Stochastic Processes II: Homework 5

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Problem I. LPW 5.4

a) We have from the proof of Theorem 5.6 that for any coordinate i, the expected coupling time is,

$$E_{x,y}(\tau_i) \leq \frac{dn^2}{4}$$

Thus, we have that, if $t = dn^2$, then,

$$\begin{aligned} P_{x,y}\{\tau_i > t\} &\leq \frac{E_{x,y}(\tau_i)}{t} \\ &\leq \frac{1}{t}\frac{dn^2}{4} \\ &= \frac{1}{4} \end{aligned}$$

Now suppose $t = 2dn^2$. By Remark 5.3, we know that this coupling is Markovian and hence, the probability that that x and y have not coupled between $t_0 = dn^2$ and $t = 2dn^2$ is still $1/4$. Thus, by induction, we have that if $t \geq kdn^2$,

$$\begin{aligned} P_{x,y}\{\tau_i > t\} &\leq P_{x,y}\{\tau_i > kdn^2\} \\ &\leq (1/4)^k \end{aligned}$$

b) We have that,

$$P\left\{\max_{1\leq i\leq d} \tau_i > kdn^2\right\} \leq P\left(\sqcup_{i=1}^d \tau_i > kdn^2\right)$$

Since the coordinates are independent of one another, our union in the second part of the above equation is disjoint. Hence, we have,

$$\begin{aligned} P\left(\sqcup_{i=1}^d \tau_i > kdn^2\right) &= \sum_{i=1}^d P\left(\tau_i > kdn^2\right) \\ &= \frac{d}{4^k} \end{aligned}$$

Now, if we take $k = \left\lceil \log_4(d/\epsilon) \right\rceil$, then we have,

$$\begin{aligned}
P\left\{\max_{1\le i\le d} \tau_i > kdn^2\right\} &\le \frac{d}{4^k} \\
&= \frac{d}{4^{\lceil \log_4(d/\epsilon)\rceil}} \\
&= \frac{d}{\lceil d/\epsilon \rceil} \\
&\le \frac{d}{d/\epsilon} \\
&= \epsilon
\end{aligned}$$

And hence by Corollary 5.5, we have that,

$$\begin{aligned}
d(t) &\le \max_{x,y\in\mathcal{X}} P_{x,y}\{\tau_{\text{couple}} > t\} \\
&\le \epsilon
\end{aligned}$$

when $t > kdn^2$ and $k = \left\lceil \log_4(d/\epsilon) \right\rceil$. Hence,

$$\begin{aligned}
t_{\text{mix}} &= kdn^2 \\
&= \left\lceil \log_4(d/\epsilon) \right\rceil dn^2
\end{aligned}$$

as required.

Problem II. LPW 5.5

Let us assume that we now have a finite $b-$ary tree with depth k. By the same coupling process as described in section 5.3.4, we have that,

$$\begin{aligned}
\tau &= \min\{t \ge 0 : X_t = Y_t\} \\
\tau_\rho &= \min\{t \ge 0 : Y_t = \rho\}
\end{aligned}$$

And $E(\tau) \le E_{y_0}(\tau_\rho)$. Moreover, the distance of Y_t from the leaves is still a birth-and-death chain, this time with $p = \frac{1}{2}\cdot\frac{1}{b} = \frac{1}{2b}$ and $q = \frac{1}{2} - p = \frac{b-1}{2b}$. Then by (2.14), we have,

$$\begin{aligned}
E_{y_0}(\tau_\rho) &\le \frac{2b}{2-b}\left[k + \frac{b-1}{2b}\left(\frac{1-(b-1)^k}{\frac{2-b}{2b}}\right)\right] \\
&= \frac{2b}{2-b}\left[k + \frac{b-1}{2-b}\left(1-(b-1)^k\right)\right] \\
&\le -2b\left[k + b\left(1-b^k\right)\right] \\
&= -2bk - 2b^2 + 2b^{k+2} \\
&= 2b(b^{k+1} - k - b) \\
&\le 2bn
\end{aligned}$$

And so,

$$
\begin{aligned}
t_{\text{mix}} &\leq 4(2bn) \\
&= 8bn
\end{aligned}
$$

Problem III. LPW 6.10

Let τ_k be the first time that vertex k has been reached.