Stochastic Processes II: Homework 3

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Problem I.

Let us define a new Markov Y_n with all possible 3 toss sequences as its states:

$$\{HHH, HHT, HTT, TTT, HTH, THH, THT, TTH\}$$

Then our transition probability matrix is given by,

$$p = \begin{array}{c} \\ {\scriptstyle HHH} \\ {\scriptstyle HHT} \\ {\scriptstyle HTT} \\ {\scriptstyle TTT} \\ {\scriptstyle HTH} \\ {\scriptstyle THH} \\ {\scriptstyle THT} \\ {\scriptstyle TTH} \end{array} \begin{array}{c} \begin{array}{cccccccc} {\scriptstyle HHH} & {\scriptstyle HHT} & {\scriptstyle HTT} & {\scriptstyle TTT} & {\scriptstyle HTH} & {\scriptstyle THH} & {\scriptstyle THT} & {\scriptstyle TTH} \end{array} \\ \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix} \end{array}$$

We'll now eliminate the row and column for HTH, yielding,

$$r = \begin{array}{c} \\ {\scriptstyle HHH} \\ {\scriptstyle HHT} \\ {\scriptstyle HTT} \\ {\scriptstyle TTT} \\ {\scriptstyle THH} \\ {\scriptstyle THT} \\ {\scriptstyle TTH} \end{array} \begin{array}{c} \begin{array}{ccccccc} {\scriptstyle HHH} & {\scriptstyle HHT} & {\scriptstyle HTT} & {\scriptstyle TTT} & {\scriptstyle THH} & {\scriptstyle THT} & {\scriptstyle TTH} \end{array} \\ \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix} \end{array}$$

Thus we have,

$$I - r = \begin{array}{c} \\ {\scriptstyle HHH} \\ {\scriptstyle HHT} \\ {\scriptstyle HTT} \\ {\scriptstyle TTT} \\ {\scriptstyle THH} \\ {\scriptstyle THT} \\ {\scriptstyle TTH} \end{array} \begin{array}{c} \begin{array}{ccccccc} {\scriptstyle HHH} & {\scriptstyle HHT} & {\scriptstyle HTT} & {\scriptstyle TTT} & {\scriptstyle THH} & {\scriptstyle THT} & {\scriptstyle TTH} \end{array} \\ \begin{pmatrix} 1/2 & -1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & -1/2 \\ -1/2 & -1/2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1/2 & -1/2 & 1 \end{pmatrix} \end{array}$$

And finally,

$$(I - r)^{-1}\mathbf{1} = \begin{pmatrix} 8 \\ 6 \\ 10 \\ 10 \\ 8 \\ 6 \\ 8 \end{pmatrix}$$

Observe that after the first three tosses, each possibility occurs with probability 1/8. Hence, we have,

$$\begin{aligned} E(\tau) &= 3 + \frac{1}{8}(0 + 8 + 6 + 10 + 10 + 8 + 6 + 8) \\ &= 10 \end{aligned}$$

Problem II. Durrett 5.9

Context from Durrett 5.8: Let $X_1, X_2, \dots$ be independent with $P(X_i = 1) = p$ and $P(X_i = -1) = q = 1 - p$ where $p < 1/2$. Let $S_n = S_0 + X_1 + \cdots + X_n$ and let $V_0 = \min\{n \geq 0 : S_n = 0\}$. (5.13) implies $\mathbb{E}_x V_0 = x/(1 - 2p)$. If we let $Y_i = X - i - (p - q)$ and note that $EY_i = 0$ and

$$\mathrm{var}(Y_i) = \mathrm{var}(X_i) = EX_i u^2 - (EX_i)^2$$

then it follows that $(S_n - (p-q)n)^2 - n(1-(p-q)^2)$ is a martingale. Moreover, when $S_0 = x$, the variance of X_0 is

$$x \cdot \frac{1 - (p - q)^2}{(q - p)^3}$$

(a)

(b)

(c)

Problem III. LPW 4.2

By Proposition 4.2, we have that,

$$\begin{aligned}
||\mu P - vP||_{TV} &= \frac{1}{2}\sum_{x\in\mathcal{X}} |\mu P(x) - vP(x)| \\
&= \frac{1}{2}\sum_{x\in\mathcal{X}} \left|\sum_{y\in\mathcal{X}} \mu(y)P(y,x) - v(y)P(y,x)\right| \\
&= \frac{1}{2}\sum_{x\in\mathcal{X}} \left|\sum_{y\in\mathcal{X}} P(y,x)[\mu(y) - v(y)]\right| \\
&\leq \frac{1}{2}\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{X}} P(y,x)|\mu(y) - v(y)| \\
&= \frac{1}{2}\sum_{y\in\mathcal{X}} |\mu(y) - v(y)| \sum_{x\in\mathcal{X}} P(y,x) \\
&= \frac{1}{2}\sum_{y\in\mathcal{X}} |\mu(y) - v(y)| \cdot 1 \\
&= \frac{1}{2}\sum_{y\in\mathcal{X}} |\mu(y) - v(y)| \\
&= ||\mu - v||_{TV}
\end{aligned}$$

Hence, we have that $||\mu P - vP||_{TV} \leq ||\mu - v||_{TV}$ as required. In particular, we have that $||\mu P^{t+1} - \pi||_{TV} \leq ||\mu P^t - \pi||_{TV}$.

Now fix $t > 0$. By the above, we must have that,

$$\begin{aligned}
&\max_{x\in\mathcal{X}} ||P^{t+1}(x,\cdot) - \pi||_{TV} \leq \max_{x\in\mathcal{X}} ||P^t(x,\cdot) - \pi||_{TV} \\
\iff &d(t+1) \leq d(t)
\end{aligned}$$

and,

$$\begin{aligned}
&\max_{x\in\mathcal{X}} ||P^{t+1}(x,\cdot) - P^{t+1}(y,\cdot)||_{TV} \leq \max_{x\in\mathcal{X}} ||P^t(x,\cdot) - P^t(y,\cdot)||_{TV} \\
\iff &\bar{d}(t+1) \leq \bar{d}(t)
\end{aligned}$$

Problem III. LPW 4.4

Let (X_i, Y_i) be the optimal coupling for each μ_i, v_i. Define $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ where X is distributed as μ and Y is distributed as v. Let (X, Y) be a coupling of these two random variables. By Proposition 4.7, we have

$$||\mu - v||_{TV} = \inf\{P\{X \neq Y\} : (X, Y) \text{ is a coupling of } \mu \text{ and } v\}$$

Since (X, Y) is a specific coupling of μ and v, it must be greater than the infimum over all couplings and we have that

$$||\mu - v||_{TV} \leq P\{X \neq Y\}$$

for this specific coupling (X, Y). This gives us,

$$\begin{aligned} ||\mu - v||_{TV} &\leq P\{X \neq Y\} \\ &\leq \sum_{i=1}^{n} P(X_i \neq Y_i) \end{aligned}$$

Since (X_i, Y_i) is the optimal coupling for each μ_i, v_i, by Proposition 4.7, we have that

$$\begin{aligned} ||\mu - v||_{TV} &\leq \sum_{i=1}^{n} P(X_i \neq Y_i) \\ &= \sum_{i=1}^{n} ||\mu_i - v_i||_{TV} \end{aligned}$$

as required.