Stochastic Processes: Homework 2

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September 9, 2020

Problem 1. Durrett, Exercise 1.2

Observe that if $0 < X_n < 5$, then there are 3 possibilities for the transition:

1. $X_{n+1} = X_n$. This can occur if two white balls are exchanged or if two black balls are exchanged

2. $X_{n+1} = X_n + 1$. This occurs if a black ball from the left urn is exchanged for a white ball from the right urn

3. $X_{n+1} = X_n - 1$. This occurs if a white ball from the left urn is exchanged for a black ball from the right urn

If $X_n = 0$, then there is only one possible transition: $X_{n+1} = X_n + 1$.

If $X_n = 5$, then there is only one possible transition: $X_{n+1} = X_n - 1$.

From these facts, we can derive the transition probability matrix:

$$
\begin{array}{c}
\\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5
\end{array}
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0.04 & 0.32 & 0.64 & 0 & 0 & 0 \\
0 & 0.16 & 0.48 & 0.36 & 0 & 0 \\
0 & 0 & 0.36 & 0.48 & 0.16 & 0 \\
0 & 0 & 0 & 0.64 & 0.32 & 0.04 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

Let $x =$ number of white balls in left urn and $y =$ number of white balls in right urn. Hence $5-x =$ number of black balls in left urn and $5-y =$ number of black balls in right urn.

Then for $0 < i < 5$, we have

$$p(i, i-1) = \frac{x}{5} \cdot \frac{5-y}{5}$$

$$p(i, i) = \frac{x}{5} \cdot \frac{y}{5} + \frac{5-x}{5} \cdot \frac{5-y}{5}$$

$$p(i, i+1) = \frac{5-x}{5} \cdot \frac{y}{5}$$

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Problem 2. Durrett, Exercise 1.3

Note that X_n can be any integer from 0 to 5. Hence, the state space is $\{0, 1, 2, 3, 4, 5\}$.

For each individual roll Y_k, the possible sums are as follows, along with possible combinations to get that sum:

$$2 : (1, 1)$$
$$3 : (2, 1), (1, 2)$$
$$4 : (2, 2), (1, 3), (3, 1)$$
$$5 : (1, 4), (4, 1), (3, 2), (2, 3)$$
$$6 : (3, 3), (4, 2), (2, 4)$$
$$7 : (4, 3), (3, 4)$$
$$8 : (4, 4)$$

We see that there are 16 possible rolls with this pair of dice. Using the possible outcomes listed above, we can derive probabilities for each sum:

$$P(Y_k = 2) = 0.0625$$
$$P(Y_k = 3) = 0.125$$
$$P(Y_k = 4) = 0.1875$$
$$P(Y_k = 5) = 0.25$$
$$P(Y_k = 6) = 0.1875$$
$$P(Y_k = 7) = 0.125$$
$$P(Y_k = 8) = 0.0625$$

Now we can show which congruence classes these sums to modulo 6:

$$2 = \bar{2}$$
$$3 = \bar{3}$$
$$4 = \bar{4}$$
$$5 = \bar{5}$$
$$6 = \bar{0}$$
$$7 = \bar{1}$$
$$8 = \bar{2}$$

Now we can convert the probabilities for Y_k using these congruence classes,

$$P(Y_k = \bar{0}) = 0.1875$$
$$P(Y_k = \bar{1}) = 0.125 + 0.125 = 0.25$$
$$P(Y_k = \bar{2}) = 0.0625 + 0.0625 = 0.125$$
$$P(Y_k = \bar{3}) = 0.125$$
$$P(Y_k = \bar{4}) = 0.1875$$
$$P(Y_k = \bar{5}) = 0.25$$

Now using these probabilities and the properties of modular arithmetic, we will derive a transition probability matrix for X_n:

$$
\begin{array}{c c c c c c c}
 & \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{5} \\
\bar{0} & 0.1875 & 0.25 & 0.125 & 0.125 & 0.1875 & 0.25 \\
\bar{1} & 0.25 & 0.1875 & 0.25 & 0.125 & 0.125 & 0.1875 \\
\bar{2} & 0.1875 & 0.25 & 0.1875 & 0.25 & 0.125 & 0.125 \\
\bar{3} & 0.125 & 0.1875 & 0.25 & 0.1875 & 0.25 & 0.125 \\
\bar{4} & 0.125 & 0.125 & 0.1875 & 0.25 & 0.1875 & 0.25 \\
\bar{5} & 0.25 & 0.125 & 0.125 & 0.1875 & 0.25 & 0.1875
\end{array}
$$

So if $j \geq i$, we have

$$p(i, j) = P(Y_k = \overline{j - i})$$

Problem 3. Durrett, Exercise 1.58

We have,

$$P(X_{n+1} = 1) - \frac{b}{a+b} = P(X_{n+1} = 1 | X_n = 1) \cdot P(X_n = 1) + P(X_{n+1} = 1 | X_n = 2) \cdot P(X_n = 2) - \frac{b}{a+b}$$

$$= (1-a)P(X_n = 1) + bP(X_n = 2) - \frac{b}{a+b}$$

$$= (1-a)P(X_n = 1) + b(1 - P(X_n = 1)) - \frac{b}{a+b}$$

$$= (1-a-b)P(X_n = 1) + b - \frac{b}{a+b}$$

$$= (1-a-b)P(X_n = 1) + \frac{b(a+b)}{a+b} - \frac{b}{a+b}$$

$$= (1-a-b)P(X_n = 1) + \frac{b(a+b) - b}{a+b}$$

$$= (1-a-b)P(X_n = 1) + \frac{b(a+b-1)}{a+b}$$

$$= (1-a-b)P(X_n = 1) - \frac{b(1-a-b)}{a+b}$$

$$= (1-a-b)\left[P(X_n = 1) - \frac{b}{a+b} \right]$$

Note that,

$$P(X_n = 1) - \frac{b}{a+b} = (1-a-b)\left[P(X_{n-1} = 1) - \frac{b}{a+b} \right]$$

$$= (1-a-b)\left[(1-a-b)\left(P(X_{n-2} = 1) - \frac{b}{a+b} \right) \right]$$

$$= (1-a-b)^2\left[P(X_{n-2} = 1) - \frac{b}{a+b} \right]$$

$$= (1-a-b)^3\left[P(X_{n-3} = 1) - \frac{b}{a+b} \right]$$

$$\vdots$$

$$= (1-a-b)^n\left[P(X_0 = 1) - \frac{b}{a+b} \right]$$

So this gives us,

$$P(X_n = 1) = \frac{b}{a+b} + (1-a-b)^n\left[P(X_0 = 1) - \frac{b}{a+b} \right]$$

Problem 4.

(A) We consider the following matrix,

$$p = \begin{array}{c} \\ \scriptstyle 1 \\ \scriptstyle 2 \\ \scriptstyle 3 \end{array} \begin{array}{ccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 \\ \left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{array} \right) \end{array}$$

Now we take $p - \lambda I$

$$p - \lambda I = \begin{array}{c} \\ \scriptstyle 1 \\ \scriptstyle 2 \\ \scriptstyle 3 \end{array} \begin{array}{ccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 \\ \left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{array} \right) \end{array} - \begin{array}{c} \\ \scriptstyle 1 \\ \scriptstyle 2 \\ \scriptstyle 3 \end{array} \begin{array}{ccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 \\ \left(\begin{array}{ccc} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{array} \right) \end{array}$$

$$= \begin{array}{c} \\ \scriptstyle 1 \\ \scriptstyle 2 \\ \scriptstyle 3 \end{array} \begin{array}{ccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 \\ \left(\begin{array}{ccc} -\lambda & 1 & 0 \\ 0 & 1/2 - \lambda & 1/2 \\ 1/2 & 0 & 1/2 - \lambda \end{array} \right) \end{array}$$

Now we need to find $det(p - \lambda I)$,

$$det(p - \lambda I) = -\lambda \begin{vmatrix} 1/2 - \lambda & 1/2 \\ 0 & 1/2 - \lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & 1/2 \\ 1/2 & 1/2 - \lambda \end{vmatrix} + 0 \begin{vmatrix} 0 & 1/2 - \lambda \\ 1/2 & 0 \end{vmatrix}$$

$$= -\lambda \begin{vmatrix} 1/2 - \lambda & 1/2 \\ 0 & 1/2 - \lambda \end{vmatrix} - \begin{vmatrix} 0 & 1/2 \\ 1/2 & 1/2 - \lambda \end{vmatrix}$$

$$= -\lambda[(1/2 - \lambda)^2 - 0] - (0 - 1/4)$$

$$= -\lambda(\lambda^2 - \lambda + 1/4) + 1/4$$

$$= -\lambda^3 + \lambda^2 - \lambda/4 + 1/4$$

Lastly, we need to solve

$$-\lambda^3 + \lambda^2 - \lambda/4 + 1/4 = 0$$

The solutions to this equation are:

$$\lambda_1 = 1$$
$$\lambda_2 = -i/2$$
$$\lambda_3 = i/2$$

(B) By diagonalization, we have that,

$$p = X\Lambda X^{-1}$$
$$p^2 = X\Lambda X^{-1}X\Lambda X^{-1} = X\Lambda^2 X^{-1}$$
$$\vdots$$
$$p^n = X\Lambda^n X^{-1}$$

where X is the matrix whose columns are the eigenvectors of p and Λ is the matrix with the eigenvalues of p on the diagonal and 0s everywhere else.

Now to find the three eigenvectors:

$$(p - \lambda_1 I)x_1 = 0$$
$$(p - \lambda_2 I)x_2 = 0$$
$$(p - \lambda_3 I)x_3 = 0$$

The solutions to these equations are:

$$x_1 = (1, 1, 1)$$
$$x_2 = (-1 - i, -\frac{1}{2} + \frac{i}{2}, 1)$$
$$x_3 = (-1 + i, -\frac{1}{2} - \frac{i}{2}, 1)$$

So,

$$X = \begin{bmatrix} 1 & -1-i & -1+i \\ 1 & -\frac{1}{2}+\frac{i}{2} & -\frac{1}{2}-\frac{i}{2} \\ 1 & 1 & 1 \end{bmatrix}$$

and the inverse is,

$$X^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ -\frac{1}{10}+\frac{3i}{10} & -\frac{1}{5}-\frac{2i}{5} & \frac{3}{10}+\frac{i}{10} \\ -\frac{1}{10}-\frac{3i}{10} & -\frac{1}{5}+\frac{2i}{5} & \frac{3}{10}-\frac{i}{10} \end{bmatrix}$$

Lastly, we have

$$\Lambda^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{i}{2} & 0 \\ 0 & 0 & \frac{i}{2} \end{bmatrix}^n$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{i}{2}^n & 0 \\ 0 & 0 & \frac{i}{2}^n \end{bmatrix}$$

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So we have,

$$p^n = X\Lambda^n X^{-}1$$

which yields

$$\frac{1}{10}\begin{pmatrix} -2i\left(\left(-\frac{i}{2}\right)^n - \left(\frac{i}{2}\right)^n + i(-i)^n\,2^{1-n} + i^{n+1}\,2^{1-n} + i\right) & (2+2i)\left(\left(-\frac{i}{2}\right)^n - i\left(\frac{i}{2}\right)^n + i(-i)^n\,2^{1-n} - i^n\,2^{1-n} + (1-i)\right) & -2\left(\left(-\frac{i}{2}\right)^n + \left(\frac{i}{2}\right)^n + i(-i)^n\,2^{1-n} - i\,i^n\,2^{1-n} - 2\right) \\ -\left(-\frac{i}{2}\right)^n - \left(\frac{i}{2}\right)^n - i(-i)^n\,2^{1-n} + i\,i^n\,2^{1-n} + 2 & (1-i)\left(\left(-\frac{i}{2}\right)^n + i(-i)^n\,2^{1-n} + i^n\,2^{1-n} + i^{n+1}\,2^{-n} + (2+2i)\right) & i\left(\left(-\frac{i}{2}\right)^n - \left(\frac{i}{2}\right)^n + i(-i)^n\,2^{1-n} + i^{n+1}\,2^{1-n} + -4i\right) \\ (1+i)\left(\left(-\frac{i}{2}\right)^n - i\left(\frac{i}{2}\right)^n + i(-i)^n\,2^{1-n} - i^n\,2^{1-n} + (1-i)\right) & -2\left(\left(-\frac{i}{2}\right)^n + \left(\frac{i}{2}\right)^n + i(-i)^n\,2^{1-n} - i\,i^n\,2^{1-n} - 2\right) & (1-i)\left(\left(-\frac{i}{2}\right)^n + i(-i)^n\,2^{1-n} + i^n\,2^{1-n} + i^{n+1}\,2^{-n} + (2+2i)\right) \end{pmatrix}$$

Hence,

$$
\begin{aligned}
p^n(1,1) &= -\frac{2i}{10}\left(\left(-\frac{i}{2}\right)^n - \left(\frac{i}{2}\right)^n + i(-i)^n 2^{1-n} + i^{n+1}2^{1-n} + i\right) \\
&= -\frac{2i}{10}\left(-\frac{i}{2}\right)^n - -\frac{2i}{10}\left(\frac{i}{2}\right)^n + \frac{2^{1-n+1}(-i)^n}{10} - \frac{2^{1-n+1}i^{n+2}}{10} + i \\
&= -\frac{2i}{10}\left(-\frac{i}{2}\right)^n - -\frac{2i}{10}\left(\frac{i}{2}\right)^n + \frac{2^{1-n+1}((-i)^n - i^{n+2})}{10} + i \\
&= -\frac{2i}{10}\left(-\frac{i}{2}\right)^n - -\frac{2i}{10}\left(\frac{i}{2}\right)^n + 1^n\left(\frac{2^{1-n+1}((-i)^n - i^{n+2})}{10} + i\right) \\
&= -\frac{2i}{10}\left(-\frac{i}{2}\right)^n - -\frac{2i}{10}\left(\frac{i}{2}\right)^n + 1^n\left(\frac{2^{1-n+1}((-i)^n - i^{n+2})}{10} + i\right) \\
&= \left(\frac{2^{1-n+1}((-i)^n - i^{n+2})}{10} + i\right)\lambda_1^n - \frac{2i}{10}\lambda_2^n - -\frac{2i}{10}\lambda_3^n \\
&= \left(\frac{2^{1-n}((-i)^n - i^{n+2})}{5} + i\right)\lambda_1^n - \frac{i}{5}\lambda_2^n - -\frac{i}{5}\lambda_3^n \\
&= \left(\frac{2^{1-n}((-1)^n(i^n - i^{n+2}))}{5} + i\right)\lambda_1^n - \frac{i}{5}\lambda_2^n - -\frac{i}{5}\lambda_3^n \\
&= \left(\frac{2^{1-n}(-1)^n(1 - i^2)i^n}{5} + i\right)\lambda_1^n - \frac{i}{5}\lambda_2^n - -\frac{i}{5}\lambda_3^n \\
&= \left(\frac{2^{1-n}(-1)^n 2i^n}{5} + i\right)\lambda_1^n - \frac{i}{5}\lambda_2^n - -\frac{i}{5}\lambda_3^n \\
&= \left(\frac{2^{1-n+1}(-1)^n i^n}{5} + i\right)\lambda_1^n - \frac{i}{5}\lambda_2^n - -\frac{i}{5}\lambda_3^n
\end{aligned}
$$

as required.

(C) We have the following equations:

$$p^0(1,1) = 1 = a + b + c$$

$$p^1(1,1) = 0 = a\lambda_1 + b\lambda_2 + c\lambda_3 = a + \frac{-i}{2}b + \frac{i}{2}c$$

$$p^2(1,1) = 0 = a\lambda_1^2 + b\lambda_2^2 + c\lambda_3^2 = a + \frac{-1}{4}b + \frac{-1}{4}c$$

We can set up an augmented matrix for this system of equations:

$$\begin{pmatrix} 1 & 1 & 1 & \bigg| & 1 \\ 1 & \frac{-i}{2} & \frac{i}{2} & \bigg| & 0 \\ 1 & \frac{-1}{4} & \frac{-1}{4} & \bigg| & 0 \end{pmatrix}$$

Row reduction yields:

$$\begin{pmatrix} 1 & 0 & 0 & \bigg| & \frac{1}{5} \\ 0 & 1 & 0 & \bigg| & \frac{2}{5} - \frac{i}{5} \\ 0 & 0 & 1 & \bigg| & \frac{2}{5} + \frac{i}{5} \end{pmatrix}$$

Hence we have,

$$p^n(1,1) = \frac{1}{5} + \left(\frac{2}{5} - \frac{i}{5}\right) \cdot \frac{-i}{2}^n + \left(\frac{2}{5} + \frac{i}{5}\right) \cdot \frac{i}{2}^n$$

(D) If n even:

$$p^n(1,1) = \frac{1}{5} + \left(\frac{2}{5} - \frac{i}{5}\right) \cdot \frac{-1}{2^n} + \left(\frac{2}{5} + \frac{i}{5}\right) \cdot \frac{-1}{2^n}$$

$$= \frac{1}{5} + \frac{-2}{5 \cdot 2^n} - \frac{-i}{5 \cdot 2^n} + \frac{-2}{5 \cdot 2^n} + \frac{-i}{5 \cdot 2^n}$$

$$= \frac{1}{5} + \frac{-4}{5 \cdot 2^n}$$

If n odd:

$$p^n(1,1) = \frac{1}{5} + \left(\frac{2}{5} - \frac{i}{5}\right) \cdot \frac{-i}{2^n} + \left(\frac{2}{5} + \frac{i}{5}\right) \cdot \frac{i}{2^n}$$

$$= \frac{1}{5} + \frac{-2i}{5 \cdot 2^n} - \frac{1}{5 \cdot 2^n} + \frac{2i}{5 \cdot 2^n} + \frac{-1}{5 \cdot 2^n}$$

$$= \frac{1}{5} + \frac{-2}{5 \cdot 2^n}$$

In either case, we see that as $n \to \infty$, the second term goes to 0. Hence, we have that,

$$p^n(1,1) \to \frac{1}{5} \text{ as } n \to \infty$$