Stochastic Processes II: Homework 6

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April 14, 2021

Problem I. LPW 6.8

If A is the set of vertices in one of the complete graphs, observe that we have chosen n of the $2n-1$ vertices of the glued graph. Hence $\pi(A) = n/(2n-1) \geq 1/2$.

For $x \notin A$, we have,

$$P^t(x, A) = 1 - \left(1 - \frac{1}{2n}[1 + o(1)]\right)^t$$

We can use this to bound the total variation distance from below as follows,

$$\begin{aligned} ||P^t(x, \cdot) - \pi||_{TV} &\geq \pi(A) - P^t(x, A) \\ &\geq (1 - \frac{1}{2n}[1 + o(1)])^t - \frac{1}{2} \end{aligned}$$

We have that

$$\log(1 - \frac{1}{2n}[1 + o(1)])^t - \frac{1}{2} \geq \frac{1}{4}$$

Now if $t < [4\frac{1}{2n}[1 + o(1)](1 - (\frac{1}{2n}[1 + o(1)])/2]^{-1}$, then

$$(1 - \frac{1}{2n}[1 + o(1)])^t - 1/2 \geq 1/4$$

which gives us that $t_{mix}(1/4) \geq \frac{n}{2}[1 + o(1)]$

Problem II. LPW 6.9

Observe that $\tau_{k+1} = T_{\tau_k} + 1$ and so $\tau_k = T_{\tau_{k-1}} + 1$. Hence,

$$\begin{aligned} m_k &= E(\tau_k) \\ &= E(T_{\tau_{k-1}} + 1) \\ &= E(T_{\tau_{k-1}}) + 1 \end{aligned}$$

Similarly,

$$\begin{aligned}
m_{k+1} &= E(T_{\tau_k}) + 1 \\
&= E(T_{\tau_{k-1}} + t_k) + 1 \\
&= E(T_{\tau_{k-1}}) + E(t_k) + 1 \\
&= E(T_{\tau_k - 1}) + 1 + E(t_k) \\
&= m_k + E(t_k)
\end{aligned}$$

Problem III. LPW 7.2

We have that,

$$\begin{aligned}
Q(S, S^C) &= \sum_{x \in S, y \in S^C} Q(x, y) \\
&= \sum_{x \in S, y \in S^C} \pi(x) P(x, y)
\end{aligned}$$

and,

$$\begin{aligned}
Q(S^C, S) &= \sum_{x \in S^C, y \in S} Q(x, y) \\
&= \sum_{x \in S, y \in S^C} \pi(x) P(x, y)
\end{aligned}$$

Observe that, for $Q(S, S^C)$, we can split the summation into two sums which yields,

$$Q(S, S^C) = \sum_{y \in S^C} \sum_{x \in C} \pi(x) P(x, y)$$

Next, note that $\sum_{x \in C} \pi(x) P(x, y) = \sum_{x \in \mathcal{X}} \pi(x) P(x, y) - \sum_{x \in S^C} \pi(x) P(x, y)$. Making this substitution in the above gives us,

$$\begin{aligned}
Q(S, S^C) &= \sum_{y \in S^C} [\sum_{x \in \mathcal{X}} \pi(x) P(x, y) - \sum_{x \in S^C} \pi(x) P(x, y)] \\
&= \sum_{y \in S^C} \sum_{x \in \mathcal{X}} \pi(x) P(x, y) - \sum_{x \in S^C} \pi(x) \sum_{y \in S^C} P(x, y) \\
&= \sum_{y \in S^C} \pi(y) - \sum_{x \in S^C} \pi(x) \left[1 - \sum_{y \in S} P(x, y) \right] \\
&= \sum_{y \in S^C} \pi(y) - \sum_{x \in S^C} \pi(x) + \sum_{x \in S^C} \sum_{y \in S} P(x, y) \pi(x) P(x, y) \\
&= \sum_{x \in S^C} \sum_{y \in S} P(x, y) \pi(x) P(x, y)
\end{aligned}$$

But note that this is exactly the definition we gave for $Q(S^C, S)$. Hence, we must have that $Q(S, S^C) = Q(S^C, S)$.

Problem IV. LPW 7.4

(a) Note that,

$$|\mathcal{X}| = (2^n - 1)(2^n - 2^1)\cdots(2^n - 2^{n-1})$$

Hence, we have,

$$\begin{aligned}\lim_{n\to\infty} \frac{|\mathcal{X}|}{2^{n^2}} &= \lim_{n\to\infty} \frac{(2^n - 1)(2^n - 2^1)\cdots(2^n - 2^{n-1})}{2^{n^2}} \\ &= \frac{\lim_{n\to\infty}(2^n - 1)(2^n - 2^1)\cdots(2^n - 2^{n-1})}{\lim_{n\to\infty} 2^{n^2}}\end{aligned}$$

(b)