1 Questions

1.1 What is the correct notion of parametric category?

There is a notion given by [HRR14] of reflexive graph category. I don't think this is strong enough. I would like a notion of "parametric category" that includes some axioms about having enough morphisms, so that the naturality results are as strong as possible. The morphisms are some subset of the edges and ideally we want the largest possible subset such that we can still make sense of composition and taking preimages of relations along morphisms.

One answer to this is to stipulate that the relations on Types are all of the form $f(x) = g(y)$ for some type Z functions $f : X \to Z$ and $g : Y \to Z$. This is equivalent to saying that the relation satisfies $\forall x_1 x_2 y_1 y_2, R(x_1, y_1) \to R(x_1, y_2) \to R(x_2, y_1) \to R(x_2, y_2)$. I think it is true that for any $F : \mathit{Type} \to \mathit{Type}$, if $R : X \to Y \to \mathit{Prop}$ satisfies the above condition then so does the corresponding relation on $F(X)$ and $F(Y)$.

Then a "parametric category" could be defined to be a category with binary pullback, such that the "edges" between objects were the pullbacks. This is better than defining an edge to be a subobject of the product both because it is simpler in a pure categorical language and because not all categories generated by parametricity have products but I cannot think of any that do not have binary pullbacks.

e.g. The category $\Sigma X : \mathit{Type}, X \times (X \to \mathit{bool})$ has pullbacks but not products.

1.2 How to put category structure on each type?

This question is hardest to answer on types where Type or Prop appears on the left of a Pi, e.g. in the Type $\mathit{Type} \to \mathit{Type}$. One natural suggestion is to say that a morphism between F and G has type $\Pi XY, (X \to Y) \to (F(X) \to G(Y))$. The problem with this definition is that the morphisms are not composable unless F and G are functors and there is no identity on F unless F is a functor. The definition $\Pi X, F(X) \to G(X)$, therefore makes more sense and these two definitions are equivalent (by parametricity) if F and G are functors.

1.3 Find the a notion of morphism of a parametric category weaker than functor, but still having some structure on the map of relations

There is a notion of a dinatural or extranatural transformation between morphisms $FG : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$. They both give a notion of naturality for something $\Pi X, F(X, X) \to G(X, X)$. However not every function $T : \mathit{Type} \to \mathit{Type}$ can be written in the $T(X) = F(X, X)$ where F is a functor $\mathit{Type}^{op} \times \mathit{Type} \to \mathit{Type}$ so this is not general enough. However it may be true that instead of functors $\mathit{Type}^{op} \times \mathit{Type} \to \mathit{Type}$ you could do include a relation between the two types, using $\Sigma AB : \mathit{Type}, A \to B \to \mathit{Prop}$, instead of

$Type^{op} \times Type$. This category is completely defined in edge_category.lean. It seems that there $group : Type \to Type$ can be written as a functor $group2 : (\Sigma AB : Type, A \to B \to Prop) \to Type$, such that $groupG = group2(G, G, eq)$. It is not clear if this generalizes or if it is useful.

1.4 Generalizing representable functors

If $F : \mathcal{C} \to Type$ is a functor, then one way of defining a representation of that functor is that it is an initial object in the category $\Sigma X : \mathcal{C}, F(X)$. This definition has the advantage that F need not be a functor.

Sometimes something has a universal property in one category and this means it is "obvious" that it has some universal property in another category. For example, it is "obvious" that the initial object in the category of pointed groups is also the free group on one generator. Can parametricity be used to generate all the equivalent UMPs of an object in different categories?

Consider the example of $\mathbb{N}$. This is a representation of the function $X \mapsto X \times (X \to X)$ by the above definition. It is also true that is the category of types, for any type X, $\mathrm{Hom}(\mathbb{N}, X) \cong X \times \mathrm{Hom}(\mathbb{N}, \mathrm{Hom}(X, X))$, which is a sort of universal property, but a bit different because $\mathrm{Hom}(\mathbb{N}, -)$ appears on both sides. Can this idea be generalized to categories other than Type?

1.5 What does parametricity have to do with recursion principles for inductive types?

References

[HRR14] Claudio Hermida, Uday S. Reddy, and Edmund P. Robinson, Logical relations and parametricity - a reynolds programme for category theory and programming languages, Electron. Notes Theor. Comput. Sci. **303** (2014), 149–180.