Stochastic Differential Equations: Final Project

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Problem 1.

We know that the probability density of the sum of two independent random variables can be computed by the convolutions of f_X and f_Y. Thus, the probability density of $X+Y$, denoted by f_Z, is

$$f_Z(z) = (f_Y * f_X) = \int_{-\infty}^{\infty} f_Y(z-x) f_X(x) dx \tag{1}$$

We know that the normal density function is given by,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Plugging this density function into equation (1) and letting $\sigma_Z = \sqrt{\sigma_X^2 + \sigma_Y^2}$ yields,

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} \left[\frac{1}{\sigma_Y\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{(z-x)-\mu_Y}{\sigma_Y}\right)^2} \frac{1}{\sigma_X\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2} \right] dx \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{\sigma_Y\sigma_X(2\pi)} e^{-\frac{1}{2}\left[\left(\frac{z-x-\mu_Y}{\sigma_Y}\right)^2 + \left(\frac{x-\mu_X}{\sigma_X}\right)^2\right]} \right] dx \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{\sigma_Y\sigma_X(2\pi)} e^{-\frac{\sigma_X^2(z-x-\mu_Y)^2 + \sigma_Y^2(x-\mu_X)^2}{2\sigma_Y^2\sigma_X^2}} \right] dx \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{(2\pi)\sigma_X\sigma_Y} e^{-\frac{x^2(\sigma_X^2+\sigma_Y^2) - 2x(\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X) + \sigma_X^2(z^2+\mu_Y^2-2z\mu_Y) + \sigma_Y^2\mu_X^2}{2\sigma_Y^2\sigma_X^2}} \right] dx \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}\sigma_Z} e^{-\frac{\sigma_Z^2\left(\sigma_X^2(z-\mu_Y)^2+\sigma_Y^2\mu_X^2\right) - \left(\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X\right)^2}{2\sigma_Z^2(\sigma_X\sigma_Y)^2}} \frac{1}{\sqrt{2\pi}\frac{\sigma_X\sigma_Y}{\sigma_Z}} e^{-\frac{\left(x-\frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2}\right)^2}{2\left(\frac{\sigma_X\sigma_Y}{\sigma_Z}\right)^2}} \right] dx \\
&= \frac{1}{\sqrt{2\pi}\sigma_Z} e^{-\frac{(z-(\mu_X+\mu_Y))^2}{2\sigma_Z^2}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}\frac{\sigma_X\sigma_Y}{\sigma_Z}} e^{-\frac{\left(x-\frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2}\right)^2}{2\left(\frac{\sigma_X\sigma_Y}{\sigma_Z}\right)^2}} \right] dx
\end{aligned}$$

The equation inside the integral symbol represents a valid normal density function for x, so we know it integrates to 1. Thus, the probability density function for $X+Y$ is given by

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma_Z} e^{-\frac{(z-(\mu_X+\mu_Y))^2}{2\sigma_Z^2}}$$

This is precisely a normal density function with mean $\mu_X + \mu_Y$ and variance $\sigma_Z^2 = (\sqrt{\sigma_X^2 + \sigma_Y^2})^2 = \sigma_X^2 + \sigma_Y^2$.

Hence, we have shown that, given $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$, $X+Y$ is distributed as $N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$.

Problem 2.

Let $Z = Y + 1$. We'll begin by finding the probability density function for Z. Since Y is a standard uniform random variable, we know that

$$f(y) = \begin{cases} 1 & 0 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Thus, the distribution function approach yields,

$$\begin{aligned} F_Z(z) = P(Z \leq z) = P(Y+1 \leq z) &= P(Y \leq z-1) \\ &= \int_{-\infty}^{z-1} f(y)dy \\ &= \int_0^{z-1} 1\,dy \\ &= z-1 \end{aligned}$$

So, since as Y ranges from 0 to 1, Z ranges from 1 to 2, we have

$$F_Z(z) = \begin{cases} 0 & z < 1 \\ z-1 & 1 \leq z \leq 2 \\ 1 & \text{elsewhere} \end{cases}$$

And the density function for Z is

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \begin{cases} 1 & 1 \leq z \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

Now we can simplify the original function to be $U = X/Z$, where we now know the probability density function for both random variables ($X \sim \text{Unif}(0,1)$ and $Z \sim \text{Unif}(1,2)$).

So the distribution function is given by,

$$\begin{aligned} F_U(u) = P(U \leq u) = P(X/Z \leq u) &= P(X \leq uZ) \\ &= \int_{-\infty}^{uz} f(x)dx \\ &= \int_{0}^{uz} 1\, dx \\ &= uz \end{aligned}$$

FINISH THIS LATER.

Problem 3.

Let Y be a standard normal variable. Then the moment-generating function of Y is given by,

$$\begin{aligned} m(t) &= E(e^{tY}) \\ &= \int_{-\infty}^{\infty} e^{ty} f(y)\, dy \end{aligned}$$

Since Y is a standard normal variable, we know that $\mu = 0$ and $\sigma = 1$. Thus, we have

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$

Plugging this into the above equation yields,

$$\begin{aligned} m(t) &= \int_{-\infty}^{\infty} e^{ty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}+ty}\, dy \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-t)^2} e^{\frac{1}{2}t^2}\, dy \\ &= e^{\frac{1}{2}t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-t)^2}\, dy \end{aligned}$$

We can see that the integral is precisely the integral for a normal random variable with $\mu = y - t$ and $\sigma = 1$. Thus, it integrates to 1, yielding,

$$m(t) = e^{\frac{1}{2}t^2}$$

Problem 4.

Suppose $g(x)$ is a monotone increasing function and X is a random variable with the probability density function f_X.

Let $U = g(X)$ where X has the above density function. Since $g(x)$ is an increasing function of x, then $g^{-1}(u)$ is an increasing function of u. Thus,

$$\begin{aligned} P(U \leq u) &= P[g(X) \leq u] \\ &= P\{g^{-1}[g(X)] \leq g^{-1}(u)\} \\ &= P[X \leq g^{-1}(u)] \end{aligned}$$

The above sequence of equalities implies that,

$$F_U(u) = F_X[g^{-1}(u)]$$

When we differentiate with respect to u, we get,

$$f_U(u) = \frac{dF_X[g^{-1}(u)]}{du} = f_X(g^{-1}(u))\frac{d[g^{-1}(u)]}{du}$$

Problem 5.

We know that the moment generating function of X is $\phi(t)$. Thus,

$$\begin{aligned} m(t) &= E(e^{tX}) \\ &= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx \\ &= \phi(t) \end{aligned}$$