Stochastic Processes II: Homework 3

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Problem I.

Problem II. Durrett 5.9

Let $X_1, X_2, \ldots$ be independent with $P(X_i = 1) = p$ and $P(X_i = -1) = q = 1 - p$ where $p < 1/2$. Let $S_n = S_0 + \xi_1 + \cdots + \xi_n$ and let $V_0 = \min\{n \geq 0 : S_n = 0\}$. (5.13) implies $\mathbb{E}_x V_0 = x/(1-2p)$. If we let $Y_i = X - i - (p - q)$ and note that $EY_i = 0$ and

$$\text{var}(Y_i) = \text{var}(X_i) = EX_i u^2 - (EX_i)^2$$

then it follows that $(S_n - (p-q)n)^2 - n(1-(p-q)^2)$ is a martingale. Moreover, when $S_0 = x$, the variance of X_0 is

$$x \cdot \frac{1-(p-q)^2}{(q-p)^3}$$

(a)

(b)

(c)

Problem III. LPW 4.2

By Proposition 4.2, we have that,

$$\begin{aligned}
||\mu P - vP||_{TV} &= \frac{1}{2}\sum_{x\in\mathcal{X}} |\mu P(x) - vP(x)| \\
&= \frac{1}{2}\sum_{x\in\mathcal{X}} \left|\sum_{y\in\mathcal{X}} \mu(y)P(y,x) - v(y)P(y,x)\right| \\
&= \frac{1}{2}\sum_{x\in\mathcal{X}} \left|\sum_{y\in\mathcal{X}} P(y,x)[\mu(y) - v(y)]\right| \\
&\leq \frac{1}{2}\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{X}} P(y,x)|\mu(y) - v(y)| \\
&= \frac{1}{2}\sum_{y\in\mathcal{X}} |\mu(y) - v(y)| \sum_{x\in\mathcal{X}} P(y,x) \\
&= \frac{1}{2}\sum_{y\in\mathcal{X}} |\mu(y) - v(y)| \cdot 1 \\
&= \frac{1}{2}\sum_{y\in\mathcal{X}} |\mu(y) - v(y)| \\
&= ||\mu - v||_{TV}
\end{aligned}$$

Hence, we have that $||\mu P - vP||_{TV} \leq ||\mu - v||_{TV}$ as required. In particular, we have that $||\mu P^{t+1} - \pi||_{TV} \leq ||\mu P^t - \pi||_{TV}$.

Now fix $t > 0$. By the above, we must have that,

$$\begin{aligned}
&\max_{x\in\mathcal{X}} ||P^{t+1}(x,\cdot) - \pi||_{TV} \leq \max_{x\in\mathcal{X}} ||P^t(x,\cdot) - \pi||_{TV} \\
\iff & d(t+1) \leq d(t)
\end{aligned}$$

and,

$$\begin{aligned}
&\max_{x\in\mathcal{X}} ||P^{t+1}(x,\cdot) - P^{t+1}(y,\cdot)||_{TV} \leq \max_{x\in\mathcal{X}} ||P^t(x,\cdot) - P^t(y,\cdot)||_{TV} \\
\iff & \bar{d}(t+1) \leq \bar{d}(t)
\end{aligned}$$

Problem III. LPW 4.4