1 Definition of Representable Functor

An object X in a category C is a representation of a functor $F : C^{op} \to Set$ if there is a natural isomorphism

$$\mathrm{Hom}(-, X) \cong F \tag{1}$$

Precisely what this means is that for any object Y in C there is an isomorphism ϕ_Y between the set $\mathrm{Hom}(X, Y)$ and the set $F(Y)$ and that this isomorphism satisfies the following equations.

For any Y and Z in C, and any $f \in \mathrm{Hom}(Z, Y)$ and g in $\mathrm{Hom}(Y, X)$, we have

$$\phi_Z(f; g) = F(f)(\phi_Y(g)) \tag{2}$$

This can also be expressed as the commutativity of the following square

$$\begin{tikzcd}
\mathrm{Hom}(Y, X) \arrow[r, "\phi_Y"] \arrow[d, "f;"'] & F(Y) \arrow[d, "F(f)"] \\
\mathrm{Hom}(Z, X) \arrow[r, "\phi_Z"'] & F(Z)
\end{tikzcd}$$

This above equation is equivalent to assuming naturality of ϕ^{-1} which is For any Y and Z in C, and any $f \in \mathrm{Hom}(Z, Y)$ and $y \in F(Y)$, we have

$$\phi_Z^{-1}(F(f)(y)) = f; \phi_Y^{-1}(y) \tag{3}$$

This can be expressed as the commutativity of the following square

$$\begin{tikzcd}
F(Y) \arrow[r, "\phi_Y^{-1}"] \arrow[d, "F(f)"'] & \mathrm{Hom}(Y, X) \arrow[d, "f;"] \\
F(X) \arrow[r, "\phi_Z^{-1}"'] & \mathrm{Hom}(Z, X)
\end{tikzcd}$$

One useful characterisation of a representation of a functor is the following. X is a representation of whenever there is the following

- An element c of $F(X)$
- For every object Y of $\mathcal{X}$, a map $r_Y : F(X) \to \mathcal{X}(Y, X)$
- For any object Y of $\mathcal{X}$, and every element $y \in F(Y)$, $F(r_Y(y))(c) = f$
- For any two morphisms $f, g : \mathcal{X}(Y, X)$, if $F(f)(c) = F(g)(c)$ then $f = g$

The above functions are enough to define a natural isomorphism between $\mathrm{Hom}(-, X)$ and F. With $f \in \mathrm{Hom}(Y, X)$, we can define $\phi_Y(f)$ to be $F(f)(c)$ and for any $y \in F(Y)$, we can define $\phi_Y^{-1}(y)$ to be $r_Y(y)$.

A corepresentation of a functor $F : C \to Set$ is defined in a similar way. An object X is a corepresentation of F if there is a natural isomorphism

$$\mathrm{Hom}(X, -) \cong F \tag{4}$$

This can be characterised by the following data about F

- An element u of $F(X)$
- For every object Y of $\mathcal{C}$, a map $e_Y : F(Y) \to \mathcal{C}(X, Y)$
- For any object Y of $\mathcal{C}$, and every element $f : F(Y)$, $F(e_Y(f))(u) = f$
- For any two morphisms $f, g : \mathcal{C}(X, Y)$, if $F(f)(u) = F(g)(u)$ then $f = g$

There are many examples of representations or corepresentations of functors in mathematics.

If C is a category, and X and Y are objects of C, then the product, $X \times Y$ is a representation of the functor $Hom(-, X) \times Hom(-, Y)$, meaning for any object Z of C, the set $\mathrm{Hom}(Z, X \times Y)$ is isomorphic to the set $\mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y)$.

In the category of sets, the set with one element, 1, is the terminal object. This means it is a representation of the constant functor $1 : Set \to Set$, i.e. the set of morphisms from any set X into 1 has exactly one element.

In the category of sets, the empty set, $\emptyset$, is the initial object. This means it is a corepresentation of the constant functor $1 : Set \to Set$, i.e. the set of morphisms from $\emptyset$ into any set X has exactly one element.

In the category of commutative rings, the polynomial ring $R[X]$ is a representation of the functor $\mathrm{Hom}(R, -) \times \mathrm{Forget}$, where Forget is the forgetful functor from commutative rings to sets. This means that for any commutative ring S a ring homomorphism $R[X] \to S$ can be constructed by taking a ring homomorphism $R \to S$ and an element of S.

In the category

2 Examples of Equality Proofs Using UMP

Using universal properties provides a convenient way of defining and proving equalities of morphisms in categories.

3 Examples of Isomorphisms made easier with UMP

4 Inheritance of UMP

4.1 Grothendieck Construction, Categories of Elements and Pi Categories

4.2 Monoidal Closed Categories

4.3 Maybe Something about Adjoints Preserving (co)limits

4.4 Something About Parametricity

5 Comparison with UMP of Inductive Types

6 The Isomorphism Problem - Transferring across Isomorphisms