# Stochastic Processes: Homework 2

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**Problem 1.** Durrett, Exercise 1.8

(d) The irreducible closed sets are: $\{1,4\},\{2,5\}$

Recurrent states: From examining the transition probabilities, we have that $\{1,4\},\{2,5\}$ are the irreducible closed sets. Hence, by Theorem 1.7, $\{1,2,4,5\}$ are all recurrent. By Theorem 1.8, these are the **only** recurrent states.

Transient states: Note that the state space $S$ is finite. Hence, by Theorem 1.8, we have that the set of transient states and irreducible closed sets (ie. recurrent states) partition $S$. Moreover, we have that these sets are disjoint. Hence, the transient states are given by

$$\begin{aligned} T &= S \setminus (\{1,4\} \cup \{2,5\}) \\ &= \{3,6\} \end{aligned}$$

(e) The irreducible closed sets are: $\{1\},\{2,4\}$

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$$\begin{aligned} T &= S \setminus (\{1\} \cup \{2,4\}) \\ &= \{3,5\} \end{aligned}$$

**Problem 2.** Durrett, Exercise 1.40

(a) The transition probability matrix is given by:

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1/3 & 2/3 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 1/3 & 2/3 \end{pmatrix} \end{array}$$

(b) Note that we are finding the stationary distribution(s) for the chain.

Hence, we need to find $\pi$ such that

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 1/3 & 2/3 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{pmatrix}$$

So we need to solve:

$$\begin{aligned} \pi_1/3 + \pi_2/3 &= \pi_1 \\ 2\pi_1/3 + \pi_3/3 &= \pi_2 \\ 2\pi_2/3 + \pi_4/3 &= \pi_3 \\ 2\pi_3/3 + 2\pi_4/3 &= \pi_4 \end{aligned}$$

Which yields,

$$\begin{pmatrix} \pi_1 & 2\pi_1 & 4\pi_1 & 8\pi_1 \end{pmatrix}$$

We must have that these probabilities add to 1, so

$$\begin{aligned} \pi_1 + 2\pi_1 + 4\pi_1 + 8\pi_1 = 1 \\ \implies \pi_1 = 1/15 \end{aligned}$$

Thus, we get

$$\pi = \begin{pmatrix} 1/15 & 2/15 & 4/15 & 8/15 \end{pmatrix}$$

Now let's check that this is indeed the stationary distribution:

$$\begin{aligned} \begin{pmatrix} 1/15 & 2/15 & 4/15 & 8/15 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 1/3 & 2/3 \end{pmatrix} &= \begin{pmatrix} 3/45 & 6/45 & 12/45 & 24/45 \end{pmatrix} \\ &= \begin{pmatrix} 1/15 & 2/15 & 4/15 & 8/15 \end{pmatrix} \end{aligned}$$

Since we have,

$$\pi \begin{pmatrix} 1/3 & 2/3 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 1/3 & 2/3 \end{pmatrix} = \pi$$

we know that $\pi = \begin{pmatrix} 1/15 & 2/15 & 4/15 & 8/15 \end{pmatrix}$ is indeed the stationary distribution.

**Problem 3.** Durrett, Exercise 1.59

(a) Let $X_n = i$. Our possible outcomes are:

(a) White ball from left urn and white ball from right urn:

$$\begin{aligned} \frac{m-i}{m} \cdot \frac{m-b+i}{m} &= \frac{(m-i)(m-b+i)}{m^2} \\ &= \frac{m^2 - mb + mi - mi + bi - i^2}{m^2} \\ &= \frac{m^2 - mb + bi - i^2}{m^2} \end{aligned}$$

Hence $X_{n+1} = i = X_n$

(b) White ball from left urn and black ball from right urn:

$$\begin{aligned} \frac{m-i}{m} \cdot \frac{b-i}{m} &= \frac{(m-i)(b-i)}{m^2} \\ &= \frac{mb - mi - bi + i^2}{m^2} \end{aligned}$$

Hence $X_{n+1} = i + 1 = X_n + 1$

(c) Black ball from left urn and black ball from right urn:

$$\frac{i}{m} \cdot \frac{b-i}{m} = \frac{bi - i^2}{m^2}$$

Hence $X_{n+1} = i = X_n$

(d) Black ball from left urn and white ball from right urn:

$$\frac{i}{m} \cdot \frac{m-b+i}{m} = \frac{mi - bi + i^2}{m^2}$$

Hence $X_{n+1} = i - 1 = X_n - 1$

So the probability that $X_{n+1} = X_n$ is given by

$$\frac{bi - i^2}{m^2} + \frac{m^2 - mb + bi - i^2}{m^2} = \frac{m^2 - mb + 2bi - 2i^2}{m^2}$$

So our transition probability matrix is:

$$p = \frac{1}{m^2}\begin{pmatrix} 0 & m^2 & 0 & 0 & \cdots & 0 & 0 \\ m-b+1 & m^2 - mb & mb - m - b + 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & & & & \vdots \\ 0 & 0 & & & & m^2 & 0 \end{pmatrix}$$

(b) Let

$$\pi(i) = \binom{b}{i}\binom{2m-b}{m-i}\Big/\binom{2m}{m}$$

So,

$$\pi = \left(\binom{2m-b}{m}\Big/\binom{2m}{m} \quad b\binom{2m-b}{m-1}\Big/\binom{2m}{m} \quad \binom{b}{2}\binom{2m-b}{m-2}\Big/\binom{2m}{m} \quad \cdots \quad (2mb - b^2)\Big/\binom{2m}{m} \quad \binom{2m-b}{m-b}\Big/\binom{2m}{m}\right)$$

(c) We are choosing $i$ black balls (which is the number of black balls in the left urn) from the $b$ total black balls. We then choose $m - i$ white balls (the number of white balls in the left urn) from the $2m - b$ total white balls. We then choose $m$ balls from the $2m$ total balls and divide by this number.

Essentially $\pi(i)$ is giving us the ratio of the number of combinations with $i$ black balls and $m - i$ white balls in the left urn to the total number of possible combinations of balls in the left urn. Another way of showing this is:

$$\pi(i) = \frac{\text{\# of combinations of balls in left urn with i black balls and m - i white balls}}{\text{total \# of possible combinations of balls in left urn}}$$

when the markov chain is in state $i$.

**Problem 4.**

Suppose $\pi$ satisfies the detailed balance condition. That is,

$$\pi(x)p(x,y) = \pi(y)p(y,x)$$

Now consider $p^2$. Note that, from Theorem 1.1, we have

$$p^2(x,y) = \sum_k p(x,k)p(k,y)$$

and

$$p^2(y,x) = \sum_k p(y,k)p(k,x)$$

Now multiplying by $\pi(x)$ in the first equation yields

$$\begin{aligned}
\pi(x)p^2(x,y) &= \pi(x)\sum_k p(x,k)p(k,y) \\
&= \sum_k \pi(x)\left[p(x,k)p(k,y)\right] \\
&= \sum_k \left[\pi(x)p(x,k)\right]p(k,y) \\
&= \sum_k \pi(k)p(k,x)p(k,y) \\
&= \sum_k p(k,x)\left[\pi(k)p(k,y)\right] \\
&= \sum_k p(k,x)\pi(y)p(y,k) \\
&= \sum_k \pi(y)\left[p(y,k)p(k,x)\right] \\
&= \pi(y)\sum_k p(y,k)p(k,x) \\
&= \pi(y)p^2(y,x)
\end{aligned}$$

Hence, we have $\pi(x)p^2(x,y) = \pi(y)p^2(y,x)$ as required.