Stochastic Processes II: Homework 8

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Problem I. LPW 8.9

Let us assume we are working with a 3 card deck. We will consider the distribution when $T = 3$.

a) Since both cards are marked on every transposition and each selection is uniform and independent, we have that each card has a 2/3 chance of being marked. At $T = 3$, there are 2 cards already marked and 1 card that is unmarked.

b) Since the right-hand card is marked on every transposition and each right-hand card is chosen uniformly, we have that each card has a 1/3 chance of being marked. At $T = 3$, there are 2 cards already marked and 1 card that is unmarked. Hence, we have a 2/3 chance of marking a marked card again, and a 1/3 chance of marking the unmarked card.

Problem II. LPW 12.1

a) By the hint, we will let $||f||_\infty = \max_{x \in \mathcal{X}} |f(x)|$. We have that,

$$||Pf||_\infty = \max_{x \in \mathcal{X}} |P(x,y)f(y)|$$

Since $0 \leq P(x,y) \leq 1$ for all $x, y \in \mathcal{X}$, we have that $|P(x,y)f(y)| < |f(y)|$ for all x, y. Hence,

$$\begin{aligned} ||Pf||_\infty &= \max_{x \in \mathcal{X}} |P(x,y)f(y)| \\ &\leq \max_{x \in \mathcal{X}} |f(x)| \\ &= ||f||_\infty \end{aligned}$$

Now suppose f is an eigenfunction with corresponding eigenvalue λ. The,

$$\begin{aligned} ||Pf||_\infty &= ||\lambda f||_\infty \\ &= \max_{x \in \mathcal{X}} |\lambda f(x)| \\ &= |\lambda| \max_{x \in \mathcal{X}} |f(x)| \\ &= |\lambda| ||f||_\infty \end{aligned}$$

From the first part of our proof, we have that,

$$\begin{aligned}||Pf||_\infty &= |\lambda|||f||_\infty \\ &\leq ||f||_\infty\end{aligned}$$

This final inequality implies that $|\lambda| \leq 1$.

b) Assume that $\mathcal{T}(x) \subset 2\mathbb{Z}$. Then every time t such that $P^t(x,x) > 0$ is a multiple of 2.

c)

Problem III. LPW 12.2

Let P be irreducible and let A be a matrix with $0 \leq A(i,j) \leq P(i,j)$ and $A \neq P$. Since $A \neq P$, we must have $A(i,j) < P(i,j)$ for some i,j. By 12.1(a), we have that every eigenvalue λ of P satisfies $|\lambda| \leq 1$. Now suppose f is an eigenfunction of A with eigenvalue λ_1 of A and λ_2 of P. We have,

$$\begin{aligned}||Af||_\infty &= |\lambda_1|||f||_\infty \\ &< ||Pf||_\infty \\ &= |\lambda_2|||f||_\infty\end{aligned}$$

Dividing through by $||f||_\infty$ yields $|\lambda_1| < |\lambda_2| \leq 1$, and so $|\lambda_1| < 1$.