_t$. In addition, we have that $\partial_t f = \cos(t)x^2$, $\partial_x f = 2\sin(t)x$, and $\partial_x^2 f = 2\sin(t)$.

Applying these identities to equation 6.2.7 yields,

$$\begin{aligned}
df(t,X_t) = d(\sin(t)W_t^2) &= \cos(t)W_t^2 dt + 2\sin(t)W_t dW_t + \frac{1}{2}(2\sin(t))(dW_t)^2 \\
&= \cos(t)W_t^2 dt + 2\sin(t)W_t dW_t + \sin(t)dt \\
&= \left[\cos(t)W_t^2 + \sin(t)\right] dt + 2\sin(t)W_t dW_t
\end{aligned}$$

Problem 21.

From Problem 8, we know that for any two independent random variables X and Y, we have $E(XY) = E(X)E(Y)$ and $\text{Var}(XY) = \text{Var}(X)\text{Var}(Y) + \text{Var}(X)\left(E(Y)\right)^2 + \text{Var}(Y)\left(E(X)\right)^2$.

Now consider two independent Brownian motions $W_t^{(1)}$ and $W_t^{(2)}$. Furthermore, consider the differences for each Brownian motion: $W_{t_1+h}^{(1)} - W_{t_1}^{(1)}$ and $W_{t_2+g}^{(2} - W_{t_2}^{(2)}$ with $h > 0$ and $g > 0$.

We know that these differences both have mean 0 and variances $|(t+h) - t| = |h| = h$ and $|(t+g) - t| = |g| = g$, respectively.

Now we will take the limit of each difference as $h \to 0^+$ and $g \to 0^+$. In this case, we get that,

$$W_{t_1+h}^{(1)} - W_{t_1}^{(1)} \to dW_t^{(1)}$$

as $h \to 0$ and,

$$\text{Var}(W_{t_1+h}^{(1)} - W_{t-1}^{(1)}) = h \to dt_1$$

as $h \to 0$.

The same holds for the differences of $W_t^{(2)}$.

Thus, $dW_t^{(1)} \sim N(0, dt_1)$ and $dW_t^{(2)}) \sim N(0, dt_2)$. From Problem 8, we have that,

$$E(dW_t^{(1)} \cdot dW_t^{(2)}) = E(dW_t^{(1)}) \cdot E(dW_t^{(2)}) = 0$$

and,

$$\begin{aligned}
\text{Var}(dW_t^{(1)} \cdot dW_t^{(2)}) &= \text{Var}(dW_t^{(1)})\text{Var}(dW_t^{(2)}) + \text{Var}(dW_t^{(1)})\left(E(dW_t^{(2)})\right)^2 + \text{Var}(dW_t^{(2)})\left(E(dW_t^{(1)})\right)^2 \\
&= dt_1 dt_2 + dt_1(0)^2 + dt_2(0)^2 \\
&= dt_1 dt_2
\end{aligned}$$

Since $dt_1 dt_2$ can be made arbitrarily close to 0, we can say that $\text{Var}(dW_t^{(1)} \cdot dW_t^{(2)})$.

Since the mean of $dW_t^{(1)} \cdot dW_t^{(2)}$ is a constant (0 in this case) and the variance approaches 0, we can say that the mean-squared limit of this quantity is 0. Thus, we have that,

$$dW_t^{(1)} \cdot dW_t^{(2)} = 0$$

Problem 22.

FINISH THIS LATER.

Problem 23.

a) We will begin by showing that the derivative of the answer is equal to the integrand.

Let $f(t,x) = 1 - e^{t/2}\cos(x)$. Then $\partial_t f = -\frac{1}{2}e^{t/2}\cos(x)$, $\partial_x f = e^{t/2}\sin(x)$, and $\partial_x^2 f = e^{t/2}\cos(x)$.

Applying these identities to equation 6.2.7 yields,

$$\begin{aligned} df(t,X_t) = d(1-e^{t/2}\cos(W_t)) &= -\frac{1}{2}e^{t/2}\cos(W_t)dt + e^{t/2}\sin(W_t)dW_t + \frac{1}{2}e^{t/2}\cos(W_t)(dW_t)^2 \\ &= -\frac{1}{2}e^{t/2}\cos(W_t)dt + e^{t/2}\sin(W_t)dW_t + \frac{1}{2}e^{t/2}\cos(W_t)dt \\ &= \frac{1}{2}\left[e^{t/2}\cos(W_t) - e^{t/2}\cos(W_t)\right]dt + e^{t/2}\sin(W_t)dW_t \\ &= \frac{1}{2}[0]dt + e^{t/2}\sin(W_t)dW_t \\ &= e^{t/2}\sin(W_t)dW_t \end{aligned}$$

So from the problem statement and the above derivation we have that,

$$\begin{aligned} \int_0^t e^{s/2}\sin(W_s)dW_s &= \int_0^t df(s,W_s) \\ &= f(t,W_t) \\ &= 1 - e^{t/2}\cos(W_t) \end{aligned}$$

b) Let us take the derivative of the integrated function. By the sum rule of the derivative, we have,

$$df(W_t) = d\left(\sin(W_t) + \frac{1}{2}\int_0^T \sin(W_t)dt\right) = d\left[\sin(W_t)\right] + \frac{1}{2}\left[d\left(\int_0^T \sin(W_t)dt\right)\right]$$

A direct result of Corollary 6.2.3 is that $d(\sin(W_t)) = \cos(W_t)dW_t - \frac{1}{2}\sin(W_t)dt$. In addition we know that the derivative of the integral over the whole domain (in this case 0 to T) is precisely the integrand. Thus,

$$\begin{aligned} df(W_t) = d\left[\sin(W_t)\right] + \frac{1}{2}\left[d\left(\int_0^T \sin(W_t)dt\right)\right] &= \cos(W_t)dW_t - \frac{1}{2}\sin(W_t)dt + \frac{1}{2}\sin(W_t)dt \\ &= \cos(W_t)dW_t \end{aligned}$$

So we have,

$$\begin{aligned} &\int_0^T df(W_t)dW_t = f(W_t) \\ \implies &\int_0^T \cos(W_t)dW_t = \sin(W_t) + \frac{1}{2}\int_0^T \sin(W_t)dt \end{aligned}$$

c) The below formulas are from pages 163 and 164 in the textbook:

Problem 24.

By Proposition 8.2.1, we know that if both the drift and volatility of a stochastic differential equation are just functions of time t, then the solution is Gaussian distributed with the mean $X_0 + \int_0^t a(s)ds$ and variance $\int_0^t b^2(s)ds$.

In the given stochastic differential equation, we see that

$$a(t) = \frac{t}{1+t^2}$$

and,

$$b(t) = t^{3/2}$$

Thus, we can apply Proposition 8.2.1 in this case. As a result, the mean is given by,

$$\begin{aligned} X_0 + \int_0^t a(s)ds &= 1 + \int_0^t \frac{s}{1+s^2}ds \\ &= 1 + \frac{1}{2}\ln(t^2+1) \end{aligned}$$

and the variance is given by,

$$\begin{aligned} \int_0^t b^2(s)ds &= \int_0^t \left(s^{3/2}\right)^2 ds \\ &= \int_0^t s^3 ds \\ &= \frac{t^4}{4} \end{aligned}$$

Thus, the distribution of the solution is given by $X_t \sim N(1 + \frac{1}{2}\ln(t^2+1), \frac{t^4}{4})$.

Moreover, the formula for the solution X_t, given by 8.1.2, is

$$\begin{aligned} X_t &= X_0 + \int_0^t a(s)ds + \int_0^t b(s)dW_s \\ &= 1 + \int_0^t \frac{s}{1+s^2}ds + \int_0^t t^{3/2}dW_s \\ &= 1 + \frac{1}{2}\ln(t^2+1) + \int_0^t t^{3/2}dW_s \end{aligned}$$

Problem 25.

a) We are given $dX_t = \cos t dt - \sin t dW_t$, $X_0 = 1$

As a result, we have $a(t) = \cos t$ and $b(t) = -\sin t$ and can use Proposition 8.2.1 as above.

Hence, the mean is given by,

$$\begin{aligned} X_0 + \int_0^t a(s)ds &= 1 + \int_0^t \cos(s)ds \\ &= 1 + [\sin(t) - \sin(0)] = 1 + \sin(t) \end{aligned}$$

and the variance is given by,

$$\begin{aligned} \int_0^t b^2(s)ds &= \int_0^t (-\sin s)^2 \, ds \\ &= \int_0^t \sin^2(s)ds \\ &= \frac{t}{2} - \frac{1}{4}\sin(2t) \\ &= \frac{1}{2}\left[t - \sin(t)\cos(t)\right] \end{aligned}$$

Thus, $X_t \sim N\left(1 + \sin(t), \frac{1}{2}\left[t - \sin(t)\cos(t)\right]\right)$ and the solution is given by,

$$\begin{aligned} X_t &= X_0 + \int_0^t a(s)ds + \int_0^t b(s)\, dW_s \\ &= 1 + \int_0^t \cos(s)ds + \int_0^t -\sin s \, dW_s \\ &= 1 + \sin(t) - \int_0^t \sin s \, dW_s \end{aligned}$$

b) We are given $dX_t = e^t dt + \sqrt{t} dW_t$, $X_0 = 0$

As a result, we have $a(t) = e^t$ and $b(t) = \sqrt{t}$ and can use Proposition 8.2.1 as above.

Hence, the mean is given by,

$$\begin{aligned} X_0 + \int_0^t a(s)ds &= 0 + \int_0^t e^s ds \\ &= e^t - e^0 \\ &= e^t - 1 \end{aligned}$$

and the variance is given by,

$$\int_0^t b^2(s)ds = \int_0^t \left(\sqrt{s}\right)^2 ds$$

$$= \int_0^t s ds$$
$$= \frac{t^2}{2}$$

Thus, $X_t \sim N\left(e^t - 1, \frac{t^2}{2}\right)$ and the solution is given by,

$$\begin{aligned} X_t &= X_0 + \int_0^t a(s)ds + \int_0^t b(s)\, dW_s \\ &= 0 + \int_0^t e^s ds + \int_0^t \sqrt{s}\, dW_s \\ &= e^t - 1 + \int_0^t \sqrt{s}\, dW_s \end{aligned}$$

Problem 26.

We have,

$$a(t,x) = 2tx^3 + 3t^2(1+x)$$

and

$$b(t,x) = 3t^2x^2 + 1$$

So the associated system is,

$$\begin{aligned} 2tx^3 + 3t^2(1+x) &= \partial_t f(t,x) + \frac{1}{2}\partial_x^2 f(t,x) \\ 3t^2x^2 + 1 &= \partial_x f(t,x) \end{aligned}$$

Integrating partially in x in the second equation yields,

$$f(t,x) = \int (3t^2x^2 + 1)dx = t^2x^3 + x + T(t)$$

Thus, $\partial_t f = 2tx^3 + T'(t)$. Using the first equation, we have

$$2tx^3 + 3t^2(1+x) = 2tx^3 + T'(t) + 3t^2x$$

This implies that $T'(t) = 3t^2$. As a result, $T(t) = t^3 + c$. Hence,

$$f(t,x) = t^2x^3 + x + t^3 + c$$

And we have,

$$X_t = f(t, W_t) = t^2(W_t)^3 + W_t + t^3 + c$$

Since $X_0 = 0$ is given, we have,

$$X_0 = f(0, W_0) = 0^2(W_0)^3 + W_0 + 0^3 + c = 0$$

Thus, $c = 0$. The solution is then,

$$X_t = t^2(W_t)^3 + W_t + t^3$$

Problem 27.

a) We have $a(t,x) = e^t$ and $b(t,x) = x^2 - t$.

So the associated system is,

$$\begin{aligned} e^t &= \partial_t f(t,x) + \frac{1}{2}\partial_x^2 f(t,x) \\ x^2 - t &= \partial_x f(t,x) \end{aligned}$$

Integrating partially in x in the second equation yields,

$$f(t,x) = \int (x^2 - t)dx = \frac{x^3}{3} - tx + T(t)$$

Thus, $\partial_t f = -x + T'(t)$. Using the first equation, we have

$$e^t = -x + T'(t) + x = T'(t)$$

This implies that $T'(t) = e^t$. As a result, $T(t) = e^t + c$. Hence,

$$f(t,x) = \frac{x^3}{3} - tx + e^t + c$$

Since $X_0 = 1$ is given, we have,

$$X_0 = f(0, W_0) = \frac{W_0^3}{3} - 0W_0 + e^0 + c = 1 + c = 1$$

Thus, $c = 0$. The solution is then,

$$X_t = \frac{W_t^3}{3} - t(W_t) + e^t$$

b) We have $a(t,x) = \sin t$ and $b(t,x) = x^2 - t$.

So the associated system is,

$$\begin{aligned} \sin t &= \partial_t f(t,x) + \frac{1}{2}\partial_x^2 f(t,x) \\ x^2 - t &= \partial_x f(t,x) \end{aligned}$$

Integrating partially in x in the second equation yields,

$$f(t,x) = \int (x^2 - t)dx = \frac{x^3}{3} - tx + T(t)$$

Thus, $\partial_t f = -x + T'(t)$. Using the first equation, we have

$$\sin t = -x + T'(t) + x = T'(t)$$

This implies that $T'(t) = \sin t$. As a result, $T(t) = -\cos t + c$. Hence,

$$f(t, x) = \frac{x^3}{3} - tx - \cos t + c$$

Since $X_0 = -1$ is given, we have,

$$X_0 = f(0, W_0) = \frac{W_0^3}{3} - 0W_0 - \cos(0) + c = -1 + c = -1$$

Thus, $c = 0$. The solution is then,

$$X_t = \frac{W_t^3}{3} - t(W_t) - \cos t$$

c) We have $a(t, x) = t^2$ and $b(t, x) = e^{x - \frac{t}{2}}$

So the associated system is,

$$t^2 = \partial_t f(t, x) + \frac{1}{2}\partial_x^2 f(t, x)$$
$$e^{x - \frac{t}{2}} = \partial_x f(t, x)$$

Integrating partially in x in the second equation yields,

$$f(t, x) = \int e^{x - \frac{t}{2}} dx = e^{x - \frac{t}{2}} + T(t)$$

Thus, $\partial_t f = -\frac{1}{2}e^{x - \frac{t}{2}} + T'(t)$. Using the first equation, we have

$$t^2 = -\frac{1}{2}e^{x - \frac{t}{2}} + T'(t) + \frac{1}{2}e^{x - \frac{t}{2}} = T'(t)$$

This implies that $T'(t) = t^2$. As a result, $T(t) = \frac{t^3}{3} + c$. Hence,

$$f(t, x) = e^{x - \frac{t}{2}} + \frac{t^3}{3} + c$$

Since $X_0 = 0$ is given, we have,

$$\begin{aligned} X_0 = f(0, W_0) &= e^{W_0 - \frac{0}{2}} + \frac{0^3}{3} + c \\ &= e^0 + 0 + c \\ &= 1 + c = 0 \end{aligned}$$

Thus, $c = -1$. The solution is then,

$$X_t = e^{W_t - \frac{t}{2}} + \frac{t^3}{3} - 1$$

d) We have $a(t,x) = t$ and $b(t,x) = e^{t/2}(\cos x)$

So the associated system is,

$$\begin{aligned} t &= \partial_t f(t,x) + \frac{1}{2}\partial_x^2 f(t,x) \\ e^{t/2}(\cos x) &= \partial_x f(t,x) \end{aligned}$$

Integrating partially in x in the second equation yields,

$$f(t,x) = \int e^{t/2}(\cos x)dx = e^{t/2}(\sin x) + T(t)$$

Thus, $\partial_t f = \frac{1}{2}e^{t/2}(\sin x) + T'(t)$. Using the first equation, we have

$$t = \frac{1}{2}e^{t/2}(\sin x) + T'(t) - \frac{1}{2}e^{t/2}(\sin x) = T'(t)$$

This implies that $T'(t) = t$. As a result, $T(t) = \frac{t^2}{2} + c$. Hence,

$$f(t,x) = e^{t/2}(\sin x) + \frac{t^2}{2} + c$$

Since $X_0 = 1$ is given, we have,

$$\begin{aligned} X_0 = f(0, W_0) &= e^{0/2}(\sin W_0) + \frac{0^2}{2} + c \\ &= e^0(\sin 0) + 0 + c \\ &= 0(0) + 0 + c = c = 1 \end{aligned}$$

Thus, $c = 1$. The solution is then,

$$X_t = e^{t/2}(\sin W_t) + \frac{t^2}{2} + 1$$

Problem 28.

a) We have $a(t,x) = x + \frac{3}{2}x^2$ and $b(t,x) = t + x^3$

We will first verify the closeness condition:

$$\partial_x a = 1 + 3x$$

and,

$$\partial_t b + \frac{1}{2}\partial_x^2 b = 1 + \frac{1}{2}(6x)$$

$$= 1 + 3x$$

Thus, we have $\partial_x a = \partial_t b + \frac{1}{2}\partial_x^2 b$ and the closeness condition is satisfied.

So the associated system is,

$$\begin{aligned} x + \frac{3}{2}x^2 &= \partial_t f(t,x) + \frac{1}{2}\partial_x^2 f(t,x) \\ t + x^3 &= \partial_x f(t,x) \end{aligned}$$

Integrating partially in x in the second equation yields,

$$f(t,x) = \int t + x^3 dx = tx + \frac{x^4}{4} + T(t)$$

Thus, $\partial_t f = x + T'(t)$. Using the first equation, we have

$$\begin{aligned} x + \frac{3}{2}x^2 &= x + T'(t) + \frac{1}{2}3x^2 \\ \implies T'(t) &= 0 \end{aligned}$$

This implies that $T'(t) = 0$. As a result, $T(t) = c$. Hence,

$$f(t,x) = tx + \frac{x^4}{4} + c$$

Since $X_0 = 0$ is given, we have,

$$\begin{aligned} X_0 = f(0, W_0) &= 0(W_0) + \frac{W_0^4}{4} + c \\ &= 0 + 0 + c \\ &= c = 0 \end{aligned}$$

Thus, $c = 0$. The solution is then,

$$X_t = t(W_t) + \frac{W_t^4}{4}$$

b) We have $a(t,x) = 2tx$ and $b(t,x) = t^2 + x$

We will first verify the closeness condition:

$$\partial_x a = 2t$$

and,

$$\partial_t b + \frac{1}{2}\partial_x^2 b = 2t + \frac{1}{2}0$$

$$= 2t$$

Thus, we have $\partial_x a = \partial_t b + \frac{1}{2}\partial_x^2 b$ and the closeness condition is satisfied.

So the associated system is,

$$\begin{aligned} 2tx &= \partial_t f(t,x) + \frac{1}{2}\partial_x^2 f(t,x) \\ t^2 + x &= \partial_x f(t,x) \end{aligned}$$

Integrating partially in x in the second equation yields,

$$f(t,x) = \int (t^2 + x)dx = t^2 x + \frac{x^2}{2} + T(t)$$

Thus, $\partial_t f = 2tx + T'(t)$. Using the first equation, we have

$$\begin{aligned} 2tx &= 2tx + T'(t) + \frac{1}{2}1 \\ \implies T'(t) &= -\frac{1}{2} \end{aligned}$$

This implies that $T'(t) = -\frac{1}{2}$. As a result, $T(t) = -\frac{1}{2}t + c$. Hence,

$$f(t,x) = t^2 x + \frac{x^2}{2} - \frac{1}{2}t + c$$

Since $X_0 = 0$ is given, we have,

$$\begin{aligned} X_0 = f(0, W_0) &= 0^2(W_0) + \frac{W_0^2}{2} - \frac{1}{2}0 + c \\ &= 0 + 0 + 0 + c \\ &= c = 0 \end{aligned}$$

Thus, $c = 0$. The solution is then,

$$X_t = t^2(W_t) + \frac{W_t^2}{2} - \frac{1}{2}t$$

c) We have $a(t,x) = e^t x + \frac{1}{2}\cos x$ and $b(t,x) = e^t + \sin x$

We will first verify the closeness condition:

$$\partial_x a = e^t - \frac{1}{2}\sin x$$

and,

$$\partial_t b + \frac{1}{2}\partial_x^2 b = e^t - \frac{1}{2}\sin x$$

Thus, we have $\partial_x a = \partial_t b + \frac{1}{2}\partial_x^2 b$ and the closeness condition is satisfied.

So the associated system is,

$$\begin{aligned} e^t x + \frac{1}{2}\cos x &= \partial_t f(t,x) + \frac{1}{2}\partial_x^2 f(t,x) \\ e^t + \sin x &= \partial_x f(t,x) \end{aligned}$$

Integrating partially in x in the second equation yields,

$$f(t,x) = \int (e^t + \sin x)dx = e^t x - \cos x + T(t)$$

Thus, $\partial_t f = e^t x + T'(t)$. Using the first equation, we have

$$\begin{aligned} e^t x + \frac{1}{2}\cos x &= e^t x + T'(t) + \frac{1}{2}\cos x \\ \Longrightarrow T'(t) &= -\frac{1}{2} \end{aligned}$$

This implies that $T'(t) = 0$. As a result, $T(t) = c$. Hence,

$$f(t,x) = e^t x - \cos x + c$$

Since $X_0 = 0$ is given, we have,

$$\begin{aligned} X_0 = f(0, W_0) &= e^0(W_0) - \cos(W_0) + c \\ &= 1(0) - 1 + c \\ &= -1 + c = 0 \end{aligned}$$

Thus, $c = 1$. The solution is then,

$$X_t = e^t(W_t) - \cos(W_t) + 1$$

d) We have $a(t,x) = e^x(1 + \frac{t}{2})$ and $b(t,x) = te^x$

We will first verify the closeness condition:

$$\partial_x a = e^x(1 + \frac{t}{2})$$

and,

$$\begin{aligned}\partial_t b + \frac{1}{2}\partial_x^2 b &= e^x + \frac{1}{2}te^x \\ &= e^x(1+\frac{t}{2})\end{aligned}$$

Thus, we have $\partial_x a = \partial_t b + \frac{1}{2}\partial_x^2 b$ and the closeness condition is satisfied.

So the associated system is,

$$\begin{aligned}e^x(1+\frac{t}{2}) &= \partial_t f(t,x) + \frac{1}{2}\partial_x^2 f(t,x) \\ te^x &= \partial_x f(t,x)\end{aligned}$$

Integrating partially in x in the second equation yields,

$$f(t,x) = \int (te^x)dx = te^x + T(t)$$

Thus, $\partial_t f = e^x + T'(t)$. Using the first equation, we have

$$\begin{aligned}& e^x(1+\frac{t}{2}) = e^x + T'(t) + \frac{1}{2}te^x \\ \Longrightarrow\ & e^x(1+\frac{t}{2}) = e^x(1+\frac{t}{2}) + T'(t) \\ \Longrightarrow\ & T'(t) = 0\end{aligned}$$

This implies that $T'(t) = 0$. As a result, $T(t) = c$. Hence,

$$f(t,x) = te^x + c$$

Since $X_0 = 2$ is given, we have,

$$\begin{aligned}X_0 = f(0, W_0) &= 0e^{W_0} + c \\ &= 0(1) + c \\ &= c = 2\end{aligned}$$

Thus, $c = 2$. The solution is then,

$$X_t = te^{W_t} + 2$$

Problem 29.

Problem 30.

Problem 31.

a) We have $dX_t = \alpha X_t dW_t$ with α a constant. The integrating factor is given by,

$$\begin{aligned}
\rho_t &= e^{-\int_0^t g(s)dW_s + \frac{1}{2}\int_0^t g^2(s)ds} \\
&= e^{-\int_0^t \alpha dW_s + \frac{1}{2}\int_0^t \alpha^2 ds} \\
&= e^{\frac{1}{2}\alpha^2 t - \alpha W_t}
\end{aligned}$$

By Ito's formula, we have that,

$$d\rho_t = \rho_t(\alpha^2 dt - \alpha dW_t)$$

Using $dt^2 = dtdW_t = 0$ and $(dW_t)^2 = dt$, we obtain,

$$\begin{aligned}
dX_t d\rho_t &= \alpha X_t dW_t[\rho_t(\alpha^2 dt - \alpha dW_t)] \\
&= \alpha X_t dW_t[\rho_t \alpha^2 dt - \rho_t \alpha dW_t] \\
&= \alpha X_t[\rho_t \alpha^2 dW_t dt - \rho_t \alpha (dW_t)^2] \\
&= \alpha X_t[-\rho_t \alpha dt] \\
&= -\alpha^2 \rho_t X_t dt
\end{aligned}$$

Multiplying by ρ_t, the initial equation becomes,

$$\rho_t dX_t - \alpha \rho_t X_t dW_t = 0$$

and adding and subtracting $\alpha^2 \rho_t X_t dt$ from the left side yields,

$$\rho_t dX_t - \alpha \rho_t X_t dW_t + \alpha^2 \rho_t X_t dt - \alpha^2 \rho_t X_t dt = 0$$

This can be written as,

$$\rho_t dX_t + X_t d\rho_t + d\rho_t dX_t = 0$$

which, by virtue of the product rule, becomes

$$d(\rho_t X_t) = 0$$

Integrating yields,

$$\begin{aligned}
\rho_t X_t = \rho_0 X_0 + \int_0^t 0ds &= \rho_0 X_0 \\
&= (e^{\frac{1}{2}\alpha^2(0) - \alpha W_0})X_0 \\
&= (e^0)X_0 = X_0
\end{aligned}$$

And hence the solution is,

$$\begin{aligned}
X_t &= \frac{X_0}{\rho_t} \\
&= \frac{X_0}{e^{\frac{1}{2}\alpha^2 t - \alpha W_t}}
\end{aligned}$$

b) We have $dX_t = X_t dt + \alpha X_t dW_t$ with α a constant. The integrating factor is given by,

$$\begin{aligned}\rho_t &= e^{-\int_0^t g(s)dW_s + \frac{1}{2}\int_0^t g^2(s)ds} \\ &= e^{-\int_0^t \alpha dW_s + \frac{1}{2}\int_0^t \alpha^2 ds} \\ &= e^{\frac{1}{2}\alpha^2 t - \alpha W_t}\end{aligned}$$

By Ito's formula, we have that,

$$d\rho_t = \rho_t(\alpha^2 dt - \alpha dW_t)$$

Using $dt^2 = dtdW_t = 0$ and $(dW_t)^2 = dt$, we obtain,

$$\begin{aligned}dX_t d\rho_t &= [X_t dt + \alpha X_t dW_t][\rho_t(\alpha^2 dt - \alpha dW_t)] \\ &= [X_t dt + \alpha X_t dW_t][\rho_t \alpha^2 dt - \rho_t \alpha dW_t] \\ &= \rho_t X_t \alpha^2 (dt)^2 - \rho_t X_t \alpha dt dW_t + \rho_t X_t \alpha^3 dt dW_t - \rho_t X_t \alpha^2 (dW_t)^2 \\ &= -\alpha^2 \rho_t X_t dt\end{aligned}$$

Multiplying by ρ_t, the initial equation becomes,

$$\rho_t dX_t - \alpha \rho_t X_t dW_t = X_t dt$$

and adding and subtracting $\alpha^2 \rho_t X_t dt$ from the left side yields,

$$\rho_t dX_t - \alpha \rho_t X_t dW_t + \alpha^2 \rho_t X_t dt - \alpha^2 \rho_t X_t dt = \rho_t X_t dt$$

which, by virtue of the product rule, becomes

$$d(\rho_t X_t) = \rho_t X_t dt$$

So, with $Y_t = \rho_t X_t$, we have

$$d(Y_t) = Y_t dt$$

c)