Stochastic Processes II: Homework 1

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Problem I.

We have that $M_n = X_n/(n+2)$. Since we start with 1 red ball in the urn, we have that $X_0 = 1$. Note that $X_n \leq n+1$ for every n because X_n will increase by at most 1 per "turn" but could stay the same, whereas n will always increase by 1 unit per turn. Thus, we have that $0 < M_n = X_n/(n+2) \leq (n+1)/(n+2) < 1$ for all $n \in \mathbb{N}_0$. Hence, $E|M_n| < \infty$ for all n.

Now, since $M_n = X_n/(n+2)$, we have that M_n is completely determined by X_n. Thus, this example satisfies the condition that M_n be determined from the values for $X_n, \ldots, X_0$ and M_0.

Now consider,

$$\begin{aligned} M_{n+1} - M_n &= X_{n+1}/((n+1)+2) - X_n/(n+2) \\ &= X_{n+1}/(n+3) - X_n/(n+2) \end{aligned}$$

Then, since X_n is given in $A_v = \{X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0, M_0 = m_0\}$, we have

$$\begin{aligned} E\left[\frac{X_{n+1}}{n+3} - \frac{X_n}{n+2} \mid A_v\right] &= 1/(n+3) \cdot E[X_{n+1} \mid A_v] - 1/(n+2) \cdot E[X_n \mid A_v] \\ &= 1/(n+3) \cdot E[X_{n+1} \mid A_v] - X_n/(n+2) \\ &= 1/(n+3) \cdot (M_n \cdot (X_n + 1) + (1 - M_n) \cdot X_n) - X_n/(n+2) \\ &= 1/(n+3) \cdot (M_n X_n + M_n + X_n - M_n X_{n+1}) - X_n/(n+2) \\ &= 1/(n+3) \cdot (M_n + X_n) - X_n/(n+2) \\ &= 1/(n+3) \cdot (X_n/(n+2) + X_n) - X_n/(n+2) \\ &= \frac{X_n}{(n+2)(n+3)} + \frac{X_n}{n+3} - \frac{X_n}{n+2} \\ &= \frac{X_n}{(n+2)(n+3)} + \frac{X_n(n+2)}{(n+2)(n+3)} - \frac{X_n(n+3)}{(n+2)(n+3)} \\ &= \frac{X_n}{(n+2)(n+3)} - \frac{X_n}{(n+2)(n+3)} \\ &= 0 \end{aligned}$$

as required. Hence, (M_n) is a martingale with respect to (X_n).

Problem II.

For each n, we have that $M_n = S_n^3 - 3nS_n = (X_1 + \cdots + X_n)^3 - 3n(X_1 + \cdots + X_n)$. Hence, M_n is an algebraic expression with a finite sum of 1s and -1s for each n, and thus $E|M_n| < \infty$. Furthermore, from the above, we see that M_n can be written as a function of $X_1, \ldots, X_n$, so M_n is fully determined by $M_0, X_0, \ldots, X_n$ as required.

Moving on to the increments, we have,

$$\begin{aligned} M_{n+1} - M_n &= ((S_{n+1})^3 - 3(n+1)(S_{n+1})) - (S_n^3 - 3nS_n) \\ &= 3nS_n - 3(n+1)S_{n+1} - S_n^3 + S_{n+1}^3 \end{aligned}$$

This gives us,

$$\begin{aligned} E\left[3nS_n - 3(n+1)S_{n+1} - S_n^3 + S_{n+1}^3 \mid A_v\right] &= 3E\left[nS_n \mid A_v\right] - 3E\left[(n+1)S_{n+1} \mid A_v\right] - \\ &\quad E\left[S_n^3 \mid A_v\right] + E\left[S_{n+1}^3 \mid A_v\right] \\ &= 3E\left[nS_n \mid A_v\right] - 3E\left[(n+1)(S_n + X_{n+1}) \mid A_v\right] \\ &\quad - E\left[S_n^3 \mid A_v\right] + E\left[S_n^3 + 3S_n^2 X_{n+1} + 3S_n X_{n+1}^2 + X_{n+1}^3 \mid A_v\right] \\ &= 3nS_n - 3(n+1)S_n - S_n^3 + S_n^3 + 3S_n^2 E\left[X_{n+1} \mid A_v\right] \\ &\quad + 3S_n E[X_{n+1}^2 \mid A_v] + E[X_{n+1}^3 \mid A_v] \\ &= -3S_n + 3S_n \\ &= 0 \end{aligned}$$

Hence (M_n) is a martingale with respect to (X_n).

Problem III.

By Lemma 5.4 in Durrett and taking $A_v = \{X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0, M_0 = m_0\}$, we have,

$$\mathbb{E}\left[\Delta M_n \Delta M_m\right] = \mathbb{E}\left[\mathbb{E}\left[\Delta M_n \Delta M_m \mid A_v\right]\right]$$

Then, by Lemma 5.1 and property (iii) from the definition of martingales, we have,

$$\begin{aligned} \mathbb{E}\left[\mathbb{E}\left[\Delta M_n \Delta M_m \mid A_v\right]\right] &= \mathbb{E}\left[\Delta M_m \, \mathbb{E}\left[\Delta M_n \mid A_v\right]\right] \\ &= \mathbb{E}\left[\Delta M_m \cdot 0\right] \\ &= \mathbb{E}\left[0\right] \\ &= 0 \end{aligned}$$

Now for the variance of M_n, we have

$$\begin{aligned} \mathbb{E}\, M_n^2 &= \mathbb{E}\left[\left((M_n - M_{n-1}) + (M_{n-1} - M_{n-2}) + \cdots + (M_1 - M_0) + M_0\right)^2\right] \\ &= \mathbb{E}[(M_n - M_{n-1})^2 + (M_{n-1} - M_{n-2})^2 + \cdots + (M_1 - M_0)^2 + M_0^2 + 2(M_n - M_{n-1})(M_{n-1} - M_{n-2}) \\ &\quad + \cdots + 2M_0(M_1 - M_0)] \end{aligned}$$

By the above arguments, all of the arguments where two intervals are multiplied together will have an expectation of 0. Hence, we get,

$$\mathbb{E}\, M_n^2 = \sum_{i=0}^{n-1} [\mathbb{E}(M_{i+1} - M_i)] + \mathbb{E}\, M_0^2$$