

FactInt

Advanced Methods for Factoring Integers

Version 1.5.0

September 19, 2007

Stefan Kohl

Stefan Kohl — Email: kohl@mathematik.uni-stuttgart.de

— Homepage: <http://www.cip.mathematik.uni-stuttgart.de/~kohlsn/>

— Address: Institut für Geometrie und Topologie

Pfaffenwaldring 57

Universität Stuttgart

70550 Stuttgart

Germany

Abstract

This package for GAP 4 provides a general-purpose integer factorization routine, which makes use of a combination of factoring methods. In particular it contains implementations of the following algorithms:

- Pollard's $p - 1$
- Williams' $p + 1$
- The Elliptic Curves Method (ECM)
- The Continued Fraction Algorithm (CFRAC)
- The Multiple Polynomial Quadratic Sieve (MPQS)

It also contains code by Frank Lübeck for making use of Richard P. Brent's tables of factors of integers of the form $b^k \pm 1$. FactInt is completely written in the GAP language and contains / requires no external binaries. It needs GAPDoc 1.0 [LN07] or higher. FactInt must be installed in the `pkg` subdirectory of the GAP distribution.

Copyright

© 1999 - 2007 by Stefan Kohl. This package is distributed under the GNU General Public License.

Acknowledgements

I would like to thank Bettina Eick and Steve Linton for their support and many interesting discussions.

Contents

1	Preface	4
2	The General Factorization Routine	5
2.1	The method for <code>Factors</code>	5
2.1.1	<code>Factors</code> (<code>FactInt</code> 's method, for integers)	5
2.1.2	<code>FactInt</code> (factorization of an integer)	6
2.2	Getting information about the factoring process	7
2.2.1	<code>IntegerFactorizationInfo</code> (<code>FactInt</code> 's <code>Info</code> class)	7
3	The Routines for Specific Factorization Methods	12
3.1	Trial Division	12
3.1.1	<code>FactorsTD</code> (trial division)	12
3.2	Pollard's $p - 1$	12
3.2.1	<code>FactorsPminus1</code> (Pollard's $p-1$)	12
3.3	Williams' $p + 1$	13
3.3.1	<code>FactorsPplus1</code> (Williams' $p+1$)	13
3.4	The Elliptic Curves Method (ECM)	14
3.4.1	<code>FactorsECM</code> (Elliptic Curves Method, ECM)	14
3.5	The Continued Fraction Algorithm (CFRAC)	15
3.5.1	<code>FactorsCFRAC</code> (Continued Fraction Algorithm, CFRAC)	15
3.6	The Multiple Polynomial Quadratic Sieve (MPQS)	16
3.6.1	<code>FactorsMPQS</code> (Multiple Polynomial Quadratic Sieve, MPQS)	16
4	How much Time does a Factorization take?	17
4.1	Timings for the general factorization routine	17
4.2	Timings for the ECM	17
4.3	Timings for the MPQS	18

Chapter 1

Preface

Factoring large integers is a computationally very difficult problem, and there is no general factorization algorithm known which can be used for factoring products of two primes with more than about 100 decimal digits each on currently existing computers. But there are methods (not algorithms in the sense that it is guaranteed that they will give the desired result after a finite number of steps) for factoring integers with prime factors being far beyond the range where trial division is feasible.

One important class of such methods is based on exponentiation in suitably chosen groups acting on subsets of the k -fold cartesian product of the set of residue classes $(\text{mod } n)$, where n denotes the number to be factored. Representatives of this class are the Elliptic Curves Method (ECM), Pollard's $p - 1$ and Williams' $p + 1$. These methods have the important property that they find smaller factors usually considerably faster than larger ones. This however comes along with the drawback of suboptimal performance for large factors.

The other important class consists of the so-called factor base methods. Their runtime depends only on the size of the number n to be factored, and not on the size of its factors. Factor base methods compute factorizations of perfect squares $(\text{mod } n)$ over an appropriately chosen factor base. A factor base is a set of small prime numbers, or of prime ideals in the case of the Generalized Number Field Sieve. They use these factorizations to determine a pair of integers (x, y) such that x^2 and y^2 are congruent $(\text{mod } n)$, but $\pm x$ and $\pm y$ are not. In this situation, taking $\text{Gcd}(n, x - y)$ will yield a non-trivial divisor of n . Representatives of this class are the Continued Fraction Algorithm (CFRAC), the Multiple Polynomial Quadratic Sieve (MPQS) and the already mentioned Generalized Number Field Sieve (GNFS). The latter has the asymptotically lowest average-case complexity of all factoring methods known today. It has however the drawback of being more efficient than the MPQS only for numbers with more than about 100 decimal digits, which is probably not within the feasible range of such a function implemented in GAP. The first two methods are implemented in this package.

The only method which I am aware of (except of the “naive” ones like trial division and some “historical” methods) that does not fit into one of the two classes mentioned above is Pollard's Rho, which is already implemented in the GAP Library.

With respect to the current state-of-the-art in integer factorization, see the [Factoring Challenge](#) of the RSA Laboratories.

Chapter 2

The General Factorization Routine

2.1 The method for `Factors`

The `FactInt` package provides a better method for the operation `Factors` for integer arguments, which supersedes the one included in the GAP Library:

2.1.1 `Factors` (`FactInt`'s method, for integers)

◇ `Factors(n)` (method)

Returns: A sorted list of the prime factors of n .

If the method fails to compute the prime factorization of n , an error is signalled. The returned factors pass the built-in probabilistic primality test of GAP (`IsProbablyPrimeInt`, see the GAP Reference Manual). The same holds for all other factorization routines provided by this package.

It follows a rough description how the factorization method works:

First of all it checks whether $n = b^k \pm 1$ for some b, k and looks for factors corresponding to polynomial factors of $x^k \pm 1$. Provided that b and k are sufficiently small, factors that do not correspond to polynomial factors are taken from Richard P. Brent's Factor Tables [Bre04]. The code for accessing these tables has been contributed by Frank Lübeck.

Then the method uses trial division and a number of cheap methods for various special cases. After the small and other “easy” factors have been found this way, `FactInt`'s method searches for “medium-sized” factors using Pollard's Rho (by the library function `FactorsRho`, see the GAP Reference Manual), Pollard's $p - 1$ (see `FactorsPminus1` (3.2.1)), Williams' $p + 1$ (see `FactorsPplus1` (3.3.1)) and the Elliptic Curves Method (ECM, see `FactorsECM` (3.4.1)) in this order.

If there is still an unfactored part remaining after that, it is factored using the Multiple Polynomial Quadratic Sieve (MPQS, see `FactorsMPQS` (3.6.1)).

The following options are interpreted:

TDHints A list of additional trial divisors. This is useful only if certain primes p are expected to divide n with probability significantly larger than $\frac{1}{p}$.

RhoSteps The number of steps for Pollard's Rho.

RhoCluster The number of steps between two gcd computations in Pollard's Rho.

Pminus1Limit1 / Pminus1Limit2 The first- / second stage limit for Pollard's $p - 1$ (see `FactorsPminus1` (3.2.1)).

Pplus1Residues The number of residues to be tried by Williams' $p + 1$ (see `FactorsPplus1` (3.3.1)).

Pplus1Limit1 / Pplus1Limit2 The first- / second stage limit for Williams' $p + 1$ (see `FactorsPplus1` (3.3.1)).

ECMCurves The number of elliptic curves to be tried by the Elliptic Curves Method (ECM) (see `FactorsECM` (3.4.1)). Also admissible: a function that takes the number n to be factored as an argument and returns the desired number of curves to be tried.

ECMLimit1 / ECMLimit2 The initial first- / second stage limit for ECM (see `FactorsECM` (3.4.1)).

ECMDelta The increment per curve for the first stage limit in ECM. The second stage limit is adjusted appropriately (see `FactorsECM` (3.4.1)).

ECMDeterministic If true, ECM chooses its curves deterministically, i.e. repeatable (see `FactorsECM` (3.4.1)).

FBMethod Specifies which of the factor base methods should be used to do the “hard work”. Currently implemented: "CFRAC" and "MPQS" (see `FactorsCFRAC` (3.5.1) and `FactorsMPQS` (3.6.1), respectively). Default: "MPQS".

For the use of the GAP Options Stack, see Chapter *Options Stack* in the GAP Reference Manual.

Setting `RhoSteps`, `Pminus1Limit1`, `Pplus1Residues`, `Pplus1Limit1`, `ECMCurves` or `ECMLimit1` equal to zero switches the respective method off. The method chooses defaults for all option values that are not explicitly set by the user. The option values are also interpreted by the routines for the particular factorization methods described in the next chapter.

Example

```
gap> Factors( Factorial(44) + 1 );
[ 694763, 9245226412016162109253, 413852053257739876455072359 ]
gap> Factors( 2^997 - 1 );
[ 167560816514084819488737767976263150405095191554732902607,
  7993430605360222292860936960123884061988016846627213757686887976005930025638\
  602973712891518592878944687759622084106508783413855778177367022158878920741413\
  700868182301410439178049533828082651513160945607018874830040978453228378816647\
  358334681553 ]
```

The above method for `Factors` calls the following function, which is the actual “working horse” of this package:

2.1.2 FactInt (factorization of an integer)

◇ **FactInt** (n) (function)

Returns: A list of two lists, where the first list contains the determined prime factors of n and the second list contains the remaining unfactored parts of n , if there are any.

This function interprets all options which are interpreted by the method for `Factors` described above. In addition, it interprets the options `cheap` and `FactIntPartial`. If the option `cheap` is set, only usually cheap factorization attempts are made. If the option `FactIntPartial` is set, the factorization process is stopped before invoking the (usually time-consuming) MPQS or CFRAC,

if the number of digits of the remaining unfactored part exceeds the bound passed as option value *MPQSLimit* or *CFRACLimit*, respectively.

`Factors(n)` is equivalent to `FactInt(n:cheap:=false, FactIntPartial:=false)[1]`.

Example

```
gap> FactInt( Factorial(300) + 1 : cheap );
[ [ 461, 259856122109, 995121825812791, 3909669044842609, 4220826953750952739,
    14841043839896940772689086214475144339 ],
  [ 10483128823176572317398383656043859405333629662907393256352061868792876450\
580106888272460615410656311193456740818340859600641445970372439235869682208979\
384309498719255615067943353399357029226058930732298505581697749539842674165663\
346174704662364145104265524709331550541782037094517458717017420005463846144727\
565841824785318809625948572758696907279733563594352516014206081210368516157890\
709802912711149521530885498556124466779020824562030140449992853222252458594688\
152833725706178959319799211283640357942345263781351 ] ]
```

2.2 Getting information about the factoring process

Optionally, the FactInt package prints information on the progress of the factorization process:

2.2.1 IntegerFactorizationInfo (FactInt's Info class)

◇ IntegerFactorizationInfo

(info class)

◇ InfoFactInt

(info class)

This Info class allows to monitor what happens during the factoring process. For a description of the Info mechanism, see Section *Info Functions* in the GAP Reference Manual.

If `InfoLevel(IntegerFactorizationInfo) = 1`, then basic information about the factoring techniques used is displayed. If this InfoLevel has value 2, then additionally all “relevant” steps in the factoring algorithms are mentioned. If it is set equal to 3, then large amounts of details of the progress of the factoring process are shown. The informational output is usually not literally the same in each factorization attempt to a given integer with given parameters.

For convenience, `FactIntInfo(level)` sets the InfoLevel of IntegerFactorizationInfo to the positive integer *level*. `FactIntInfo(level);` is equivalent to `SetInfoLevel(IntegerFactorizationInfo, level);`. There is a shorthand `FactInfo` for `FactIntInfo`.

The timings in the example below are given for a Pentium 200 under Windows as a reference machine (such a machine was a fast PC at the time the first version of this package has been written).

Example

```
gap> FactIntInfo(2);
gap> Factors(1459^60-1);
#I
#I Check for n = b^k +/- 1
#I
697136005439518321339364341462731479643931164718338822169985761598533184155968\
687683267309071831767783053294661533309653344501590527857911683460652704454392\
2114173712945384179913214273324400 = 1459^60 - 1
#I The factors corresponding to polynomial factors are
```

```

[ 1458, 1460, 2127223, 2128682, 2130141, 4528179181405, 4531280671081,
  4534390675481, 20518450806493943781446161, 20532514165758816624282961,
  20546596816316767296268561,
  421584732115767299535558510031830342788459718258721 ]
#I Intermediate result : [ [ 4531280671081, 20546596816316767296268561 ],
  [ 1458, 1460, 2127223, 2128682, 2130141, 4528179181405, 4534390675481,
    20518450806493943781446161, 20532514165758816624282961,
    421584732115767299535558510031830342788459718258721 ] ]
#I
#I Factors already found : [ 4531280671081, 20546596816316767296268561 ]
#I
#I Trial division by all primes p < 1000
#I Intermediate result : [ [ 2, 3, 3, 3, 3, 3, 3 ], [ ] ]
#I Intermediate result : [ [ 2, 2, 5, 73 ], [ ] ]
#I Intermediate result : [ [ 7, 303889 ], [ ] ]
#I Intermediate result : [ [ 2, 1064341 ], [ ] ]
#I Intermediate result : [ [ 3, 13, 193, 283 ], [ ] ]
#I Intermediate result : [ [ 5, 11, 521, 158024051 ], [ ] ]
#I Intermediate result : [ [ 31, 146270666951 ], [ ] ]
#I Intermediate result : [ [ 61 ], [ 336368046008097439040101 ] ]
#I
#I Factors already found : [ 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 5, 5, 7, 11,
  13, 31, 61, 73, 193, 283, 521, 303889, 1064341, 158024051, 146270666951,
  4531280671081, 20546596816316767296268561 ]
#I
#I Trial division by some already known primes
#I
#I Check for perfect powers
#I
#I Pollard's Rho
Steps = 16384, Cluster = 1638
Number to be factored :
336368046008097439040101
#I Intermediate result : [ [ 2251, 149430495783250750351 ], [ ] ]
#I
#I Pollard's Rho
Steps = 16384, Cluster = 1638
Number to be factored :
20532514165758816624282961
#I
#I Pollard's Rho
Steps = 16384, Cluster = 1638
Number to be factored :
421584732115767299535558510031830342788459718258721
#I
#I Factors already found : [ 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 5, 5, 7, 11,
  13, 31, 61, 73, 193, 283, 521, 2251, 303889, 1064341, 158024051,
  146270666951, 4531280671081, 149430495783250750351,
  20546596816316767296268561 ]
#I
#I Pollard's p - 1
Limit1 = 10000, Limit2 = 400000
Number to be factored :

```



```

20532514165758816624282961
#I Initializing prime differences list, PrimeDiffLimit = 1000000
#I p-1 for n = 20532514165758816624282961
a : 2, Limit1 : 10000, Limit2 : 400000
#I First stage
#I Second stage
#I
#I Pollard's p - 1
Limit1 = 10000, Limit2 = 400000
Number to be factored :
421584732115767299535558510031830342788459718258721
#I p-1 for n = 421584732115767299535558510031830342788459718258721
a : 2, Limit1 : 10000, Limit2 : 400000
#I First stage
#I Second stage
#I
#I Williams' p + 1
Residues = 2, Limit1 = 2000, Limit2 = 80000
Number to be factored :
20532514165758816624282961
#I p+1 for n = 20532514165758816624282961
Residues : 2, Limit1 : 2000, Limit2 : 80000
#I Residue no. 1
#I Residue no. 2
#I
#I Williams' p + 1
Residues = 2, Limit1 = 2000, Limit2 = 80000
Number to be factored :
421584732115767299535558510031830342788459718258721
#I p+1 for n = 421584732115767299535558510031830342788459718258721
Residues : 2, Limit1 : 2000, Limit2 : 80000
#I Residue no. 1
#I Residue no. 2
#I
#I Elliptic Curves Method (ECM)
Curves = <func.>
Init. Limit1 = <func.>, Init. Limit2 = <func.>, Delta = <func.>
Number to be factored :
20532514165758816624282961
#I ECM for n = 20532514165758816624282961
#I Digits : 26, Curves : 3
#I Initial Limit1 : 200, Initial Limit2 : 20000, Delta : 200
#I Curve no.      1 (      3), Limit1 :      200, Limit2 :      20000
#I Curve no.      2 (      3), Limit1 :      400, Limit2 :      40000
#I 11-digit factor 12077430061 was found in second stage
#I Intermediate result : [ [ 12077430061, 1700073116718901 ], [ ] ]
#I
#I Elliptic Curves Method (ECM)
Curves = <func.>
Init. Limit1 = <func.>, Init. Limit2 = <func.>, Delta = <func.>
Number to be factored :
421584732115767299535558510031830342788459718258721
#I ECM for n = 421584732115767299535558510031830342788459718258721

```

```

#I Digits : 51, Curves : 27
#I Initial Limit1 : 200, Initial Limit2 : 20000, Delta : 200
#I Curve no.      1 ( 27), Limit1 : 200, Limit2 : 20000
#I Curve no.      2 ( 27), Limit1 : 400, Limit2 : 40000
#I Curve no.      3 ( 27), Limit1 : 600, Limit2 : 60000
#I               [ ... ]
#I Curve no.     25 ( 27), Limit1 : 5000, Limit2 : 500000
#I Curve no.     26 ( 27), Limit1 : 5200, Limit2 : 520000
#I Curve no.     27 ( 27), Limit1 : 5400, Limit2 : 540000
#I
#I Factors already found : [ 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 5, 5, 7, 11,
13, 31, 61, 73, 193, 283, 521, 2251, 303889, 1064341, 158024051,
12077430061, 146270666951, 4531280671081, 1700073116718901,
149430495783250750351, 20546596816316767296268561 ]
#I
#I Multiple Polynomial Quadratic Sieve (MPQS)
Number to be factored :
421584732115767299535558510031830342788459718258721
#I MPQS for n = 421584732115767299535558510031830342788459718258721
#I Digits : 51
#I Multiplier : 1
#I Size of factor base : 1326
#I Prime powers to be sieved : 1349
#I Length of sieving interval : 131072
#I Small prime limit : 117
#I Large prime limit : 3500186
#I Number of used a-factors : 4
#I Size of a-factors pool : 52
#I Initialization time : 11 sec.
#I
#I Sieving
#I
#I Complete factorizations over the factor base : 50
#I Relations with a large prime factor : 5
#I Relations remaining to be found : 1343
#I Total factorizations with a large prime factor : 614
#I Used polynomials : 67
#I Elapsed runtime : 39 sec.
#I Progress (relations) : 3 %
#I
#I               [ ... ]
#I
#I Complete factorizations over the factor base : 651
#I Relations with a large prime factor : 826
#I Relations remaining to be found : 0
#I Total factorizations with a large prime factor : 9164
#I Used polynomials : 1021
#I Elapsed runtime : 455 sec.
#I Progress (relations) : 100 %
#I
#I Creating the exponent matrix
#I Doing Gaussian Elimination, #rows = 1477, #columns = 1378
#I Processing the zero rows

```

```

#I Dependency no. 1 yielded no factor
    [ ... ]
#I Dependency no. 8 yielded no factor
#I Dependency no. 9 yielded factor 26725378467920336641
#I The factors are
[ 26725378467920336641, 15774696422795817479975816558881 ]
#I Digit partition : [ 20, 32 ]
#I MPQS runtime : 473.507 sec.

#I Intermediate result :
[ [ 26725378467920336641, 15774696422795817479975816558881 ], [ ] ]
#I
#I The result is
[ [ 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3, 5, 5, 7, 11, 13, 31, 61, 73, 193, 283,
    521, 2251, 303889, 1064341, 158024051, 12077430061, 146270666951,
    4531280671081, 1700073116718901, 26725378467920336641,
    149430495783250750351, 20546596816316767296268561,
    15774696422795817479975816558881 ], [ ] ]

#I The total runtime was 799.408 sec

[ 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3, 5, 5, 7, 11, 13, 31, 61, 73, 193, 283,
  521, 2251, 303889, 1064341, 158024051, 12077430061, 146270666951,
  4531280671081, 1700073116718901, 26725378467920336641,
  149430495783250750351, 20546596816316767296268561,
  15774696422795817479975816558881 ]

```

Chapter 3

The Routines for Specific Factorization Methods

Descriptions of the factoring methods implemented in this package can be found in [Bre89] and [Coh93]. The last book contains also descriptions of the other methods mentioned in the preface.

3.1 Trial Division

3.1.1 FactorsTD (trial division)

◇ `FactorsTD(n[, Divisors])` (function)

Returns: A list of two lists: The first list contains the prime factors found, and the second list contains remaining unfactored parts of n , if there are any.

This function tries to factor n using Trial Division. The optional argument *Divisors* is the list of trial divisors. If not given, it defaults to the list of primes $p < 1000$.

Example

```
gap> FactorsTD(12^25+25^12);  
[ [ 13, 19, 727 ], [ 5312510324723614735153 ] ]
```

3.2 Pollard's $p - 1$

3.2.1 FactorsPminus1 (Pollard's p-1)

◇ `FactorsPminus1(n[, a], Limit1[, Limit2])` (function)

Returns: A list of two lists: The first list contains the prime factors found, and the second list contains remaining unfactored parts of n , if there are any.

This function tries to factor n using Pollard's $p - 1$. It uses a as base for exponentiation, *Limit1* as first stage limit and *Limit2* as second stage limit. If the function is called with three arguments, these arguments are interpreted as n , *Limit1* and *Limit2*. Defaults are chosen for all arguments which are omitted.

Pollard's $p - 1$ is based on the fact that exponentiation (mod n) can be done efficiently enough to compute $a^{k!} \bmod n$ for sufficiently large k in a reasonable amount of time. Assume that p is a prime

factor of n which does not divide a , and that $k!$ is a multiple of $p - 1$. Then Lagrange's Theorem states that $a^{k!} \equiv 1 \pmod{p}$. If $k!$ is not a multiple of $q - 1$ for another prime factor q of n , it is likely that the factor p can be determined by computing $\gcd(a^{k!} - 1, n)$. A prime factor p is usually found if the largest prime factor of $p - 1$ is not larger than $Limit2$, and the second-largest one is not larger than $Limit1$. (Compare with `FactorsPplus1` (3.3.1)" and `FactorsECM` (3.4.1).)

Example

```
gap> FactorsPminus1( Factorial(158) + 1, 100000, 1000000 );
[ [ 2879, 5227, 1452486383317, 9561906969931, 18331561438319,
    483714299709483760811581110341732950506493218122654853400674921345082310\
    906370452295654816571305041217323052879842924826121333143254713674832962773107\
    806789945715570386038565256719614524924705165110048148716160964980629081176057\
    0095669 ], [ ] ]
gap> List( last[ 1 ][[ 3, 4, 5 ]], p -> Factors( p - 1 ) );
[ [ 2, 2, 3, 3, 81937, 492413 ], [ 2, 3, 3, 3, 5, 7, 7, 1481, 488011 ],
  [ 2, 3001, 7643, 399613 ] ]
```

3.3 Williams' $p + 1$

3.3.1 FactorsPplus1 (Williams' $p+1$)

◇ `FactorsPplus1(n [, Residues], Limit1[, Limit2])` (function)

Returns: A list of two lists: The first list contains the prime factors found, and the second list contains remaining unfactored parts of n , if there are any.

This function tries to factor n using Williams' $p + 1$. It tries *Residues* different residues, and uses *Limit1* as first stage limit and *Limit2* as second stage limit. If the function is called with three arguments, these arguments are interpreted as n , *Limit1* and *Limit2*. Defaults are chosen for all arguments which are omitted.

Williams' $p + 1$ is very similar to Pollard's $p - 1$ (see `FactorsPminus1` (3.2.1)). The difference is that the underlying group here can either have order $p + 1$ or $p - 1$, and that the group operation takes more time. A prime factor p is usually found if the largest prime factor of the group order is at most *Limit2* and the second-largest one is not larger than *Limit1*. (Compare also with `FactorsECM` (3.4.1).)

Example

```
gap> FactorsPplus1( Factorial(55) - 1, 10, 10000, 100000 );
[ [ 73, 39619, 277914269, 148257413069 ],
  [ 106543529120049954955085076634537262459718863957 ] ]
gap> List( last[ 1 ], p -> [ Factors( p - 1 ), Factors( p + 1 ) ] );
[ [ [ 2, 2, 2, 3, 3 ], [ 2, 37 ] ],
  [ [ 2, 3, 3, 31, 71 ], [ 2, 2, 5, 7, 283 ] ],
  [ [ 2, 2, 2207, 31481 ], [ 2, 3, 5, 9263809 ] ],
  [ [ 2, 2, 47, 788603261 ], [ 2, 3, 5, 13, 37, 67, 89, 1723 ] ] ]
```

3.4 The Elliptic Curves Method (ECM)

3.4.1 FactorsECM (Elliptic Curves Method, ECM)

◇ `FactorsECM(n[, Curves[, Limit1[, Limit2[, Delta]]]])` (function)

◇ `ECM(n[, Curves[, Limit1[, Limit2[, Delta]]]])` (function)

Returns: A list of two lists: The first list contains the prime factors found, and the second list contains remaining unfactored parts of n , if there are any.

This function tries to factor n using the Elliptic Curves Method (ECM). The argument *Curves* is the number of curves to be tried. The argument *Limit1* is the initial first stage limit, and *Limit2* is the initial second stage limit. The argument *Delta* is the increment per curve for the first stage limit. The second stage limit is adjusted appropriately. Defaults are chosen for all arguments which are omitted.

`FactorsECM` recognizes the option *ECMDeterministic*. If set, the choice of the curves is deterministic. This means that in repeated runs of `FactorsECM` the same curves are used, and hence for the same n the same factors are found after the same number of trials.

The Elliptic Curves Method is based on the fact that exponentiation in the elliptic curve groups $E(a,b)/n$ can be performed fast enough to compute for example $g^{k!}$ for k large enough (e.g. 100000 or so) in a reasonable amount of time and without using much memory, and on Lagrange's Theorem. Assume that p is a prime divisor of n . Then Lagrange's Theorem states that if $k!$ is a multiple of $|E(a,b)/p|$, then for any elliptic curve point g , the power $g^{k!}$ is the identity element of $E(a,b)/p$. In this situation – under reasonable circumstances – the factor p can be determined by taking an appropriate gcd.

In practice, the algorithm chooses in some sense “better” products P_k of small primes rather than $k!$ as exponents. After reaching the first stage limit with P_{Limit1} , it considers further products $P_{Limit1}q$ for primes q up to the second stage limit *Limit2*, which is usually set equal to something like 100 times the first stage limit. The prime q corresponds to the largest prime factor of the order of the group $E(a,b)/p$.

A prime divisor p is usually found if the largest prime factor of the order of one of the examined elliptic curve groups $E(a,b)/p$ is at most *Limit2* and the second-largest one is at most *Limit1*. Thus trying a larger number of curves increases the chance of factoring n as well as choosing a larger value for *Limit1* and/or *Limit2*. It turns out to be not optimal either to try a large number of curves with very small *Limit1* and *Limit2* or to grind on with a single curve and very large limits. (Compare with `FactorsPminus1` (3.2.1).)

The elements of the group $E(a,b)/n$ are the points (x,y) given by the solutions of $y^2 = x^3 + ax + by$ in the residue class ring $(\text{mod } n)$, and an additional point ∞ at infinity, which serves as identity element. To turn this set into a group, define the product (although elliptic curve groups are usually written additively, I prefer using the multiplicative notation here to retain the analogy to `FactorsPminus1` (3.2.1) and `FactorsPplus1` (3.3.1)) of two points p_1 and p_2 as follows: If $p_1 \neq p_2$, let l be the line through p_1 and p_2 , otherwise let l be the tangent to the curve C given by the above equation in the point $p_1 = p_2$. The line l intersects C in a third point, say p_3 . If l does not intersect the curve in a third affine point, then set $p_3 := \infty$. Define $p_1 \cdot p_2$ by the image of p_3 under the reflection across the x -axis. Define the product of any curve point p and ∞ by p itself. This – more or less obviously, checking associativity requires some calculation – turns the set of points on the given curve into an abelian group $E(a,b)/n$.

However, the calculations are done in projective coordinates to have an explicit representation of the identity element and to avoid calculating inverses $(\text{mod } n)$ for the group operation. Otherwise this

would require using an $O((\log n)^3)$ -algorithm, while multiplication (mod n) is only $O((\log n)^2)$. The corresponding equation is given by $bY^2Z = X^3 + aX^2Z + XZ^2$. This form allows even more efficient computations than the Weierstrass model $Y^2Z = X^3 + aXZ^2 + bZ^3$, which is the projective equivalent of the affine representation $y^2 = x^3 + ax + by$ mentioned above. The algorithm only keeps track of two of the three coordinates, namely X and Z . The curves are chosen in a way that ensures the order of the corresponding group to be divisible by 12. This increases the chance that it is smooth enough to find a factor of n . The implementation follows the description of R. P. Brent given in [Bre96], pp. 5 – 8. In terms of this paper, for the second stage the “improved standard continuation” is used.

Example

```
gap> FactorsECM(2^256+1,100,10000,1000000,100);
[ [ 1238926361552897,
    9346163971535797769163558199606896584051237541638188580280321 ], [ ] ]
```

3.5 The Continued Fraction Algorithm (CFRAC)

3.5.1 FactorsCFRAC (Continued Fraction Algorithm, CFRAC)

◇ FactorsCFRAC(n)

(function)

◇ CFRAC(n)

(function)

Returns: A list of the prime factors of n .

This function tries to factor n using the Continued Fraction Algorithm (CFRAC), also known as Brillhart-Morrison Algorithm. In case of failure an error is signalled. The same warning applies as to the Quadratic Sieve (see FactorsMPQS (3.6.1)).

The Continued Fraction Algorithm tries to find integers x and y such that $x^2 \equiv y^2 \pmod{n}$, but not $\pm x \equiv \pm y \pmod{n}$. In this situation, taking $\gcd(x - y, n)$ yields a nontrivial factor of n . For determining such a pair (x, y) , the algorithm uses the continued fraction expansion of the square root of n . If x_i/y_i is a continued fraction approximation of the square root of n , then $c_i := x_i^2 - ny_i^2$ is bounded by a small constant times the square root of n . The algorithm tries to find as many c_i as possible which factor completely over a chosen factor base (a list of small primes) or with only one factor not in the factor base. The latter ones can be used if and only if a second c_i with the same “large” factor is found. Once enough values c_i have been factored, as a final stage Gaussian Elimination over GF(2) is used to determine which of the congruences $x_i^2 \equiv c_i \pmod{n}$ have to be multiplied together to obtain a congruence of the desired form $x^2 \equiv y^2 \pmod{n}$. Let M be the corresponding matrix. Then the entries of M are given by $M_{ij} = 1$ if an odd power of the j -th element of the factor base divides the i -th usable factored value, and $M_{ij} = 0$ otherwise. To obtain the desired congruence, it is necessary that the rows of M are linearly dependent. In other words, this means that the number of factored c_i needs to be larger than the rank of M , which is approximately given by the size of the factor base. (Compare with FactorsMPQS (3.6.1).)

Example

```
gap> FactorsCFRAC( Factorial(34) - 1 );
[ 10398560889846739639, 28391697867333973241 ]
```

3.6 The Multiple Polynomial Quadratic Sieve (MPQS)

3.6.1 FactorsMPQS (Multiple Polynomial Quadratic Sieve, MPQS)

◇ FactorsMPQS(*n*)

(function)

◇ MPQS(*n*)

(function)

Returns: A list of the prime factors of *n*.

This function tries to factor *n* using the Single Large Prime Variation of the Multiple Polynomial Quadratic Sieve (MPQS). In case of failure an error is signalled.

Caution: The runtime of this function depends only on the size of *n*, not on the size of the factors. Thus if a small factor is not found during the preprocessing which is done before invoking the sieving process, the runtime is as long as if *n* would have two prime factors of roughly equal size.

The intermediate results of a computation can be saved by interrupting it with [Ctrl][C] and calling `Pause()`; from the break loop. This causes all data needed for resuming the computation again to be pushed as a record *MPQSTmp* on the options stack. When called again with the same argument *n*, `FactorsMPQS` takes the record from the options stack and uses the previously computed factorization data. For continuing the factorization process in another session, one needs to write this record to a file. This can be done by the function `SaveMPQSTmp(filename)`. The file written by this function can be read by the standard `Read`-function of GAP.

The Multiple Polynomial Quadratic Sieve tries to find integers *x* and *y* such that $x^2 \equiv y^2 \pmod{n}$, but not $\pm x \equiv \pm y \pmod{n}$. In this situation, taking $\gcd(x - y, n)$ yields a non-trivial factor of *n*. For determining such a pair (*x*, *y*), the algorithm chooses polynomials f_a of the form $f_a(r) = ar^2 + 2br + c$ with suitably chosen coefficients *a*, *b* and *c* which satisfy $b^2 \equiv n \pmod{a}$ and $c = (b^2 - n)/a$. The identity $a \times f_a(r) = (ar + b)^2 - n$ yields a congruence (mod *n*) with a perfect square on one side and $a \times f_a(r)$ on the other. The algorithm uses a sieving technique similar to the Sieve of Eratosthenes over an appropriately chosen sieving interval to search for factorizations of values $f_a(r)$ over a chosen factor base. Any two factorizations with the same single “large” factor which does not belong to the factor base can also be used. Taking more polynomials and hence shorter sieving intervals yields the advantage of having to factor smaller values $f_a(r)$ over the factor base.

Once enough values $f_a(r)$ have been factored, as a final stage Gaussian Elimination over GF(2) is used to determine which congruences have to be multiplied together to obtain a congruence of the desired form $x^2 \equiv y^2 \pmod{n}$. Let *M* be the corresponding matrix. Then the entries of *M* are given by $M_{ij} = 1$ if an odd power of the *j*-th element of the factor base divides the *i*-th usable factored value, and $M_{ij} = 0$ otherwise. To obtain the desired congruence, it is necessary that the rows of *M* are linearly dependent. In other words, this means that the number of usable factorizations of values $f_a(r)$ needs to be larger than the rank of *M*. The latter is approximately equal to the size of the factor base. (Compare with `FactorsCFRAC` (3.5.1).)

Example

```
gap> FactorsMPQS( Factorial(38) + 1 );
[ 14029308060317546154181, 37280713718589679646221 ]
```


Chapter 4

How much Time does a Factorization take?

4.1 Timings for the general factorization routine

A few words in advance: In general, it is not possible to give a precise prediction for the CPU time needed for factoring a given integer. This time depends heavily on the sizes of the factors of the given number and on some other properties which cannot be tested before actually doing the factorization. But nevertheless, rough runtime estimates can be given for numbers with factors of given orders of magnitude.

After casting out the small and other “easy” factors – which should not take more than at most a few minutes for numbers of “reasonable” size – the general factorization routine uses first ECM (see `FactorsECM` (3.4.1)) for finding factors very roughly up to the third root of the remaining composite and then the MPQS (see `FactorsMPQS` (3.6.1)) for doing the “rest” of the work. The latter is often the most time-consuming part.

In the sequel, some timings for the ECM and for the MPQS are given. These methods are by far the most important ones with respect to runtime statistics (the $p \pm 1$ -methods (see `FactorsPminus1` (3.2.1) and `FactorsPplus1` (3.3.1)) are only suitable for finding factors with certain properties and CFRAC (see `FactorsCFRAC` (3.5.1)) is just a slower predecessor of the MPQS). All absolute timings are given for a Pentium 200 under Windows as a reference machine (such a machine was a fast PC at the time the first version of this package has been written).

4.2 Timings for the ECM

The runtime of `FactorsECM` depends mainly on the size of the factors of the input number. On average, finding a 12-digit factor of a 100-digit number takes about 1 min 40 s, finding a 15-digit factor of a 100-digit number takes about 10 min and finding an 18-digit factor of a 100-digit number takes about 50 min. A general rule of thumb is the following: one digit more increases the runtime by a bit less than a factor of two. These timings are very rough, and they may vary by a factor of 10 or more. You can compare trying an elliptic curve with throwing a couple of dice, where a success corresponds to the case where all of them show the same side – it is possible to be successful with the first trial, but it is also possible that this takes much longer. In particular, all trials are independent of one another. In general, ECM is superior to Pollard’s Rho for finding factors with at least 10 decimal digits. In the same time needed by Pollard’s Rho for finding a 13-digit factor one can reasonably

expect to find a 17-digit factor when using ECM, for which Pollard's Rho in turn would need about 100 times as long as ECM. For larger factors this difference grows rapidly. From theory it can be said that finding a 20-digit factor requires about 500 times as much work as finding a 10-digit factor, finding a 30-digit factor requires about 160 times as much work as finding a 20-digit factor and finding a 40-digit factor requires about 80 times as much work as finding a 30-digit factor.

The default parameters are optimized for finding factors with about 15 – 35 digits. This seems to be a sensible choice, since this is the most important range for the application of ECM. The function `FactorsECM` usually gives up when the input number n has two factors which are both larger than its third root. This is of course only a “probabilistic” statement. Sometimes – but seldom – the remaining composite has 3 factors, 4 factors should occur (almost) never.

The user can of course specify other parameters than the default ones, but giving timings for all possible choices is obviously impossible. The interested reader should follow the references given in the bibliography at the end of this manual for getting information on how many curves with which parameters are usually needed for finding factors of a given size. This depends mainly on the distribution of primes, respectively of numbers with prime factors not exceeding a certain bound.

For benchmarking purposes, the amount of time needed for trying a single curve with given smoothness bounds for a number of given size is suited best. A typical example is the following: one curve with $(Limit1, Limit2) = (100000, 10000000)$ applied to a 100-digit integer requires a total of 10 min 20 s, where 6 min 45 s are spent for the first stage and 3 min 35 s are spent for the second stage. The time needed for the first stage is approximately linear in $Limit1$ and the time needed for the second stage is a bit less than linear in $Limit2$.

4.3 Timings for the MPQS

The runtime of `FactorsMPQS` depends only on the size of the input number, and not on the size of its factors. Rough timings are as follows: 90 s for a 40-digit number, 10 min for a 50-digit number, 2 h for a 60-digit number, 20 h for a 70-digit number and 100 h for a 75-digit number. These timings are much more precise than those given for ECM, but they may also vary by a factor of 2 or 3 depending on whether a good factor base can be found without using a large multiplier or not. A general rule of thumb is the following: 10 digits more cause 10 times as much work. For benchmarking purposes, precise timings for some integers are given: $38! + 1$ (45 digits, good factor base with multiplier 1): 2 min 22 s, $40! - 1$ (48 digits, not so good factor base even with multiplier 7): 8 min 58 s, cofactor of $1093^{33} + 1$ (61 digits, good factor base with multiplier 1): 1 h 12 min.

References

- [Bre89] D. M. Bressoud. *Factorization and Primality Testing*. Springer-Verlag, 1989. 12
- [Bre96] R. P. Brent. Factorization of the tenth and eleventh Fermat numbers, 1996. <ftp://ftp.comlab.ox.ac.uk/pub/Documents/techpapers/Richard.Brent/rpb161tr.dvi.gz>. 15
- [Bre04] R. P. Brent. Factor tables, 2004. <http://web.comlab.ox.ac.uk/oucl/work/richard.brent/factors.html>. 5
- [Coh93] H. Cohen. *A Course in Computational Algebraic Number Theory*. Springer-Verlag, 1993. 12
- [LN07] F. Luebeck and M. Neunhoeffter. *GAPDoc (Version 1.0)*. RWTH Aachen, 2007. GAP package, <http://www.gap-system.org/Packages/gapdoc.html>. 2

Index

CFRAC

shorthand for FactorsCFRAC, [15](#)

Continued Fraction Algorithm (CFRAC), [15](#)

continued fraction approximation, [15](#)

ECM

shorthand for FactorsECM, [14](#)

elliptic curve groups, [14](#)

elliptic curve point, [14](#)

Elliptic Curves Method (ECM), [14](#)

FactInt

factorization of an integer, [6](#)

factor base, [15](#)

large factors, [15](#)

Factors

FactInt's method, for integers, [5](#)

FactorsCFRAC

Continued Fraction Algorithm, CFRAC,
[15](#)

FactorsECM

Elliptic Curves Method, ECM, [14](#)

FactorsMPQS

Multiple Polynomial Quadratic Sieve,
MPQS, [16](#)

FactorsPminus1

Pollard's $p-1$, [12](#)

FactorsPplus1

Williams' $p+1$, [13](#)

FactorsTD

trial division, [12](#)

first stage limit, [14](#)

Gaussian Elimination, [15](#)

Generalized Number Field Sieve, [4](#)

InfoFactInt

shorthand for IntegerFactorizationInfo, [7](#)

information about factoring process, [7](#)

IntegerFactorizationInfo

FactInt's Info class, [7](#)

Lagrange's Theorem, [13](#)

MPQS

shorthand for FactorsMPQS, [16](#)

Multiple Polynomial Quadratic Sieve (MPQS),
[16](#)

Pollard's $p-1$, [12](#)

Pollard's Rho, [4](#)

primality of the factors, [5](#)

prime ideal, [4](#)

projective coordinates, [14](#)

RSA Factoring Challenge, [4](#)

second stage limit, [14](#)

sieving interval, [16](#)

Trial Division, [12](#)

Weierstrass model, [15](#)

Williams' $p+1$, [13](#)