Stochastic Processes: Homework 11

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December 4, 2020

Problem 1. Durrett, Exercise 3.4

Let g_i be the random fair from distribution G paid at time i. Then we have that $W(t) = \sum_{i=1}^{N(t)} g_i$. So

$$\begin{aligned} EW(t) &= E\left[\sum_{i=1}^{N(t)} g_i\right] \\ &= \sum_{i=1}^{N(t)} Eg_i \\ &= \sum_{i=1}^{N(t)} \mu_G \\ &= N(t) \cdot \mu_G \end{aligned}$$

So we get,

$$\lim_{t\to\infty} EW(t)/t = \mu_G \lim_{t\to\infty} N(t)/t$$

Then, using Theorem 3.1, we get,

$$\begin{aligned} \mu_G \lim_{t\to\infty} N(t)/t &= \mu_G \cdot 1/\mu_F \\ &= \frac{\mu_G}{\mu_F} \end{aligned}$$

Problem 2. Durrett, Exercise 3.11

a)

b)

c)

Problem 3. Durrett, Exercise 3.17

Let w_i denote the time that the i-th person waits until the start of the tour (in minutes). Note that we need k people to start the tour, that each tour costs \$20, and that it costs \$0.10 per minute that each person waits. Hence, we have that the cost per person can be formulated as,

$$f(k) = \frac{0.1 \cdot (w_1 + w_2 + \cdots + w_k) + 20}{k}$$

To find the average, let us take the expectation:,

$$E\left[\frac{0.1 \cdot (w_1 + w_2 + \cdots + w_k) + 20}{k}\right] = \frac{0.1 \cdot (Ew_1 + Ew_2 + \cdots + Ew_k) + 20}{k}$$

Since people arrive at a rate of 1 per minute, we get the following expected waiting times,

$$Ew_i = k - i$$

This yields,

$$\begin{aligned}
\frac{0.1 \cdot (Ew_1 + Ew_2 + \cdots + Ew_k) + 20}{k} &= \frac{0.1 \cdot 1/2 \cdot (k-1)k + 20}{k} \\
&= \frac{0.05 \cdot (k-1)k + 20}{k} \\
&= \frac{0.05k^2 - 0.05k + 20}{k}
\end{aligned}$$

Now we want to minimize this equation. Let us take the derivative with respect to k and set it to 0, yielding,

$$0 = 0.05 - \frac{20}{k^2}$$

This gives $k = 20$ or $k = -20$. We can't have a negative number of people, so our only valid critical point is $k = 20$. We can now use the second derivative test to ensure that this is a local minimum:

$$\frac{d}{dk}\left[0.05 - \frac{20}{k^2}\right] = \frac{40}{k^3}$$

So $f''(20) = 40/20^3 > 0$, and thus 20 is a local minimum as required.

Problem 4.

A)

B) Fix x, n. Then we need to show that,

$$\int_{-\infty}^{x} F_n(x-y)g(y) \leq F_n(x)$$

C)