Stochastic Processes II: Homework 8

Chris Hayduk

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Note: Worked with Raghu and George.

Problem I. LPW 8.9

Let us assume we are working with a 3 card deck. We will consider the distribution when $T = 3$.

a) Suppose we have two of the three cards checked. Without loss of generality, assume card 3 is unchecked in the third position. We only check the third card if it is part of a transposition. Hence, the only possible transpositions we can introduce where we mark the third card is by transposing it with one of the first two cards. That is, if A_i is the i-th position in the deck, we only mark card 3 if it ends up in A_1 or A_2 (that is, it is transposed with 1 or with 2). Hence, the resulting permutation given that two cards are already checked only has positive probability if 3 is in one of the first two positions. The other 2 possible permutations of 3 cards (where card 3 is in position 3) have probability 0. Hence, this is not uniform.

b) Suppose we have two of the three cards checked. Without loss of generality, assume card 3 is unchecked and in the third position. We only check the third card if it is the right-hand card of a transposition. Hence, in order to get a permutation where 3 is in the third position, we would need to swap it with card 1 or card 2 as the left-hand size of a transposition and then choose 3 as the right-hand side of a transposition that sends it to A_3. Hence, it requires at least 2 steps from our initial set up for 3 to be marked in position A_3. However, we can get 3 marked in position A_1 or A_2 in only 1 step by swapping 3 as the right-hand side of a transposition with one of the first two cards. Hence, since it requires more steps to get 3 marked in the final position, the probability of that permutation must be lower than the probability of a permutation where 3 ends up in A_1 or A_2. Thus, this is not uniform.

Problem II. LPW 12.1

a) By the hint, we will let $||f||_\infty = \max_{x\in\mathcal{X}} |f(x)|$. We have that,

$$||Pf||_\infty = \max_{x\in\mathcal{X}} |P(x,y)f(y)|$$

Since $0 \leq P(x,y) \leq 1$ for all $x, y \in \mathcal{X}$, we have that $|P(x,y)f(y)| < |f(y)|$ for all x, y. Hence,

$$\begin{aligned}||Pf||_\infty &= \max_{x\in\mathcal{X}} |P(x,y)f(y)| \\ &\leq \max_{x\in\mathcal{X}} |f(x)| \\ &= ||f||_\infty\end{aligned}$$

Now suppose f is an eigenfunction with corresponding eigenvalue λ. The,

$$\begin{aligned}||Pf||_\infty &= ||\lambda f||_\infty \\ &= \max_{x\in\mathcal{X}} |\lambda f(x)| \\ &= |\lambda| \max_{x\in\mathcal{X}} |f(x)| \\ &= |\lambda| ||f||_\infty\end{aligned}$$

From the first part of our proof, we have that,

$$\begin{aligned}||Pf||_\infty &= |\lambda| ||f||_\infty \\ &\leq ||f||_\infty\end{aligned}$$

This final inequality implies that $|\lambda| \leq 1$.

b) Assume that $\mathcal{T}(x) \subset 2\mathbb{Z}$. -1 is a 2-nd root of unity because $-1^2 = 1$. Let $\mathcal{C}_j = \{x \in \mathcal{X} : P^{2m+j}(x_0, x) > 0 \text{ for some } m\}$ for $j = 0, 1, 2$. We have that there is a unique $j(x) \in \{0, 1\}$ such that $x \in \mathcal{C}_{j(x)}$ and if $P(x,y) > 0$ then $j(y) = j(x) + 1 \mod 2$.

Let $f : \mathcal{X} \to \mathbb{C}$ be defined by $f(x) = -1^{j(x)}$. We have that, for some $\ell \in \mathbb{Z}$,

$$\begin{aligned}Pf(x) = \sum_{y\in\mathcal{X}} P(x,y)(-1)^{j(y)} &= -1^{j(x)+1 \mod 2} \\ &= (-1)^{j(x)+1+2\ell} = -1(-1)^{j(x)} = -1f(x)\end{aligned}$$

Thus, $f(x)$ is an eigenfunction of P with eigenvalue -1.

Now let -1 be an eigenvalue of P. Choose x such that $|f(x)| = r := \max_{y\in\mathcal{X}} |f(y)|$. Since

$$-1f(x) = Pf(x) = \sum_{y\in\mathcal{X}} P(x,y)f(y)$$

taking absolute values shows that

$$r \leq \sum_{y \in X} P(x,y)|f(y)| \leq r$$

We conclude that if $P(x,y) > 0$, then $|f(y)| = r$. By irreducibility, $|f(y)| = r$ for all $y \in \mathcal{X}$.

Since the average of complex numbers of norm r has norm r if and only if all the values have the same angle, it follows that $f(y)$ has the same value for all y with $P(x,y) > 0$. Therefore, if $P(x,y) > 0$, then $f(y) = -1 f(x)$. Now fix $x_0 \in \mathcal{X}$ and define for $j = 0, 1$,

$$\mathcal{C}_j = \{z \in \mathcal{X} : f(z) = -1^j f(x_0)\}$$

It is clear that if $P(x,y) > 0$ and $x \in \mathcal{C}_j$, then $x \in \mathcal{C}_{j+1 \mod 2}$. It is clear that if $t \in \mathcal{T}(x_0)$, then 2 divides t and hence $\mathcal{T}(x_0) \subset 2\mathbb{Z}$, as required.

c) We now generalize the previous proof. Assume that a divides $\mathcal{T}(x)$. If b is the gcd of $\mathcal{T}(x)$, then a divides b. If ω is an a-th root of unity, then $\omega^b = 1$. Let $\mathcal{C}_j = \{x \in \mathcal{X} : P^{mb+j}(x_0, x) > 0 \text{ for some } m\}$ for $j = 0, \ldots, b$. We have that there is a unique $j(x) \in \{0, \ldots, b-1\}$ such that $x \in \mathcal{C}_{j(x)}$ and if $P(x,y) > 0$ then $j(y) = j(x) + 1$ mod b.

Let $f : \mathcal{X} \to \mathbb{C}$ be defined by $f(x) = \omega^{j(x)}$. We have that, for some $\ell \in \mathbb{Z}$,

$$\begin{aligned} Pf(x) = \sum_{y \in \mathcal{X}} P(x,y)\omega^{j(y)} &= \omega^{j(x)+1 \mod b} \\ &= \omega^{j(x)+1+b\ell} = \omega\omega^{j(x)} = \omega f(x) \end{aligned}$$

Thus, $f(x)$ is an eigenfunction of P with eigenvalue ω.

Now let ω be an a-th root of unity and suppose that $\omega f = Pf$ for some f. Choose x such that $|f(x)| = r := \max_{y \in \mathcal{X}} |f(y)|$. Since

$$\omega f(x) = Pf(x) = \sum_{y \in \mathcal{X}} P(x,y) f(y)$$

taking absolute values shows that

$$r \leq \sum_{y \in X} P(x,y)|f(y)| \leq r$$

We conclude that if $P(x,y) > 0$, then $|f(y)| = r$. By irreducibility, $|f(y)| = r$ for all $y \in \mathcal{X}$.

Since the average of complex numbers of norm r has norm r if and only if all the values have the same angle, it follows that $f(y)$ has the same value for all y with $P(x,y) > 0$. Therefore, if $P(x,y) > 0$, then $f(y) = \omega f(x)$. Now fix $x_0 \in \mathcal{X}$ and define for $j = 0, \ldots, k-1$,

$$\mathcal{C}_j = \{z \in \mathcal{X} : f(z) = \omega^j f(x_0)\}$$

It is clear that if $P(x,y) > 0$ and $x \in \mathcal{C}_j$, then $x \in \mathcal{C}_{j+1 \mod k}$. It is clear that if $t \in \mathcal{T}(x_0)$, then k divides t and hence $\mathcal{T}(x_0) \subset k\mathbb{Z}$, as required.

Problem III. LPW 12.2

Let P be irreducible and let A be a matrix with $0 \leq A(i,j) \leq P(i,j)$ and $A \neq P$. Since $A \neq P$, we must have $A(i,j) < P(i,j)$ for some i,j. By 12.1(a), we have that $||Pf||_\infty \leq ||f||_\infty$ where $||f||_\infty = \max_{x\in\mathcal{X}} |f(x)|$. Let $f'(x_0)$ be the largest eigenfunction of P and let us define f and λ to be an eigenfunction and its corresponding eigenvalue of A, respectively. Define $||f||_\infty = |f(x_1)|$ so that $|f(y)| \leq |f(x_1)|$ for all $y \in \mathcal{X}$. Thus,

$$\begin{aligned}
||Af||_\infty = ||\lambda f||_\infty &= \max_{x\in\mathcal{X}} \left| \sum_{y\in\mathcal{X}} A(x,y) f(y) \right| \\
&\leq \max_{x\in\mathcal{X}} \left| \sum_{y\in\mathcal{X}} A(x,y) f(x_1) \right| \\
&= |f(x_1)| \max_{x\in\mathcal{X}} \sum_{y\in\mathcal{X}} A(x,y) \\
&< |f(x_1)| \max_{x\in\mathcal{X}} \sum_{y\in\mathcal{X}} P(x,y) \\
&= \frac{|f(x_1)|}{|f'(x_0)|} \max_{x\in\mathcal{X}} \sum_{y\in\mathcal{X}} P(x,y) \cdot |f'(x_0)| \\
&= \frac{|f(x_1)|}{|f'(x_0)|} \max_{x\in\mathcal{X}} \sum_{y\in\mathcal{X}} |P(x,y) \cdot f'(x_0)| \\
&= \frac{|f(x_1)|}{|f'(x_0)|} ||f'||_\infty \\
&= \frac{|f(x_1)|}{|f'(x_0)|} |f'(x_0)|
\end{aligned}$$

Since $||\lambda f||_\infty = \lambda |f(x_1)| < \frac{|f(x_1)|}{|f'(x_0)|} |f'(x_0)| = |f(x_1)|$, then $\lambda < 1$ as desired.