1 Cocompletion of a Category

If $\mathcal{C}$ is a small category then $\mathcal{C}^{op} \to$ Type is its cocompletion. Intuitively this is because every element of $\mathcal{C}^{op} \to$ Type is a small colimit of a diagram in $\mathcal{C}$. If $\mathcal{C}$ is a large category, then the cocompletion is the subcategory of $\mathcal{C}^{op} \to$ Type consisting of elements which are a small colimit of objects in $\mathcal{C}$. (Check this works, does the map from this subcategory preserve colimits)? We use the notation ΣC for the cocompletion of $\mathcal{C}$ and $\Pi\mathcal{C}$ for the completion.

2 Partial Cocompletion of a Category

We have large categories $\mathcal{C}$ and $\mathcal{D}$ and a fully faithful cocontinuous functor $F : \mathcal{C} \to \mathcal{D}$, such that everything in $\mathcal{D}$ is a limit of objects in $\mathcal{C}$. Then ΣF is a cocomplete category such that there is a cocontinuous map $i_{\mathcal{C}} : \mathcal{C} to \Sigma F$, and a functor $i_{\mathcal{D}} : \mathcal{D} \to \Sigma F$ such that $i_{\mathcal{D}} \circ F = i_{\mathcal{C}}$, and for any category $\mathcal{E}$ with a cocontinous map $j_{\mathcal{C}} : \mathcal{C} \to \mathcal{E}$ and a map $j_{\mathcal{D}} : \mathcal{D} \to \mathcal{E}$ such that $j_{\mathcal{D}} \circ F = j_{\mathcal{C}}$, then there is a unique cocontinuous map $\Sigma F \to \mathcal{E}$ making everything commute.

$$
\begin{tikzcd}
& \Sigma F \arrow[dd, bend left, "\exists! cocon"] \\
\mathcal{C} \arrow[ur, "cocon"] \arrow[r, hook, two heads, "F(cocon)"] \arrow[dr, "cocon"'] & \mathcal{D} \arrow[u] \arrow[d] \\
& \mathcal{E}
\end{tikzcd}
$$

When we construct ΣF it should be obvious that $i_{\mathcal{D}}$ is fully faithful and continuous. It will be constructed as a subcategory of $PSh(\mathcal{D})$.

2.1 Constructing the Partial Cocompletion

We define an adjunction between $\Sigma\mathcal{D}$ and $\Pi\mathcal{D}$. It is inspired by Isbell conjugation, if F is the identity functor, then it is Isbell conjugation.