Stochastic Processes: Midterm Exam

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Problem 1.

(A) We know the deck contains 10 cards and that the draws are independent and uniform. Hence, the probability of drawing any single card is $\frac{1}{10}$. So, if we suppose that we have seen a certain card i times in a row, the probability of selecting that card for the $i+1$ time in a row would be $\frac{1}{10}$. Thus, it is clear that, for $n \geq 1$, we have $P(X_{n+1} = i+1 | X_n = i, X_{n-1} = i_{n_1}, \cdots X_0 = 0) = \frac{1}{10} = P(X_{n+1} = i+1 | X_n = i)$.

In the case of $n = 0$, we have $X_0 = 0$ by definition. In addition, note that $P(X_{n+1} = 0 | X_n = i) = \frac{9}{10}$ for $i \neq 0$. Since $X_{n+1} = i+1, X_{n+1} = 0$ are the only possible transitions from $X_n = i, i \neq 0$, we have that they add up to 1 as required. Moreover, we can define any other transitions as 0 probability, so $p(i,j) \geq 0$ for any i, j in the state space as required. In the case of $i = 0$, these conditions still hold, as the only possible transition is $P(X_{n+1} = 1 | X_n = 0) = 1$.

As a result, we have that this defines a valid Markov chain with the following transition probability matrix,

$$p = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 9/10 & 0 & 1/10 & 0 & 0 & \cdots \\ 9/10 & 0 & 0 & 1/10 & 0 & \cdots \\ 9/10 & 0 & 0 & 0 & 1/10 & \cdots \\ & & \vdots & & & \cdots \end{pmatrix} \end{array}$$

That is, we have,

$$p(0,1) = 1$$

and for $i > 0$, we have,

$$p(i,0) = 9/10$$
$$p(i,i+1) = 1/10$$

(B) Let us consider the transition matrix $\tilde{p}$, where we eliminate every state after 9 and consider state 9 an absorbing state:

$$\tilde{p} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9/10 & 0 & 1/10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9/10 & 0 & 0 & 1/10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9/10 & 0 & 0 & 0 & 1/10 & 0 & 0 & 0 & 0 & 0 \\ 9/10 & 0 & 0 & 0 & 0 & 1/10 & 0 & 0 & 0 & 0 \\ 9/10 & 0 & 0 & 0 & 0 & 0 & 1/10 & 0 & 0 & 0 \\ 9/10 & 0 & 0 & 0 & 0 & 0 & 0 & 1/10 & 0 & 0 \\ 9/10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/10 & 0 \\ 9/10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

We are able to make this change because we have not altered any of the states which lead to 9 and for the purposes of this analysis, we do not need to consider any of the states which come after 9. Let us now construct the matrix r by removing the rows and columns corresponding to state 9:

$$r = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9/10 & 0 & 1/10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9/10 & 0 & 0 & 1/10 & 0 & 0 & 0 & 0 & 0 \\ 9/10 & 0 & 0 & 0 & 1/10 & 0 & 0 & 0 & 0 \\ 9/10 & 0 & 0 & 0 & 0 & 1/10 & 0 & 0 & 0 \\ 9/10 & 0 & 0 & 0 & 0 & 0 & 1/10 & 0 & 0 \\ 9/10 & 0 & 0 & 0 & 0 & 0 & 0 & 1/10 & 0 \\ 9/10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/10 \\ 9/10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Now we must evaluate $I - r$, where I is the 9×9 identity matrix,

$$I - r = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -9/10 & 1 & -1/10 & 0 & 0 & 0 & 0 & 0 & 0 \\ -9/10 & 0 & 1 & -1/10 & 0 & 0 & 0 & 0 & 0 \\ -9/10 & 0 & 0 & 1 & -1/10 & 0 & 0 & 0 & 0 \\ -9/10 & 0 & 0 & 0 & 1 & -1/10 & 0 & 0 & 0 \\ -9/10 & 0 & 0 & 0 & 0 & 1 & -1/10 & 0 & 0 \\ -9/10 & 0 & 0 & 0 & 0 & 0 & 1 & -1/10 & 0 \\ -9/10 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1/10 \\ -9/10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Once we take the inverse and multiply by the column vector of all 1s, we get,

$$(I-r)^{-1}\mathbb{1} = \begin{pmatrix} 211,111,110 \\ \vdots \end{pmatrix}$$

So the expected number of moves starting from state 0 is $211,111,110$. For a brief sanity check, we see that the probability of seeing a specific card 9 times in a row is

$$\begin{aligned} 1 \cdot \frac{1}{10^8} &= \frac{1}{100,000,000} \\ &= 1 \times 10^{-8} \end{aligned}$$

So our answer makes sense given the extremely low probability of reaching state 9.

Problem 2.

The system of equations for f_1 is,

$$\begin{aligned} f_1(a) &= \frac{1}{3} \\ f_1(b) &= \frac{1}{6} \\ f_1(c) &= \frac{1}{3} \\ f_1(d) &= 0 \\ f_1(e) &= \frac{1}{6} \end{aligned}$$

Then for f_2 we have,

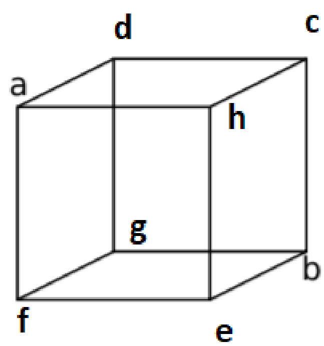
$$\begin{aligned} f_2(a) &= \frac{1}{6} \\ f_2(b) &= \frac{1}{6} \\ f_2(c) &= \frac{1}{12} \\ f_2(d) &= \frac{1}{4} \\ f_2(e) &= \frac{1}{6} \end{aligned}$$

And for f_3,

$$
\begin{aligned}
f_3(a) &= \frac{1}{12} \\
f_3(b) &= \frac{1}{8} \\
f_3(c) &= \frac{5}{24} \\
f_3(d) &= \frac{1}{8} \\
f_3(e) &= \frac{5}{12}
\end{aligned}
$$

Problem 3.

Let us label the vertices as follows:



Since each vertex is connected to three other vertices, we have that the transition probability is 1/3 between any two vertices sharing an edge and 0 between all others. This yields the following transition probability matrix:

$$
p = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{array}
\begin{array}{c}
\begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 1/3 & 0 & 1/3 & 0 & 1/3 \\
0 & 0 & 1/3 & 0 & 1/3 & 0 & 1/3 & 0 \\
0 & 1/3 & 0 & 1/3 & 0 & 0 & 0 & 1/3 \\
1/3 & 0 & 1/3 & 0 & 0 & 0 & 1/3 & 0 \\
0 & 1/3 & 0 & 0 & 0 & 1/3 & 0 & 1/3 \\
1/3 & 0 & 0 & 0 & 1/3 & 0 & 1/3 & 0 \\
0 & 1/3 & 0 & 1/3 & 0 & 1/3 & 0 & 0 \\
1/3 & 0 & 1/3 & 0 & 1/3 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Now we need to find $P(T_a < T_b | X_0 = a)$. That is, the probability that the first return to a will be sooner than the first return to b when we start from state a.

We have three distinct cases here: $X_1 = d$, $X_1 = h$, and $X_1 = f$. We will consider all three cases with both a and b as absorbing states. Thus, the probability $P(T_a < T_b | X_0 = a)$ will be equivalent to $\frac{1}{3}(\lim_{n\to\infty} \tilde{p}^n(d,a) + \lim_{n\to\infty} \tilde{p}^n(h,a) + \lim_{n\to\infty} \tilde{p}^n(f,a))$. This is true because, since a and b are both absorbing states, $\lim_{n\to\infty} \tilde{p}^n(d,a)$ will show the probability of reaching a (and therefore not reaching b) starting from state d. The case of the other two states is the same. We multiply by 1/3 because each possibility for $X_1 = d, h, f$ has a 1/3 chance of occurring.

Let us now set up the new transition probability matrix:

$$\tilde{p} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 1/3 & 0 & 0 & 0 & 1/3 \\ 1/3 & 0 & 1/3 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 0 & 0 & 1/3 & 0 & 1/3 \\ 1/3 & 0 & 0 & 0 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 1/3 & 0 & 1/3 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Note that we can reformulate our above discussion such that,

$$\begin{aligned} P(T_a < T_b | X_0 = a) &= \frac{1}{3}(\lim_{n\to\infty} \tilde{p}^n(d,a) + \lim_{n\to\infty} \tilde{p}^n(h,a) + \lim_{n\to\infty} \tilde{p}^n(f,a)) \\ &= \frac{1}{3}(P(V_a < V_b | X_0 = d) + P(V_a < V_b | X_0 = h) + P(V_a < V_B | X_0 = f)) \end{aligned}$$

This is possible since a and b are both absorbing states now.

Now let $P_x(V_a < V_b) =: j(x)$. Let $j(a) = 1$ and $j(b) = 0$. Then we have the following system of equations,

$$\begin{aligned} j(a) &= 1 \\ j(b) &= 0 \\ j(c) &= \frac{1}{3}(j(d) + j(h)) \\ j(d) &= \frac{1}{3}(1 + j(c) + j(g)) \\ j(e) &= \frac{1}{3}(j(f) + j(h)) \\ j(f) &= \frac{1}{3}(1 + j(e) + j(g)) \\ j(g) &= \frac{1}{3}(j(d) + j(f)) \\ j(h) &= \frac{1}{3}(1 + j(c) + j(e)) \end{aligned}$$

Note also that we must have $j(d) = j(h) = j(f)$ and $j(g) = j(e) = j(c)$ because each of these vertices are the same number of moves away from a and b and each move has the same probability. In addition, each vertex that they can possibly move to is the same number of moves from a or b. So let's revise this system of equations, using one representative from each equivalence class,

$$\begin{aligned}
j(a) &= 1 \\
j(b) &= 0 \\
j(c) &= \frac{2j(d)}{3} \\
j(d) &= \frac{1}{3}(1 + 2j(g)) \\
j(e) &= \frac{2j(d)}{3} \\
j(f) &= \frac{1}{3}(1 + 2j(g)) \\
j(g) &= \frac{2j(d)}{3} \\
j(h) &= \frac{1}{3}(1 + 2j(g))
\end{aligned}$$

Plugging $j(g)$ into the equation for $j(d)$ yields,

$$j(d) = \frac{3}{5}$$

and

$$j(g) = \frac{2}{5}$$

Hence, our system of equations is,

$$\begin{aligned}
j(a) &= 1 \\
j(b) &= 0 \\
j(c) &= \frac{2}{5} \\
j(d) &= \frac{3}{5} \\
j(e) &= \frac{2}{5} \\
j(f) &= \frac{3}{5} \\
j(g) &= \frac{2}{5} \\
j(h) &= \frac{3}{5}
\end{aligned}$$

Thus, we have,

$$\begin{aligned}
P(T_a < T_b | X_0 = a) &= \frac{1}{3}(P(V_a < V_b | X_0 = d) + P(V_a < V_b | X_0 = h) + P(V_a < V_B | X_0 = f)) \\
&= \frac{1}{3}(\frac{3}{5} + \frac{3}{5} + \frac{3}{5}) \\
&= \frac{3}{5}
\end{aligned}$$

For a quick sanity check, we compute the matrix powers of $\tilde{p}$ (done in the R programming language) and find,

$$\tilde{p}^{1000} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Hence, we have $P(V_a < V_b | X_0 = d) = P(V_a < V_b | X_0 = h) = P(V_a < V_B | X_0 = f) = 0.6 = \frac{3}{5}$ as required.

Problem 4.

Fix $i \in \mathcal{S}$. Suppose $X_n = i$ for some n. Now consider the state $j \in S$ such that $j < i$. It is possible to reach j from i because $p(i, 0) = 1/4$. Once we are at 0, we have that $p(0, 1) = 3/4$, $p(1, 2) = 3/4$. Hence, by induction, we have $p(j - 1, j) = 3/4$.

Now assume $j > i$. We can perform the same process, except without returning to 0. We have $p(i, i+1) = p(i+1, i+2) = \cdots = p(j-1, j) = 3/4$. Thus, we can reach j from i when $j > i$.

Lastly, when $j = i$, we need to perform the same process as $j < i$. We can return to 0 with probability $p(i, 0) = 1/4$, and then we have, $p(0, 1) = p(1, 2) = \cdots p(i - 1, i) = 3/4$. Hence, we have that we can reach i from state i.

Since i was arbitrary and we covered every possible case for another state j, this holds for any state in $\mathcal{S}$. Thus, every state in $\mathcal{S}$ communicates with every other state, and so the chain is irreducible.

Now we need to show that there is a positive recurrent state in the chain. Let us consider state 0. Consider $P_0(T_0 = \infty) = 1 - P_0(T_0 < \infty)$. We have that,

$$\begin{aligned}
P_0(T_0 = \infty) &= p(0, 1) \cdot p(1, 2) \cdot p(2, 3) \cdot p(3, 4) \cdots \\
&= \sum_{i=0}^{\infty} p(i, i+1)
\end{aligned}$$

Since $p(i, i+1) = 3/4$ for every $i \in \mathcal{S}$, this sum is equivalent to,

$$\lim_{n\to\infty} 3n/4$$

Thus, we have that,

$$\begin{aligned} P_0(T_0 < \infty) &= 1 - P_0(T_0 = \infty) \\ &= 1 - \lim_{n\to\infty} 3n/4 \\ &= 1 - 0 = 1 \end{aligned}$$

Hence, 0 is recurrent in the sense that we will certainly return. Now consider $E_0 T_0$. Note that from each state x, we have probability $1/4$ of returning to 0. Define $g(x)$ as the function which denotes the expected time to return to 0 from state x. Then we have,

$$g(x) = 1 + \frac{3}{4} g(x+1)$$

So, we have,

$$g(0) = 1 + \frac{3}{4} \sum_{i=1}^{\infty} g(i)$$