Stochastic Processes: Homework 2

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Problem 1. Durrett, Exercise 1.2

Observe that if $0 < X_n < 5$, then there are 3 possibilities for the transition:

1. $X_{n+1} = X_n$. This can occur if two white balls are exchanged or if two black balls are exchanged
2. $X_{n+1} = X_n + 1$. This occurs if a black ball from the left urn is exchanged for a white ball from the right urn
3. $X_{n+1} = X_n - 1$. This occurs if a white ball from the left urn is exchanged for a black ball from the right urn

If $X_n = 0$, then there is only one possible transition: $X_{n+1} = X_n + 1$.

If $X_n = 5$, then there is only one possible transition: $X_{n+1} = X_n - 1$.

From these facts, we can derive the transition probability matrix:

$$\begin{array}{c c}
 & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0.04 & 0.32 & 0.64 & 0 & 0 & 0 \\
0 & 0.16 & 0.48 & 0.36 & 0 & 0 \\
0 & 0 & 0.36 & 0.48 & 0.16 & 0 \\
0 & 0 & 0 & 0.64 & 0.32 & 0.04 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\end{array}$$

Let x = number of white balls in left urn and y = number of white balls in right urn. Hence $5-x$ = number of black balls in left urn and $5-y$ = number of black balls in right urn.

Then for $0 < i < 5$, we have

$$\begin{aligned}
p(i, i-1) &= \frac{x}{5} \cdot \frac{5-y}{5} \\
p(i, i) &= \frac{x}{5} \cdot \frac{y}{5} + \frac{5-x}{5} \cdot \frac{5-y}{5} \\
p(i, i+1) &= \frac{5-x}{5} \cdot \frac{y}{5}
\end{aligned}$$

Problem 2. Durrett, Exercise 1.3

Note that X_n can be any integer from 0 to 5. Hence, the state space is $\{0, 1, 2, 3, 4, 5\}$.

For each individual roll Y_k, the possible sums are as follows, along with possible combinations to get that sum:

$$\begin{aligned}
2 :&(1,1)\\
3 :&(2,1),(1,2)\\
4 :&(2,2),(1,3),(3,1)\\
5 :&(1,4),(4,1),(3,2),(2,3)\\
6 :&(3,3),(4,2),(2,4)\\
7 :&(4,3),(3,4)\\
8 :&(4,4)
\end{aligned}$$

We see that there are 16 possible rolls with this pair of dice. Using the possible outcomes listed above, we can derive probabilities for each sum:

$$\begin{aligned}
P(Y_k = 2) &= 0.0625\\
P(Y_k = 3) &= 0.125\\
P(Y_k = 4) &= 0.1875\\
P(Y_k = 5) &= 0.25\\
P(Y_k = 6) &= 0.1875\\
P(Y_k = 7) &= 0.125\\
P(Y_k = 8) &= 0.0625
\end{aligned}$$

Now we can show which congruence classes these sums to modulo 6:

$$\begin{aligned}
2 &= \bar{2}\\
3 &= \bar{3}\\
4 &= \bar{4}\\
5 &= \bar{5}\\
6 &= \bar{0}\\
7 &= \bar{1}\\
8 &= \bar{2}
\end{aligned}$$

Now we can convert the probabilities for Y_k using these congruence classes,

$$\begin{aligned}
P(Y_k = \bar{0}) &= 0.1875 \\
P(Y_k = \bar{1}) &= 0.125 + 0.125 = 0.25 \\
P(Y_k = \bar{2}) &= 0.0625 + 0.0625 = 0.125 \\
P(Y_k = \bar{3}) &= 0.125 \\
P(Y_k = \bar{4}) &= 0.1875 \\
P(Y_k = \bar{5}) &= 0.25
\end{aligned}$$

Now using these probabilities and the properties of modular arithmetic, we will derive a transition probability matrix for X_n:

$$\begin{array}{c c}
 & \begin{array}{cccccc} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{5} \end{array} \\
\begin{array}{c} \bar{0} \\ \bar{1} \\ \bar{2} \\ \bar{3} \\ \bar{4} \\ \bar{5} \end{array} &
\begin{pmatrix}
0.1875 & 0.25 & 0.125 & 0.125 & 0.1875 & 0.25 \\
0.25 & 0.1875 & 0.25 & 0.125 & 0.125 & 0.1875 \\
0.1875 & 0.25 & 0.1875 & 0.25 & 0.125 & 0.125 \\
0.125 & 0.1875 & 0.25 & 0.1875 & 0.25 & 0.125 \\
0.125 & 0.125 & 0.1875 & 0.25 & 0.1875 & 0.25 \\
0.25 & 0.125 & 0.125 & 0.1875 & 0.25 & 0.1875
\end{pmatrix}
\end{array}$$

So if $j \geq i$, we have

$$p(i,j) = P(Y_k = \overline{j-i})$$

Problem 3. Durrett, Exercise 1.58

Problem 4.