Stochastic Processes: Homework 6

Chris Hayduk

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Problem 1. Durrett, Exercise 1.51

We have the following transition matrix,

$$p = \begin{array}{c} \\ HHH \\ HHT \\ HTH \\ THH \\ HTT \\ THT \\ TTH \\ TTT \end{array} \begin{array}{c} \begin{array}{cccccccc} HHH & HHT & HTH & THH & HTT & THT & TTH & TTT \end{array} \\ \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix} \end{array}$$

Case 1:

Let us define $h(x)$ to be the probability of visiting THH before HHH, satisfied by the following equations.,

$$\begin{aligned}
h(HHH) &= 0 \\
h(THH) &= 1 \\
h(HHT) &= \frac{1}{2}(h(HTH) + h(HTT)) \\
h(HTH) &= \frac{1}{2}(h(THH) + h(THT)) = \frac{1}{2}(1 + h(THT)) \\
h(HTT) &= \frac{1}{2}(h(TTH) + h(TTT)) = \frac{1}{2}(2h(TTH)) = h(TTH) \\
h(THT) &= \frac{1}{2}(h(HTH) + h(HTT)) \\
h(TTH) &= \frac{1}{2}(h(THH) + h(THT)) = \frac{1}{2}(1 + h(THT)) \\
h(TTT) &= \frac{1}{2}(h(TTH) + h(TTT)) = h(TTH)
\end{aligned}$$

We get that $h(x) = 1$ for every $x \neq HHH$.

Now note that each state is equally probable (with probability 1/8), so we get the following,

$$\frac{1}{8}(0+1+1+1+1+1+1+1) = \frac{7}{8}$$

Case 2:

We need to define $h(x)$ as the probability of visiting THH before HHT. This time, we get the following system of equations,

$$\begin{aligned}
h(HHT) &= 0 \\
h(THH) &= 1 \\
h(HHH) &= \frac{1}{2}(h(HHH) + h(HHT)) = \frac{1}{2}h(HHH) = 0 \\
h(HTH) &= \frac{1}{2}(h(THH) + h(THT)) = \frac{1}{2}(1 + h(THT)) \\
h(HTT) &= \frac{1}{2}(h(TTH) + h(TTT)) = \frac{1}{2}(2h(TTH)) = h(TTH) \\
h(THT) &= \frac{1}{2}(h(HTH) + h(HTT)) \\
h(TTH) &= \frac{1}{2}(h(THH) + h(THT)) = \frac{1}{2}(1 + h(THT)) \\
h(TTT) &= \frac{1}{2}(h(TTH) + h(TTT)) = h(TTH)
\end{aligned}$$

Here we have $h(x) = 1$ for every $x \neq HHT, HHH$. This yields,

$$\begin{aligned}
\frac{1}{8}(0+1+0+1+1+1+1+1) &= \frac{6}{8} \\
&= \frac{2}{3}
\end{aligned}$$

Case 3:

We need to define $h(x)$ as the probability of visiting HHT before HTH. This time, we get the following system of equations,

$$
\begin{aligned}
h(HTH) &= 0 \\
h(HHT) &= 1 \\
h(HHH) &= \frac{1}{2}(h(HHH) + h(HHT)) = \frac{1}{2}h(HHH) + \frac{1}{2} = 1 \\
h(THH) &= \frac{1}{2}(h(HHH) + h(HHT)) = \frac{1}{2}(h(HHH) + 1) = 1 \\
h(HTT) &= \frac{1}{2}(h(TTH) + h(TTT)) = \frac{1}{2}(2h(TTH)) = h(TTH) = \frac{2}{3} \\
h(THT) &= \frac{1}{2}(h(HTH) + h(HTT)) = \frac{1}{2}h(HTT) = \frac{1}{3} \\
h(TTH) &= \frac{1}{2}(h(THH) + h(THT)) = \frac{1}{2}(1 + h(THT)) = \frac{2}{3} \\
h(TTT) &= \frac{1}{2}(h(TTH) + h(TTT)) = h(TTH) = \frac{2}{3}
\end{aligned}
$$

Using the above values, we get,

$$\frac{1}{8}\left(0 + 1 + 1 + 1 + \frac{2}{3} + \frac{1}{3} + \frac{2}{3} + \frac{2}{3}\right) = \frac{2}{3}$$

Case 4:

We need to define $h(x)$ as the probability of visiting HHT before HTT. This time, we get the following system of equations,

$$
\begin{aligned}
h(HTT) &= 0 \\
h(HHT) &= 1 \\
h(HHH) &= \frac{1}{2}(h(HHH) + h(HHT)) = \frac{1}{2}h(HHH) + \frac{1}{2} = 1 \\
h(THH) &= \frac{1}{2}(h(HHH) + h(HHT)) = 1 \\
h(HTH) &= \frac{1}{2}(h(THH) + h(THT)) = \frac{1}{2}h(THT) + \frac{1}{2} = \frac{2}{3} \\
h(THT) &= \frac{1}{2}(h(HTH) + h(HTT)) = \frac{1}{2}h(HTH) = \frac{1}{3} \\
h(TTH) &= \frac{1}{2}(h(THH) + h(THT)) = \frac{2}{3} \\
h(TTT) &= \frac{1}{2}(h(TTH) + h(TTT)) = h(TTH) = \frac{2}{3}
\end{aligned}
$$

Using the above values, we get,

$$\frac{1}{8}\left(0 + 1 + 1 + 1 + \frac{2}{3} + \frac{1}{3} + \frac{2}{3} + \frac{2}{3}\right) = \frac{2}{3}$$

Problem 2. Durrett, Exercise 1.52

(a) Larry starts with 1 coupon. That is, $X_0 = 1$. We also have the following transition probability:

$$p = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

We have $h(0) = 0$ and $h(3) = 1$ with the remaining finite set of states $\{1, 2\}$. Hence, we can apply Theorem 1.28, which gives us $h(x) = \sum_y p(x, y)h(y)$. In addition, since $P_x(V_{\{0\}} \wedge V_{\{3\}} < \infty) > 0$ for $x = 1, 2$, we also have that $h(x) = P_x(V_3 < V_0)$. We have $q = p = 0.5$, so for $0 < x < 3$, we get,

$$h(x) = 0.5h(x + 1) + 0.5h(x - 1)$$

We get,

$$\begin{aligned} h(1) &= 0.5h(2) \\ h(2) &= 0.5 + 0.5h(1) \end{aligned}$$

Combining equation (1) and (2) yields $h(2) = 0.5/0.75 \approx 0.667$. Hence, we have,

$$\begin{aligned} h(1) = P_1(V_3 < V_0) &= \frac{0.25}{0.75} \\ &\approx 0.333 \end{aligned}$$

(b) We have the following system of equations,

$$\begin{aligned} g(1) &= 1 + 0.5g(2) \\ g(2) &= 1 + 0.5g(1) \end{aligned}$$

So we get that $g(2) = 1 + 0.5 + 0.25g(2) = 2$ and thus $g(1) = 1 + 0.5(2) = 2$ as well.

Hence, starting from 1 ticket, Larry will need 2 plays on average in order to win or lose the game.

Problem 3. Durrett, Exercise 1.67

(a) Observe that for $X_n = n$ with $0 \le n \le 5$, we there are n sides of the die which have already been seen and $6 - n$ sides which have not been seen. Since each side is equally probable to show up, we have that,

$$P(X_{n+1} = n) = n/6$$
$$P(X_{n+1} = n+1) = (6-n)/6$$

When $X_n = 6$, we have $P(X_{n+1} = 6) = 1$.

Thus, the transition probability matrix is,

$$p = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/6 & 5/6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2/6 & 4/6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3/6 & 3/6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4/6 & 2/6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

(b) Let $T = \min\{n : X_n = 6\}$. We need to find ET. That is, ET the expected minimum number of rolls to see all 6 sides of the die.

Consider the matrix after deleting the row and column corresponding to 6:

$$r = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1/6 & 5/6 & 0 & 0 & 0 \\ 0 & 0 & 2/6 & 4/6 & 0 & 0 \\ 0 & 0 & 0 & 3/6 & 3/6 & 0 \\ 0 & 0 & 0 & 0 & 4/6 & 2/6 \\ 0 & 0 & 0 & 0 & 0 & 5/6 \end{pmatrix} \end{array}$$

So we have,

$$I - r = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 5/6 & -5/6 & 0 & 0 & 0 \\ 0 & 0 & 4/6 & -4/6 & 0 & 0 \\ 0 & 0 & 0 & 3/6 & -3/6 & 0 \\ 0 & 0 & 0 & 0 & 2/6 & -2/6 \\ 0 & 0 & 0 & 0 & 0 & 1/6 \end{pmatrix} \end{array}$$

This yields,

$$(I-r)^{-1} = \begin{array}{c c} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 1 & 6/5 & 3/2 & 2 & 3 & 6 \\ 0 & 6/5 & 3/2 & 2 & 3 & 6 \\ 0 & 0 & 3/2 & 2 & 3 & 6 \\ 0 & 0 & 0 & 2 & 3 & 6 \\ 0 & 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix} \end{array}$$

Lastly, we have

$$(I-r)^{-1}\mathbb{1} = \begin{pmatrix} 14.7 \\ 13.7 \\ 12.5 \\ 11 \\ 9 \\ 6 \end{pmatrix}$$

Thus, starting from state 0, we would expect to make 14.7 moves on average before seeing all 6 sides of the die.