Stochastic Processes: Homework 1

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Problem 1.

Let A_1 = "Donna drew a fair coin", A_2 = "Donna drew the two-headed coin" and let B = "Donna flipped 6 heads in a row".

Note that

$$\begin{aligned} P(A_1) &= \frac{64}{65} \\ P(B|A_1) &= \frac{1/2^6}{64/65} = \frac{65}{4096} \approx 0.01587 \end{aligned}$$

and

$$\begin{aligned} P(A_2) &= \frac{1}{65} \\ P(B|A_2) &= 1 \end{aligned}$$

Hence,

$$\begin{aligned} \sum P(A_j)P(B|A_j) &= \frac{64}{65} \cdot \frac{65}{4096} + \frac{1}{65} \cdot 1 \\ &= \frac{129}{4160} \approx 0.031 \end{aligned}$$

Thus, we have,

$$\begin{aligned} P(A_2|B) &= \frac{P(A_2)P(B|A_2)}{\sum P(A_j)P(B|A_j)} \\ &= \frac{1/65}{129/4160} \\ &= \frac{64}{129} \approx 0.4961 \end{aligned}$$

So if Donna flips 6 heads in a row, there is about a 49.61% chance that she chose the two-headed coin.

Problem 2.

We have that X_1, X_2 are two independent uniform (0, 1) random variables on the same probability space. In addition, we have that $Y = \min\{X_1, X_2\}$.

In order to find the CDF of Y, we must find $P(Y \leq y)$. Note that,

$$P(Y \leq y) = P(\min\{X_1, X_2\} \leq y)$$

In addition, note that,

$$\begin{aligned} P(Y \leq y) &= 1 - P(Y > y) \\ &= 1 - P(\min\{X_1, X_2\} > y) \end{aligned}$$

Since we are taking a minimum, $\min\{X_1, X_2\} > y$ if and only if we have both $X_1 > y$ and $X_2 > y$. Since X_1 and X_2 are independent, we can separate this joint probability as follows:

$$\begin{aligned} P(Y \leq y) &= 1 - P(Y > y) \\ &= 1 - P(\min\{X_1, X_2\} > y) \\ &= 1 - P(X_1 > y) \cdot P(X_2 > y) \\ &= 1 - P(X_1 > y)^2 \\ &= 1 - (1 - P(X_1 \leq y))^2 \end{aligned}$$

We know from p. 258 in the text that for a uniform random variable X, we have $P(X \leq x) = (x - a)/(b - a)$. Thus, in our case, we have

$$\begin{aligned} 1 - P(X_1 \leq y) &= 1 - (y - 0)/(1 - 0) \\ &= 1 - x \end{aligned}$$

Hence,

$$P(Y \leq y) = 1 - (1 - y)^2$$

and as a result, the CDF of Y is,

$$F_Y(y) = \begin{cases} 0 & y \leq 0 \\ 1 - (1 - y)^2 & 0 < y < 1 \\ 1 & y \geq 1 \end{cases}$$

Problem 3.

Let X and Y be independent, integer-valued random variables and fix $n \in \mathbb{Z}$. Then we have,

$$\begin{aligned}
P(X+Y=n) &= \sum_{m\in\mathbb{Z}} P(X=m, Y=n-m) \\
&= \sum_{m\in\mathbb{Z}} P(Y=n-m \,|\, X=m)P(X=m)
\end{aligned}$$

Since X and Y are independent, we have

$$\begin{aligned}
P(X+Y=n) &= \sum_{m\in\mathbb{Z}} P(Y=n-m \,|\, X=m)P(X=m) \\
&= \sum_{m\in\mathbb{Z}} P(X=m)P(Y=n-m)
\end{aligned}$$

Problem 4.

(A) We have

$$\begin{aligned}
\mathbb{E}[YZ] &= \int_0^1 \sin(2\pi x)\cdot\cos(2\pi x)dx \\
&= \int_0^1 \frac{1}{2}\sin(4\pi x)dx \\
&= \frac{1}{2}\int_0^1 \sin(4\pi x)dx \\
&= \frac{1}{2}\left[-\frac{\cos(4\pi\cdot 1)}{4\pi} - \left(-\frac{\cos(4\pi\cdot 0)}{4\pi}\right)\right] \\
&= \frac{1}{2}\left[-\frac{\cos(4\pi)}{4\pi} + \frac{\cos(0)}{4\pi}\right] \\
&= \frac{1}{2}\left[-\frac{1}{4\pi} + \frac{1}{4\pi}\right] \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[Y]\cdot\mathbb{E}[Z] &= \int_0^1 \sin(2\pi x)dx \cdot \int_0^1 \cos(2\pi x)dx \\
&= \left[-\frac{\cos(2\pi)}{2\pi} + \frac{\cos(0)}{2\pi}\right]\cdot\left[\frac{\sin(2\pi)}{2\pi} - \frac{\sin(0)}{2\pi}\right] \\
&= \left[-\frac{1}{2\pi} + \frac{1}{2\pi}\right]\cdot\left[\frac{0}{2\pi} - \frac{0}{2\pi}\right] \\
&= 0\cdot 0 = 0
\end{aligned}$$

Hence, $\mathbb{E}[YZ] = \mathbb{E}[Y]\cdot\mathbb{E}[Z]$ and, as a result, we have that Y and Z are uncorrelated.

(B) Y and Z are not independent variables. To show this, we need to find sets A and B such that

$$P(Y \in A, Z \in B) \neq P(Y \in A) \cdot P(Z \in B)$$

Let $A = B = [0.9, 1]$.

We have,

$$\begin{aligned} P(Y \in A) &= P(0.9 \leq Y \leq 1) \\ &= P(0.9 \leq \sin(2\pi x) \leq 1) \\ &= P\left(\frac{\sin^{-1}(0.9)}{2\pi} \leq x \leq \frac{\sin^{-1}(1)}{2\pi}\right) \\ &\approx P(0.17821 \leq x \leq 0.25) \\ &= 0.25 - 0.17821 = 0.07179 \end{aligned}$$

We also have,

$$\begin{aligned} P(Z \in B) &= P(0.9 \leq Z \leq 1) \\ &= P(0.9 \leq \cos(2\pi x) \leq 1) \\ &= P\left(\frac{\cos^{-1}(0.9)}{2\pi} \leq x \leq \frac{\cos^{-1}(1)}{2\pi}\right) \\ &\approx P(0.07178 \leq x \leq 0) \\ &= 0 \end{aligned}$$

And thus,

$$\begin{aligned} P(Y \in A) \cdot P(Z \in B) &= P(0.9 \leq Y \leq 1) \cdot P(0.9 \leq Z \leq 1) \\ &= 0 \end{aligned}$$

However,

$$\begin{aligned} P(Y \in A, Z \in B) &= P(0.9 \leq Y \leq 1, 0.9 \leq Z \leq 1) \\ \\ &= \int_{0.9}^{1} \sin(2\pi x) \cdot \cos(2\pi x) dx \\ &\approx -0.0274933 \end{aligned}$$

So $P(Y \in A, Z \in B) \neq P(Y \in A) \cdot P(Z \in B)$, and thus Y and Z are not independent.