Stochastic Processes: Homework 9

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Problem 1. Durrett, Exercise 2.30

(a) The total number of calls in an hour is Poisson with mean 4. Hence, by Theorem 2.11, the number of men calling in an hour is Poisson with mean $4 \cdot (3/4) = 3$ and the number of women calling in an hour is Poisson with mean $4 \cdot (1/4) = 1$. These two Poisson processes are independent as well. Thus, the probability of seeing exactly two men and three women is given by,

$$e^{-3} \cdot \frac{3^2}{2!} \cdot e^{-1} \cdot \frac{1^3}{3!} \approx 0.014$$

(b) The sex of the caller is independent of the time of the call, so we can consider this binomial distribution with probability 3/4 for male and 1/4 for female. The probability for 3 males in the first 3 calls is thus given by,

$$\begin{aligned} (3/4)^3 &= 27/64 \\ &\approx 0.422 \end{aligned}$$

Problem 2. Durrett, Exercise 2.33

a By Theorem 2.15, the set of arrival times $\{T_1, T_2\}$ has the same distribution as $\{U_1, U_2\}$, which are independent and uniformly distributed on $[0, t]$. Given that the customers arrived in the first 5 minutes, the arrival times are uniformly distributed on $[0, 5]$, and so the probability that each of them arrived in the first 2 minutes is 2/5. The arrival times for customers are independent, hence the probability that both customers arrived in the first 2 minutes is given by,

$$\begin{aligned} (2/5) \cdot (2/5) &= 4/25 \\ &= 0.16 \end{aligned}$$

b The probability that at least 1 customer arrived in the first 2 minutes is given by 1 minus the probability that both arrived in the last 3 minutes. Each customer has probability 3/5 of arriving in the last 3 minutes and, as before, their arrivals are independent. Hence, the probability that at least 1 arrived in the first 2 minutes is given by,

$$\begin{aligned} 1 - (3/5)^2 &= 16/25 \\ &= 0.64 \end{aligned}$$

Problem 3. Durrett, Exercise 2.54

We have,

$$\begin{aligned} ES(t) &= E\left[S_0 \prod_{i=1}^{N(t)} X_i\right] \\ &= S_0 E\left[\prod_{i=1}^{N(t)} X_i\right] \end{aligned}$$

By the independence of the X_i variables and law of total expectation, we get,

$$\begin{aligned} ES(t) &= S_0 \prod_{i=1}^{\lambda t} EX_i \\ &= S_0 \prod_{i=1}^{\lambda t} \mu \\ &= S_0 \cdot \mu^{\lambda t} \end{aligned}$$

Now note that since $\text{Var}(X_i) = \sigma^2$ for each i, we have,

$$\begin{aligned} &\text{Var}(X_i) = E[X_i^2] - E[X_i]^2 \\ \implies &\sigma^2 = E[X_i^2] - \mu^2 \\ \implies &E[X_i^2] = \sigma^2 + \mu^2 \end{aligned}$$

Then the variance is given by,

$$\begin{aligned} \text{Var}S(t) &= ES(t)^2 - (ES(t))^2 \\ &= E\left[\left(S_0 \prod_{i=1}^{N(t)} X_i\right)^2\right] - (S_0 \cdot \mu^{\lambda t})^2 \\ &= E\left[S_0^2 \prod_{i=1}^{N(t)} X_i^2\right] - S_0^2 \cdot \mu^{2\lambda t} \\ &= S_0^2 \prod_{i=1}^{\lambda t} E\left[X_i^2\right] - S_0^2 \cdot \mu^{2\lambda t} \\ &= S_0^2 \prod_{i=1}^{\lambda t} \left[\sigma^2 + \mu^2\right] - S_0^2 \cdot \mu^{2\lambda t} \\ &= S_0^2 \left[(\sigma^2 + \mu^2)^{\lambda t} - \mu^{2\lambda t}\right] \end{aligned}$$

Problem 4. Durrett, Exercise 2.58

We have that the sum of independent exponential random variables is a gamma distribution. Hence,

$$T = t_1 + \cdots + t_N \sim \text{gamma}(N, \lambda)$$

Notice that N is a random variable. We can circumvent this by conditioning on $N = n$, which yields,

$$T = t_1 + \cdots + t_n \sim \text{gamma}(n, \lambda)$$

Now note that if we let $A =$ "$T = t$" and $B =$ "$N = n$", we get,

$$P(A \mid B) \cdot P(B) = P(A \cap B)$$

We already know that $P(A \mid B) = \text{gamma}(n, \lambda)$, and from the problem we know that $P(N = n) = (1-p)^{n-1}$. Hence, we have that $P(B) = (1-p)^{n-1}$ and so,

$$\begin{aligned} P(A \cap B) &= \text{gamma}(n, \lambda) \cdot (1-p)^{n-1} \\ &= \lambda e^{-\lambda t} \cdot \frac{(\lambda \cdot t)^{n-1}}{(n-1)!} \cdot (1-p)^{n-1} \end{aligned}$$