1 Representable Functor

Let $\mathcal{C}$ be a category and let $F : C \to \text{Type}$ be a co-presheaf on $\mathcal{C}$. Then we will define a corepresentation of F to be the following.

1. An object X of $\mathcal{C}$
2. An element x of $F(X)$
3. For every object Y of $\mathcal{C}$, a map $e_Y : F(Y) \to \mathcal{C}(X, Y)$
4. For any object Y of $\mathcal{C}$, and every element $f : F(Y)$, $F(e_Y(f))(x) = f$
5. For any object Y of $\mathcal{C}$, and every morphism $f : \mathcal{C}(X, Y)$, $e_Y(F(f)(x)) = f$

A representation of a functor $F : C^{op} \to \text{Type}$ is the dual concept to a corepresentable functor. We define it explicitly here.

1. An object X of $\mathcal{C}$
2. An element x of $F(X^{op})$
3. For every object Y of $\mathcal{C}$, a map $r_Y : F(Y^{op}) \to \mathcal{C}(Y, X)$
4. For any object Y of $\mathcal{C}$, and every element $f : F(Y^{op})$, $F(r_Y^{op}(f))(x) = f$.
5. For any object Y of $\mathcal{C}$, and every morphism $f : \mathcal{C}(Y, X)$, $r_Y(F(f^{op})(x)) = f$

We now prove that this definition is equivalent to the more usual definition. We need to show that

$$\mathcal{C}(X, -) \cong F \tag{1}$$

Given a map $f : \mathcal{C}(X, Y)$, then $F(f)(x)$ is an element of $F(Y)$. This gives one direction of the isomorphism. The other direction is given by e_Y, the map that is Axiom 3 of our definition of corepresentation. Axioms 4 and 5 say that these are two sided inverses of each other. We only need to prove naturality.

To prove naturality we need to prove that given $f : \mathcal{C}(X, Y)$ and $g : \mathcal{C}(Y, Z)$, that $F(g \circ f)(x) = F(g)(F(f)(x))$. This follows from functoriality of F.

Lemma 1.0.1. Extensionality If X is a corepresentation of F then any two maps $f, g : \mathcal{C}(X, Y)$ are equal if $F(f)(x) = F(g)(x)$. This follows from the fact that $f \mapsto F(f)(x)$ is injective because it is an isomorphism.

Extensionality is in fact equivalent to Axiom 5 of a corepresentation and could be used instead of Axiom 5 as part of the definition.

2 Adjunction

Given a functor $F : \mathcal{C} \to \mathcal{D}$ we will define a left adjoint of F to a corepresention of $\mathcal{D}(F(A), -)$ for every object A of $\mathcal{C}$. We will call this map of object sets G. We can now prove that G is a functor. Explicitly, this is the following data.

1. A map of object set $G : \mathrm{Obj}(\mathcal{D}) \to \mathrm{Obj}(\mathcal{C})$
2. For every object X of $\mathcal{D}$, a map $\eta_X : \mathcal{D}(X, F(G(X)))$
3. For every object X of $\mathcal{D}$ and Y of $\mathcal{C}$, a map of sets $e_{X,Y} : \mathcal{D}(X, F(Y)) \to \mathcal{C}(G(X), Y)$
4. For every object X of $\mathcal{D}$ and Y of $\mathcal{C}$, and every element $f : \mathcal{D}(X, F(Y))$, $F(e_{X,Y}(f)) \circ \eta_X = f$
5. For every object X of $\mathcal{D}$ and Y of $\mathcal{C}$, and every element $f : \mathcal{C}(G(X), Y)$, $e_{X,Y}(F(f) \circ \eta_X) = f$

Lemma 2.0.1. Extensionality Two maps $f, g : \mathcal{C}(G(X), Y)$ are equal iff $F(f) \circ \eta_X = F(g) \circ \eta_X$.

We can prove that G is in fact a functor. Given objects A and B of $\mathcal{D}$ and a map $f : \mathcal{D}(, B)$,A then define $G(f)$ to be $e_{A,G(B)}(\eta_B \circ f)$.

Then

$$G(\mathrm{id}_A) = e_{A,G(A)}(\eta_A \circ \mathrm{id}_A) = e_{A,G(A)}(F(\mathrm{id}_{G(A)}) \circ \eta_A) = \mathrm{id}_{G(A)} \tag{2}$$

Also for $f : \mathcal{D}(A, B)$ and $g : \mathcal{C}(B, C)$ then we apply the extensionality lemma

$$\begin{array}{cr} & F(G(g \circ f)) \circ \eta_A \\ = & F(e_{A,G(C)}(\eta_C \circ g \circ f)) \circ \eta_A \\ = & \eta_C \circ g \circ f \\ = & F(e_{A,G(B)}(\eta_C \circ g)) \circ \eta_B \circ f \\ = & F(e_{A,G(B)}(\eta_C \circ g)) \circ F(e_{B,G(C)}(\eta_B \circ f)) \\ = & F(G(g) \circ G(f)) \end{array} \tag{3}$$

This works similarly for right adjoints.

3 Method for Checking Equalities

The basic method for checking equalities of morphisms is to write every morphism in terms of the universal property whenever possible, and then use extensionality and then repeatedly rewrite using Axiom 4 of the corepresentable or representable functor axioms.

4 Polynomial Associativity Example

The Free Module functor which we will call F is the left adjoint to the forgetful functor $\text{Forget} : \text{Mod}_R \to \text{Type}$.

We will call the map $A \to \text{Forget}(F(A))$, X and use subscripts for application. We might not always write the forgetful functor explicitly.

The map $(A \to \text{Forget}(B)) \to \text{Mod}_R(F(A), B)$ will be called *extend*.

We will define multiplication on $F(\mathbb{N})$ as a morphism of Type

$$\text{Mod}_R(F(\mathbb{N}), [F(\mathbb{N}), F(\mathbb{N})]) \tag{4}$$

Square brackets indicate the hom object in Mod_R.

The definition of multiplication is as follows

$$\textit{extend}(m \mapsto \textit{extend}(n \mapsto X_{m+n})) \tag{5}$$

We would like to use our extensionality lemma to prove associativity of multiplication. In order to do this, we have to state associativity as an equality of morphisms, as opposed to an equality of elements of the free module. We use two operations to do this, both of which are versions of linear map composition as a linear map.

For modules A, B, and C we have two versions of linear map composition which we call R and L.

$$\begin{aligned} R &: \text{Mod}_R([A, B], [[B, C], [A, C]]) \\ L &: \text{Mod}_R([B, C], [[A, B], [A, C]]) \end{aligned} \tag{6}$$

Then the map $a, b, c \mapsto \text{mul}(\text{mul}(a)(b))(c)$ can be written as

$$\text{Forget}(R)(\text{mul}) \circ \text{mul} \tag{7}$$

Similarly the map $a, b, c \mapsto \text{mul}(a)(\text{mul}(b)(c))$ can be written as

$$\text{Forget}(L)(\text{mul}) \circ (R \circ \text{mul}) \tag{8}$$

These linear maps both have Type $\text{Mod}_R(F(\mathbb{N}), [F(\mathbb{N}), F(\mathbb{N})])$.

We can apply the extensionality lemma three times (using functional extensionality as well).

We then have to check that for any $i, j, k : \mathbb{N}$ that

$$(R)(\text{mul}) \circ \text{mul}(X_i)(X_j)(X_k) = (L)(\text{mul}) \circ (R \circ \text{mul})(X_i)(X_j)(X_k) \tag{9}$$

Unfolding the definitions of linear map composition and applying Axiom 4 several times gives the following equality.

$$X_{(i+j)+k} = X_{i+(j+k)} \tag{10}$$

The associativity of multiplication of polynomials was reduced to the associativity of addition of natural numbers.

5 Potential Improvements

Having to unfold the definition of linear map composition is unsatisfying as well as having to directly apply funext. Probably it would be better to express the universal property as a representation of a functor $\mathrm{Mod}_R \to \mathrm{Mod}_R$ and to develop some theory of representable functors in enriched categories.

Given a functor $F : \mathrm{Mod}_R \to \mathrm{Mod}_R$, then if X is a corepresentation of F, the object $[X, A]$ is a representation of the functor $B \mapsto [B, F(A)]$. The hom object inherits a universal property from X and linear composition can probably be written in terms of this universal property.