1 Free Module

We start by defining the free module. The free module $F(\alpha)$ over a Type α is the module such that there is a natural isomorphism of functors, natural in both M and α.

$$\mathrm{Hom}(F(\alpha), M) \cong \mathrm{Hom}(\alpha, \mathrm{Forget}(M)) \tag{1}$$

1.1 Definition in Lean

The universal property in Lean is stated using the following four definitions or lemmas.

Definition 1.1.1. There is a map

$$X : \mathrm{Hom}(\alpha, Forget(F(\alpha))) \tag{2}$$

Definition 1.1.2. There is a map

$$extend : \mathrm{Hom}(\alpha, \mathrm{Forget}(M)) \to \mathrm{Hom}(F(\alpha), M) \tag{3}$$

There are two properties proven

Lemma 1.1.3. Let f be a map of sets, $\mathrm{Hom}(\alpha, \mathrm{Forget}(M))$. Then

$$Forget(extend(f)) \circ X = f \tag{4}$$

Lemma 1.1.4. Extensionality

Let f and g be module homomorphisms, $\mathrm{Hom}(F(\alpha), M)$. Then these two maps are equal whenever

$$Forget(f) \circ X = Forget(g) \circ X \tag{5}$$

The universal property follows from these two definitions and two lemmas. We need to prove the natural isomorphism of functors $\mathrm{Hom}(F(\alpha), M) \cong \mathrm{Hom}(\alpha, \mathrm{Forget}(M))$.

The map $\mathrm{Hom}(F(\alpha), M) \to \mathrm{Hom}(\alpha, \mathrm{Forget}(M))$ is given by composition with X. If $f : \mathrm{Hom}(F(\alpha), M)$, then $Forget(f) \circ X : \mathrm{Hom}(\alpha, \mathrm{Forget}(M))$.

The other direction of the isomorphism is *extend*.

To prove this is an isomorphism we need to prove two equalities.

To prove this is an isomorphism we need to prove that for any $f : \mathrm{Hom}(\alpha, Forget(M))$, $Forget(extend(f)) \circ X = f$ and for any $f : \mathrm{Hom}(F(\alpha), M)$, $extend(Forget(f) \circ X) = f$. The first equality is part of our Lean definition. The second we prove by applying our extensionality lemma so we have to prove $Forget(extend(Forget(f) \circ X)) \circ X = Forget(f) \circ X$. Apply the first lemma to the right hand side proves the equality.

We also need to prove that F is a functor. Given Types α, β and a map $f : \alpha \to \beta$, the map $F(f) : \mathrm{Hom}(F(\alpha), F(\beta))$ is given by $F(f) = extend(X \circ f)$.

To prove $F(id) = id$ we apply the extensionality lemma and we now need to prove $Forget(extend(X \circ id)) \circ X = Forget(id) \circ X$. This is a direction application of Lemma 1.1.3.

To prove $F(g \circ f) = F(g) \circ F(f)$, then first apply extensionality and we need to prove $Forget(extend(X \circ g \circ f)) \circ X = Forget(extend(X \circ g) \circ extend(X \circ f)) \circ X$. Three applications of Lemma 1.1.3 can prove this. The first three equalities in the below expression follow from Lemma 1.1.3, the final one is functoriality of the forgetful functor.

$$\begin{aligned} & Forget(extend(X \circ g \circ f)) \circ X \\ = \quad & X \circ g \circ f \\ = \quad & Forget(extend(X \circ g)) \circ X \circ f \\ = \quad & Forget(extend(X \circ g)) \circ Forget(extend(X \circ f)) \circ X \\ = \quad & Forget(extend(X \circ g) \circ extend(X \circ f)) \circ X \end{aligned} \tag{6}$$

We now prove naturality in both arguments. There are two equalities we need to prove

Lemma 1.1.5. For any Types α, β, modules M, N and any map $f : \beta \to \alpha$, and any module homomorphism $g : \mathrm{Hom}(M, N)$, and any map $h : \mathrm{Hom}(F(\alpha), M)$

$$Forget(g \circ h \circ F(f)) \circ X = Forget(g) \circ Forget(h) \circ X \circ f \tag{7}$$

Replacing $F(f)$ by its definition $extend(X \circ f)$ and applying Lemma 1.1.3 will prove this.

The other direction of naturality is given bibliography

Lemma 1.1.6. For any Types α, β, modules M, N and any map $f : \beta \to \alpha$, and any module homomorphism $g : \mathrm{Hom}(M, N)$, and any map $h : \mathrm{Hom}(\alpha, Forget(M))$.

$$extend(Forget(g) \circ h \circ f) = g \circ extend(h) \circ F(f) \tag{8}$$

We can apply extensionality to see that it suffices to prove

$$Forget(extend(Forget(g) \circ h \circ f)) \circ X = Forget(g \circ extend(h) \circ F(f)) \circ X \tag{9}$$

We apply Lemma 1.1.3 to the left hand side and replace $F(f)$ with its definition and then we see it suffices to prove

$$Forget(g) \circ h \circ f = Forget(g) \circ Forget(extend(h)) Forget(extend(X \circ f))) \circ X \tag{10}$$

Rewriting twice with Lemma 1.1.3 on the right hand side proves the desired result.