Stochastic Processes II: Homework 9

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Problem I. LPW 12.3

Let $P_L = (P+I)/2$ be the transition matrix of the lazy version of the chain with transition matrix P. Suppose P_L has eigenvalue $\lambda < 0$ with corresponding eigenfunction f. Then,

$$\begin{aligned} P_L f &= (P/2 + I/2)f \\ &= (Pf + f)/2 \\ &= \lambda f \end{aligned}$$

From the above equalities, we have,

$$\begin{aligned} & Pf/2 + f/2 = \lambda f \\ \iff & Pf/2 = \lambda f - f/2 \\ \iff & Pf = 2\lambda f - f \\ \iff & Pf = (2\lambda - 1)f \end{aligned}$$

Hence, $2\lambda - 1$ is an eigenvalue of P and so, by Lemma 12.1(i), we have that,

$$\begin{aligned} & 2\lambda - 1 \geq -1 \\ \iff & \lambda \geq 0 \end{aligned}$$

We assumed that $\lambda < 0$, and hence we have a contradiction. Thus, there are no eigenvalues λ of P_L such that $\lambda < 0$, as required.

Problem II. LPW 12.4

We have that,

$$\begin{aligned}
E_\pi(P^t f) &= \pi P^t f \\
&= \pi f \\
&= E_\pi(f)
\end{aligned}$$

We have that the first eigenfunction $f_1 \equiv 1$, and so we have,

$$\begin{aligned}
P^t f - E_\pi(P^t f) &= \sum_{j=1}^{|\mathcal{X}|} \langle f, f_j \rangle_\pi f_j \lambda_j^t \\
&= \sum_{j=2}^{|\mathcal{X}|} \langle f, f_j \rangle_\pi f_j \lambda_j^t
\end{aligned}$$

Because the f_j's are an orthonormal basis, this gives us,

$$\begin{aligned}
\mathrm{Var}_\pi(f) &= ||P^t f - E_\pi(P^t f)||^2_{\ell^2(\pi)} \\
&= \sum_{j=2}^{|\mathcal{X}|} \langle f, f_j \rangle_\pi^2 f_j \lambda_j^{2t}
\end{aligned}$$

Now note that for all λ_j, we have that $1 - \gamma_* \geq \lambda_j$. Thus, we have,

$$\begin{aligned}
\mathrm{Var}_\pi(f) &= \sum_{j=2}^{|\mathcal{X}|} \langle f, f_j \rangle_\pi^2 f_j \lambda_j^{2t} \\
&\leq \sum_{j=2}^{|\mathcal{X}|} \langle f, f_j \rangle_\pi^2 f_j (1 - \gamma_*)^{2t} \\
&= (1 - \gamma_*)^{2t} \sum_{j=2}^{|\mathcal{X}|} \langle f, f_j \rangle_\pi^2 f_j
\end{aligned}$$

Moreover, we have that,

$$\begin{aligned}
\sum_{j=2}^{|\mathcal{X}|} \langle f, f_j \rangle_\pi^2 &= E_\pi(f^2) - E_\pi^2(f) \\
&= \mathrm{Var}_\pi(f)
\end{aligned}$$

And so,

$$\mathrm{Var}_\pi(f) \leq (1 - \gamma_*)^{2t} \mathrm{Var}_\pi(f)$$

as required.

Problem III. LPW 18.1

Suppose that the n-th chain in a sequence of Markov chains satisfies,

$$\lim_{n\to\infty} d_n(ct^n_{\text{mix}}) - \begin{cases} 1 & \text{if } c < 1, \\ 0 & \text{if } c > 1 \end{cases}$$

Then for any $\gamma > 0$ and for n large enough, we have,

$$\begin{aligned} t_{\text{mix}}(\epsilon) &\leq (1+\gamma)t^n_{\text{mix}} \\ t_{\text{mix}}(1-\epsilon) &\geq (1-\gamma)t^n_{\text{mix}} \end{aligned}$$

These equations together yield,

$$\frac{t_{\text{mix}}(\epsilon)}{t_{\text{mix}}(1-\epsilon)} \leq \frac{1+\gamma}{1-\gamma}$$

If we let γ approach 0, then n must approach ∞ and we get,

$$\lim_{n\to\infty} \frac{t_{\text{mix}}(\epsilon)}{t_{\text{mix}}(1-\epsilon)} = 1$$

as required.

Now suppose that we have,

$$\lim_{n\to\infty} \frac{t_{\text{mix}}(\epsilon)}{t_{\text{mix}}(1-\epsilon)} = 1$$

Fix $\gamma > 0$. Then for any $\epsilon > 0$ and for n large enough, we have $t_{\text{mix}}(\epsilon) \leq (1+\gamma)t^n_{\text{mix}}$. That is, $\lim_{n\to\infty} d_n((1+\gamma)t^n_{\text{mix}}) \leq \epsilon$. Since this holds for all ϵ,

$$\lim_{n\to\infty} d_n((1+\gamma)t^n_{\text{mix}}) = 0$$

Moreover, $\lim_{n\to\infty} d_n((1-\gamma)t^n_{\text{mix}}) \geq 1-\epsilon$ since $t_{\text{mix}}(1-\epsilon) \geq (1-\gamma)t^n_{\text{mix}}$ for n sufficiently large. Thus, we have that,

$$\lim_{n\to\infty} d_n((1-\gamma)t^n_{\text{mix}}) = 1$$