HW01

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1 Homework 1 - MA 797 (Mathematics of Radar Imaging)

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1.1 Problem 1.

For the linear chirp

$$P(t) = A(t)e^{-i\omega t}, \quad \text{where} \quad A(t) = \chi_\tau e^{-i\alpha t^2}$$

&

$$\chi_\tau(t) = \begin{cases} 1, & -\frac{\tau}{2} \leq t \leq \frac{\tau}{2} \\ 0, & \text{otherwise} \end{cases}$$

find its spectral representation given by the Fourier integral:

$$\tilde{P}(\omega) = \int_{-\infty}^{\infty} P(t)e^{i\omega t}dt = \int_{-\infty}^{\infty} \chi_\tau(t)e^{-i\alpha t^2}e^{-i\omega_o t}e^{i\omega t}dt.$$

Evaluate the Fourier integral approximately using the method of stationary phase & derive the following expression (see eqn. (2.108) in your text [1]):

$$\tilde{P}(\omega) \approx \sqrt{\frac{\pi}{\alpha}}e^{-i\frac{\pi}{4}}e^{i\frac{(\omega-\omega_o)^2}{4\alpha}}\chi_B(\omega-\omega_o),$$

where

$$\chi_B(\omega-\omega_o) = \begin{cases} 1, & |\omega-\omega_o| \leq \frac{B}{2} \\ 0, & \text{otherwise} \end{cases}.$$

When applying the method of stationary phase, disregard the transient behavior that corresponds to having the stationary point of the phase located close to or exactly at one of the endpoints of the integration interval.

Using the method of stationary phase from [4], we see that a more common way to state the principle in terms of the integral is

$$I(x) = \int_a^b g(t)e^{ixf(t)}dt,$$

where a, b, and the function $f(t)$ are real, and x is a large real parameter. And in terms of our parameters, we can rewrite the above equation as:

$$I(\omega) = \int_a^b g(t)e^{i\omega f(t)}dt \quad (1).$$

We can then rewrite $\tilde{P}(\omega)$ as, where $a = -\frac{\tau}{2}$ and $b = \frac{\tau}{2}$ from $\chi_\tau(t)$,

$$\tilde{P}(\omega) = \int_a^b 1 \cdot e^{i\omega t(1-\frac{\omega_o}{\omega}-\frac{\alpha t}{\omega})}dt,$$

where

$$g(t) = 1 \quad \text{and} \quad f(t) = t(1 - \frac{\omega_o}{\omega} - \frac{\alpha t}{\omega}).$$

We can show this is equivalent to our original $\tilde{P}(\omega)$:

$$\tilde{P}(\omega) = \int_a^b 1 \cdot e^{i\omega t(1-\frac{\omega_o}{\omega}-\frac{\alpha t}{\omega})}dt = \int_a^b e^{it(\omega-\omega_o-\alpha t)}dt = \int_a^b e^{-i\alpha t^2}e^{-i\omega_o t}e^{i\omega t}dt$$

From [4], we see that the underlying principle of stationary phase is the assertion that the major asymptotic contribution to the integral (1) comes from points where the phase function, $f(t)$ is stationary, i.e., where $f'(t)$ vanishes. We suppose that $f(t)$ has only one such point, say c, in (a, b), and that $f''(t)$ is negative there, i.e., $f(c)$ is a simple maximum. Then, heuristiaclly, we have,

$$I(\omega) \sim g(c)\sqrt{\frac{2\pi}{-\omega f''(c)}}e^{i[\omega f(c)-\frac{\pi}{4}]} \quad (2) \quad .$$

Let us prove that our $f(c)$ is a simple maximum at this point, by computing $f''(t)$:

$$f'(t) = \frac{df(t)}{dt} \quad \text{and} \quad f''(t) = \frac{d^2f(t)}{dt^2},$$

where

$$f'(t) = 1 - \frac{\omega_o}{\omega} - \frac{2\alpha t}{\omega} \quad \text{and} \quad f''(t) = -\frac{2\alpha}{\omega}.$$

We can clearly see that $f''(t)$ is negative regardless of the input t.

Additionally, since we know that the width of suitable ω that are non-zero is of width B centered at ω_o, where B is the bandwidth of the signal $P(\omega)$, so $\omega \in \left[\omega_o - \frac{B}{2}, \omega_o + \frac{B}{2}\right]$. We can see this translates to,

$$\chi_B(\omega - \omega_o) = \begin{cases} 1, & |\omega - \omega_o| \leq \frac{B}{2} \\ 0, & \text{otherwise} \end{cases} .$$

Where we can see that our vanishing point, c, becomes $c = \omega - \omega_o$. Thus, we see trivial answers for evaluating $g(c)$ and $f''(c)$; $g(c) = g(\omega - \omega_o) = 1$ & $f''(c) = f''(\omega - \omega_o) = -\frac{2\alpha}{\omega}$.

Evaluating $f(c)$ we see,

$$f(c) = c(1 - \frac{\omega_o}{\omega} - \frac{\alpha c}{\omega}) = -\frac{\alpha c^2}{\omega} + c(1 - \frac{\omega_o}{\omega}) = -\frac{\alpha(\omega^2 - 2\omega_o\omega + \omega_o^2)}{\omega} + \omega - 2\omega_o + \frac{\omega_o^2}{\omega}$$

Plugging in our evaluations for $g(c)$, $f(c)$, & $f''(c)$ into (2),

$$\begin{aligned}
I(\omega) &\sim 1 \cdot \sqrt{\frac{2\pi}{-\omega(-\frac{2\alpha}{\omega})}} \cdot e^{i[\omega(-\frac{\alpha(\omega-\omega_o)^2}{\omega}) + \omega - 2\omega_o + \frac{\omega_o^2}{\omega} - \frac{\pi}{4}]} \\
&= \sqrt{\frac{\pi}{\alpha}} \cdot e^{i[-\alpha(\omega-\omega_o)^2 + \omega^2 - 2\omega_o\omega + \omega_o^2 - \frac{\pi}{4}]} \\
&= \sqrt{\frac{\pi}{\alpha}} \cdot e^{-\frac{i\pi}{4}} e^{-i\alpha(\omega-\omega_o)^2 + i(\omega-\omega_o)^2} \\
&= \sqrt{\frac{\pi}{\alpha}} \cdot e^{-i\frac{\pi}{4}} e^{-i\alpha(\omega-\omega_o)^2} e^{i(\omega-\omega_o)^2} \\
&= \sqrt{\frac{\pi}{\alpha}} \cdot e^{-i\frac{\pi}{4}} e^{i\frac{(\omega-\omega_o)^2}{\alpha}} \chi_B(\omega - \omega_o)
\end{aligned}$$

1.2 Problem 2.

1.2.1 A)

Provide a full derivation for eqn. (2.18) in your text [1]. Specifically, show where the quadratic term $\frac{(z_1'-x_1)^2}{2|\underline{z}-\underline{x}|^2} sin^2(\gamma)$ comes from. To do so, you will need to take into account the first three terms (rather than only two) of the Taylor expansion of the square root in eqn. (2.18) and then carefully assemble all the terms quadratic with regards to the small quantity $\frac{(z_1'-x_1)}{|\underline{z}-\underline{x}|}$.

$$\begin{aligned}
|\underline{z} - (z_1', x_2, x_3)| &= \sqrt{|\underline{z}-\underline{x}|^2 + (z_1'-x_1)^2 - 2(z_1'-x_1)\,|\underline{z}-\underline{x}|\, cos(\gamma)} \qquad (2.18) \\
&\approx |\underline{z}-\underline{x}|\,(1 - \frac{(z_1'-x_1}{|\underline{z}-\underline{x}|} cos(\gamma) + \frac{(z_1'-x_1)^2}{2\,|\underline{z}-\underline{x}|^2} sin^2(\gamma))
\end{aligned}$$

By the Kirchhoff integral we can solve for $u^{(0)}(t, \underline{z})$, the impinging field, by:

$$u^{(0)}(t, \underline{z}) = \frac{1}{4\pi} \iiint\limits_{\mathbb{R}^3} \frac{f(t - \frac{|\underline{z}-\underline{z}'|}{c}, \underline{z}')}{|\underline{z}-\underline{z}'|} d\underline{z}'.$$

Where $f(t, \underline{z})$ is the function of the antenna source & $f(t - \frac{|\underline{z}-\underline{z}'|}{c}, \underline{z}')$ is the electromagnetic function of the antenna after some time from the target $\underline{z}$ back to the antenna at position $\underline{z}'$. And $|\underline{z}-\underline{z}'|$ in the denominator is used to normalize the antenna power as it degrades with time from the original position.

We define $f\left(t - \frac{|\underline{z}-\underline{z}'|}{c}, \underline{z}'\right)$ as a linear 1D antenna by,

$$f\left(t-\frac{|\underline{z}-\underline{z}'|}{c},\underline{z}'\right)=P\left(t-\frac{|\underline{z}-\underline{z}'|}{c}\right)\chi_D(z_1'-x_1)\delta(z_2'-x_2)\delta(z_3'-x_3),$$

where

$$\chi_D(x)=\begin{cases}1, & x_1-\frac{D}{2}\leq x\leq x_1+\frac{D}{2}\\ 0, & \text{otherwise}\end{cases}.$$

Where D is the length of the antenna and $\delta(z_2'-x_2)$, $\delta(z_3-x_3)$ indicate that the antenna function is not 0, when the received signal is on the same 1D track as the 1D linear antenna, therefore,

$$u^{(0)}(t,\underline{z})=\frac{1}{4\pi}\iiint\limits_{\mathbb{R}^3}\frac{\chi_D(z_1'-x_1)\delta(z_2'-x_2)\delta(z_3'-x_3)}{|\underline{z}-\underline{z}'|}P\left(t-\frac{|\underline{z}-\underline{z}'|}{c}\right)d\underline{z}'.$$

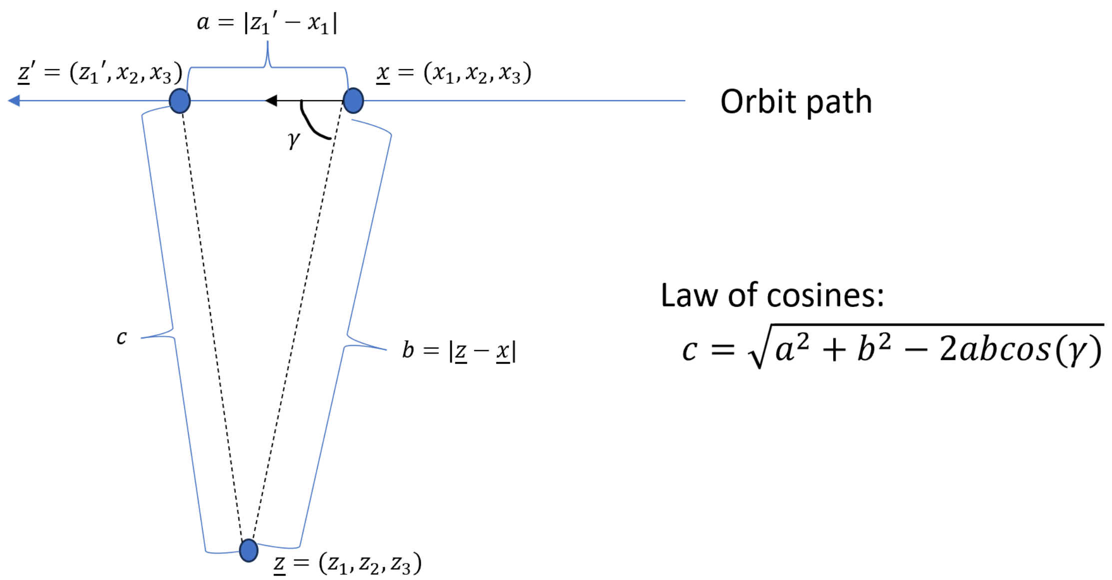
Then, we can change the bounds of integration due to χ_D and the δ functions to,

$$u^{(0)}(t,\underline{z})=\frac{1}{4\pi}\int_{x_1-\frac{D}{2}}^{x_1+\frac{D}{2}}\frac{P\left(t-\frac{|\underline{z}-\underline{z}'|}{c}\right)}{|\underline{z}-\underline{z}'|}dz_1',$$

where the center of the antenna is located at $\underline{x}=(x_1,x_2,x_3)$, and, since, x_2 & x_3 are kept constant while x_1 varies by z_1', we get $\underline{z}'=(z_1',z_2',z_3')$, therefore,

$$u^{(0)}(t,\underline{z})=\frac{1}{4\pi}\int_{x_1-\frac{D}{2}}^{x_1+\frac{D}{2}}\frac{P\left(t-\frac{|\underline{z}-(z_1',x_2,x_3)|}{c}\right)}{|\underline{z}-(z_1',x_2,x_3)|}dz_1',$$

Then, we donote γ as the angle between the positive direction x_1 & the vector $\underline{z}-\underline{x}$. From the below figure, we see that we will use the law of cosines in solving for $|\underline{z}-(z_1',x_2,x_3)|$, where we also assume the orbit of the satellite is in a straight line, which is a reasonable approximation as the relative scale between z_1' & x_1 is much smaller than the distance between $\underline{z}$ & $\underline{x}$. We also employ the Start-Stop Approximation to give us definitive points where we fully emit the impinging field and fully receive the reflected field.



Henceforth,

$$
\begin{aligned}
|\underline{z} - (z_1', x_2, x_3)| &= \sqrt{|z_1' - x_1|^2 + |\underline{z} - \underline{x}|^2 - 2\,|z_1' - x_1|\,|\underline{z} - \underline{x}|\,cos(\gamma)} \\
&= |\underline{z} - \underline{x}|\sqrt{1 + \frac{|z_1' - x_1|^2}{|\underline{z} - \underline{x}|^2} - 2\frac{|z_1' - x_1|}{|\underline{z} - \underline{x}|}cos(\gamma)}
\end{aligned}.
$$

Then, we see the values under the square root look similar to the Taylor series for $\sqrt{1+\alpha}$, so we will approximate what's under the square root using the first three terms of the Taylor series expansion following $\sqrt{1+\alpha}$, where

$$
\alpha = \frac{|z_1' - x_1|^2}{|\underline{z} - \underline{x}|^2} - 2\frac{|z_1' - x_1|}{|\underline{z} - \underline{x}|}cos(\gamma).
$$

We also know, the first three terms of this Taylor series expansion will have the form $\sqrt{1+\alpha} \approx 1 + \frac{1}{2}\alpha - \frac{1}{8}\alpha^2$, so solving for each term independently we get:

$$
\frac{1}{2}\alpha = \frac{|z_1' - x_1|^2}{2\,|\underline{z} - \underline{x}|^2} - \frac{|z_1' - x_1|}{|\underline{z} - \underline{x}|}cos(\gamma)
$$

$$
\begin{aligned}
-\frac{1}{8}\alpha^2 &= -\frac{1}{8}\left(\frac{|z_1' - x_1|^2}{|\underline{z} - \underline{x}|^2} - 2\frac{|z_1' - x_1|}{|\underline{z} - \underline{x}|}cos(\gamma)\right)^2 \\
&= -\frac{1}{8}\left(\frac{|z_1' - x_1|^4}{|\underline{z} - \underline{x}|^4} - 4\frac{|z_1' - x_1|^3}{|\underline{z} - \underline{x}|^3}cos(\gamma) + 4\frac{|z_1' - x_1|^2}{|\underline{z} - \underline{x}|^2}cos^2(\gamma)\right) \\
&= -\frac{1}{8}\frac{|z_1' - x_1|^4}{|\underline{z} - \underline{x}|^4} + \frac{1}{2}\frac{|z_1' - x_1|^3}{|\underline{z} - \underline{x}|^3}cos(\gamma) - \frac{1}{2}\frac{|z_1' - x_1|^2}{|\underline{z} - \underline{x}|^2}cos^2(\gamma)
\end{aligned}
$$

Since $|\underline{z} - \underline{x}| \gg |z_1' - x_1|$, because $|\underline{z} - \underline{x}|$ is the distance between the satellite in orbit and the ground, therefore, is on the scale of 10's of kilometers. And $|z_1' - x_1|$ is the distance between the start & stop of the satellite in between emitting the linear chirp and receiving the reflected wave, therefore, on the scale of 10's of meters. Then we will assume the terms with $|\underline{z} - \underline{x}|^r$, where $r > 2$, in the numerator will tend to 0, therefore, we get a new approximation of the $-\frac{1}{8}\alpha^2$ term as:

$$
-\frac{1}{8}\alpha^2 \approx 0 + 0 - \frac{|z_1' - x_1|^2}{2\,|\underline{z} - \underline{x}|^2}cos^2(\gamma)
$$

Using the following trignometric identity: $cos^2(\gamma) = 1 - sin^2(\gamma)$

$$
-\frac{1}{8}\alpha^2 \approx -\frac{|z_1' - x_1|^2}{2\,|\underline{z} - \underline{x}|^2} + \frac{|z_1' - x_1|^2}{2\,|\underline{z} - \underline{x}|^2}sin^2(\gamma).
$$

Plugging our values back into $\sqrt{1+\alpha} \approx 1 + \frac{1}{2}\alpha - \frac{1}{8}\alpha^2$, we get:

$$
\begin{aligned}
\sqrt{1+\alpha} \approx 1 + \frac{|z_1' - x_1|^2}{2\,|\underline{z} - \underline{x}|^2} &= \frac{|z_1' - x_1|}{|\underline{z} - \underline{x}|}cos(\gamma) - \frac{|z_1' - x_1|^2}{2\,|\underline{z} - \underline{x}|^2} + \frac{|z_1' - x_1|^2}{2\,|\underline{z} - \underline{x}|^2}sin^2(\gamma) \\
&\approx 1 - \frac{|z_1' - x_1|}{|\underline{z} - \underline{x}|}cos(\gamma) + \frac{|z_1' - x_1|^2}{2\,|\underline{z} - \underline{x}|^2}sin^2(\gamma)
\end{aligned}
$$

Therefore, plugging this back in to where we were originally deriving the Taylor series expansion, we see:

$$
|\underline{z} - (z_1', x_2, x_3)| \approx |\underline{z} - \underline{x}|\left(1 - \frac{|z_1' - x_1|}{|\underline{z} - \underline{x}|}cos(\gamma) + \frac{|z_1' - x_1|^2}{2\,|\underline{z} - \underline{x}|^2}sin^2(\gamma)\right)
$$

And since, earlier, we assumed that z_1' was in the positive direction from x_1, we can rewrite our expression with $|z_1' - x_1| = (z_1' - x_1)$, and get the same result as in eqn. (2.18) from [1]:

$$|\underline{z}-(z_1', x_2, x_3)| \approx |\underline{z}-\underline{x}| \left(1 - \frac{(z_1'-x_1)}{|\underline{z}-\underline{x}|}cos(\gamma) + \frac{(z_1'-x_1)^2}{2\,|\underline{z}-\underline{x}|^2}sin^2(\gamma)\right)$$

Then, assembling all of the terms quadratic with respect to $\frac{(z_1'-x_1)}{|\underline{z}-\underline{x}|}$, we will see, from the quadratic equation,

$$ax^2 + bx + c = 0,$$

where

$$x = \frac{(z_1'-x_1)}{|\underline{z}-\underline{x}|}, \quad a = \frac{1}{2}sin^2(\gamma), \quad b = -cos(\gamma), \quad c = 1.$$

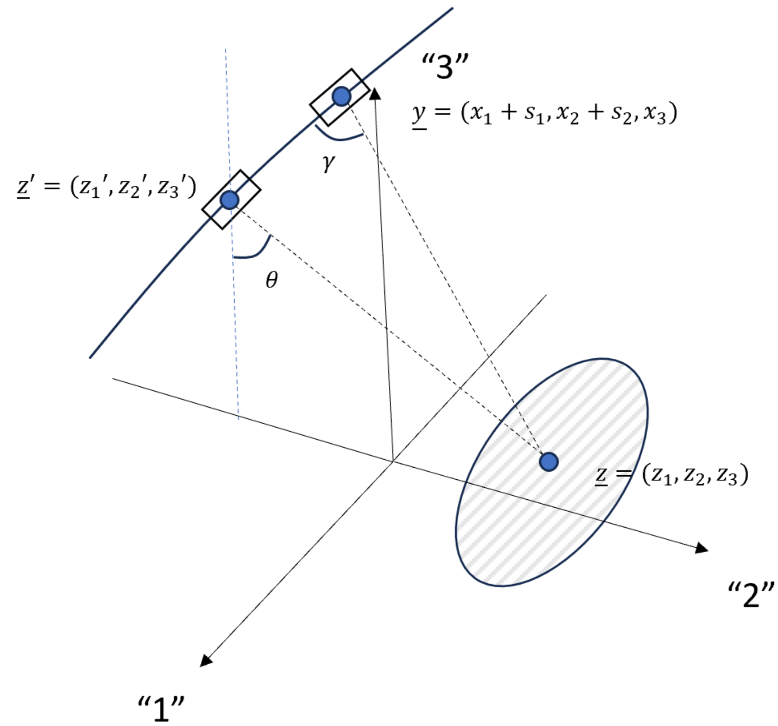
Where the quadratic equation forms the solution for $\frac{(z_1'-x_1)}{|\underline{z}-\underline{x}|}$ inside the parentheses as,

$$\frac{(z_1'-x_1)}{|\underline{z}-\underline{x}|} = \frac{cos(\gamma) \pm \sqrt{cos^2(\gamma) - 2sin^2(\gamma)}}{sin^2(\gamma)}.$$

1.2.2 B)

Beyond the case considered in Section 2.2 of your text [1] of a linear antenna of size D aligned with the flight direction, analyze the case of a two-dimensional antenna and derive the corresponding radiation pattern. Assume that the antenna has the shape of a planar rectangle with one side parallel to the flight track (orbit) and the other side perpendicular to the flight track. Also assume that the normal to the plane of this rectangle (which is also perpendicular to the orbit) is pointing downward with the look angle θ, as in Figure 2.1, see [1, page 25]. You should expect to get both horizontal and vertical radiation patterns of the $sinc(\cdot)$ type. The horizontal pattern will be the same as the one we have derived in class while the vertical one will be similar and due to the finite size of the antenna in the other direction. You can use the analysis of [5, Section 2] as guidelines.

First, let us visualize the two-dimensional antenna as it would be in orbit using notation from [5]:



where "1" is the flight track axis, "2" is the range axis, and "3" is the elevation axis. We also express the original antenna position as $\underline{y} = \underline{x} \cdot \underline{q} = (x_1 + s_1, x_2 + s_2, x_3)$, where $\underline{q} = s_1 \cdot \hat{e}_1 + s_2 \cdot \hat{e}_2$ indicates some shift from the center position of the 2D antenna at position $\underline{x} = (x_1, x_2, x_3)$. Where $\hat{e}_1$ & $\hat{e}_2$ are unit vectors along the length & width of the antenna, respectively. The vector $\hat{e}_1$ points along the direction of flight. For side-looking systems, $\hat{e}_2$ is tilted with regards to the range axis so that a vector perpendicular to the antenna points to the side of the flight track.

Using Kirchhoff's integral, as done so in part A), we define the impinging field as:

$$u^{(0)}(t, \underline{z}) = \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{f(t - \frac{|\underline{z}-\underline{z}'|}{c}, \underline{z}')}{|\underline{z} - \underline{z}'|} d\underline{z}',$$

where

$$f\left(t - \frac{|\underline{z} - \underline{z}'|}{c}, \underline{z}'\right) = P\left(t - \frac{|\underline{z} - \underline{z}'|}{c}\right) \chi_D(z_1' - x_1)\chi_L(z_2' - x_2)\delta(z_3' - x_3)$$

$$\chi_D(x) = \begin{cases} 1, & x_1 - \frac{D}{2} \leq x \leq x_1 + \frac{D}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\chi_L(x) = \begin{cases} 1, & x_2 - \frac{L}{2} \leq x \leq x_2 + \frac{L}{2} \\ 0, & \text{otherwise} \end{cases}.$$

Where χ_D denotes the variation in antenna length & χ_L denotes the variation of the antenna width.

$$
\begin{aligned}
u^{(0)}(t,\underline{z}) &= \frac{1}{4\pi}\iiint\limits_{\mathbb{R}^3} \frac{\chi_D((z_1'-x_1)\cdot s_1)\chi_L((z_2'-x_2)\cdot s_2)\delta(z_3'-x_3)}{|\underline{z}-\underline{z}'|} P\left(t-\frac{|\underline{z}-\underline{z}'|}{c}\right) d\underline{z}' \\
&= \frac{1}{4\pi}\int_{x_1\cdot s_1-\frac{D}{2}}^{x_1\cdot s_1+\frac{D}{2}} \int_{x_2\cdot s_2-\frac{L}{2}}^{x_2\cdot s_2+\frac{L}{2}} \frac{P\left(t-\frac{|\underline{z}-\underline{z}'|}{c}\right)}{|\underline{z}-\underline{z}'|} dz_2' dz_1' \\
&= \frac{1}{4\pi}\int_{x_1\cdot s_1-\frac{D}{2}}^{x_1\cdot s_1+\frac{D}{2}} \int_{x_2\cdot s_2-\frac{L}{2}}^{x_2\cdot s_2+\frac{L}{2}} \frac{P\left(t-\frac{|\underline{z}-(z_1',z_2',x_3)|}{c}\right)}{|\underline{z}-(z_1',z_2',x_3)|} dz_2' dz_1'
\end{aligned}
$$

Then, we can use the results from part A), where we employed the Taylor series on $|\underline{z}-\underline{z}'|$,

$$
|\underline{z}-\underline{z}'| = |\underline{z}-\underline{x}|\left(1-\frac{(z_1'-x_1)\cdot s_1}{|\underline{z}-\underline{x}|}cos(\gamma)-\frac{(z_2'-x_3)\cdot s_2}{|\underline{z}-\underline{x}|}cos(\theta)+\frac{1}{2}\frac{((z_1'-x_1)\cdot s_1)^2}{|\underline{z}-\underline{x}|^2}sin^2(\gamma)+\frac{1}{2}\frac{((z_2'-x_2)\cdot s_2)^2}{|\underline{z}-\underline{x}|^2}.\right.
$$

where we will drop the higher order terms due to the them being $\ll 1$, and get:

$$
|\underline{z}-\underline{z}'| = |\underline{z}-\underline{x}| - ((z_1'-x_1)\cdot s_1)cos(\gamma) - ((z_2'-x_2)\cdot s_2)cos(\theta).
$$

Thus, defining $P(t)$:

$$
\begin{gathered}
P(t) = \chi_\tau(t)e^{-i\alpha t^2}e^{-i\omega_o t} \\
A(t) = \chi_\tau(t)e^{-i\alpha t^2} \\
\chi_\tau(t) = \begin{cases} 1, & -\frac{\tau}{2}\le t\le \frac{\tau}{2} \\ 0, & \text{otherwise} \end{cases}.
\end{gathered}
$$

Now we can rewrite the impinging field using these definitions,

$$
\begin{aligned}
u^{(o)}(t,\underline{z}) &= \frac{1}{4\pi}\int_{x_1\cdot s_1-\frac{D}{2}}^{x_1\cdot s_1+\frac{D}{2}} \int_{x_2\cdot s_2-\frac{L}{2}}^{x_2\cdot s_2+\frac{L}{2}} \frac{A\left(t-\frac{|\underline{z}-\underline{z}'|}{c}\right)}{|\underline{z}-\underline{z}'|} e^{-i\omega_o(t-\frac{|\underline{z}-\underline{z}'|}{c})} dz_2' dz_1' \\
&= \frac{1}{4\pi}\int_{-\frac{D}{2}}^{\frac{D}{2}} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{A\left(t-\frac{|\underline{z}-\underline{z}'|}{c}\right)}{|\underline{z}-\underline{z}'|} e^{-i\omega_o\left(t-\frac{|\underline{z}-\underline{x}|}{c}-\frac{(z_1'-x_1)\cdot s_1}{c}cos(\gamma)-\frac{(z_2'-x_2)\cdot s_1}{c}cos(\theta)\right)} ds_1 ds_2 \\
&= \frac{1}{4\pi}\int_{-\frac{D}{2}}^{\frac{D}{2}} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{A\left(t-\frac{|\underline{z}-\underline{z}'|}{c}\right)}{|\underline{z}-\underline{z}'|} e^{-i\omega_o(t-\frac{|\underline{z}-\underline{x}|}{c})} e^{i\omega_o\frac{(z_1'-x_1)\cdot s_1}{c}cos(\gamma)} e^{i\omega_o\frac{(z_2'-x_2)\cdot s_2}{c}cos(\theta)} ds_2 ds_1 \\
&= \frac{P\left(t-\frac{|\underline{z}-\underline{x}|}{c}\right)}{4\pi|\underline{z}-\underline{x}|}\int_{-\frac{D}{2}}^{\frac{D}{2}} e^{ik((z_1'-x_1)\cdot s_1)cos(\gamma)} ds_1 \int_{-\frac{L}{2}}^{\frac{L}{2}} e^{ik((z_2'-x_2)\cdot s_2)cos(\theta)} ds_2
\end{aligned}
.
$$

Where $k = \frac{\omega_o}{c}$,

$$\int_{-\frac{D}{2}}^{\frac{D}{2}} e^{ik((z_1'-x_1)\cdot s_1)cos(\gamma)} ds_1 = 2Dsinc(k(z_1'-x_1)\frac{D}{2})$$

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} e^{ik((z_2'-x_2)\cdot s_2)cos(\theta)} ds_2 = 2Lsinc(k(z_2'-x_2)\frac{L}{2})$$

$$u^{(o)}(t,\underline{z}) = \frac{P\left(t - \frac{|\underline{z}-\underline{x}|}{c}\right)}{4\pi\left|\underline{z}-\underline{x}\right|} 2Dsinc(k(z_1'-x_1)\frac{D}{2}) 2Lsinc(k(z_2'-x_2)\frac{L}{2})$$