Gap interface to Cdd package

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Introduction

1.1 Why CddInterface

We know that every convex polyhedron has two representations, one as the intersection of finite half-spaces and the other as Minkowski sum of the convex hull of finite points and the nonnegative hull of finite directions. These are called H-representation and V-representation, respectively. CddInterface is basicly written to translate between these two representations.

1.2 H-representation and V-representation of polyhedra

Let us start by introducing the H-representation. Let A be $m \times d$ matrix and let b be a column m-vector. The H-representation of the polyhedron defined by the system $b + Ax \ge 0$ of m inequalities and d variables $x = (x_1, \ldots, x_d)$ is as follows:

H-representation

linearity t, $[i_1, i_2, \ldots, i_t]$

begin

 $m \times (d+1)$ numbertype

b A

end

The linearity line is added when we want to specify that some rows of the system b+Ax are equalities. That is, $k \in \{i_1, i_2, \dots, i_t\}$ means that the row k of the system b+Ax is specified to be equality. For example, the H-representation of the polyhedron defined by the following system:

$$4 - 3x_1 + 6x_2 - 5x_4 = 0$$
$$1 + 2x_1 - 2x_2 - 7x_3 \ge 0$$

$$-3x_2 + 5x_4 = 0$$

is as follows:

H-representation

linearity 2, [1,3]

begin

 3×5 rational

end

Next we define Polyhedra V-format. Let P be represented by n gerating points and s generating directions (rays) as

$$P = conv(v_1, \dots, v_n) + nonneg(r_{n+1}, \dots, r_{n+s}).$$

Then the Polyhedra V-format is for *P* is:

```
V-representation
```

```
linearity t, [i_1, i_2, \dots, i_t]
begin
(n+s) \times (d+1) numbertype
1 \quad v_1
\vdots \quad \vdots
1 \quad v_n
0 \quad r_{n+1}
\vdots \quad \vdots
0 \quad r_{n+s}
end
```

In the above format the generating points and generating rays may appear mixed in arbitrary order. Linearity for V-representation specifies a subset of generators whose coefficients are relaxed to be free. That is, $k \in \{i_1, i_2, ..., i_t\}$ specifies that the k-th generator is specified to be free. This means for each such a ray r_k , the line generated by r_k is in the polyhedron, and for each such a vertex v_k , its coefficient is no longer nonnegative but still the coefficients for all v_i 's must sum up to one.

For example the V-representation of the polyhedron defined as

$$P := conv((2,3), (-2,-3), (3,5)) + nonneg((1,2), (-1,-2), (2,11))$$

V-representation

| linearity 2, [1,3] | begin | 4×3 numbertype | 1 | 2 | 3 | 1 | 3 | 5 | 5 | 0 | 1 | 2 | 0 | 2 | 11 | end |

Creating polyhedra and their Operations

2.1 Creating a polyhedron

2.1.1 Cdd_PolyhedronByInequalities

```
▷ Cdd_PolyhedronByInequalities(arg)

Returns: a CddPolyhedron Object
```

The function takes a list in which every entry represents an inequality (or equality). In case we want some entries to represent equalities we should refer in a second list to their indices.

(function)

```
_ Example .
gap> A:= Cdd_PolyhedronByInequalities([ [ 0, 1, 0 ], [ 0, 1, -1 ] ] );
<Polyhedron given by its H-representation>
gap> Display( A );
H-representation
begin
   2 X 3 rational
   0
     1
          0
gap> B:= Cdd_PolyhedronByInequalities( [ [ 0, 1, 0 ], [ 0, 1, -1 ] ], [ 2 ] );
<Polyhedron given by its H-representation>
gap> Display( B );
H-representation
linearity 1, [2]
begin
   2 X 3 rational
      1
   0
          0
   0
       1 -1
end
```

2.1.2 Cdd_PolyhedronByGenerators

```
▷ Cdd_PolyhedronByGenerators(arg)

Returns: a CddPolyhedron Object

(function)
```

The function takes a list in which every entry represents a vertex in the ambient vector space. In case we want some vertices to be free (the vertex and its negative belong to the polyhedron) we should refer in a second list to their indices .

```
_{\scriptscriptstyle -} Example _{\scriptscriptstyle -}
gap> A:= Cdd_PolyhedronByGenerators( [ [ 0, 1, 3 ], [ 1, 4, 5 ] ] );
<Polyhedron given by its V-representation>
gap> Display( A );
V-representation
begin
   2 X 3 rational
   0 1 3
   1 4 5
gap> B:= Cdd_PolyhedronByGenerators( [ [ 0, 1, 3 ] ], [ 1 ] );
<Polyhedron given by its V-representation>
gap> Display( B );
V-representation
linearity 1, [1]
begin
   1 X 3 rational
   0 1 3
end
```

2.2 Some operations on a polyhedron

2.2.1 Cdd_FourierProjection (for IsCddPolyhedron, IsInt)

```
▷ Cdd_FourierProjection(P, i) (operation)

Returns: a CddPolyhedron Object
```

The function returns the Fourier projection of the polyhedron in the subspace $(O, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ after applying the Fourier elemination algorithm to get rid of the variable x_i .

To illustrate this projection, Let P = Conv((1,2),(4,5)) in \mathbb{Q}^2 .

To find its projection on the subspace (O, x_1) , we apply the Fourier elemination to get rid of x_2

```
____ Example -
gap> P := Cdd_PolyhedronByGenerators([[1, 1, 2], [1, 4, 5]]);
<Polyhedron given by its V-representation>
gap> H := Cdd_H_Rep( P );
<Polyhedron given by its H-representation>
gap> Display( H );
H-representation
linearity 1, [3]
begin
   3 X 3 rational
    4 -1
           0
   -1 1
   -1 -1
           1
end
```

```
gap> P_x1 := Cdd_FourierProjection( H, 2);
<Polyhedron given by its H-representation>
gap> Display( P_x1 );
H-representation
linearity 1, [3]
begin
  3 X 3 rational
           0
      -1
   -1
       1
           0
gap> Display( Cdd_V_Rep( P_x1 ) );
V-representation
begin
  2 X 3 rational
  1 1 0
   1 4 0
end
```

Let again Q = Conv((2,3,4),(2,4,5)) + nonneg((1,1,1)), and let us compute its projection on (O,x_2,x_3)

```
\_ Example \_
gap> Q := Cdd_PolyhedronByGenerators( [ [1,2,3,4],[1,2,4,5], [0,1,1,1] ] );
<Polyhedron given by its V-representation>
gap> R := Cdd_H_Rep( Q );
<Polyhedron given by its H-representation>
gap> Display( R );
H-representation
linearity 1, [4]
begin
   4 X 4 rational
   2 1 -1
               0
   -2 1 0
               0
          1
   -1 -1
               0
   -1
       0 -1
gap> P_x2_x3 := Cdd_FourierProjection( R, 1);
<Polyhedron given by its H-representation>
gap> Display( P_x2_x3 );
H-representation
linearity 2, [1, 3]
begin
  3 X 4 rational
       0 -1
   -1
                1
   -3
       0
          1
               0
           0
               0
gap> Display( Cdd_V_Rep( last ) );
V-representation
```

```
begin
2 X 4 rational
0 0 1 1
1 0 3 4
end
```

2.3 Some operations on two polyhedrons

2.3.1 Cdd_IsContained (for IsCddPolyhedron, IsCddPolyhedron)

The function returns *true* if P1 is contained in P2, otherwise returns *false*.

```
_ Example .
gap> A := Cdd_PolyhedronByInequalities( [ 10, -1, 1, 0 ],
> [ -24, 9, 2, 0 ], [ 1, 1, -1, 0 ], [ -23, -12, 1, 11 ] ], [ 4 ] );
<Polyhedron given by its H-representation>
gap> B := Cdd_PolyhedronByInequalities( [ [ 1, 0, 0, 0 ],
> [ -4, 1, 0, 0 ], [ 10, -1, 1, 0 ], [ -3, -1, 0, 1 ] ], [ 3, 4 ] );
<Polyhedron given by its H-representation>
gap> Cdd_IsContained( B, A );
gap> Display( Cdd_V_Rep( A ) );
V-representation
begin
   3 X 4 rational
       2
               4
           3
       4 -6
              7
gap> Display( Cdd_V_Rep( B ) );
V-representation
begin
   2 X 4 rational
       4 -6
               7
   0
           1
end
```

2.3.2 Cdd_Intersection (for IsCddPolyhedron, IsCddPolyhedron)

```
Cdd_Intersection(P1, P2) (operation)
Returns: a CddPolyhedron
The function returns the intersection of P1 and P2
```

```
gap> A := Cdd_PolyhedronByInequalities([[3, 4, 5]], [1]);;
gap> B := Cdd_PolyhedronByInequalities([[9, 7, 2]], [1]);;
gap> C := Cdd_Intersection(A, B);;
```

```
gap> Display( Cdd_V_Rep( A ) );
V-representation
linearity 1, [2]
begin
   2 X 3 rational
  1 -3/4
       -5
end
gap> Display( Cdd_V_Rep( B ) );
V-representation
linearity 1, [2]
begin
  2 X 3 rational
  1 -9/7
     -2
              7
  0
gap> Display( Cdd_V_Rep( C ) );
V-representation
begin
  1 X 3 rational
  1 -13/9
              5/9
end
```

2.3.3 \+ (for IsCddPolyhedron, IsCddPolyhedron)

Returns: a *CddPolyhedron*

The function returns the Minkuwski sum of *P1* and *P2*.

```
_ Example .
gap> P:= Cdd_PolyhedronByGenerators( [ [1, 2, 5], [0, 1, 2] ] );
< Polyhedron given by its V-representation >
gap> Q:= Cdd_PolyhedronByGenerators( [ [ 1, 4, 6 ], [ 1, 3, 7 ], [ 0, 3, 1 ] ] );
< Polyhedron given by its V-representation >
gap> S:= P+Q;
< Polyhedron given by its H-representation >
gap> V:= Cdd_V_Rep( S );
< Polyhedron given by its V-representation >
gap> Display( V );
V-representation
begin
  4 X 3 rational
      3 1
   1
      6 11
  1
      5 12
  0
      1
         2
gap> Cdd_GeneratingVertices( P );
[[2,5]]
```

```
gap> Cdd_GeneratingVertices( Q );
[ [ 3, 7 ], [ 4, 6 ] ]
gap> Cdd_GeneratingVertices( S );
[ [ 5, 12 ], [ 6, 11 ] ]
gap> Cdd_GeneratingRays( P );
[ [ 1, 2 ] ]
gap> Cdd_GeneratingRays( Q );
[ [ 3, 1 ] ]
gap> Cdd_GeneratingRays( S );
[ [ 1, 2 ], [ 3, 1 ] ]
```

Linear Programs

3.1 Creating and solving linear programs

3.1.1 Cdd_LinearProgram (for IsCddPolyhedron, IsString, IsList)

```
\triangleright Cdd_LinearProgram(P, str, obj) (operation)
```

Returns: a *CddLinearProgram* Object

The function takes three variables. The first is a polyhedron *poly*, the second *str* should be "max" or "min" and the third *obj* is the objective function.

3.1.2 Cdd_SolveLinearProgram (for IsCddLinearProgram)

```
▷ Cdd_SolveLinearProgram(1p)
```

(operation)

Returns: a list if the program is optimal, otherwise returns the value 0

The function takes a linear program. If the program is optimal, the function returns a list of two entries, the solution vector and the optimal value of the objective, otherwise it returns fail.

To illustrate the using of these functions, let us solve the linear program given by: **Maximize** P(x,y) = 1 - 2x + 5y, with

```
100 \le x \le 200

80 \le y \le 170

y \ge -x + 200
```

We bring the inequalities to the form $b + AX \ge 0$, we get:

```
-100 + x \ge 0
200 - x \ge 0
-80 + y \ge 0
170 - y \ge 0
-200 + x + y > 0
```

```
Example

gap> A:= Cdd_PolyhedronByInequalities( [ [ -100, 1, 0 ], [ 200, -1, 0 ],

> [ -80, 0, 1 ], [ 170, 0, -1 ], [ -200, 1, 1 ] ] );

<Polyhedron given by its H-representation>

gap> lp1:= Cdd_LinearProgram( A, "max", [1, -2, 5 ] );

<Linear program>
```

```
gap> Display( lp1 );
Linear program given by:
H-representation
begin
   5 X 3 rational
   -100
                  0
            1
    200
           -1
                  0
    -80
            0
                  1
    170
            0
                 -1
   -200
            1
                  1
end
max [1, -2, 5]
gap> Cdd_SolveLinearProgram( lp1 );
[ [ 100, 170 ], 651 ]
gap> lp2:= Cdd_LinearProgram( A, "min", [ 1, -2, 5 ] );
<Linear program>
gap> Display( lp2 );
Linear program given by:
H-representation
begin
   5 X 3 rational
   -100
                  0
            1
    200
           -1
                  0
    -80
            0
                  1
    170
            0
                 -1
   -200
            1
                  1
end
min [1, -2, 5]
gap> Cdd_SolveLinearProgram( lp2 );
[ [ 200, 80 ], 1 ]
gap> B:= Cdd_V_Rep( A );
<Polyhedron given by its V-representation>
gap> Display( B );
V-representation
begin
   5 X 3 rational
   1 100 170
   1
      100
           100
            80
   1
      120
      200
            80
   1
      200 170
   1
end
```

So the optimal solution for 1p1 is (x = 100, y = 170) with optimal value p = 1 - 2(100) + 5(170) = 651 and for 1p2 is (x = 200, y = 80) with optimal value p = 1 - 2(200) + 5(80) = 1.

Attributes and properties

4.1 Attributes and properties of polyhedron

4.1.1 Cdd_Canonicalize (for IsCddPolyhedron)

```
ightharpoonup Cdd_Canonicalize(P) (attribute)
```

Returns: a *CddPolyhedron* Object

The function takes a polyhedron and reduces its defining inequalities (generators set) by deleting all redundant inequalities (generators).

```
gap> A:= Cdd_PolyhedronByInequalities( [ [ 0, 2, 6 ], [ 0, 1, 3 ], [1, 4, 10 ] ] );

<Polyhedron given by its H-representation>
gap> B:= Cdd_Canonicalize( A );

<Polyhedron given by its H-representation>
gap> Display( B );
H-representation
begin
    2 X 3 rational

    0    1    3
        1    4    10
end
```

4.1.2 Cdd_V_Rep (for IsCddPolyhedron)

```
ightharpoonup Cdd_V_{Rep}(P) (attribute)
```

Returns: a CddPolyhedron Object

The function takes a polyhedron and returns its reduced V-representation.

4.1.3 Cdd_H_Rep (for IsCddPolyhedron)

```
ightharpoonup Cdd_H_Rep(P) (attribute)
```

Returns: a CddPolyhedron Object

The function takes a polyhedron and returns its reduced H-representation.

```
gap> A:= Cdd_PolyhedronByInequalities([[0, 1, 1], [0, 5, 5]]);
<Polyhedron given by its H-representation>
```

```
gap> B:= Cdd_V_Rep( A );
<Polyhedron given by its V-representation>
gap> Display( B );
V-representation
linearity 1, [2]
begin
   2 X 3 rational
          0
      1
   0 -1
end
gap> C:= Cdd_H_Rep( B );
<Polyhedron given by its H-representation>
gap> Display( C );
H-representation
begin
   1 X 3 rational
  0 1 1
end
gap> D:= Cdd_PolyhedronByInequalities( [ [ 0, 1, 1, 34, 22, 43 ],
> [ 11, 2, 2, 54, 53, 221 ], [33, 23, 45, 2, 40, 11 ] ]);
<Polyhedron given by its H-representation>
gap> Cdd_V_Rep( D );
<Polyhedron given by its V-representation>
gap> Display( last );
V-representation
linearity 2, [ 5, 6 ]
begin
  6 X 6 rational
               369/14
   1 -743/14
                         11/14
                                     0
                                               0
   0
       -1213
                619
                          22
                                     0
                                               0
                                      0
                                               0
   0
          -1
                   1
                            0
  0
                                     0
                                               0
         764
                 -390
                           -11
  0
      -13526
                 6772
                           99
                                    154
                                               0
   0 -116608
                59496
                          1485
                                             154
end
```

4.1.4 Cdd_AmbientSpaceDimension (for IsCddPolyhedron)

▷ Cdd_AmbientSpaceDimension(P)

(attribute

Returns: The dimension of the ambient space of the polyhedron(i.e., the space that contains *P*).

4.1.5 Cdd_Dimension (for IsCddPolyhedron)

▷ Cdd_Dimension(P)

(attribute)

Returns: The dimension of the polyhedron, where the dimension, $\dim(P)$, of a polyhedron P is the maximum number of affinely independent points in P minus 1.

4.1.6 Cdd_GeneratingVertices (for IsCddPolyhedron)

▷ Cdd_GeneratingVertices(P)

(attribute)

Returns: The reduced generating vertices of the polyhedron

4.1.7 Cdd_GeneratingRays (for IsCddPolyhedron)

▷ Cdd_GeneratingRays(P)

(attribute)

Returns: list

The output is the reduced generating rays of the polyhedron

4.1.8 Cdd_Equalities (for IsCddPolyhedron)

▷ Cdd_Equalities(P)

(attribute)

Returns: a list

The output is the reduced equalities of the polyhedron.

4.1.9 Cdd_Inequalities (for IsCddPolyhedron)

▷ Cdd_Inequalities(P)

(attribute)

Returns:

The output is the reduced inequalities of the polyhedron.

4.1.10 Cdd_InteriorPoint (for IsCddPolyhedron)

▷ Cdd_InteriorPoint(P)

(attribute)

Returns: a list

The output is an interior point in the polyhedron

4.1.11 Cdd_Faces (for IsCddPolyhedron)

▷ Cdd_Faces(P)

(attribute)

Returns: a list

This function takes a H-represented polyhedron *poly* and returns a list. Every entry in this list is a again a list, contains the dimension and linearity of the face defined as a polyhedron over the same system of inequalities.

4.1.12 Cdd_FacesWithInteriorPoints (for IsCddPolyhedron)

▷ Cdd_FacesWithInteriorPoints(P)

(attribute)

Returns: a list

This function takes a H-represented polyhedron *P* and returns a list. Every entry in this list is a again a list, contains the dimension, linearity of the face defined as a polyhedron over the same system of inequalities and an interior point in the face.

4.1.13 Cdd_Facets (for IsCddPolyhedron)

Returns: a list

This function takes a H-represented polyhedron P and returns a list. Every entry in this list is a again a list, contains the dimension, linearity of the facet defined as a polyhedron over the same system of inequalities.

4.1.14 Cdd_Lines (for IsCddPolyhedron)

 $ightharpoonup Cdd_Lines(P)$ (attribute)

Returns: a list

This function takes a H-represented polyhedron *P* and returns a list. Every entry in this list is the linearity of a line defined as a polyhedron over the same system of inequalities.

4.1.15 Cdd_Vertices (for IsCddPolyhedron)

▷ Cdd_Vertices(P) (attribute)

Returns: a list

This function takes a H-represented polyhedron *P* and returns a list. Every entry in this list is the linearity of a vertex defined as a polyhedron over the same system of inequalities.

4.1.16 Cdd_IsEmpty (for IsCddPolyhedron)

 \triangleright Cdd_IsEmpty(P) (property)

Returns: true or false

The output is *true* if the polyhedron is empty and *false* otherwise

4.1.17 Cdd_IsCone (for IsCddPolyhedron)

 $ightharpoonup Cdd_IsCone(P)$ (property)

Returns: true or false

The output is *true* if the polyhedron is cone and *false* otherwise

4.1.18 Cdd_IsPointed (for IsCddPolyhedron)

 \triangleright Cdd_IsPointed(P) (property)

Returns: true or false

The output is true if the polyhedron is pointed and false otherwise

```
Example

gap> poly:= Cdd_PolyhedronByInequalities([[1, 3, 4, 5, 7], [1, 3, 5, 12, 34],

> [9, 3, 0, 2, 13]], [1]);

<Polyhedron given by its H-representation>
gap> Cdd_InteriorPoint( poly );
[-194/75, 46/25, -3/25, 0]
gap> Cdd_FacesWithInteriorPoints( poly );
[[3, [1], [-194/75, 46/25, -3/25, 0]], [2, [1, 2],
[-62/25, 49/25, -7/25, 0]], [1, [1, 2, 3],
[-209/75, 56/25, -8/25, 0]], [2, [1, 3], [-217/75, 53/25, -4/25, 0]]]
gap> Cdd_Dimension( poly );
```

```
gap> Cdd_IsPointed( poly );
false
gap> Cdd_IsEmpty( poly );
false
gap> Cdd_Faces( poly );
[[3,[1]],[2,[1,2]],[1,[1,2,3]],[2,[1,3]]]
gap> poly1:= Cdd_ExtendLinearity( poly, [1,2,3] );
<Polyhedron given by its H-representation>
gap> Display( poly1 );
H-representation
linearity 3, [ 1, 2, 3 ]
begin
  3 X 5 rational
             5
                 7
      3
          4
          5 12 34
  1
      3
  9
      3
          0
              2 13
gap> Cdd_Dimension( poly1 );
gap> Cdd_Facets( poly );
[[2,[1,2]],[2,[1,3]]]
gap> Cdd_GeneratingVertices( poly );
[ [ -209/75, 56/25, -8/25, 0 ] ]
gap> Cdd_GeneratingRays( poly );
[[-97, 369, -342, 75], [-8, -9, 12, 0],
[ 23, -21, 3, 0 ], [ 97, -369, 342, -75 ] ]
gap> Cdd_Inequalities( poly );
[[1, 3, 5, 12, 34], [9, 3, 0, 2, 13]]
gap> Cdd_Equalities( poly );
[[1,3,4,5,7]]
gap> P := Cdd_FourierProjection( poly, 2);
<Polyhedron given by its H-representation>
gap> Display( P );
H-representation
linearity 1, [3]
begin
  3 X 5 rational
        3
             0
                 2
                     13
       -3
   -1
             0
                23 101
   0
        0
                 0
                      0
             1
end
```

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