Stochastic Processes II: Homework 2

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Problem I. Durrett 5.1

Let (X_n) be the state of the Markov chain at time n. Define M_n to be the sum of the digits of the state for X_n. Note that for every n, we have that $0 \leq M_n \leq 4$ since the only possible states are: $22, 21, 20, 11, 10, 00$. Hence, for a fixed n, we have that,

$$|M_n| \leq 4$$

Hence, $E|M_n| < \infty$ for any fixed $n \geq 0$. In addition, from the definition of M_n, we have that M_n can be determined from the values for $X_n, \ldots, X_0$ and M_0 since it is simply a function of X_n. Lastly, we need to show that

$$E(M_{n+1} - M_n \mid A_v) = 0$$

for all $n \geq 0$ where $A_v = \{X_n = x_n, \ldots, X_0 = x_0, M_0 = m_0\}$.

Note that

$$M_{n+1} - M_n = \text{sum of digits of } X_{n+1} - \text{ sum of digits of } X_n$$

Since the sum of the digits of X_n is a constant when conditioning on A_v, let us write it as c:

$$\begin{aligned} E(M_{n+1} - M_n \mid A_v) &= E(M_{n+1} \mid A_v) - E(M_n \mid A_v) \\ &= E(M_{n+1} \mid A_v) - c \end{aligned}$$

Now note that since A_v is given, we have the information on both digits of X_n. We know that if x is the number of As in parent 1 in X_n and y is the number of As in parent 2 in X_n, then the number of As in X_n is $c = x + y$. We have that probability of choosing an A from parent 1 is $2 \cdot \left[\binom{x}{1} / \binom{2}{1}\right]$ since we are choosing one A from the total number of As in parent 1 (given by x). We perform this choice twice, once for each child, which is where the coefficient of 2 comes from. Finally, the denominator gives us the total number of choices we can make from parent 1. Similarly, the probability of choosing an A from parent 2 is $2 \cdot \left[\binom{y}{1} / \binom{2}{1}\right]$.

Hence, the expected number of As in X_{n+1}, given by M_{n+1}, is,

$$\begin{aligned}
E(M_{n+1} \mid A_v) &= 2 \cdot \left[\binom{x}{1} / \binom{2}{1}\right] + 2 \cdot \left[\binom{y}{1} / \binom{2}{1}\right] \\
&= 2 \frac{\binom{x}{1} + \binom{y}{1}}{\binom{2}{1}} \\
&= 2 \frac{\binom{x}{1} + \binom{y}{1}}{2} \\
&= \binom{x}{1} + \binom{y}{1} \\
&= x + y \\
&= c
\end{aligned}$$

So we have,

$$\begin{aligned}
E(M_{n+1} - M_n \mid A_v) &= E(M_{n+1} \mid A_v) - E(M_n \mid A_v) \\
&= E(M_{n+1} \mid A_v) - c \\
&= c - c \\
&= 0
\end{aligned}$$

Thus, M_n is a martingale with respect to the sum of the digits of the state of X_n.

Problem II. Durrett 5.4

(a) For a fixed n, we have,

$$\begin{aligned}
2^n Y_n &= 2^n U_n \cdots U_0 \\
&< 2^n 1 \cdots 1 \\
&= 2^n < \infty
\end{aligned}$$

Hence, we have that

$$E|M_n| < \infty$$

Now note that we have,

$$\begin{aligned}
M_{n+1} &= 2^{n+1} Y_{n+1} \\
&= 2
\end{aligned}$$

Observe that,

$$\begin{aligned}
M_{n+1} - M_n &= 2^{n+1} Y_{n+1} - 2^n Y_n \\
&= 2^n (2U_{n+1} \cdots U_0 - U_n \cdots U_0) \\
&= 2^n U_n \cdots U_0 (2U_{n+1} - 1) \\
&= 2^n Y_n (2U_{n+1} - 1) \\
&= M_n (2U_{n+1} - 1)
\end{aligned}$$

So we have by the fact that $E(U_n) = 0.5$ for every n,

$$\begin{aligned} E(M_{n+1} - M_n \mid A_v) &= M_n E(2U_{n+1} - 1 \mid A_v) \\ &= M_n \left[2E(U_{n+1} \mid A_v) - 1\right] \\ &= M_n \left[2(0.5) - 1\right] \\ &= M_n \cdot 0 \\ &= 0 \end{aligned}$$

(b) We have,

$$(1/n) \log Y_n = 1/n(\log U_1 + \cdots + \log U_n)$$

Since $U_n \sim \text{Uniform}(0, 1)$, we have that $-\log U_n \sum \text{Exponential}(\lambda = 1)$ for every n. Hence, $E(\log U_n) = -1$ for every n. Thus, by the Strong Law of Large Numbers, we have that,

$$(1/n) \log Y_n = 1/n(\log U_1 + \cdots + \log U_n) \to -1$$

as $n \to \infty$.

(c) Recall that $M_n = 2^n Y_n$. We have that $Y_n = e^{\log Y_n}$

Problem III. Durrett 5.6

Let $S_n = X_1 + \cdots + X_n$ where the X_i are independent and $EX_i = 0$ and $\text{var}(X_i) = \sigma^2$. By Example 5.3, $S_n^2 - n\sigma^2$ is a martingale. Let $T = \{n : |S_n| > a\}$. Then by Theorem 5.10, we have that

$$EM_{T \wedge n} = EM_0$$