Stochastic Processes II: Homework 7

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Problem I. LPW 8.1

Fix $\eta \in S_n$. We want to show that $\sigma_{n-1}(j)$ is uniformly distributed on $\mathcal{S}_n$. Note that $\mathcal{S}_n$ has size $n!$, so we are looking to show that the probability that $\sigma_n = \eta$ with $\eta \in \mathcal{S}_n$ is $1/n!$ for any $\eta \in \mathcal{S}_n$.

Let us start with the base case, σ_1. We have that the size of $\mathcal{S}_n$ is $1/n!$. Observe that, to construct σ_1, we select an integer uniformly from $\{1, 2, \ldots, n\}$. That is, we have $1/n$ probability of picking any integer. If 1 is selected to be J_1, then we have that $\sigma_1 = \sigma_0 \circ (11) = \text{id}$. For any other number selected, we have that $\sigma_1 = \sigma_0 \circ (1J_1) = (1J_1)$. Note that there are n such permutations (if we include the identity) in $\mathcal{S}_n$. Hence, the probability that $\sigma_1 = \eta$ is $1/n = \prod_{i=0}^{k-1} 1/(n-i)$.

Now suppose we know the probability that σ_{k-1} is uniform on $\mathcal{S}_n$ with probability $\prod_{i=0}^{k-2} 1/(n-i)$. Consider σ_k. Let us select J_k from $\{k, \ldots, n\}$. We have $n-k+1$ choices, each with probability $(1/(n-k+1)$. Note that if $J_k = k$, then $\sigma_k = \sigma_{k-1} \circ (kk) = \sigma_{k-1}$. So let us consider $J_k \in \{k+1, \ldots, n\}$. For any choice J_k in this set, we have that $\sigma_k = \sigma_{k-1} \circ (kJ_k)$. Any two cycle of the form (kJ_k) cannot already be in σ_{k-1} because, by definition, k had not yet been reached, so σ_k is distinct from σ_{k-1}. Hence, there are $n-k-1$ possibilities for σ_k given σ_{k-1}. So we have the probability that $\sigma_k = \eta$ is,

$$\prod_{i=0}^{k-2} 1/(n-i) \cdot 1/(n-k+1) = \prod_{i=0}^{k-1} 1/(n-i)$$

Hence, by the induction above, we have that the probability that $\sigma_n = \eta$ for $\eta \in \mathcal{S}_n$ is

$$\prod_{i=0}^{n-1} 1/(n-i) = 1/n!$$

Since $\mathcal{S}_n$ has size $n!$, we have that σ_n is uniform on $\mathcal{S}_n$.

Problem II. LPW 8.4

a) Let us consider permutations of the blocks $\{1, \ldots, 15\}$. Note that we want to swap 14 and 15. Any such permutation will be odd because $\sigma(15) - \sigma(14) = 14 - 15 = -1$. However, for the empty tile to end up back in the bottom right corner, the permutation must be even. Thus, there is no way to swap the tiles 14 and 15 while leaving all of the other tiles fixed.

b)

Problem III. LPW 8.6