# Stochastic Processes: Homework 5

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**Problem 1.** Durrett, Exercise 1.61

(A) We have that the transition matrix for this Markov chain is,

$$p = \begin{array}{cc} & \begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} & \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 \end{pmatrix} \end{array}$$

Now we need to find the stationary distribution $\pi$ such that,

$$\pi p = \pi$$

Note that $p$ is doubly stochastic, so by Theorem 1.14, we have that $\pi(y) = 1/12$ for every $y$ in the state space. Observe also that $p$ is irreducible because we can follow the path $1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 8 \to 9 \to 10 \to 11 \to 12 \to 1$. Hence, every state communicates with every state.

So now we can apply Theorem 1.12, which says,

$$\pi(y) = 1/E_y T_y$$

Note that we are looking for $E_y T_y$, so we want $1/\pi(y)$. Thus, we have,

$$\begin{aligned} E_y T_y &= 1/\pi(y) \\ &= 12 \end{aligned}$$

for every $y$ in the state space.

(B) Without loss of generality, suppose we start at 12 and our first step is to 1. We will now treat 11 and 12 as absorbing states. By doing this, the only way to reach 11 is via state 10. Hence, we can simply look at the probability of reaching 11 to ensure that we have hit every state.

So we can re-write our transition probability as follows:

$$\tilde{p} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{matrix} \\ \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

We find numerically that,

$$\tilde{p}^{100000} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.09090909 & 0.90909091 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.18181818 & 0.81818182 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.27272727 & 0.72727273 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.36363636 & 0.63636364 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.45454545 & 0.54545455 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.54545455 & 0.45454545 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.63636364 & 0.36363636 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.72727273 & 0.27272727 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.81818182 & 0.18181818 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.90909091 & 0.09090909 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$= \tilde{p}^{100001}$$

And so, we find that,

$$\lim_{n\to\infty} p^n(1,11) \approx 0.09090909$$

This is the probability that the chain will visit all 11 states before returning to state 12.

Since the structure of the graph is the same regardless of the starting state, this probability holds for any choice of state $X_0$.

**Problem 2.**

Suppose $p$ is the transition probability matrix for a Markov chain on $\mathcal{S}$. Let $\delta$ be defined on $\mathcal{S}$ such that $d(x, y) = 1$ if $x = y$ and $d(x, y) = 0$ otherwise.

Now let $q = (p + \delta)/2$. Fix $x \in \mathcal{S}$. Let us consider the row $x$ in the matrix $q$. We have that,

$$q(x, y) = p(x, y)/2$$

for every $y \neq x$. Also, we have,

$$\begin{aligned} q(x, x) &= (1 + p(x, x))/2 \\ &= 1/2 + p(x, x) \end{aligned}$$

Since $p$ is a transition matrix, we have that,

$$p(x, y_1) + p(x, y_2) + p(x, y_3) + \cdots + p(x, x) + \cdots + p(x, y_n) = 1$$

So,

$$\begin{aligned} & p(x, y_1)/2 + p(x, y_2)/2 + \cdots + p(x, x)/2 + \cdots + p(x, y_n)/2 = 1/2 \\ \iff & p(x, y_1)/2 + p(x, y_2)/2 + \cdots + 1/2 + p(x, x)/2 + \cdots + p(x, y_n)/2 = 1/2 + 1/2 \\ \iff & p(x, y_1)/2 + p(x, y_2)/2 + \cdots + (1 + p(x, x))/2 + \cdots + p(x, y_n)/2 = 1 \\ \iff & \sum_{y \in \mathcal{S}} q(x, y) = 1 \end{aligned}$$

So every row of $q$ sums to 1 as required.

Now fix $y \neq x \in \mathcal{S}$ as well. Since $p$ is a transition probability matrix, we have that,

$$\begin{aligned} & 0 \leq p(x, y) \leq 1 \\ \implies & 0 \leq q(x, y) = p(x, y)/2 \leq 1/2 < 1 \end{aligned}$$

So $0 \leq q(x, y) \leq 1$ when $x \neq y$. Now,

$$\begin{aligned} & 0 \leq p(x, x) \leq 1 \\ \implies & 1 \leq 1 + p(x, x) \leq 2 \\ \implies & 0 < 1/2 \leq (1 + p(x, x))/2 \leq 1 \end{aligned}$$

Thus, we also have $0 \leq q(x, x) \leq 1$. So every row of $q$ sums to 1 and every entry of $q$ is greater than or equal to 0 and less than or equal to 1. Hence, $q$ defines a valid transition matrix.

Now observe that $q$ is aperiodic if every state has period 1. In addition, by Lemma 1.18, if $q(x, x) > 0$ for a state $x$, then $x$ has period 1. Consider $q(x, x)$ for some state $x$. We have that,

$$q(x, x) = 1/2 + p(x, x)/2$$

Since $p$ is a valid transition probability matrix, we have $p(x,x) \geq 0$. Hence,

$$q(x,x) \geq 1/2$$

Thus, for every state $x$, $q(x,x) > 0$ and therefore has period 1. Hence, $q$ is aperiodic.

**Problem 3.**

(A) Fix $x \in \mathcal{S}$ and $n \in \mathbb{N}$. Then,

$$(\mu_n p)(x) = \frac{1}{n}(q(x)p(x) + q(x)p^2(x) + \cdots + q(x)p^n(x))$$

So,

$$\begin{aligned}
|(\mu_n p)(x) - \mu_n(x)| &= \left|\frac{1}{n}((qp)(x) + (qp^2)(x) + \cdots + (qp^n)(x)) - \frac{1}{n}(q(x) + (qp)(x) + \cdots + qp^{n-1})(x))\right| \\
&= \frac{1}{n}|(qp)(x) + (qp^2)(x) + \cdots + (qp^n)(x) - q(x) - (qp)(x) - \cdots - (qp^{n-1})(x)| \\
&= \frac{1}{n}\,|(qp^n)(x) - q(x)|
\end{aligned}$$

Observe that $0 \leq q(x), (qp^n)(x) \leq 1$, and so $-1 \leq (qp^n)(x) - q(x) \leq 1$. Thus,

$$\begin{aligned}
|(\mu_n p)(x) - \mu_n(x)| &= \frac{1}{n}\,|(qp^n)(x) - q(x)| \\
&\leq \frac{1}{n}
\end{aligned}$$

(B) Observe that, for every $x$ and for any $k \in \mathbb{N}$, we have,

$$0 \leq (qp^k)(x) \leq 1$$

since $q$ defines a valid initial distribution and $p$ defines a valid transition probability matrix.

Thus, for any $n \in \mathbb{N}$ and any $x \in \mathcal{S}$, we have,

$$\begin{aligned}
\mu_n(x) &= \frac{1}{n}(q(x) + (qp)(x) + \cdots + (qp^{n-1})(x)) \\
&\geq \frac{1}{n}(0 + 0 + \cdots 0) \\
&= 0
\end{aligned}$$

as well as,

$$\begin{aligned}\mu_n(x) &= \frac{1}{n}(q(x) + (qp)(x) + \cdots + (qp^{n-1})(x)) \\ &\leq \frac{1}{n}(1 + 1 + \cdots 1) \\ &= \frac{n}{n} \\ &= 1\end{aligned}$$

Hence, we have $0 \leq \mu_n(x) \leq 1$, and so the sequence $\{\mu_n(x)\}$ is bounded. Note also that $\mu_n(x) \in \mathbb{R}$ for every $n$. As a result, we have that $\{\mu_n(x)\}$ is a bounded sequence in $\mathbb{R}$ for any choice of $x$. Thus, we can apply the Bolzano-Weierstrass theorem, which states that there exists a convergent subsequence $\mu_{n_k}(x)$.

Consider $\mu_{n_k}$. That is, the convergent subsequence above without evaluating it at $x$. As shown above, every state $x \in \mathcal{S}$ has a corresponding convergent subsequence $\mu_{n_{k_x}}$. As such, we have $N$ convergent subsequences - one for each state in $\mathcal{S}$.

(C) Fix $x \in \mathcal{S}$ and define $\pi(x) = \lim_{k\to\infty} \mu_{n_k}(x)$. Then,

$$\begin{aligned}|(\pi p)(x) - \pi(x)| &= |\lim_{k\to\infty} \mu_{n_k} p - \lim_{k\to\infty} \mu_{n_k}| \\ &= \frac{1}{n}|\lim_{k\to\infty}((qp)(x) + \cdots + (qp^{n_k})(x)) - \frac{1}{n}\lim_{k\to\infty}(q(x) + (qp)(x) \cdots + (qp^{n_k-1})(x))| \\ &= \frac{1}{n}\lim_{k\to\infty}|((qp)(x) + \cdots + (qp^{n_k})(x)) - (q(x) + (qp)(x) \cdots + (qp^{n_k-1})(x))| \\ &= \frac{1}{n}\lim_{k\to\infty}|(qp^{n_k})(x) - q(x)|\end{aligned}$$

From part $(A)$, we have that $|(qp^{n_k})(x) - q(x)| \leq \frac{1}{n_k}$ for every $k$. So,

$$\begin{aligned}|(\pi p)(x) - \pi(x)| &= \frac{1}{n}\lim_{k\to\infty}|(qp^{n_k})(x) - q(x)| \\ &\leq \frac{1}{n}\lim_{k\to\infty}\frac{1}{n_k} \\ &= 0\end{aligned}$$

Hence, we have $|(\pi p)(x) - \pi(x)| = 0$, which implies that $(\pi p)(x) = \pi(x)$. Since $x \in \mathcal{S}$ was arbitrary, this holds for every state. Thus,

$$\pi p = \pi$$

and so $\pi$ is a stationary distribution for the chain.

**Problem 4.**

Let state 0 denote that the musician is playing and state 1 denote that the musician is upset. Then we have the following transition probability,

$$p = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 1/2 & 1/2 \\ \sqrt{x/1000} & 1 - \sqrt{x/1000} \end{pmatrix} \end{array}$$

where $0 \leq x \leq 1000$. We need to maximize the profit function $h(x) = 750 \cdot \frac{N_n(0)}{n} - x \cdot \frac{N_n(1)}{n}$.

Note that $p$ is irreducible when $x \neq 0, 1000$ because both states communicate with themselves and each other. Note also that $p$ is closed under the same conditions for $x$. In addition, the state space is finite. So, by Theorem 1.7, all states in $p$ are recurrent. Thus, by Theorem 1.20, we have,

$$\frac{N_n(0)}{n} \to \frac{1}{E_0 T_0}$$

and

$$\frac{N_n(1)}{n} \to \frac{1}{E_1 T_1}$$

If we can find a stationary distribution $\pi$ for $p$, then we can apply Theorem 1.21 here. Hence, we will now attempt to compute a stationary distribution $\pi$. This yields,

$$\begin{aligned} & \pi p = \pi \\ \iff & \begin{pmatrix} \pi_0 & \pi_1 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ \sqrt{x/1000} & 1 - \sqrt{x/1000} \end{pmatrix} = \begin{pmatrix} \pi_0 & \pi_1 \end{pmatrix} \end{aligned}$$

So we need to solve,

$$\begin{aligned} 1/2\pi_0 + \pi_1 \sqrt{x/1000} &= \pi_0 \\ 1/2\pi_0 + \pi_1 (1 - \sqrt{x/1000}) &= \pi_1 \end{aligned}$$

Observe that $\sqrt{x/1000}$ is a bijection between the intervals $(0, 1000)$ and $(0, 1)$. So let us replace $\sqrt{x/1000}$ with $y \in (0, 1)$ in the above equations:

$$\begin{aligned} 1/2\pi_0 + \pi_1 y &= \pi_0 \\ 1/2\pi_0 + \pi_1 (1 - y) &= \pi_1 \end{aligned}$$

From these equations, we get,

$$\begin{aligned} \pi_0 &= \pi_0 \\ \pi_1 &= \frac{1}{2y}\pi_0 \end{aligned}$$

We also have the condition that $\pi_1 + \pi_0 = 1$, so this yields,

$$\begin{aligned} &\pi_0 + \pi_1 = 1 \\ \iff &\pi_0 + \frac{1}{2y}\pi_0 = 1 \\ \iff &\pi_0(1 + \frac{1}{2y}) = 1 \\ \iff &\pi_0 = \frac{1}{1 + \frac{1}{2y}} \end{aligned}$$

This yields,

$$\begin{aligned} \pi_0 &= \frac{1}{1 + \frac{1}{2y}} = \frac{2y}{2y+1} \\ \pi_1 &= \frac{1}{2y+1} \end{aligned}$$

Now we have,

$$\begin{aligned} \pi p &= \begin{pmatrix} \frac{2y}{2y+1} & \frac{1}{2y+1} \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ y & 1-y \end{pmatrix} \\ &= \begin{pmatrix} \frac{2y}{2y+1} & \frac{1}{2y+1} \end{pmatrix} \end{aligned}$$

So indeed, we have that,

$$\begin{aligned} \pi &= \begin{pmatrix} \frac{2y}{2y+1} & \frac{1}{2y+1} \end{pmatrix} \\ &= \begin{pmatrix} \frac{2\sqrt{x/1000}}{2\sqrt{x/1000}+1} & \frac{1}{2\sqrt{x/1000}+1} \end{pmatrix} \end{aligned}$$

We can now use Theorem 1.21 together with Theorem 1.20 to state that,

$$\frac{N_n(y)}{n} \to \frac{1}{E_y T_y} = \pi(y)$$

We can now re-write equation $h(x)$ using these relations,

$$h(x) = 750 \cdot \frac{2\sqrt{x/1000}}{2\sqrt{x/1000}+1} - x \cdot \frac{1}{2\sqrt{x/1000}+1}$$

Taking the derivative yields,

$$h'(x) = \frac{-25(-750\sqrt{10} + 100\sqrt{x} + \sqrt{10}x)}{50 + \sqrt{10}\sqrt{x})^2\sqrt{x}}$$

Setting $h'(x) = 0$ yields $x = 250$. We have $h(250) = 250$. We also need to check the endpoints of the interval, $x = 0$ and $x = 1000$. In these instances, we have $h(0) = 0$ and $h(1000) = \frac{500}{3} \approx 166.67$.

Hence, we have that $h(x)$ attains its maximum when $x = 250$. Thus, in the long run, the manager should spend \$250 on each gift in order to maximize his overall net earnings.