Stochastic Processes: Homework 4

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September 30, 2020

Problem 1. Durrett, Exercise 1.38

(a) Let us look at each case. If $X_n = 0$, then she cannot bring any umbrellas with her and all three are at the next location. So $X_{n+1} = 3$.

If $X_n = 1$, she brings the umbrella with her with probability 0.2. So with probability 0.2, $X_{n+1} = 3$ and with probability 0.8, $X_{n+1} = 2$.

If $X_n = 2$, she brings the umbrella with her with probability 0.2. So with probability 0.2, $X_{n+1} = 2$ and with probability 0.8, $X_{n+1} = 1$.

If $X_n = 3$, she brings the umbrella with her with probability 0.2. So with probability 0.2, $X_{n+1} = 1$ and with probability 0.8, $X_{n+1} = 0$.

This yields the following transition probability matrix:

$$p = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0.8 & 0.2 \\ 0 & 0.8 & 0.2 & 0 \\ 0.8 & 0.2 & 0 & 0 \end{pmatrix} \end{array}$$

(b) Note that she gets wet when $X_n = 0$ and it rains outside.

We have that the above Markov chain is finite since there are only 4 possible states. Moreover, p is closed as well because there is no way out of the chain. Thus, by Theorem 1.7, we have that all states p are recurrent. In addition, p is irreducible because every state communicates with every other state in the chain. Hence, we can apply Theorem 1.20 which states,

$$\frac{N_n(0)}{n} \to \frac{1}{E_0 T_0}$$

where $\frac{N_n(0)}{n}$ is the fraction of visits to 0 up to time n.

By Theorem 1.21, we also have,

$$\pi(0) = \frac{1}{E_0 T_0}$$

So we just need to find the stationary distribution and its entry for state 0 in order to find the limiting fraction of visits to state 0.

Let's compute the stationary distribution π,

$$\begin{aligned}
&\pi p = \pi \\
\iff &\begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & \pi_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0.8 & 0.2 \\ 0 & 0.8 & 0.2 & 0 \\ 0.8 & 0.2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & \pi_3 \end{pmatrix}
\end{aligned}$$

So we need to solve:

$$\begin{aligned}
0.8\pi_3 &= \pi_0 \\
0.8\pi_2 + 0.2\pi_3 &= \pi_1 \\
0.8\pi_1 + 0.2\pi_2 &= \pi_2 \\
\pi_0 + 0.2\pi_1 &= \pi_3
\end{aligned}$$

Which yields,

$$\begin{aligned}
\pi_0 &= \pi_0 \\
\pi_1 &= 5\pi_0/4 \\
\pi_2 &= 5\pi_0/4 \\
\pi_3 &= 5\pi_0/4
\end{aligned}$$

With the additional constraint that they must sum to 1, we get,

$$\pi_0 = \frac{4}{19} \approx 0.2105$$

Note that this is the fraction of times where she is left with 0 umbrellas. With probability $0.2 = 1/5$, it will rain when she has no umbrellas left. So the limiting fraction of times she gets wet is,

$$\begin{aligned}
\frac{4}{19} \cdot \frac{1}{5} &= \frac{4}{95} \\
&\approx 0.042105
\end{aligned}$$

Problem 2. Durrett, Exercise 1.62

(a) Let u be in the set of vertices V of the chessboard. Then we have that the degree of u is $d(u) = \sum_{v \in V} A(u, v)$, where $A(u, v) = 1$ if there is an edge connecting u to v and 0 otherwise. The degrees for each vertex are given by the following matrix,

$$
\begin{array}{cccc|cccc}
3 & 5 & 5 & 5 & 5 & 5 & 5 & 3 \\
5 & 8 & 8 & 8 & 8 & 8 & 8 & 5 \\
5 & 8 & 8 & 8 & 8 & 8 & 8 & 5 \\
5 & 8 & 8 & 8 & 8 & 8 & 8 & 5 \\
\hline
5 & 8 & 8 & 8 & 8 & 8 & 8 & 5 \\
5 & 8 & 8 & 8 & 8 & 8 & 8 & 5 \\
5 & 8 & 8 & 8 & 8 & 8 & 8 & 5 \\
3 & 5 & 5 & 5 & 5 & 5 & 5 & 3
\end{array}
$$

We know that $A(u, v) = A(v, u)$ because if a king can move to any spot on the board, then it can move back to the previous spot. In addition, we have that,

$$p(u, v) = \frac{A(u, v)}{d(u)}$$

defines a transition probability. So if $\pi(u) = cd(u)$ for some constant $c > 0$, we have that,

$$
\begin{aligned}
\pi(u)p(u, v) &= cd(u) \cdot \frac{A(u, v)}{d(u)} \\
&= cA(u, v) \\
&= cA(v, u) \\
&= (cA(v, u)) \cdot \frac{d(v)}{d(v)} \\
&= cd(v) \cdot \frac{A(v, u)}{d(v)} \\
&= \pi(v)p(v, u)
\end{aligned}
$$

If we take $c = 1/\sum_{u \in V} d(u)$, then clearly $\sum_{u \in V} cd(u) = 1$ and so cd defines a stationary probability distribution.

The sum of the degrees is $4 \cdot 3 + 5 \cdot 24 + 8 \cdot 36 = 420$, so $c = 1/420$ and the stationary distribution of the chain is,

$$\pi = 1/420 \cdot \begin{array}{cccc|cccc} 3 & 5 & 5 & 5 & 5 & 5 & 5 & 3 \\ 5 & 8 & 8 & 8 & 8 & 8 & 8 & 5 \\ 5 & 8 & 8 & 8 & 8 & 8 & 8 & 5 \\ 5 & 8 & 8 & 8 & 8 & 8 & 8 & 5 \\ \hline 5 & 8 & 8 & 8 & 8 & 8 & 8 & 5 \\ 5 & 8 & 8 & 8 & 8 & 8 & 8 & 5 \\ 5 & 8 & 8 & 8 & 8 & 8 & 8 & 5 \\ 3 & 5 & 5 & 5 & 5 & 5 & 5 & 3 \end{array}$$

(b) Now we need to find the expected number of moves to return to corner $(1, 1)$ when we start there.

We have that there is a stationary distribution π for this Markov chain.

In addition, we will show that p is irreducible. Imagine the king is on vertex u and we want to move to vertex v. If u is in the same row as v, we can move the piece horizontally until it reaches v. If u is in the same column as v, we can move the piece vertically until it reaches v. If neither are true, we can move the piece horizontally until it is in the same column as v and then move it vertically until it reaches v. If $u = v$, then move vertically and make the opposite move to get back to u.

Hence, we have shown that for any two vertices $u, v \in V$, u communicates with v. As a result, p is irreducible. Then we can apply Theorem 1.21, which states,

$$\pi(y) = 1/E_y T_y$$

Note that the value we are looking for is $E_{(1,1)}T_{(1,1)}$, so we take,

$$\begin{aligned} E_{(1,1)}T_{(1,1)} &= 1/\pi((1, 1)) \\ &= 1/(3/420) \\ &= 420/3 \\ &= 140 \end{aligned}$$

Hence, the expected number of moves to return to corner $(1, 1)$ is 140.

Problem 3. Durrett, Exercise 1.64

Let $p(x,y) = \binom{N}{y}(\rho_x)^y(1-\rho_x)^{N-y}$ where $\rho_x = (1-u)x/N + v(N-x)/N$.

(a) Suppose $u, v > 0$ and $\max\{u, v\} < 1$. Hence, $0 < u, v < 1$. Then,

$$\begin{aligned}
\rho_x &= (1-u)x/N + v(N-x)/N \\
&< x/N + (N-x)/N \\
&= N/N = 1
\end{aligned}$$

In addition, $\rho_x > 0$ because, if $x = 0$, we have,

$$\rho_x = v > 0$$

and if $x > 0$, we have,

$$\rho_x = (1-u)x/N + v(N-x)/N$$

where $(1-u), x, N, v, (N-x) > 0$.

So then,

$$\begin{aligned}
0 < p(x,y) &= \binom{N}{y}(\rho_x)^y(1-\rho_x)^{N-y} \\
&< \binom{N}{y}
\end{aligned}$$

Since $p(x,y) > 0$ for arbitrary x, y, we have that p is irreducible. In addition, if we let $y = x$, then $p(x,x) > 0$ and x has period 1. Since this holds for every x, we have that p is aperiodic.

Now we will check if the Kolmogorov Cycle Condition holds. Consider a cycle of states $x_0, x_1, \cdots, x_n = x_0$ with $p(x_{i-1}, x_i) > 0$ for $1 \le i \le n$. Then we have,

$$\begin{aligned}
&\prod_{i=1}^{n} p(x_{i-1}, x_i) = \prod_{i=1}^{n} p(x_i, x_{i-1}) \\
\iff &\prod_{i=1}^{n} \binom{N}{x_i}(\rho_{x_{i-1}})^{x_i}(1-\rho_{x_{i-1}})^{N-x_i} = \prod_{i=1}^{n} \binom{N}{x_{i-1}}(\rho_{x_i})^{x_{i-1}}(1-\rho_{x_i})^{N-x_{i-1}} \\
\iff &\prod_{i=1}^{n} (\rho_{x_{i-1}})^{x_i}(1-\rho_{x_{i-1}})^{N-x_i} = \prod_{i=1}^{n} (\rho_{x_i})^{x_{i-1}}(1-\rho_{x_i})^{N-x_{i-1}} \\
\iff &(\rho_{x_0})^{x_1-x_{n-1}}(1-\rho_{x_0})^{x_{n-1}+x_1} + (\rho_{x_1})^{x_2-x_0}(1-\rho_{x_1})^{x_0+x_2} + \cdots \\
&+ (\rho_{x_{n-1}})^{x_n-x_{n-2}}(1-\rho_{x_{n-1}})^{x_n+x_{n-2}} = 1
\end{aligned}$$

Suppose this holds. Then by Theorem 1.16, there is a stationary distribution for this Markov chain that satisfies detailed balance. We can now apply Theorem 1.19 (Convergence Theorem) in order to assert,

$$\lim_{n\to\infty} p^n(x,y) = \pi(y)$$

as required.

(b) We need to find the mean,

$$v = \sum_y y\pi(y) = \lim_{n\to\infty} E_x X_n$$

Not sure how to approach this one.

Problem 4.

We have that, from a starting distribution σ, a transition to any state (including σ itself) has probability 1/52

(A) It is clear that $|S| = 52!$ because there are 52! ways to shuffle a deck of cards.

Moreover, fix a state $v_j\sigma$ in the column of p. That is, $v_j\sigma = X_{n+1}$. Let $v_j\sigma = (v_j, \sigma_1, \cdots, \sigma_5 1)$. Observe that it is possible to reach this state from the following states,

$$\begin{gathered}(v_j, \sigma_1, \cdots, \sigma_{51}) \\ (\sigma_1, v_j, \sigma_2, \cdots, \sigma_{51}) \\ \vdots \\ (\sigma_1, \cdots, v_j, \sigma_{51}) \\ (\sigma_1, \cdots, \sigma_{51}, v_j)\end{gathered}$$

So there 52 possible initial states that can transition to $v_j\sigma$, and each will do so with probability 1/52 by our assumption of the transition probability. Hence, the column sum for any fixed $v_j\sigma$ will be 1, and so the transition matrix is doubly stochastic. Thus, we have that

$$\pi(\sigma) = \frac{1}{52!}$$

for every σ.

(B) We have that there exists a stationary distribution. Now we need to show that p is irreducible. Fix states $\sigma_1, \sigma_2 \in S$. The following algorithm will provide a path from σ_1 to σ_2:

Examine $\sigma_{2_{52}}$, the card on the bottom of σ_2. Find the equivalent card in σ_1, and move this card to the top of the deck. Next, find the equivalent of $\sigma_{2_{51}}$ in σ_1, and move this card to the top of the deck. After these steps, we are left with,

$$\sigma_1 = (\sigma_{2_{51}}, \sigma_{2_{52}}, \sigma_{1_1}, \cdots, \sigma_{1_{52}})$$

These moves are legal because any move where we take a card and move it to the top of the deck has probability $1/52$.

If we continue in this manner, it is clear that we can transition from σ_1 to σ_2. Since both of these states were arbitrary, we have that any two states in the state space communicate, and so p is irreducible. Hence, we can apply Theorem 1.21 and get,

$$\pi(\sigma) = 1/E_\sigma T_\sigma$$

Hence, we have,

$$\begin{aligned} E_\sigma T_\sigma &= 1/\pi(\sigma) \\ &= 1/(1/52!) = 52! \end{aligned}$$

for every σ.

We are assuming that each step takes 1 second, and we must give the answer in years. Since there are $31,536,000$ seconds in 1 year, we get,

$$52!/31536000 = 2.56 \cdot 10^{60}$$

This is the expected number of years it would take to return to the initial ordering σ if each transition takes 1 second.