Stochastic Processes: Homework 1

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Problem 1.

Let A_1 = "Donna drew a fair coin", A_2 = "Donna drew the two-headed coin" and let B = "Donna flipped 6 heads in a row".

Note that

$$\begin{aligned} P(A_1) &= \frac{64}{65} \\ P(B|A_1) &= \frac{1/2^6}{64/65} = \frac{65}{4096} \approx 0.01587 \end{aligned}$$

and

$$\begin{aligned} P(A_2) &= \frac{1}{65} \\ P(B|A_2) &= 1 \end{aligned}$$

Hence,

$$\begin{aligned} \sum P(A_j)P(B|A_j) &= \frac{64}{65} \cdot \frac{65}{4096} + \frac{1}{65} \cdot 1 \\ &= \frac{129}{4160} \approx 0.031 \end{aligned}$$

Thus, we have,

$$\begin{aligned} P(A_2|B) &= \frac{P(A_2)P(B|A_2)}{\sum P(A_j)P(B|A_j)} \\ &= \frac{1/65}{129/4160} \\ &= \frac{64}{129} \approx 0.4961 \end{aligned}$$

So if Donna flips 6 heads in a row, there is about a 49.61% chance that she chose the two-headed coin.

Problem 2.

We have that X_1, X_2 are two independent uniform (0, 1) random variables on the same probability space. In addition, we have that $Y = \min\{X_1, X_2\}$.

In order to find the CDF of Y, we must find $P(Y \leq y)$. Note that,

$$P(Y \leq y) = P(\min\{X_1, X_2\} \leq y)$$

In addition, note that,

$$\begin{aligned} P(Y \leq y) &= 1 - P(Y > y) \\ &= 1 - P(\min\{X_1, X_2\} > y) \end{aligned}$$

Since we are taking a minimum, $\min\{X_1, X_2\} > y$ if and only if we have both $X_1 > y$ and $X_2 > y$. Since X_1 and X_2 are independent, we can separate this joint probability as follows:

$$\begin{aligned} P(Y \leq y) &= 1 - P(Y > y) \\ &= 1 - P(\min\{X_1, X_2\} > y) \\ &= 1 - P(X_1 > y) \cdot P(X_2 > y) \\ &= 1 - P(X_1 > y)^2 \\ &= 1 - (1 - P(X_1 \leq y))^2 \end{aligned}$$

We know from p. 258 in the text that for a uniform random variable X, we have $P(X \leq x) = (x - a)/(b - a)$. Thus, in our case, we have

$$\begin{aligned} 1 - P(X_1 \leq y) &= 1 - (y - 0)/(1 - 0) \\ &= 1 - x \end{aligned}$$

Hence,

$$P(Y \leq y) = 1 - (1 - y)^2$$

and as a result, the CDF of Y is,

$$F_Y(y) = \begin{cases} 0 & y \leq 0 \\ 1 - (1 - y)^2 & 0 < y < 1 \\ 1 & y \geq 1 \end{cases}$$

Problem 3.

Problem 4.

(A)

(B)