Chapter 8 - Vector and Cartesian Spaces

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1 Exercises

Problem 8.A. If V is a vector space and if $x + z = x$ for some x and z in V, show that $z = 0$. Hence, the zero element in V is unique.

Suppose $x + z = x$. Then by the properties of addition on a vector space, we have,

$$\begin{aligned} & x + z = x \\ \iff & x + z + (-x) = x + (-x) \\ \iff & x + (-x) + z = 0 \\ \iff & 0 + z = 0 \\ \iff & z = 0 \end{aligned}$$

All of these statements follow directly from the definition of a vector space (Definition 8.1 in the text).

Problem 8.B. If $x + y = 0$ for some x and y in V, show that $y = -x$.

Again using the properties of addition on a vector space, we have,

$$\begin{aligned} & x + y = 0 \\ \iff & x + y + (-x) = 0 + (-x) \\ \iff & x + (-x) + y = -x \\ \iff & 0 + y = -x \\ \iff & y = -x \end{aligned}$$

Problem 8.C. Let $S = \{1, 2, \cdots, p\}$ for some $p \in \mathbb{N}$. Show that the vector space $\mathbb{R}^S$ is "essentially the same" as the space $\mathbb{R}^p$.

We know from Example 8.2d in the text that $\mathbb{R}^S$ denotes the collection of all functions u with domain S and range in $\mathbb{R}$.

We assert that there is a bijection between $\mathbb{R}^p$ and $\mathbb{R}^S$ defined using the functions in $\mathbb{R}^S$. Namely, $f : \mathbb{R}^S \to \mathbb{R}^p$ with $f(u) = (u(1), u(2), \cdots, u(k))$.

First we will show that f is injective. So suppose $u, v \in \mathbb{R}^S$ and $x \in \mathbb{R}^p$ with $f(u) = x$ and $f(v) = x$. That is, $u(k) = x_k$ and $v(k) = x_k$ for every k. Then clearly $u(k) = v(k)$ for every k and hence the functions are equal.

Now to show that f is surjective. Let $x \in \mathbb{R}^p$. Then $x = (x_1, x_2, \cdots, x_p)$.

We know that $\mathbb{R}^S$ denotes the collection of all functions u with domain S and range in $\mathbb{R}$, and we know that $x_k \in \mathbb{R}$ for every k. Hence, there clearly exists a function $u \in \mathbb{R}^S$ such that $u(k) = x_k$ for every k. Thus, f is surjective.

Now we need to show that f preserves addition and scalar multiplication. That is, $f(u+v) = f(u) + f(v)$ and $f(cu) = cf(u)$.

Let $u, v \in \mathbb{R}^S$ with $f(u) = x$ and $f(v) = y$. Then,

$$\begin{aligned} f(u+v) &= (u(1)+v(1), u(2)+v(2), \cdots, u(p)+v(p)) \\ &= (x_1+y_1, \cdots, x_p+y_p) \\ &= (x_1, \cdots, x_p) + (y_1, \cdots, y_p) \\ &= x + y \\ &= f(u) + f(v) \end{aligned}$$

So f preserves addition. Now to check scalar multiplication:

$$\begin{aligned} f(cu) &= (cu(1), \cdots, cu(p)) \\ &= c(u(1), \cdots, u(p)) \\ &= cf(u) \end{aligned}$$

So f preserves scalar multiplication.

Since f is a bijection between the elements of $\mathbb{R}^S$ and $\mathbb{R}^p$ which preserves addition and scalar multiplication, these two vector spaces are isomorphic (ie. “essentially the same”).

Problem 8.D. If w_1 and w_2 are strictly positive, show that the definition $(x_1, x_2) \cdot (y_1, y_2) = x_1 y_1 w_1 + x_2 y_2 w_2$ yields an inner product on $\mathbb{R}^2$. Generalize this to $\mathbb{R}^p$.

We need to show that this potential norm satisfies the five conditions of Definition 8.3 in the text:

1.

$$\begin{aligned} x \cdot x = (x_1, x_2) \cdot (x_1, x_2) &= x_1 x_1 w_1 + x_2 x_2 w_2 \\ &= x_1^2 w_1 + x_2^2 w_2 \\ &\geq 0 \end{aligned}$$

2.

$$\begin{aligned}
&(x_1, x_2) \cdot (x_1, x_2) = 0 \\
\iff &x_1^2 w_1 + x_2^2 w_2 = 0 \\
\iff &x_1^2 w_1 = -x_2^2 w_2
\end{aligned}$$

We know that $x_1^2 w_1 \geq 0$ and $x_2^2 w_2 \geq 0$. Hence, $x_1^2 w_1 = -x_2^2 w_2$ iff $x_1^2 w_1 = 0 = x_2^2 w_2$.

3.

$$\begin{aligned}
x \cdot y = (x_1, x_2) \cdot (y_1, y_2) &= x_1 y_1 w_1 + x_2 y_2 w_2 \\
&= y_1 x_1 w_1 + y_2 x_2 w_2 \\
&= (y_1, y_2) \cdot (x_1, x_2) \\
&= y \cdot x
\end{aligned}$$

4.

$$\begin{aligned}
x \cdot (y + z) = (x_1, x_2) \cdot (y_1 + z_1, y_2 + z_2) &= x_1(y_1 + z_1)w_1 + x_2(y_2 + z_2)w_2 \\
&= (x_1 y_1 + x_1 z_1)w_1 + (x_2 y_2 + x_2 z_2)w_2 \\
&= x_1 y_1 w_1 + x_1 z_1 w_1 + x_2 y_2 w_2 + x_2 z_2 w_2 \\
&= (x_1 y_1 w_1 + x_2 y_2 w_2) + (x_1 z_1 w_1 + x_2 z_2 w_2) \\
&= (x_1, x_2) \cdot (y_1, y_2) + (x_1, x_2) \cdot (z_1, z_2) \\
&= x \cdot y + x \cdot z
\end{aligned}$$

and,

$$\begin{aligned}
(x + y) \cdot z = (x_1 + y_1, x_2 + y_2) \cdot (z_1, z_2) &= (x_1 + y_1)z_1 w_1 + (x_2 + y_2)z_2 w_2 \\
&= (x_1 z_1 + y_1 z_1)w_1 + (x_2 z_2 + y_2 z_2)w_2 \\
&= x_1 z_1 w_1 + y_1 z_1 w_1 + x_2 z_2 w_2 + y_2 z_2 w_2 \\
&= (x_1 z_1 w_1 + x_2 z_2 w_2) + (y_1 z_1 w_1 + y_2 z_2 w_2) \\
&= (x_1, x_2) \cdot (z_1, z_2) + (y_1, y_2) \cdot (z_1, z_2) \\
&= x \cdot z + y \cdot z
\end{aligned}$$

5.

$$\begin{aligned}
(ax) \cdot y &= (ax_1, ax_2) \cdot (y_1, y_2) \\
&= ax_1 y_1 w_1 + ax_2 y_2 w_2 \\
&= a(x_1 y_1 w_1 + x_2 y_2 w_2) = a(x \cdot y) \\
&= x_1(ay_1)w_1 + x_2(ay_2)w_2 \\
&= x \cdot (ay)
\end{aligned}$$

Hence, the proposed norm satisfies all of the necessary properties and thus defines a valid norm on $\mathbb{R}^2$.

In order to generalize this norm to $\mathbb{R}^p$, we need to fix $w \in \mathbb{R}^p$ such that each component of w is strictly positive. Then the corresponding norm is given by,

$$x \cdot y = \sum_{i=1}^{p} x_i y_i w_i$$

Problem 8.E. The definition $(x_1, x_2) \cdot (y_1, y_2) = x_1 y_1$ is *not* an inner product on $\mathbb{R}^2$. Why?

Let $x = (0, 1)$. Then $x \cdot x = 0(0) = 0$, but $x \neq (0, 0)$. Hence, property 2 of Definition 8.3 is violated and this does not define a valid norm.

Problem 8.F. If $x = (x_1, x_2, \cdots, x_p) \in \mathbb{R}^p$, define $||x||_1$ by $||x||_1 = |x_1| + |x_2| + \cdots + |x_p|$. Prove that $x \to ||x||_1$ is a norm on $\mathbb{R}^p$.

We need to show that this proposed norm satisfies all four properties of Definition 8.5 in the text.

1. We have that $||x||_1 = |x_1| + |x_2| + \cdots + |x_p|$ for every $x \in \mathbb{R}^p$. Note that, for each $x_k \in \mathbb{R}$, $|x_k| \geq 0$. Hence, the sum of these terms must also be greater than or equal to 0.

   As a result, we have that $||x|| \geq 0$ for every $x \in \mathbb{R}^p$.

2. Suppose $||x||_1 = 0$. Then $|x_1| + |x_2| + \cdots + |x_p| = 0$. Since each $|x_k| \geq 0$ for every k, the sum of the terms is 0 iff each term is 0. Hence, x = 0.

   Now suppose $x = 0$. Then $||x||_1 = |x_1| + |x_2| + \cdots + |x_p| = |0| + \cdots |0| = 0$.

3. Let $a \in \mathbb{R}$ and $x \in \mathbb{R}^p$. Then,

$$\begin{aligned} ||ax||_1 &= |ax_1| + |ax_2| + \cdots |ax_p| \\ &= |a| \cdot |x_1| + |a| \cdot |x_2| + \cdots |a| \cdot |x_p| \\ &= |a| \cdot (|x_1| + \cdots |x_p|) \\ &= |a| \cdot ||x||_1 \end{aligned}$$

4. Let $x, y \in \mathbb{R}^p$. We have that,

$$||x + y|| = |x_1 + y_1| + |x_2 + y_2| + \cdots |x_p + y_p|$$

   By Theorem 5.12 in the text, we know that $|x_k + y_k| \leq |x_k| + |y_k|$ since $x_k, y_k \in \mathbb{R}$ for every k. Hence, we have,

$$\begin{aligned} ||x + y|| &\leq |x_1| + |y_1| + |x_2| + |y_2| + \cdots |x_p| + |y_p| \\ &= |x_1| + \cdots |x_p| + |y_1| + \cdots + |y_p| \\ &= ||x|| + ||y|| \end{aligned}$$

Thus, all of the properties hold and this defines a valid norm on $\mathbb{R}^p$.

Problem 8.G. If $x = (x_1, x_2, \cdots, x_p) \in \mathbb{R}^p$, define $||x||_\infty$ by $||x||_\infty = \sup\{|x_1|, |x_2|, \cdots, |x_p|\}$. Prove that $x \to ||x||_\infty$ is a norm on $\mathbb{R}^p$.

Once again, we need to show that this proposed norm satisfies all four properties of Definition 8.5 in the text.

1. Note that the supremum of a finite set is just the maximum element of that set. As we noted above $|x_k| \geq 0$ for every k and every $x \in \mathbb{R}^p$.

   Hence, let $k \in \{1, \cdots, p\}$ such that $x_k = \sup\{|x_1|, |x_2|, \cdots, |x_p|\}$. Then we have that $||x||_\infty = \sup\{|x_1|, |x_2|, \cdots, |x_p|\} = |x_k| \geq 0$ as required.

2. Suppose $||x||_\infty = 0$ and let $k \in \{1, \cdots, p\}$ such that $x_k = \sup\{|x_1|, |x_2|, \cdots, |x_p|\}$. Then,

$$\begin{aligned} &||x||_\infty = 0 \\ \implies &||x||_\infty = \sup\{|x_1|, |x_2|, \cdots, |x_p|\} = |x_k| = 0 \end{aligned}$$

   Since $|x_k|$ is the supremum of the above set, we have that $|x_j| \leq |x_k|$ for every $j \in \{1, \cdots p\}$. Again note that $|y| \geq 0 \; \forall y \in \mathbb{R}$.

   Thus, we have that $0 \leq |x_j| \leq |x_k| = 0$ for every $j \in \{1, \cdots p\}$.

   As a result, we have that $|x_j| = 0$ for every $j \in \{1, \cdots p\}$ and, hence, $x = 0$.

   Now suppose $x = 0$. Then,

$$\begin{aligned} ||x||_\infty &= \sup\{|x_1|, |x_2|, \cdots, |x_p|\} \\ &= \sup\{|0|, \cdots, |0|\} \\ &= |0| = 0 \end{aligned}$$

3. Let $a \in \mathbb{R}$ and $x \in \mathbb{R}^p$. Also let $k \in \{1, \cdots, p\}$ such that $\sup\{|ax_1|, |ax_2|, \cdots, |ax_p|\} = |ax_k|$. Then,

$$\begin{aligned} ||ax||_\infty &= \sup\{|ax_1|, |ax_2|, \cdots, |ax_p|\} \\ &= |ax_k| \\ &= |a| \cdot |x_k| \end{aligned}$$

   Now we need to show that $|x_k| = ||x||_\infty$.

   Suppose for contradiction that $\exists j \in \{1, \cdots, p\}$ such that $|x_j| > |x_k|$.

Suppose $a \neq 0$. Since $|a| > 0$ for every $a \neq 0$, multiplying both sides of this inequality by $|a|$ preserves it. This yields,

$$\begin{aligned} &|x_j||a| > |x_k||a| \\ \iff &|ax_j| > |ax_k| \end{aligned}$$

which is a contradiction since $|ax_k| = \sup\{|ax_1|, |ax_2|, \cdots, |ax_p|\}$. Hence, if $a \neq 0$, then $||x||_\infty = |x_k|$ and $||ax||_\infty = |a| \cdot ||x||_\infty$.

Moreover, if $a = 0$, then each $|ax_i| = 0$ as shown in part 2, so $||ax||_\infty = |ax_k| = 0$. In addition, regardless of the value of $||x||_\infty$, we have that $|a| \cdot ||x||_\infty = |0| \cdot ||x||_\infty = 0 = ||ax||_\infty$. So when $a = 0$, we have that $||ax||_\infty = 0 = |a| \cdot ||x||_\infty$

Hence, we have that for any $a \in \mathbb{R}$, $x \in \mathbb{R}^p$, $||ax||_\infty = |a| \cdot ||x||_\infty$

4. Let $x, y \in \mathbb{R}^p$. In addition, let $k \in \{1, \cdots, p\}$ be such that $|x_k + y_k| = \sup\{|x_1 + y_1|, |x_2 + y_2|, \cdots, |x_p + y_p|\}$. Then,

$$\begin{aligned} ||x + y||_\infty &= \sup\{|x_1 + y_1|, |x_2 + y_2|, \cdots, |x_p + y_p|\} \\ &= |x_k + y_k| \end{aligned}$$

Since $x_k, y_k \in \mathbb{R}$, we have by the triangle inequality that $|x_k + y_k| \leq |x_k| + |y_k|$.

Note that $||x||_\infty + ||y||_\infty$ is defined as,

$$\begin{aligned} ||x||_\infty + ||y||_\infty &= \sup\{|x_1|, |x_2|, \cdots, |x_p|\} + \sup\{|y_1|, |y_2|, \cdots, |y_p|\} \\ &= |x_j| + |y_i| \end{aligned}$$

for some $i, j \in \{1, \cdots, p\}$. Note, by the definition of supremum, we have that $|x_k| \leq |x_j|$ and $|y_k| \leq |y_i|$. Hence, it follows that,

$$|x_k + y_k| \leq |x_k| + |y_k| \leq |x_j| + |y_i|$$

which gives us,

$$||x + y||_\infty \leq ||x||_\infty + ||y||_\infty$$

This proposed norm satisfies all of the above properties and is thus a valid norm on $\mathbb{R}^p$.

Problem 8.H. In the set $\mathbb{R}^2$, describe the sets $S_1 = \{x \in \mathbb{R}^2 : ||x||_1 < 1\}$ and $S_\infty = \{x \in \mathbb{R}^2 : ||x||_\infty < 1\}$.

2 Project