2 - Discrete-Time Signals

Energy $E_x = \|x\|_2^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$

Power $P_x = \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N} |x[n]|^2$

Finite support $\exists M$ s.t $x[n] = 0,\ \forall n < M$ and $n > M + N - 1$

3 - Signals and Hilbert Spaces

Vector Space

Commutativity $x + y = y + x$

Associativity $(x + y) + z = x + (y + z)$

Distributivity $\alpha(x + y) = \alpha x + \alpha y$

$(\alpha + \beta)x = \alpha x + \beta y$

Null vector $\exists 0 \in V$ s.t $x + 0 = x \forall x$

Inverse $x + y = y + x$

Identity element $\exists 1$ s.t $x.1 = 1.x = x$

Inner Product Space

Distributivity $\langle x + y,\, z\rangle$

Scaling property $\langle x,\, \alpha y\rangle = \alpha^* \langle x,\, y\rangle$

Commutative $\langle x,\, y\rangle = \langle y,\, x\rangle *$

Self-product positive $\langle x,\, x\rangle \geq 0$

$\langle x,\, x\rangle = 0 \Leftrightarrow x = 0$ An inner product space is a vector space equipped with an inner product.

Complete space sequences converge in the space

Hilbert space space is a complete inner product space

Subspace

- Closure under additions :

$\forall x, y \in P \Rightarrow x + y \in P$

- Closure under scalar multiplication:

$\forall x \in P, \forall \alpha \in S \Rightarrow \alpha x \in P$

Span set of all linear combinations of the vectors in W.

Basis set linearly independant, its span covers the region, ie span$(W) = P$

Norm : ≥ 0, $||\alpha x|| = |\alpha|\ ||x||$. triangle inequality

Orthogonal/Orthonormal Basis

Parseval's Identity

$$\|y\| = \sum_{k=0}^{K-1} \left|\left\langle x^{(k)},\, y\right\rangle\right|^2$$

Bassel's Identity

$$\|y\| \geq \sum_{l=0}^{L-1} \left|\left\langle g^{(l)},\, y\right\rangle\right|^2, \text{ with } G \subset P$$

Best Approximations

$$\hat{y} = \sum_{k=0}^{K-1} \left\langle x^{(k)},\, y\right\rangle x^{(k)}$$

Notations

- ℓ_1 : absolutely summable sequence
- ℓ_2 : square summable sequence
- L_1 : absolutely integrable function

Cauchy Sequence

$\forall \epsilon > 0, \exists N \in \mathbb{N} \mid \forall n, m \geq N, |x_n - x_m| < \epsilon$

4 - Fourier Analysis

Discrete-Time Fourier Transform (DTFT)

used for:	infinite, two sided signals ($x[n] \in \ell_2(\mathbb{Z})$)
analysis formula:	$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$
synthesis formula:	$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\mathrm{d}\omega$
symmetries:	$x[-n] \overset{DTFT}{\longleftrightarrow} X(e^{-j\omega})$
	$x^*[n] \overset{DTFT}{\longleftrightarrow} X^*(e^{-j\omega})$
shifts:	$x[n - n_0] \overset{DTFT}{\longleftrightarrow} e^{-j\omega n_0}X(e^{j\omega})$
	$e^{-j\omega_0 n}x[n] \overset{DTFT}{\longleftrightarrow} X(e^{j(\omega-\omega_0)})$
Parseval:	$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 \mathrm{d}\omega$

Discrete Fourier Series(DFS)

used for:	periodic signals ($\tilde{x}[n] \in \tilde{\mathbb{C}}^N$)
analysis formula:	$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j\frac{2\pi}{N}nk}, k = 0, \ldots, N-1$
synthesis formula:	$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]e^{j\frac{2\pi}{N}nk}, n = 0, \ldots, N-1$
symmetries:	$\tilde{x}[-n] \overset{DFS}{\longleftrightarrow} \tilde{X}[-k]$
	$\tilde{x}^*[n] \overset{DFS}{\longleftrightarrow} \tilde{X} * [-k]$
shifts:	$\tilde{x}[n - n_0] \overset{DFS}{\longleftrightarrow} e^{-j\frac{2\pi}{N}kn_0}\tilde{X}[k]$
	$e^{j\frac{2\pi}{N}nk_0}\tilde{x}[n] \overset{DFS}{\longleftrightarrow} \tilde{X}[k - k_0]$
Parseval:	$\sum_{n=-0}^{N-1} |\tilde{x}[n]|^2 = \frac{1}{N}\sum_{0}^{N-1} \left|\tilde{X}[k]\right|^2 \mathrm{d}\omega$

Some DTFT pairs

$x[n] = \delta[n - k]$	$X(e^{j\omega}) = e^{-j\omega k}$
$x[n] = 1$	$X(e^{j\omega k}) = \tilde{\delta}(\omega)$
$x[n] = u[n]$	$X(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} + \frac{1}{2}\tilde{\delta}(\omega)$
$x[n] = a^n u[n], |a| < 1$	$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$
$x[n] = e^{j\omega_0 n}$	$X(e^{j\omega}) = \tilde{\delta}(\omega - \omega_0)$
$x[n] = \cos(\omega_0 n + \phi)$	$X(e^{j\omega}) = \frac{1}{2}[e^{j\phi}\tilde{\delta}(\omega - \omega_0) + e^{-j\phi}\tilde{\delta}(\omega + \omega_0)]$
$x[n] = \sin(\omega_0 n + \phi)$	$X(e^{j\omega}) = \frac{-j}{2}[e^{j\phi}\tilde{\delta}(\omega - \omega_0) - e^{-j\phi}\tilde{\delta}(\omega + \omega_0)]$

$$x[n] = \begin{cases} 1 & \text{for } 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases} \qquad X(e^{j\omega}) = \frac{\sin((N/2)\omega)}{\sin(\omega/2)}e^{-j\frac{N-1}{2}\omega}$$

Discrete Fourier Transform(DFT)

used for:	finite support signals ($x[n] \in \mathbb{C}^N$)
analysis formula:	$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}nk}, k = 0, \ldots, N-1$
synthesis formula:	$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}nk}, n = 0, \ldots, N-1$
symmetries:	$x[-n \mod N] \overset{DFT}{\longleftrightarrow} X[-k \mod N]$
	$x^*[n] \overset{DFT}{\longleftrightarrow} X^*[-k \mod N]$
shifts:	$x[(n - n_0) \mod N] \overset{DFT}{\longleftrightarrow} e^{-j\frac{2\pi}{N}kn_0}X[k]$
	$e^{j\frac{2\pi}{N}nk_0}x[n] \overset{DFT}{\longleftrightarrow} X[(k - k_0) \mod N]$
Parseval:	$\sum_{n=-0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{0}^{N-1} |X[k]|^2 \mathrm{d}\omega$

Some DFT pairs for length-N signals

$$\begin{array}{ll} x[n]=\delta[n-k] & X[k]=e^{-j\frac{2\pi}{N}k} \\ x[n]=1 & X[k]=N\delta[k] \\ x[n]=e^{j\frac{2\pi}{N}L} & X[k]=N\delta[k-L] \end{array}$$

$$\begin{array}{ll} x[n]=\cos(\frac{2\pi}{N}Ln+\phi) & X[k]=\frac{N}{2}[e^{j\phi}\delta[k-L]+e^{-j\phi}\delta[k-N+L] \\ x[n]=\sin(\frac{2\pi}{N}Ln+\phi) & X[k]=\frac{-jN}{2}[e^{j\phi}\delta[k-L]-e^{-j\phi}\delta[k-N+L] \end{array}$$

$$x[n]=\begin{cases} 1 & \text{for } n\le M-1 \\ 0 & \text{for } M\le n\le N-1 \end{cases} \qquad X[k]=\frac{\sin((\pi/N)Mk)}{\sin((\pi/N)k)}e^{-j\frac{\pi}{N}(M-1)k}$$

5 - Discrete-Time Filters

Linear Time-Invariant Systems (LTI)

Time Invariance:
$y[n]=\mathscr{H}\{x[n]\}\Leftrightarrow y[n-n_0]=\mathscr{H}\{x[n-n_0]\}$

LTI: $y[n]=\mathscr{H}\{x[n]\}=\displaystyle\sum_{k=-\infty}^{\infty}x[k]h[n-k]$

Filtering in the Time Domain

Convolution : $\tilde{x}[n]*\tilde{y}[n]=\displaystyle\sum_{k=0}^{N-1}\tilde{x}[k]\tilde{y}[n-k]$

$X(e^{j\omega})*Y(e^{j\omega})=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\sigma})*Y(e^{j(\omega-\sigma)})\mathrm{d}\sigma$

Properties of the Impulse Response

IIR filters: Infinite Impulse Response.

FIR filters: Finite Impulse Response.

Causality: outputs doesn't depend on future values of input.

Stability: A system is called BIBO stable if its output is bounded for all bounded input sequences.

sufficient condition : absolutely summable

Filtering in the Frequency Domain

Properties of the Frequency response

Magnitude : lowpass, highpass, bandpass or allpass filter.

Phase : the phase response acts as a generalized delay.

Linear Phase : $\measuredangle H(e^{j\omega})=\omega d$

Filtering by Example : Frequency Domain

Moving Average $H(e^{j\omega})=\frac{1}{N}\frac{\sin(\omega N/2)}{\sin(\omega/2)}e^{-j\frac{N-1}{2}\omega}$

Leaky Integrator $H(e^{j\omega})=\frac{1-\lambda}{1-\lambda e^{-j\omega}}$

Ideal Filters

Ideal Lowpass: $h_{lp}[n]=\frac{\sin(\omega_c n)}{\pi n}$; $\quad \mathrm{sinc}(x)=\frac{\sin(\pi x)}{\pi x}$

Ideal Highpass: $h_{hp}[n]=\delta[n]-\frac{\omega_c}{\pi}\,\mathrm{sinc}\left(\frac{\omega_c}{\pi}n\right)$

Ideal Bandpass: $h_{bp}[n]=2\cos(\omega_0 n)\frac{\omega_b}{2\pi}\,\mathrm{sinc}\left(\frac{\omega_b}{2\pi}n\right)$

Hilbert Filter: $H(e^{j\omega})=\begin{cases} -j & 0\le\omega\le\pi \\ +j & -\pi\le\omega\le 0 \end{cases}$

$$h[n]=\frac{2\sin^2(\pi n/2)}{\pi n}=\begin{cases} 0 & \text{for } n \text{ even} \\ \frac{2}{n\pi} & \text{for } n \text{ odd} \end{cases}$$

6 - The Z-transform

For DFTs : $z=e^{j\omega}$

Signal	**Transform**	**ROC**
$\delta[n]$	1	all z
$\delta[n-n_0]$	z^{-n_0}	$z\neq 0$
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert>1$
$e^{-\alpha n}u[n]$	$\frac{1}{1-e^{-\alpha}z^{-1}}$	$\lvert z\rvert>\lvert e^{-\alpha}\rvert$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert<1$
$nu[n]$	$\frac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert>1$
$-nu[-n-1]$	$\frac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert<1$
$n^2u[n]$	$\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$	$\lvert z\rvert>1$
$-n^2u[-n-1]$	$\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$	$\lvert z\rvert<1$
$n^3u[n]$	$\frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$	$\lvert z\rvert>1$
$-n^3u[-n-1]$	$\frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$	$\lvert z\rvert<1$
$a^nu[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert>\lvert a\rvert$
$-a^nu[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert<\lvert a\rvert$
$na^nu[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert>\lvert a\rvert$
$-na^nu[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert<\lvert a\rvert$
$n^2a^nu[n]$	$\frac{az^{-1}(1+az^{-1})}{(1-az^{-1})^3}$	$\lvert z\rvert>\lvert a\rvert$
$-n^2a^nu[-n-1]$	$\frac{az^{-1}(1+az^{-1})}{(1-az^{-1})^3}$	$\lvert z\rvert<\lvert a\rvert$
$\cos(\omega_0 n)u[n]$	$\frac{1-z^{-1}\cos(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\lvert z\rvert>1$
$\sin(\omega_0 n)u[n]$	$\frac{z^{-1}\sin(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\lvert z\rvert>1$
$a^n\cos(\omega_0 n)u[n]$	$\frac{1-az^{-1}\cos(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\lvert z\rvert>\lvert a\rvert$
$a^n\sin(\omega_0 n)u[n]$	$\frac{az^{-1}\sin(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\lvert z\rvert>\lvert a\rvert$

Filter Analysis

Z-transform: $X(z)=\mathscr{Z}\{x[n]\}=\sum_{n=-\infty}^{\infty}x[n]z^{-n}$

Inverse Z-transform: $\mathscr{Z}^{-1}\{X(z)\}=\frac{1}{2\pi j}\oint_C X(z)z^{n-1}\mathrm{d}z$

Time-shift $\mathscr{Z}\{x[n-N](z)\}=z^{-N}X(z)$

CCDE: Constant-Coefficient Difference Equation

$$y[n]=\sum_{k=0}^{M-1}b_kx[n-k]-\sum_{k=0}^{N-1}a_ky[n-k], \quad H(z)=\frac{\sum_{k=0}^{M-1}b_kz^{n-k}}{1+\sum_{k=0}^{N-1}a_kz^{n-k}}$$

Region pf Convergence

The ROC has circular geometry

Anticausal ROC: Disk Causal ROC: plane \disk

Stability: a system is BIBO stable if its ROC includes the unit circle.

The Pole-Zero plot

$$H(z)=b_0\frac{\prod_{n=0}^{M-1}(1-z_nz^{-1})}{\prod_{n=0}^{N-1}(1-p_nz^{-1})}$$

Sketching the Transfer Function from the Pole-Zero Plot

1- Check for the zeros on the unit circle; these correspond to points on the frequency axis in which the magnitude response is exactly zero

2 - Draw a line from the origin of the complex plane to each pole and each zero. The point of intersection of each line with the unit circle gives the location of a local extremum for the magnitude response.

3 - The effect of each pole and each zero is made stronger by their proximity to the unit circle

7 - Filter Design

Filter Specifications and Tradeoffs
Transition Band. Range of frequencies between passband and stopband **Tolerances.** Min and Max frequency response over passband and stopband.
FIR Filter Design
FIR Filter Design By Windowing *Impulse Response Truncation*

$$\hat{h}[n] = \begin{cases} h[n] & -N \leq n \leq N \\ 0 & \text{otherwise} \end{cases}$$

The Rectangular Window

$$\hat{h}[n] = h[n]w[n] \quad \text{with : } w[n] = \text{rect}\left(\frac{n}{N}\right) = \begin{cases} 1 & -N \leq n \leq N \\ 0 & \text{otherwise} \end{cases}$$

Gibbs phenomenon: The maximum error does not decrease with increasing N and, therefore, there are no means to meet a set of specifications which require less tan 9% error in either stopband or passband.

This truncation produces 2 main effects :
- Sharp transition from passband to stopband is smoothed by the convolution with the main lobe of width Δ.
- Ripples appear both in the stopband and the passband due to the convolution with the sidelobes (the largest ripple being the Gibbs phenomenon).

8 - Stochastic Signal Processing

Random Variables
Expectation: $E[X] = \int_{-\infty}^{\infty} x f_X(x)dx$
$E[g(X)] = \int_{-\infty}^{\infty} g(x)f_X(x)dx$
Correlation: $R_{XY} = E[XY]$
Covariance: $K_{XY} = E[XY] - E[X]E[Y]$
Variance: $\sigma_X^2 = E[(W - m_x)^2]$

Random Vectors
Expectation:
$E[\mathbf{X}] = [E[X_0], E[X_1], ..., E[X_{N-1}]]^T$
Correlation: $\mathbf{R_{XY}} = E[\mathbf{XY}^T]$
Cov: $\mathbf{K_{XY}} = E[(\mathbf{X} - \mathbf{m_X})(\mathbf{Y} - \mathbf{m_Y})]$

Random Processes
Probability Distribution:
$f_{X[i_0]X[i_1]...X[i_{k-1}]}(x_0, x_1, ..., x_{k-1})$
Second Order Description:
$R_X[l,k] = E[X[l]X[k]], \quad l,k \in \mathbb{Z}$
$K_X[l,k] = E\left[\left(X[l] - m_{X[l]}\right)\left(X[k] - m_{X[k]}\right)\right]$
$R_{XY}[l,k] = E[X[l]Y[k]]$ (cross-correlation)
Stationary Processes
Strict sense: The full probabilistic description of the process is time invariant
WSS: mean and variance are constant over time and autocorrelation and covariance only depend on the time lag $(l-k)$
Ergodicity
It is legitimate to estimate expectations from a single realization

Spectral Rep of Statio Rand Processes
$P_X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} r_X[k]e^{-j\omega k}$

Stochastic Signal Processing
$Y[n] = \sum_{k=-\infty}^{\infty} h[k]X[n-k]$ with stable LTI filter and WSS input process
Time-Domain Analysis
$m_{Y[n]} = \sum_{n=-\infty}^{\infty} h[k]m_{n-k} = m_X H(e^{j0})$
$r_Y[n] = h[n] * h[-n] * r_X[n]$
$r_{XY}[N] = h[n] * r_X[n]$
Frequency-Domain Analysis
$P_Y(e^{j\omega}) = \left|H(e^{j\omega})\right|^2 P_X(e^{j\omega})$
$P_{XY}(e^{j\omega}) = H(e^{j\omega})P_X(e^{j\omega})$

9 - Interpolation ans Sampling

Continuous-Time Fourier Transform
$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$
$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega$

Bandlimited Signals
$\exists\Omega_N$ s.t $X(j\Omega) = 0$ for $|\Omega| \geq |\Omega_N|$
Polynomial Interpolation

$$L_n^{(N)}(t) = \prod_{\substack{k=-N \\ k\neq n}}^{N} \frac{(t - t_k)}{(t_n - t_k)} = \prod_{\substack{k=-N \\ k\neq n}}^{N} \frac{(t/T_s - k)}{(n-k)}$$

$$P(t) = \sum_{n=-N}^{N} x[n]L_n^{(N)}(t)$$

Sinc Interpolation

$$x(t) = \sum_{n=-\infty}^{\infty} x[n]\,\text{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

The Sampling Theorem
If $x(t)$ is a Ω_N-bandlimited continuous-time signal, a sufficient representation of $x(t)$ is given by the discrete time signal $x[n] = x(nT_s)$, with $T_s = \pi/\Omega_N$. The continuous time signal $x(t)$ can be exactly reconstruced form the discrete-time signal $x[n]$ as :

$$x(t) = \sum_{n=-\infty}^{\infty} x[n]\,\text{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

Observation: limited bandwidth in a domain $\Leftrightarrow$ illimited in the other

10 - A/D and D/A Conversions

Quantization
$\mathscr{Q}\{x[n]\} = \{k | x[n] \in I_k\}$ with $I_k = [i_k, i_{k+1}]$
Quantization Error
Quantization is highly non-linear.

$$\mathscr{Q}\{\underbrace{x[n]}_{i_m} + \epsilon[n]\} = \begin{cases} k-1 & \text{if } \epsilon[n] \leq 0 \\ k & \text{if } \epsilon[n] > 0 \end{cases}$$

$\mathbb{P}\left(\tilde{X} = \tilde{x} = \frac{i}{2^r}\right) = p\left\{\frac{i}{2^r} \leq x \leq \frac{i+1}{2^r}\right\}$
$p_e = \int_{\Omega}(x - \tilde{x})^2 f_x(x)dx$

11 - Multirate Signal Processing

Downsampling

$x_{ND}[n] = \mathscr{D}\{x[n]\} = x[nN]$;

$$X_{ND}(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\frac{\omega}{N}-\frac{2\pi}{N}k)}) = \frac{1}{N}\sum_{k=0}^{N-1} X(z^{\frac{1}{N}}e^{-j\frac{2\pi}{N}k})$$

Upsampling

$$x_{NU}[n] = \mathscr{U}\{x[n]\} = \begin{cases} x[k] & \text{for } n = kN, k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} ;$$

$X_{NU}(e^{j\omega}) = X(e^{j\omega N}) = X(z^N)$

Rappels

Series

$$\sum_{i=0}^{n-1} q^i = \frac{1-q^n}{1-q}; \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2};$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Trigonometry

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$
$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$
$$\tan(a+b) = \frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}$$

$$\cos(p)+\cos(q) = 2\cos(\frac{p+q}{2})\cos(\frac{p-q}{2})$$
$$\cos(p)-\cos(q) = -2\sin(\frac{p+q}{2})\sin(\frac{p-q}{2})$$
$$\sin(p)+\sin(q) = 2\sin(\frac{p+q}{2})\cos(\frac{p-q}{2})$$
$$\sin(p)-\sin(q) = 2\sin(\frac{p-q}{2})\cos(\frac{p+q}{2})$$

$$\cos(a)\cos(b) = \frac{1}{2}(\cos(a+b)+\cos(a-b))$$
$$\sin(a)\sin(b) = \frac{1}{2}(\cos(a-b)-\cos(a+b))$$
$$\sin(a)\cos(b) = \frac{1}{2}(\sin(a+b)+\sin(a-b))$$

$$\cos(a) = \frac{1-\tan^2(\frac{a}{2})}{1+\tan^2(\frac{a}{2})}$$
$$\sin(a) = \frac{2\tan(\frac{a}{2})}{1+\tan^2(\frac{a}{2})}$$
$$\tan(a) = \frac{2\tan(\frac{a}{2})}{1-\tan^2(\frac{a}{2})}$$

Residues

$Res(X(z)z^{n-1})\,|_{z=z_i} = \frac{A(z_i)}{B'(z_i)}z_i^{n-1}$

Taylor series

$\log(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n$
$\frac{1}{1-x} = \sum_{n=1}^{\infty}x^n$

Bayes' Theorem

$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$

Chebychev Polynomials

Used to prove Gibbs Phenomenon

$$T_0(x) = 1, T_1(x) = x, T_2(x) = 2x^2 - 1$$
$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)$$
$$\cos(n\omega) = T_n(cos(\omega))$$

Gaussian Distribution

$f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{\frac{(x-\mu)^2}{2\sigma^2}}$

Integral Derivation

if $f(t) = \int_{a(t)}^{b(t)} g(x,t)\mathrm{d}x$ then

$$\frac{\partial f(t)}{\partial t} = \int_{a(t)}^{b(t)}\frac{g(x,t)}{t}\mathrm{d}x + \frac{b(t)}{t}g(b(t),t) - \frac{a(t)}{t}g(a(t),t)$$

Period, pulsation, frequence

$T = \frac{1}{f}, \quad \omega = 2\pi$

Multirate identities (Noble identities)

- Downsampling by N followed by filtering by $H(z)$ is equivalent to filtering by $H(z^N)$ followed by downsampling by N.
- Filtering by $H(z)$ followed by upsampling by N is equivalent to upsampling by N followed by filtering by $H(z^N)$.

DTFT

$$x[n] = \begin{cases} 1 & \text{if } |n| \le T \\ 0 & \text{otherwise} \end{cases}$$

$\overset{DTFT}{\longleftrightarrow} X(e^{j\omega}) = \frac{\sin(\omega(T+\frac{1}{2}))}{\sin(\omega/2)}$