Stochastic Processes: Homework 2

Chris Hayduk

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Problem 1. Durrett, Exercise 1.2

Observe that if $0 < X_n < 5$, then there are 3 possibilities for the transition:

1. $X_{n+1} = X_n$. This can occur if two white balls are exchanged or if two black balls are exchanged
2. $X_{n+1} = X_n + 1$. This occurs if a black ball from the left urn is exchanged for a white ball from the right urn
3. $X_{n+1} = X_n - 1$. This occurs if a white ball from the left urn is exchanged for a black ball from the right urn

If $X_n = 0$, then there is only one possible transition: $X_{n+1} = X_n + 1$.

If $X_n = 5$, then there is only one possible transition: $X_{n+1} = X_n - 1$.

From these facts, we can derive the transition probability matrix:

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} \;\;0\;\; & \;\;1\;\; & \;\;2\;\; & \;\;3\;\; & \;\;4\;\; & \;\;5\;\; \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0.04 & 0.32 & 0.64 & 0 & 0 & 0 \\ 0 & 0.16 & 0.48 & 0.36 & 0 & 0 \\ 0 & 0 & 0.36 & 0.48 & 0.16 & 0 \\ 0 & 0 & 0 & 0.64 & 0.32 & 0.04 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Let $x =$ number of white balls in left urn and $y =$ number of white balls in right urn. Hence $5-x =$ number of black balls in left urn and $5-y =$ number of black balls in right urn.

Then for $0 < i < 5$, we have

$$\begin{aligned} p(i, i-1) &= \frac{x}{5} \cdot \frac{5-y}{5} \\ p(i, i) &= \frac{x}{5} \cdot \frac{y}{5} + \frac{5-x}{5} \cdot \frac{5-y}{5} \\ p(i, i+1) &= \frac{5-x}{5} \cdot \frac{y}{5} \end{aligned}$$

Problem 2. Durrett, Exercise 1.3

Note that X_n can be any integer from 0 to 5. Hence, the state space is $\{0, 1, 2, 3, 4, 5\}$.

For each individual roll Y_k, the possible sums are as follows, along with possible combinations to get that sum:

$$\begin{aligned}
2 :&(1,1)\\
3 :&(2,1),(1,2)\\
4 :&(2,2),(1,3),(3,1)\\
5 :&(1,4),(4,1),(3,2),(2,3)\\
6 :&(3,3),(4,2),(2,4)\\
7 :&(4,3),(3,4)\\
8 :&(4,4)
\end{aligned}$$

We see that there are 16 possible rolls with this pair of dice. Using the possible outcomes listed above, we can derive probabilities for each sum:

$$\begin{aligned}
P(Y_k = 2) &= 0.0625\\
P(Y_k = 3) &= 0.125\\
P(Y_k = 4) &= 0.1875\\
P(Y_k = 5) &= 0.25\\
P(Y_k = 6) &= 0.1875\\
P(Y_k = 7) &= 0.125\\
P(Y_k = 8) &= 0.0625
\end{aligned}$$

Now we can show which congruence classes these sums to modulo 6:

$$\begin{aligned}
2 &= \bar{2}\\
3 &= \bar{3}\\
4 &= \bar{4}\\
5 &= \bar{5}\\
6 &= \bar{0}\\
7 &= \bar{1}\\
8 &= \bar{2}
\end{aligned}$$

Now we can convert the probabilities for Y_k using these congruence classes,

$$\begin{aligned}
P(Y_k = \bar{0}) &= 0.1875 \\
P(Y_k = \bar{1}) &= 0.125 + 0.125 = 0.25 \\
P(Y_k = \bar{2}) &= 0.0625 + 0.0625 = 0.125 \\
P(Y_k = \bar{3}) &= 0.125 \\
P(Y_k = \bar{4}) &= 0.1875 \\
P(Y_k = \bar{5}) &= 0.25
\end{aligned}$$

Now using these probabilities and the properties of modular arithmetic, we will derive a transition probability matrix for X_n:

$$\begin{array}{c c}
 & \begin{array}{cccccc} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{5} \end{array} \\
\begin{array}{c} \bar{0} \\ \bar{1} \\ \bar{2} \\ \bar{3} \\ \bar{4} \\ \bar{5} \end{array} &
\begin{pmatrix}
0.1875 & 0.25 & 0.125 & 0.125 & 0.1875 & 0.25 \\
0.25 & 0.1875 & 0.25 & 0.125 & 0.125 & 0.1875 \\
0.1875 & 0.25 & 0.1875 & 0.25 & 0.125 & 0.125 \\
0.125 & 0.1875 & 0.25 & 0.1875 & 0.25 & 0.125 \\
0.125 & 0.125 & 0.1875 & 0.25 & 0.1875 & 0.25 \\
0.25 & 0.125 & 0.125 & 0.1875 & 0.25 & 0.1875
\end{pmatrix}
\end{array}$$

So if $j \geq i$, we have

$$p(i,j) = P(Y_k = \overline{j-i})$$

Problem 3. Durrett, Exercise 1.58

We have,

$$\begin{aligned}
P(X_{n+1}=1)-\frac{b}{a+b} &= P(X_{n+1}=1|X_n=1)\cdot P(X_n=1)+P(X_{n+1}=1|X_n=2)\cdot P(X_n=2)-\frac{b}{a+b}\\
&=(1-a)P(X_n=1)+bP(X_n=2)-\frac{b}{a+b}\\
&=(1-a)P(X_n=1)+b(1-P(X_n=1))-\frac{b}{a+b}\\
&=(1-a-b)P(X_n=1)+b-\frac{b}{a+b}\\
&=(1-a-b)P(X_n=1)+\frac{b(a+b)}{a+b}-\frac{b}{a+b}\\
&=(1-a-b)P(X_n=1)+\frac{b(a+b)-b}{a+b}\\
&=(1-a-b)P(X_n=1)+\frac{b(a+b-1)}{a+b}\\
&=(1-a-b)P(X_n=1)-\frac{b(1-a-b)}{a+b}\\
&=(1-a-b)\left[P(X_n=1)-\frac{b}{a+b}\right]
\end{aligned}$$

Note that,

$$\begin{aligned}
P(X_n=1)-\frac{b}{a+b} &= (1-a-b)\left[P(X_{n-1}=1)-\frac{b}{a+b}\right]\\
&=(1-a-b)\left[(1-a-b)\left(P(X_{n-2}=1)-\frac{b}{a+b}\right)\right]\\
&=(1-a-b)^2\left[P(X_{n-2}=1)-\frac{b}{a+b}\right]\\
&=(1-a-b)^3\left[P(X_{n-3}=1)-\frac{b}{a+b}\right]\\
&\vdots\\
&=(1-a-b)^n\left[P(X_0=1)-\frac{b}{a+b}\right]
\end{aligned}$$

So this gives us,

$$P(X_n=1)=\frac{b}{a+b}+(1-a-b)^n\left[P(X_0=1)-\frac{b}{a+b}\right]$$

Problem 4.

(A) We consider the following matrix,

$$p = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix} \end{array}$$

Now we take $p - \lambda I$

$$p - \lambda I = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix} \end{array} - \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \end{array}$$

$$= \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & 1/2 - \lambda & 1/2 \\ 1/2 & 0 & 1/2 - \lambda \end{pmatrix} \end{array}$$

Now we need to find $det(p - \lambda I)$,

$$\begin{aligned} det(p - \lambda I) &= -\lambda \begin{vmatrix} 1/2 - \lambda & 1/2 \\ 0 & 1/2 - \lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & 1/2 \\ 1/2 & 1/2 - \lambda \end{vmatrix} + 0 \begin{vmatrix} 0 & 1/2 - \lambda \\ 1/2 & 0 \end{vmatrix} \\ &= -\lambda \begin{vmatrix} 1/2 - \lambda & 1/2 \\ 0 & 1/2 - \lambda \end{vmatrix} - \begin{vmatrix} 0 & 1/2 \\ 1/2 & 1/2 - \lambda \end{vmatrix} \\ &= -\lambda[(1/2 - \lambda)^2 - 0] - (0 - 1/4) \\ &= -\lambda(\lambda^2 - \lambda + 1/4) + 1/4 \\ &= -\lambda^3 + \lambda^2 - \lambda/4 + 1/4 \end{aligned}$$

Lastly, we need to solve

$$-\lambda^3 + \lambda^2 - \lambda/4 + 1/4 = 0$$

The solutions to this equation are:

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= -i/2 \\ \lambda_3 &= i/2 \end{aligned}$$

(B) By diagonalization, we have that,

$$
\begin{aligned}
p &= X\Lambda X^{-1} \\
p^2 &= X\Lambda X^{-1}X\Lambda X^{-1} = X\Lambda^2 X^{-1} \\
&\vdots \\
p^n &= X\Lambda^n X^{-1}
\end{aligned}
$$

where X is the matrix whose columns are the eigenvectors of p and Λ is the matrix with the eigenvalues of p on the diagonal and 0s everywhere else.

Now to find the three eigenvectors:

$$
\begin{aligned}
(p - \lambda_1 I)x_1 &= 0 \\
(p - \lambda_2 I)x_2 &= 0 \\
(p - \lambda_3 I)x_3 &= 0
\end{aligned}
$$

The solutions to these equations are:

$$
\begin{aligned}
x_1 &= (1, 1, 1) \\
x_2 &= (-1 - i, -\frac{1}{2} + \frac{i}{2}, 1) \\
x_3 &= (-1 + i, -\frac{1}{2} - \frac{i}{2}, 1)
\end{aligned}
$$

So,

$$
X = \begin{bmatrix} 1 & -1 - i & -1 + i \\ 1 & -\frac{1}{2} + \frac{i}{2} & -\frac{1}{2} - \frac{i}{2} \\ 1 & 1 & 1 \end{bmatrix}
$$

and the inverse is,

$$
X^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ -\frac{1}{10} + \frac{3i}{10} & -\frac{1}{5} - \frac{2i}{5} & \frac{3}{10} + \frac{i}{10} \\ -\frac{1}{10} - \frac{3i}{10} & -\frac{1}{5} + \frac{2i}{5} & \frac{3}{10} - \frac{i}{10} \end{bmatrix}
$$

Lastly, we have

$$
\begin{aligned}
\Lambda^n &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{i}{2} & 0 \\ 0 & 0 & \frac{i}{2} \end{bmatrix}^n \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{i}{2}^n & 0 \\ 0 & 0 & \frac{i}{2}^n \end{bmatrix}
\end{aligned}
$$

So we have,

$$p^n = X\Lambda^n X^-1$$

which yields

$$\frac{1}{10}\begin{pmatrix} -2\,i\left(\left(-\frac{i}{2}\right)^n - \left(\frac{i}{2}\right)^n + i\,(-i)^n\,2^{1-n} + i^{n+1}\,2^{1-n} + i\right) & (2+2\,i)\left(\left(-\frac{i}{2}\right)^n - i\left(\frac{i}{2}\right)^n + i\,(-i)^n\,2^{1-n} - i^n\,2^{1-n} + (1-i)\right) & -2\left(\left(-\frac{i}{2}\right)^n + \left(\frac{i}{2}\right)^n + i\,(-i)^n\,2^{1-n} - i\,i^n\,2^{1-n} - 2\right) \\ -\left(-\frac{i}{2}\right)^n - \left(\frac{i}{2}\right)^n - i\,(-i)^n\,2^{1-n} + i\,i^n\,2^{1-n} + 2 & (1-i)\left(\left(-\frac{i}{2}\right)^n + i\,(-i)^n\,2^{1-n} + i^n\,2^{1-n} + i^{n+1}\,2^{-n} + (2+2\,i)\right) & i\left(\left(-\frac{i}{2}\right)^n - \left(\frac{i}{2}\right)^n + i\,(-i)^n\,2^{1-n} + i^{n+1}\,2^{1-n} + -4\,i\right) \\ (1+i)\left(\left(-\frac{i}{2}\right)^n - i\left(\frac{i}{2}\right)^n + i\,(-i)^n\,2^{1-n} - i^n\,2^{1-n} + (1-i)\right) & -2\left(\left(-\frac{i}{2}\right)^n + \left(\frac{i}{2}\right)^n + i\,(-i)^n\,2^{1-n} - i\,i^n\,2^{1-n} - 2\right) & (1-i)\left(\left(-\frac{i}{2}\right)^n + i\,(-i)^n\,2^{1-n} + i^n\,2^{1-n} + i^{n+1}\,2^{-n} + (2+2\,i)\right) \end{pmatrix}$$

Hence,

$$\begin{aligned}
p^n(1,1) &= -\frac{2i}{10}\left(\left(-\frac{i}{2}\right)^n - \left(\frac{i}{2}\right)^n + i(-i)^n 2^{1-n} + i^{n+1}2^{1-n} + i\right) \\
&= -\frac{2i}{10}\left(-\frac{i}{2}\right)^n - -\frac{2i}{10}\left(\frac{i}{2}\right)^n + \frac{2^{1-n+1}(-i)^n}{10} - \frac{2^{1-n+1}i^{n+2}}{10} + i \\
&= -\frac{2i}{10}\left(-\frac{i}{2}\right)^n - -\frac{2i}{10}\left(\frac{i}{2}\right)^n + \frac{2^{1-n+1}((-i)^n - i^{n+2})}{10} + i \\
&= -\frac{2i}{10}\left(-\frac{i}{2}\right)^n - -\frac{2i}{10}\left(\frac{i}{2}\right)^n + 1^n\left(\frac{2^{1-n+1}((-i)^n - i^{n+2})}{10} + i\right) \\
&= -\frac{2i}{10}\left(-\frac{i}{2}\right)^n - -\frac{2i}{10}\left(\frac{i}{2}\right)^n + 1^n\left(\frac{2^{1-n+1}((-i)^n - i^{n+2})}{10} + i\right) \\
&= \left(\frac{2^{1-n+1}((-i)^n - i^{n+2})}{10} + i\right)\lambda_1^n - \frac{2i}{10}\lambda_2^n - -\frac{2i}{10}\lambda_3^n
\end{aligned}$$

as required.

(C) We have the following equations:

$$
\begin{aligned}
p^0(1,1) &= 1 = a + b + c \\
p^1(1,1) &= 0 = a\lambda_1 + b\lambda_2 + c\lambda_3 = a + \frac{-i}{2}b + \frac{i}{2}c \\
p^2(1,1) &= 0 = a\lambda_1^2 + b\lambda_2^2 + c\lambda_3^2 = a + \frac{-1}{4}b + \frac{-1}{4}c
\end{aligned}
$$

We can set up an augmented matrix for this system of equations:

$$
\left(\begin{array}{ccc|c}
1 & 1 & 1 & 1 \\
1 & \frac{-i}{2} & \frac{i}{2} & 0 \\
1 & \frac{-1}{4} & \frac{-1}{4} & 0
\end{array}\right)
$$

Row reduction yields:

$$
\left(\begin{array}{ccc|c}
1 & 0 & 0 & \frac{1}{5} \\
0 & 1 & 0 & \frac{2}{5} - \frac{i}{5} \\
0 & 0 & 1 & \frac{2}{5} + \frac{i}{5}
\end{array}\right)
$$

Hence we have,

$$
p^n(1,1) = \frac{1}{5} + \left(\frac{2}{5} - \frac{i}{5}\right) \cdot \frac{-i}{2}^n + \left(\frac{2}{5} + \frac{i}{5}\right) \cdot \frac{i}{2}^n
$$

(D) If n even:

$$
\begin{aligned}
p^n(1,1) &= \frac{1}{5} + \left(\frac{2}{5} - \frac{i}{5}\right) \cdot \frac{-1}{2^n} + \left(\frac{2}{5} + \frac{i}{5}\right) \cdot \frac{-1}{2^n} \\
&= \frac{1}{5} + \frac{-2}{5 \cdot 2^n} - \frac{-i}{5 \cdot 2^n} + \frac{-2}{5 \cdot 2^n} + \frac{-i}{5 \cdot 2^n} \\
&= \frac{1}{5} + \frac{-4}{5 \cdot 2^n}
\end{aligned}
$$

If n odd:

$$
\begin{aligned}
p^n(1,1) &= \frac{1}{5} + \left(\frac{2}{5} - \frac{i}{5}\right) \cdot \frac{-i}{2^n} + \left(\frac{2}{5} + \frac{i}{5}\right) \cdot \frac{i}{2^n} \\
&= \frac{1}{5} + \frac{-2i}{5 \cdot 2^n} - \frac{1}{5 \cdot 2^n} + \frac{2i}{5 \cdot 2^n} + \frac{-1}{5 \cdot 2^n} \\
&= \frac{1}{5} + \frac{-2}{5 \cdot 2^n}
\end{aligned}
$$

In either case, we see that as $n \to \infty$, the second term goes to 0. Hence, we have that,

$$
p^n(1,1) \to \frac{1}{5} \text{ as } n \to \infty
$$