Stochastic Processes II: Homework 4

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Problem I. LPW 5.2

Since $P\{\tau_{\text{couple}} \leq t_0\} \geq \alpha$, we have

$$P\{\tau_{\text{couple}} > t_0\} \leq 1 - \alpha$$

Note that since this coupling is Markovian, the probability of not coupling on any length t_0 is the same. Hence

$$P\{\tau_{\text{couple}} > kt_0\} \leq (1 - \alpha)^k$$

Problem II.

Observe that for (X_n), we have,

$$\begin{aligned} S_X &= \{g \in G \ : \ \mu(g) > 0\} \\ &= \{3\} \end{aligned}$$

This set generates the following group:

$$\langle 3, 1 \rangle$$

Hence, by Prop 2.13, we have that (X_n) is not irreducible. On the other hand, we have

$$\begin{aligned} S_Y &= \{g \in G \ : \ \mu(g) > 0\} \\ &= \{3, 5\} \end{aligned}$$

and so the set S_Y generates the following group:

$$\langle 1, 3, 5, 7 \rangle$$

Thus, (Y_t) is in fact irreducible. Note that in G, every element is its own inverse. That is $g = g^{-1}$ for all $g \in G$. Hence we have,

$$\mu(g) = \mu(g^{-1})$$

and

$$\nu(g) = \nu(g^{-1})$$

Thus, both μ and ν are symmetric.

Problem III.

We have $X \sim \text{Poisson}(a)$, $Z \sim \text{Poisson}(b-a)$, and $Y = X + Z$. Hence, $Y \sim \text{Poisson}(b)$. We have that (X, Y) is a coupling of μ and ν. Observe that $X = Y$ iff $Z = 0$. Thus, $P\{X \neq Y\} = P\{Z \neq 0\}$. We have,

$$\begin{aligned}
||\mu - \nu||_{TV} &\leq \inf\{P\{X \neq Y\} : \ (X, Y) \text{ is a coupling of } \mu \text{ and } \nu\} \\
&\leq P\{X \neq Y\} \\
&= P\{Z \neq 0\} \\
&= \frac{(b-a)^0 e^{-0}}{0!} \\
&= 1
\end{aligned}$$