HW03

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1 Homework 3 - MA 797 (Mathematics of Radar Imaging)

1.0.1 Christopher Kelton 03/19/2024

1.1 Problem 1

Derive [1, equation (6.57)] in your text. Consider Taylor expansion of R_{Dop} with respect to $\frac{\tilde{x}}{R}$, $\frac{d}{R}$ up to the linear terms around $\tilde{x} = d = 0$ and a given l.

Since we are taking into account the movement of the platform during the transmission of the radar signals emitted, we will utilize the Lorentz transform in order to transform a moving object to a coordinate system that is stationary, while keeping the wave equation and Maxwell's equations invariant. Thus, we will have $(t, z_1, z_2, z_3) \mapsto (\sigma, \zeta_1, z_2, z_3)$. Where,

$$\sigma = \frac{1}{\beta}\left(t - \frac{vz_1}{c^2}\right), \quad \zeta_1 = \frac{1}{\beta}\left(-vt + z_1\right),$$

and

$$\beta = \sqrt{1 - \frac{v^2}{c^2}}.$$

Thus, leading to the following wave equation:

$$\begin{aligned}\frac{1}{c^2}\frac{\partial^2 u}{\partial \sigma^2} - \frac{\partial^2 u}{\partial \zeta_1^2} - \frac{\partial^2 u}{\partial z_2^2} - \frac{\partial^2 u}{\partial z_3^2} &= P\left(\frac{\sigma}{\beta} + \frac{\frac{v}{c}}{\beta}\frac{\zeta_1}{c}\right)\delta\left(\beta\zeta_1\right)\delta\left(z_2 + L\right)\delta\left(z_3 - H\right) \\ &= f\left(\sigma, \zeta_1, z_2, z_3\right).\end{aligned}$$

Thus, using Kirchhoff's integral theorem to define the impinging field, we get

$$u^{(0)}\left(\sigma, \zeta_1, z_2, z_3\right) = \frac{1}{4\pi}\int\int\int \frac{f\left(\sigma - \frac{\rho'}{c}, \zeta_1', z_2', z_3'\right)}{\rho'} d\zeta_1' dz_2' dz_3',$$

where

$$\rho' = \left(\left(\zeta_1 - \zeta_1'\right)^2 + \left(z_2 - z_2'\right)^2 + \left(z_3 - z_3'\right)^2\right)^{\frac{1}{2}}.$$

Thus, defining $\zeta_1' = 0$, as we have transformed into a coordinate system where our antenna is no longer moving, $z_2' = -L$, as this is the projection distance of the antenna, $z_3' = H$, as this is the height of the antenna from the ground. Thus, we are saying that the antenna is at rest at $x = (0, -L, H)$. Integrating $u^{(0)}$ we get,

$$u^{(0)} = \frac{1}{4\pi\beta}\frac{P\left(\frac{\sigma-\frac{\rho}{c}}{\beta}\right)}{\rho},$$

where

$$\rho = \left(\zeta_1^2 + (z_2+L)^2 + (z_3-H)^2\right)^{\frac{1}{2}}.$$

Notice, if we drop β, then we get the straightforward retarded potential in the new coordinate system. Solving for ρ,

$$\rho = \left(\zeta_1^2 + (z_2+L)^2 + (z_3-H)^2\right)^{\frac{1}{2}}$$

$$\begin{aligned}
\zeta_1^2 &= \left(\frac{1}{\beta}(-vt+z_1)\right)^2 \\
&= \frac{1}{\beta^2}(z_1-vt)^2 \\
&= \frac{1}{\beta^2}\left(z_1^2 - 2z_1vt + (vt)^2\right) \\
&= \frac{z_1^2}{\beta^2} + \frac{1}{\beta^2}\left(-2z_1vt + (vt)^2\right)
\end{aligned},$$

where

$$\begin{aligned}
\frac{z_1^2}{\beta^2} &= z_1^2 - z_1^2 + \frac{z_1^2}{\beta^2} \\
&= z_1^2 + z_1^2\left(\frac{1}{\beta^2} - 1\right) \\
&= z_1^2 + \frac{1-\beta^2}{\beta^2}z_1^2 \\
&= z_1^2.
\end{aligned}$$

Where we dropped any $\frac{c^2}{v^2} \gg 1$ terms. Thus,

$$\begin{aligned}
\zeta_1^2 &= z_1^2 + \frac{1}{\beta^2}\left(-2z_1vt + (vt)^2\right) \\
&= z_1^2 - 2z_1vt + (vt)^2 + \frac{1-\beta^2}{\beta^2}(z_1-vt)^2 \\
&= z_1^2 - 2z_1vt + (vt)^2
\end{aligned}$$

$$\rho \approx \left(z_1^2 - 2z_1vt + (vt)^2 + (z_2+L)^2 + (z_3-H)^2\right)^{\frac{1}{2}}$$

$$r = R_z\Big|_{t=0,z_3=0} = \sqrt{z_1^2 + (z_2+L)^2 + H^2}.$$

Where r defines the moment of time $t = 0$ when the center of the pulse is emitted corresponds to the location $x(0) = (0, -L, H)$ on the orbit. Therefore, when $z_3 = 0$,

$$\rho \approx \left(r^2 - 2z_1 vt + (vt)^2\right)^{\frac{1}{2}}.$$

Furthermore, the timescale we are interested in is the travel time of the impinging signal to propagate to the target or $t \sim \frac{r}{c}$. Therefore, the 3rd term on the right-hand side is only taken into account if we have a very narrow sub-angle or $|z_1|\,|vt| \sim (vt)^2$ as $(vt)^2 = (v\frac{r}{c})^2 = (\frac{v}{c}r)^2 = \frac{v^2}{c^2}r^2$, so we would assume to drop this term as well due to $\frac{v^2}{c^2}$. Henceforth,

$$\begin{aligned}
\rho &\approx \sqrt{r^2 - 2z_1 vt} \\
&= r\sqrt{1 - \frac{2z_1 vt}{r^2}} \\
&\approx r\left(1 - \frac{1}{2}\frac{2z_1 vt}{r^2} + \frac{1}{4}\frac{4z_1^2 (vt)^2}{r^4}\right) \\
&= r - \frac{z_1 vt}{r}.
\end{aligned}$$

Substituting this into $u^{(0)}$ and dropping the $\beta \sim 1$ terms,

$$\begin{aligned}
u^{(0)}(t,z) &\approx \frac{1}{4\pi}\frac{P\left(\sigma - \frac{\rho}{c}\right)}{\rho} \\
&\approx \frac{1}{4\pi}\frac{P\left(t - \frac{vz_1}{c^2} - \frac{r}{c} - \frac{v}{c}\frac{z_1 t}{r}\right)}{\rho} \\
&= \frac{1}{4\pi}\frac{P\left(t\left(1 - \frac{v}{c}\frac{z_1}{r}\right) - \frac{vz_1}{c^2} - \frac{r}{c}\right)}{\rho} \\
&= \frac{1}{4\pi}\frac{P\left(\left(t - \frac{r}{c}\right)\left(1 + \frac{v}{c}\frac{z_1}{r}\right)\right)}{\rho}.
\end{aligned}$$

Integrating ρ with respect to t, we get $\left|\frac{\partial\rho}{\partial t}\right| \approx \frac{z_1 v}{r}$ and since we are not considering a narrow sub-angle this will result in a small amplitude and we can simply replace the slowly oscillating ρ with r. Thus,

$$u^{(0)}(t,z) \approx \frac{1}{4\pi}\frac{P\left(\left(t - \frac{r}{c}\right)\left(1 + \frac{v}{c}\frac{z_1}{r}\right)\right)}{r},$$

and $\frac{z_1}{r}$ defines the look angle of the satellite, which, by the law of cosines, will replace with $\cos(\gamma_z(0))$. Where γ_z is constantly changing as the satellite is moving, however, we will only take the moment of emission at $t = 0$, as, from previous derivations, conclude that the change in range position of the satellite is very small. We will denote $\cos(\gamma_z(0)) = \cos(\gamma_z)$. Therefore,

$$u^{(0)}(t,z) \approx \frac{1}{4\pi}\frac{P\left(\left(t - \frac{r}{c}\right)\left(1 + \frac{v}{c}\cos(\gamma_z)\right)\right)}{r}.$$

Then, the receiving signal, $u^{(1)}$ will become:

$$u^{(1)}(t,x') = \int\int\int \nu(z) P\left(\left(t - \frac{r}{c} - \frac{|x'-z|}{c}\right)\left(1 + \frac{v}{c} cos(\gamma_z)\right)\right) dz.$$

Note, if $v = 0$, we get back the solution using the Start-Stop approximation. Now, we need to perform another Lorentz transform on the received, scattered, field at the moving antenna receiver. Therefore, we must perform the mapping $(t, x_1', x_2', x_3') \mapsto (\vartheta, \xi_1, x_2', x_3')$, where

$$\vartheta = \frac{1}{\beta}\left(t - \frac{v x_1'}{c^2}\right), \quad \xi_1 = \frac{1}{\beta}\left(-vt + x_1'\right).$$

Note, $t - \frac{r}{c}$ is the moment the pulse is reflected at the target, defining $\breve{t} = \frac{r}{c}$. Therefore, the moment the emitting pulse is re-emitted at the target z, event of reflection, is

$$\breve{\vartheta} = \frac{1}{\beta}\left(\breve{t} - \frac{v z_1}{c}\right), \quad \breve{\xi}_1 = \frac{1}{\beta}\left(-v\breve{t} + z_1\right).$$

We also know, from derivations in earlier homeworks, that the maximum of the *sinc* argument in the impinging field, $u^{(0)}(t,z)$, definition is $\frac{\lambda_0}{D}$. Where λ_0 is the wavelength of the emitting radiation pattern and D is the length the of synthetic aperture. Thus the upper bound on the emitting angle is $\frac{\lambda_0}{D}$, where this produces the maximum level of radiation emitted by a linear antenna, when parallel to the orbit. Henceforth, the condition where the moment of reflection of the point z is within the antenna beam is expressed as:

$$\frac{\left|\breve{\xi}_1\right|}{R} \leq \frac{\lambda_0}{D}.$$

For $t > \breve{t} = \frac{r}{c}$, the scattered field is obtained by:

$$u^{(1)}(t,x') \approx \int \nu(z) P\left(\left(t - \breve{t} - \frac{r'}{c}\right)\left(1 + \frac{v}{c}\cos(\gamma_z)\right)\right) \cdot \chi_\Theta\left(\frac{z_1 - \frac{vr}{c}}{\beta R}\right) dz.$$

Where $t - \breve{t} - \frac{r'}{c}$ comes from the event of reflection at the target, $\breve{t}$, and the modified distance between the moving antenna, x', and the target, z. Which is a doppler shift of the retarded time. Where,

$$\begin{aligned} r' &= \sqrt{\left(x_1' - z_1\right)^2 + \left(z_2 + L\right)^2 + H^2} \\ &= |x' - z|. \end{aligned}$$

The indicator function, χ_Θ comes from $\frac{|\breve{\xi}_1|}{R} \leq \frac{\lambda_0}{D}$, where this is the permissible range of the angle of the emitting radiation.

Performing the inverse transform of ϑ & ξ_1, we get the following expressions:

$$t = \frac{1}{\beta}\left(\vartheta + \frac{v\xi_1}{c^2}\right), \quad x_1' = \frac{1}{\beta}\left(v\vartheta + \xi_1\right).$$

Thus, substituting all values into $t - \breve{t} - \frac{r'}{c}$:

$$t - \breve{t} - \frac{r'}{c} = \frac{1}{\beta}\left(\vartheta + \frac{v\xi_1}{c^2}\right) - \frac{1}{\beta}\left(\breve{\vartheta} + \frac{c\breve{\xi}_1}{c^2}\right) - \frac{1}{c}\sqrt{\left(\frac{1}{\beta}\left(v\vartheta + \xi_1\right) - z_1\right)^2 + \left(z_2 + L\right)^2 + H^2}.$$

Then, taking $\beta \approx 1$ and finding the inverse of z_1 with regards to $\breve{\vartheta}$ & $\breve{\xi}_1$, we get

$$t - \breve{t} - \frac{r'}{c} \approx \left(\vartheta - \breve{\vartheta}\right) + \frac{v}{c^2}\left(\xi_1 - \breve{\xi}_1\right) - \frac{1}{c}\sqrt{\left(\xi_1 - \left(\breve{\xi}_1 + v\breve{\vartheta}\right) + v\vartheta\right)^2 + \left(z_2 + L\right)^2 + H^2}.$$

Solving for $\left(\xi_1 - \left(\breve{\xi}_1 + v\breve{\vartheta}\right) + v\vartheta\right)^2$:

$$\begin{aligned}
\left(\xi_1 - \left(\breve{\xi}_1 + v\breve{\vartheta}\right) + v\vartheta\right)^2 &= \left(\xi_1 - \breve{\xi}_1 - v\breve{\vartheta} + v\vartheta\right)^2 \\
&= \xi_1^2 - \xi_1\breve{\xi}_1 - \xi_1 v\breve{\vartheta} + \xi_1 v\vartheta - \xi_1\breve{\xi}_1 + \breve{\xi}_1^{\,2} + \breve{\xi}_1 v\breve{\vartheta} - \breve{\xi}_1 v\vartheta - \xi_1 v\breve{\vartheta} + \breve{\xi}_1 v\breve{\vartheta} + \left(v\breve{\vartheta}\right)^2 - v^2\breve{\vartheta}\vartheta + \xi_1 v \\
&= \left(\xi_1^2 - 2\xi_1\breve{\xi}_1 + \breve{\xi}_1^{\,2}\right) + \left(2\xi_1 v\vartheta - 2\breve{\xi}_1 v\vartheta - 2\xi_1 v\breve{\vartheta} + 2\breve{\xi}_1 v\breve{\vartheta}\right) + \left(\left(v\vartheta\right)^2 - 2v^2\vartheta\breve{\vartheta}\right) \\
&= \left(\xi_1 - \breve{\xi}_1\right)^2 + 2v\left(\vartheta - \breve{\vartheta}\right)\left(\xi_1 - \breve{\xi}_1\right) + v^2\left(\vartheta - \breve{\vartheta}\right)^2
\end{aligned}$$

Therefore,

$$t - \breve{t} - \frac{r'}{c} = \left(\vartheta - \breve{\vartheta}\right) + \frac{v}{c^2}\left(\xi_1 - \breve{\xi}_1\right) - \frac{1}{c}\sqrt{\left(\xi_1 - \breve{\xi}_1\right)^2 + 2v\left(\vartheta - \breve{\vartheta}\right)\left(\xi_1 - \breve{\xi}_1\right) + v^2\left(\vartheta - \breve{\vartheta}\right)^2 + \left(z_2 + L\right)^2 + H^2}.$$

Introducing, $\varrho = \sqrt{\left(\xi_1 - \breve{\xi}_1\right)^2 + \left(z_2 + L\right)^2 + H^2}$

$$t - \breve{t} - \frac{r'}{c} = \left(\vartheta - \breve{\vartheta}\right) + \frac{v}{c^2}\left(\xi_1 - \breve{\xi}_1\right) - \frac{1}{c}\sqrt{\varrho^2 + 2v\left(\vartheta - \breve{\vartheta}\right)\left(\xi_1 - \breve{\xi}_1\right) + v^2\left(\vartheta - \breve{\vartheta}\right)^2}.$$

Analyzing the third term containing the square root,

$$\frac{1}{c}\sqrt{\varrho^2 + 2v\left(\vartheta - \breve{\vartheta}\right)\left(\xi_1 - \breve{\xi}_1\right) + v^2\left(\vartheta - \breve{\vartheta}\right)^2} = \sqrt{\frac{\varrho^2}{c^2} + \frac{2v}{c^2}\left(\vartheta - \breve{\vartheta}\right)\left(\xi_1 - \breve{\xi}_1\right) + \frac{v^2}{c^2}\left(\vartheta - \breve{\vartheta}\right)^2}.$$

The $\frac{v^2}{c^2}$ term is approximately zero. Therefore,

$$\frac{1}{c}\sqrt{\varrho^2 + 2v\left(\vartheta - \breve{\vartheta}\right)\left(\xi_1 - \breve{\xi}_1\right) + v^2\left(\vartheta - \breve{\vartheta}\right)^2} \approx \frac{1}{c}\sqrt{\varrho^2 + 2v\left(\vartheta - \breve{\vartheta}\right)\left(\xi_1 - \breve{\xi}_1\right)}.$$

Applying the Taylor series to the expression under the square root where $\alpha = \frac{2v}{\varrho^2}\left(\vartheta - \breve{\vartheta}\right)\left(\xi_1 - \breve{\xi}_1\right)$ and is sufficiently small, so $\sqrt{1+\alpha} \approx 1 + \frac{1}{2}\alpha - \frac{1}{4}\alpha^2 + \cdots$ and taking the first two terms:

$$\begin{aligned}\frac{1}{c}\sqrt{\varrho^2 + 2v\left(\vartheta - \breve{\vartheta}\right)\left(\xi_1 - \breve{\xi}_1\right) + v^2\left(\vartheta - \breve{\vartheta}\right)^2} &\approx \frac{\varrho}{c} + \frac{\varrho}{c}\frac{1}{2}\left(\frac{2v}{\varrho^2}\left(\vartheta - \breve{\vartheta}\right)\left(\xi_1 - \breve{\xi}_1\right)\right) \\ &= \frac{\varrho}{c} + v\left(\vartheta - \breve{\vartheta}\right)\frac{1}{c}\frac{\xi_1 - \breve{\xi}_1}{\varrho}.\end{aligned}$$

Thus,

$$\begin{aligned}t - \breve{t} - \frac{r'}{c} &\approx \left(\vartheta - \breve{\vartheta}\right) + \frac{v}{c^2}\left(\xi_1 - \breve{\xi}_1\right) - \frac{\varrho}{c} - v\left(\vartheta - \breve{\vartheta}\right)\frac{1}{c}\frac{\xi_1 - \breve{\xi}_1}{\varrho} \\ &= \left(\vartheta - \breve{\vartheta} - \frac{\varrho}{c}\right)\left(1 + \frac{v}{c}\frac{\breve{\xi}_1 - \xi_1}{\varrho}\right).\end{aligned}$$

The quantity $\frac{\breve{\xi}_1 - \xi_1}{\varrho}$ can be interpreted as cosine of the angle between the velocity and the direction of the observation point, where the scattered field is received on the antenna, to the pulse re-emission location $\breve{\xi}_1$ at the re-emission time $\breve{\vartheta}$, thus,

$$\frac{\breve{\xi}_1 - \xi_1}{\varrho} = \cos(\gamma_z').$$

Therefore,

$$t - \breve{t} - \frac{r'}{c} = \left(\vartheta - \breve{\vartheta} - \frac{\varrho}{c}\right)\left(1 + \frac{v}{c}\cos(\gamma_z')\right).$$

Thus, substituting the above expression into the defintion of $u^{(1)}$ we get:

$$u^{(1)}\left(t, x'\right) \approx \int \nu(z) P\left(\left(\vartheta - \breve{\vartheta} - \frac{\varrho}{c}\right)\left(1 + \frac{v}{c}\cos(\gamma_z')\right)\left(1 + \frac{v}{c}\cos(\gamma_z)\right)\right) \cdot \chi_\Theta\left(\frac{z_1 - \frac{vr}{c}}{R}\right) dz.$$

The overall Doppler effect for the scattered field received at the antenna is a combination of two contributions. One, where the field is emitted by a moving antenna onto a motionless target, which is given by $1 + \frac{v}{c}\cos(\gamma_z)$. The second, where the impinging field is scattered by a motionless target onto a moving antenna, which is given by $1 + \frac{v}{c}\cos(\gamma_z')$. Expanding our expression for $u^{(1)}$ and neglecting the $\frac{v^2}{c^2}$ terms, we get:

$$u^{(1)}(t, x') \approx \int \nu(z) P\left(\left(\vartheta - \breve{\vartheta} - \frac{\varrho}{c}\right)\left(1 + \frac{v}{c}\cos(\gamma_z') + \frac{v}{c}\cos(\gamma_z)\right)\right) \cdot \chi_\Theta\left(\frac{z_1 - \frac{vr}{c}}{R}\right) dz.$$

This expression of the scattered field is still in the Lorentz-transformed coordinates, so we will want to convert the coordinate system back into our original cartesian coordinate system,

$$u^{(1)}(t,x') \approx \int \nu(z) P\left(\left(t - \frac{vx_1'}{c^2} - \frac{r}{c} + \frac{vz_1}{c^2} - \frac{\varrho}{c}\right)\left(1 + \frac{v}{c}\cos(\gamma_z') + \frac{v}{c}\cos(\gamma_z)\right)\right) \cdot \chi_\Theta\left(\frac{z_1 - \frac{vr}{c}}{R}\right) dz.$$

When the scattered signal is received at the antenna, $\xi_1 = 0$, therefore,

$$\begin{aligned}
\varrho &= \sqrt{\breve{\xi}_1^{\,2} + (z_2 + L)^2 + H^2} \\
&= \sqrt{\left(z_1 - \frac{v}{c}r\right)^2 + (z_2 + L)^2 + H^2} \\
&= \sqrt{z_1^2 - 2\frac{v}{c}z_1 r + \frac{v^2}{c^2}r^2 + (z_2 + L)^2 + H^2} \\
&= \sqrt{z_1^2 - 2\frac{v}{c}z_1 r + (z_2 + L)^2 + H^2} \\
&= \sqrt{r^2 - 2z_1\frac{v}{c}r} \\
&= r\sqrt{1 - 2z_1\frac{v}{cr}} \\
&\approx r\left(1 - \frac{1}{2}\left(2z_1\frac{v}{cr}\right) + \frac{1}{4}\left(2z_1\frac{v}{cr}\right)^2\right) \\
&\approx r - z_1\frac{v}{c},
\end{aligned}$$

where we neglected the $\frac{v^2}{c^2}$ terms. Substituting this expression into $\cos(\gamma_z')$ at $\xi_1 = 0$ and the inverse of $\breve{\xi}_1$, we get

$$\begin{aligned}
\cos(\gamma_z') = \left.\frac{\breve{\xi}_1 - \xi_1}{\varrho}\right|_{\xi_1=0} &\approx \frac{z_1 - \frac{v}{c}r}{r - z_1\frac{v}{c}} \\
&\approx \frac{z_1 - \frac{v}{c}r}{r\left(1 - \frac{z_1}{r}\frac{v}{c}\right)} \\
&\approx \frac{z_1 - \frac{v}{c}r}{r}\left(1 + \frac{z_1}{r}\frac{v}{c}\right) \\
&= \frac{z_1 - \frac{v}{c}r}{r}\left(1 + \frac{v}{c}\cos(\gamma_z)\right) \\
&= \left(\cos(\gamma_z) - \frac{v}{c}\right)\left(1 + \frac{v}{c}\cos(\gamma_z)\right).
\end{aligned}$$

Henceforth,

$$
\begin{aligned}
1+\frac{v}{c}\cos(\gamma_z')+\frac{v}{c}\cos(\gamma_z) &= 1+\frac{v}{c}\left(\cos(\gamma_z)-\frac{v}{c}\right)\left(1+\frac{v}{c}\cos(\gamma_z)\right)+\frac{v}{c}\cos(\gamma_z)\\
&= 1+\frac{v}{c}\left(\cos(\gamma_z)+\frac{v}{c}\cos^2(\gamma_z)-\frac{v}{c}-\frac{v^2}{c^2}\cos(\gamma_z)\right)+\frac{v}{c}\cos(\gamma_z)\\
&\approx 1+\frac{v}{c}\cos(\gamma_z)+\frac{v}{c}\cos(\gamma_z)\\
&= 1+2\frac{v}{c}\cos(\gamma_z).
\end{aligned}
$$

Substituting these expressions into $u^{(1)}$, we get:

$$
\begin{aligned}
u^{(1)}(t,x') &\approx \int \nu(z)P\left(\left(t-\frac{vx_1'}{c}-\frac{r}{c}+\frac{vz_1}{c^2}-\frac{r-z_1\frac{v}{c}}{c}\right)\left(1+2\frac{v}{c}\cos(\gamma_z)\right)\right)\cdot\chi_\Theta\left(\frac{z_1-\frac{vr}{c}}{R}\right)\\
&= \int \nu(z)P\left(\left(t-\frac{vx_1'}{c}-\frac{r}{c}+\frac{vz_1}{c^2}-\frac{r}{c}+z_1\frac{v}{c^2}\right)\left(1+2\frac{v}{c}\cos(\gamma_z)\right)\right)\cdot\chi_\Theta\left(\frac{z_1-\frac{vr}{c}}{R}\right)\\
&= \int \nu(z)P\left(\left(t-\frac{vx_1'}{c}-2\frac{r}{c}+2\frac{vz_1}{c^2}\right)\left(1+2\frac{v}{c}\cos(\gamma_z)\right)\right)\cdot\chi_\Theta\left(\frac{z_1-\frac{vr}{c}}{R}\right).
\end{aligned}
$$

Within the $P(\cdot)$ expression, x_1' denotes the first coordinate of the antenna when it receives the signal initially, which is the pulse round-trip time (RTT) of the antenna to the target, from the antenna, and reflected back to the antenna. Therefore, we will say $x=(0,-L,H)$ and $x'=(vt,-L,H)$, where the orbit of the antenna is estimated as a straight line. Using the Law of Cosines:

$$
\begin{aligned}
t &= \frac{r}{c}+\frac{1}{c}\sqrt{r^2+(vt)^2-2rvt\cos(\gamma_z)}\\
&= \frac{2r\left(c-v\cos(\gamma_z)\right)}{c^2-v^2}\\
&\approx \frac{2r}{c}\left(1-\frac{v}{c}\cos(\gamma_z)\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
vt &\approx v\left(\frac{2r}{c}\left(1-\frac{v}{c}\cos(\gamma_z)\right)\right)\\
&= 2r\frac{v}{c}-2r\frac{v^2}{c^2}\cos(\gamma_z),
\end{aligned}
$$

and

$$
x_1' \approx 2r\frac{v}{c}\left(1-\frac{v}{c}\cos(\gamma_z)\right).
$$

Then, within $P(\cdot)$, $-2\frac{r}{c}-\frac{vx_1'}{c^2}$ becomes

$$
\begin{aligned}
-2\frac{r}{c} - \frac{vx_1'}{c^2} &= -2\frac{r}{c} - \frac{2r}{c}\frac{v^2}{c^2}\left(1 - \frac{v}{c}\cos(\gamma_z)\right) \\
&= -2\frac{r}{c} - 2\frac{r}{c}\frac{v^2}{c^2} + 2r\frac{v^3}{c^4}\cos(\gamma_z) \\
&\approx -2\frac{r}{c}\left(1 + \frac{v^2}{c^2}\right) \\
&\approx -2\frac{r}{c}.
\end{aligned}
$$

Now, substituting this expression into $u^{(1)}$ we get:

$$
\begin{aligned}
u^{(1)}(t,x') &\approx \int \nu(z) P\left(\left(t - 2\frac{r}{c} + 2\frac{vz_1}{c^2}\right)\left(1 + 2\frac{v}{c}\cos(\gamma_z)\right)\right) \cdot \chi_\Theta\left(\frac{z_1 - \frac{vr}{c}}{R}\right) dz \\
&= \int \nu(z) P\left(\left(t - 2\frac{r}{c}\left(1 - \frac{vz_1}{cr}\right)\right)\left(1 + 2\frac{v}{c}\cos(\gamma_z)\right)\right) \cdot \chi_\Theta\left(\frac{z_1 - \frac{vr}{c}}{R}\right) dz \\
&= \int \nu(z) P\left(\left(t - 2\frac{r}{c}\left(1 - \frac{v}{c}\cos(\gamma_z)\right)\right)\left(1 + 2\frac{v}{c}\cos(\gamma_z)\right)\right) \cdot \chi_\Theta\left(\frac{z_1 - \frac{vr}{c}}{R}\right) dz \\
&\approx \int \nu(z) P\left(t\left(1 + 2\frac{v}{c}\cos(\gamma_z)\right) - 2\frac{r}{c}\left(1 + \frac{v}{c}\cos(\gamma_z)\right)\right) \cdot \chi_\Theta\left(\frac{z_1 - \frac{vr}{c}}{R}\right) dz.
\end{aligned}
$$

This provides a mathematical model for the SAR raw data while taking into account the physical linear Doppler effect in fast time. We also recall that r is the distance between the emitting location of the antenna x and the target z: $r = |z - x| = R_z$. Where $2\frac{r}{c}$ is the retarded moment of time due to travel delay.

Next, we need to apply the matched filter to obtain the imaging kernel in order to define our image. The matched filter portion will have an overbar over it in order to maximize the filter response of $\nu(z)$,

$$
I_x(y) = \int dz \nu(z) \chi_\Theta\left(\frac{z_1 - x_1}{R}\right) \cdot \int_\chi dt \overline{P\left(t\left(1 + 2\frac{v}{c}\cos(\gamma_y) - 2\frac{R_y}{c}\left(1 + \frac{v}{c}\cos(\gamma_y)\right)\right)\right)} \cdot P\bigg(t\left(1 + 2\frac{v}{c}\cos(\gamma_z)\right) - 2\frac{R_z}{c}
$$

where the interior integral is the Point Spread Function (PSF) $W_x(y,z)$. The x_1 argument in $\chi_\Theta(\cdot)$ denotes where the radiation signal was emitted from on the antenna, where we will now be looking at multiple emitting locations $x^n = (x_1^n, -L, H)$. The formula for the single-look angles is given by:

$$
I_{x^n}(y) = \int dz \nu(z) \chi_\Theta\left(\frac{z_1 - x_1^n}{R}\right) \cdot \int_\chi dt \overline{P\left(t\left(1 + 2\frac{v}{c}\cos(\gamma_y^n) - 2\frac{R_y^n}{c}\left(1 + \frac{v}{c}\cos(\gamma_y^n)\right)\right)\right)} \cdot P\bigg(t\left(1 + 2\frac{v}{c}\cos(\gamma_z^n)\right) - 2
$$

Where,

$$
R_y^n = |y - x^n|, \quad \cos(\gamma_y^n) = \frac{y_1 - x_1^n}{R_y^n},
$$

and

$$R_z^n = |z - x^n|\,, \quad \cos(\gamma_z^n) = \frac{z_1 - x_1^n}{R_z^n}.$$

Where x^n is the sequence of emitting locations of the antenna on the orbit that form the synthetic aperture for a given image point y. Then, defining the overall image using the individual contributions of each $I_{x^n}(y)$ where there is overlap between the image, y_1, and target location, z_1, with reference to the emission locations, x_1^n, we obtain:

$$\begin{aligned}
I(y) &= \sum_n \chi_{L_{SA}}\left(y_1 - x_1^n\right) I_{x^n}(y) \\
&= \int \left[\sum_n \chi_{L_{SA}}\left(y_1 - x_1^n\right) \chi_{L_{SA}}\left(z_1 - x_1^n\right) W_{x^n}(y,z)\right] \nu(z) dz \\
&= \int W(y,z)\nu(z)dz.
\end{aligned}$$

Where $W(y,z)$ is the imaging kernel, which is a coherent sum of the PSFs, $W_{x^n}(y,z)$ given by the dt integral, previously. Now, we introduce new notation to help with compactness of the formula:

$$\kappa_y^n = 1 + 2\frac{v}{c}\cos(\gamma_y^n) \quad \text{and} \quad \kappa_z^n = 1 + 2\frac{v}{c}\cos(\gamma_z^n).$$

Substituting κ_y^n into the y term in $P(\cdot)$,

$$t\left(1 + 2\frac{v}{c}\cos(\gamma_y^n)\right) - 2\frac{R_y^n}{c}\left(1 + \frac{v}{c}\cos(\gamma_y^n)\right) = \left(1 + 2\frac{v}{c}\cos(\gamma_y^n)\right)\left(t - 2\frac{R_y^n}{c}\left(1 + \frac{v}{c}\cos(\gamma_y^n)\right)\left(1 + 2\frac{v}{c}\cos(\gamma_y^n)\right)^{-1}\right).$$

Since, we know γ_y^n & γ_z^n needs to be as close to $\frac{\pi}{2}$ in order to maximize the response of the kernel, we know that $\cos(\gamma_y^n)$ will be very small and the $cos(\gamma_y^n)$ term also has a factor of $\frac{v}{c}$, where $c \gg v$. Therefore, $2\frac{v}{c}\cos(\gamma_y^n) \ll 1$, and using the property $\frac{1}{1+x} \simeq 1 - x$ when $x \ll 1$. The multiplication on the right-hand side of the second parenthesis term will become:

$$\begin{aligned}
-2\frac{R_y^n}{c}\left(1 + \frac{v}{c}\cos(\gamma_y^n)\right)\left(1 + 2\frac{v}{c}\cos(\gamma_y^n)\right)^{-1} &\approx -2\frac{R_y^n}{c}\left(1 + \frac{v}{c}\cos(\gamma_y^n)\right)\left(1 - 2\frac{v}{c}\cos(\gamma_y^n)\right) \\
&= -2\frac{R_y^n}{c}\left(1 - 2\frac{v}{c}\cos(\gamma_y^n) + \frac{v}{c}\cos(\gamma_y^n) - 2\frac{v^2}{c^2}\cos^2(\gamma_y^n)\right) \\
&\approx -2\frac{R_y^n}{c}\left(1 - \frac{v}{c}\cos(\gamma_y^n)\right),
\end{aligned}$$

where we dropped the $\frac{v^2}{c^2}$ term. Continuing the substitution into the y term in $P(\cdot)$,

$$\begin{aligned}t\left(1+2\frac{v}{c}\cos(\gamma_y^n)\right)-2\frac{R_y^n}{c}\left(1+\frac{v}{c}\cos(\gamma_y^n)\right) &\approx \left(1+2\frac{v}{c}\cos(\gamma_y^n)\right)\left(t-2\frac{R_y^n}{c}\left(1-\frac{v}{c}\cos(\gamma_y^n)\right)\right)\\ &= \kappa_y^n\left(t-t_y^n\right),\end{aligned}$$

where

$$t_y^n = 2\frac{R_y^n}{c}\left(1-\frac{v}{c}\cos(\gamma_y^n)\right) \quad \text{and} \quad t_z^n = 2\frac{R_z^n}{c}\left(1-\frac{v}{c}\cos(\gamma_z^n)\right).$$

Where the interrogation of the z term in $P(\cdot)$ shows a similar result. Therefore,

$$I_{x^n}(y) = \int dz\nu(z)\chi_\Theta\left(\frac{z_1-x_1^n}{R}\right)\int_\chi dt\overline{P\left(\left(t-t_y^n\right)\kappa_y^n\right)}P\left(\left(t-t_z^n\right)\kappa_z^n\right),$$

where

$$W_{x^n}(y,z) = \int_\chi dt\overline{P\left(\left(t-t_y^n\right)\kappa_y^n\right)}P\left(\left(t-t_z^n\right)\kappa_z^n\right).$$

Using the following definition, derived in a previous homework,

$$\begin{aligned}P(t) &= A(t)e^{-i\omega_o t}\\ &= \chi_\tau(t)e^{-i\alpha t^2}e^{-i\omega_o t},\end{aligned}$$

we can define $W_{x^n}(y,z)$ as:

$$\begin{aligned}W_{x^n}(y,z) &= \int_\chi dt\overline{A\left(\left(t-t_y^n\right)\kappa_y^n\right)}A\left(\left(t-t_z^n\right)\kappa_z^n\right)e^{i\omega_0(t-t_y^n)\kappa_y^n}e^{-i\omega_0(t-t_z^n)\kappa_z^n}\\ &= \int_\chi dt\chi_\tau\left(\left(t-t_y^n\right)\kappa_y^n\right)e^{i\alpha\left((t-t_y^n)\kappa_y^n\right)^2}\chi_\tau\left(\left(t-t_z^n\right)\kappa_z^n\right)e^{-i\alpha\left((t-t_z^n)\kappa_z^n\right)^2}e^{i\omega_0(t-t_y^n)\kappa_y^n}e^{-i\omega_0(t-t_z^n)\kappa_z^n}.\end{aligned}$$

Finally, defining the entire imaging kernel, $W(y,z)$:

$$\begin{aligned}W(y,z) &= \sum_n \chi_{L_{SA}}(y_1-x_1^n)\chi_{L_{SA}}(z_1-x_1^n)\cdot W_{x^n}(y,z)\\ &= \sum_n \chi_{L_{SA}}(y_1-x_1^n)\chi_{L_{SA}}(z_1-x_1^n)\cdot\int_\chi dt\chi_\tau\left(\left(t-t_y^n\right)\kappa_y^n\right)e^{i\alpha\left((t-t_y^n)\kappa_y^n\right)^2}\chi_\tau\left(\left(t-t_z^n\right)\kappa_z^n\right)e^{-i\alpha\left((t-t_z^n)\kappa_z^n\right)^2}e^{i\omega_0(t-}\end{aligned}$$

Notice the indicator functions, χ_τ, have different arguments in comparison to the Generalized Ambiguity Function (GAF) derived for the non-doppler corrected GAF. This indicates to us the stretching of time that occurs during the RTT of the emitted pulse from the antenna and the reception of the scattered field at the moving antenna. Thus, these intervals that define non-trivial contributions to the imaging kernel will have different lengths between the y and z terms. We define these indicator functions below:

$$\chi_\tau\left(\left(t-t_y^n\right)\kappa_y^n\right)=\begin{cases}1, & \text{if } t\in\left[t_y^n-\frac{\tau}{2\kappa_y^n}, t_y^n+\frac{\tau}{2\kappa_y^n}\right]\\ 0, & \text{otherwise}\end{cases}$$

$$\chi_\tau\left(\left(t-t_z^n\right)\kappa_z^n\right)=\begin{cases}1, & \text{if } t\in\left[t_z^n-\frac{\tau}{2\kappa_z^n}, t_z^n+\frac{\tau}{2\kappa_z^n}\right]\\ 0, & \text{otherwise.}\end{cases}$$

The integration is performed over the intersection of these two intervals, i.e., over $\chi_\tau\left(\left(t-t_y^n\right)\kappa_y^n\right)\cdot\chi_\tau\left(\left(t-t_z^n\right)\kappa_z^n\right)$. Which would be either

$$\left[t_y^n-\frac{\tau}{2\kappa_y^n}, t_z^n+\frac{\tau}{2\kappa_z^n}\right]\approx\left[t_y^n-\frac{\tau}{2}\left(1-2\frac{v}{c}\cos(\gamma_y^n)\right), t_z^n+\frac{\tau}{2}\left(1-2\frac{v}{c}\cos(\gamma_z^n)\right)\right]$$

or

$$\left[t_z^n-\frac{\tau}{2\kappa_z^n}, t_y^n+\frac{\tau}{2\kappa_y^n}\right]\approx\left[t_z^n-\frac{\tau}{2}\left(1-2\frac{v}{c}\cos(\gamma_z^n)\right), t_y^n+\frac{\tau}{2}\left(1-2\frac{v}{c}\cos(\gamma_y^n)\right)\right].$$

The analysis is the same for both cases, so we shall continue with the first interval definition, corresponding to $t_z^n < t_y^n$. We also introduce the following notation:

$$T^n=\frac{t_y^n-t_z^n}{2}.$$

Performing similar derivations of the range, $W_R(y,z)$, and azimuthal, $W_{\sum}(y,z)$, factors of $W(y,z)$ as the case without the doppler corrections, we get the following expressions:

$$W_{\sum}(y,z)=\sum_n \chi_{L_{SA}}(y_1-x_1^n)\chi_{L_{SA}}(z_1-x_1^n)e^{-i\omega_0 T^n(\kappa_y^n+\kappa_z^n)}$$

and

$$W_R(y,z)=\tau^c sinc\left(\tau^c\left(\alpha T^c\left(\left(\kappa_y^c\right)^2+\left(\kappa_z^c\right)^2\right)-\omega_0\frac{\left(\kappa_y^c-\kappa_z^c\right)}{2}\right)\right),$$

where τ^c is the center values of the slowly varying envelope portion of the range factor, so all center values would be constants and taken outside of the summation with respect to n. Solving for $T^n\left(\kappa_y^n+\kappa_z^n\right)$ using the previously defined expressions that we will rewrite here for ease of use:

$$\kappa_y^n=1+2\frac{v}{c}\cos(\gamma_y^n)\quad\text{and}\quad\kappa_z^n=1+2\frac{v}{c}\cos(\gamma_z^n),$$

$$T^n = \frac{t_y^n - t_z^n}{2},$$

$$t_y^n = 2\frac{R_y^n}{c}\left(1 - \frac{v}{c}\cos(\gamma_y^n)\right) \quad \text{and} \quad t_z^n = 2\frac{R_z^n}{c}\left(1 - \frac{v}{c}\cos(\gamma_z^n)\right).$$

$$\begin{aligned}
T^n\left(\kappa_y^n + \kappa_z^n\right) &= \frac{t_y^n - t_z^n}{2}\left(1 + 2\frac{v}{c}\cos(\gamma_y^n) + 1 + 2\frac{v}{c}\cos(\gamma_z^n)\right) \\
&= \frac{t_y^n - t_z^n}{2}\left(2 + 2\frac{v}{c}\cos(\gamma_y^n) + 2\frac{v}{c}\cos(\gamma_z^n)\right) \\
&= \left(t_y^n - t_z^n\right)\left(1 + \frac{v}{c}\cos(\gamma_y^n) + \frac{v}{c}\cos(\gamma_z^n)\right) \\
&= \left(2\frac{R_y^n}{c}\left(1 - \frac{v}{c}\cos(\gamma_y^n)\right) - 2\frac{R_z^n}{c}\left(1 - \frac{v}{c}\cos(\gamma_z^n)\right)\right)\left(1 + \frac{v}{c}\cos(\gamma_y^n) + \frac{v}{c}\cos(\gamma_z^n)\right) \\
&= \frac{2}{c}\left(R_y^n\left(1 - \frac{v}{c}\cos(\gamma_y^n)\right) - R_z^n\left(1 - \frac{v}{c}\cos(\gamma_z^n)\right)\right)\left(1 + \frac{v}{c}\cos(\gamma_y^n) + \frac{v}{c}\cos(\gamma_z^n)\right).
\end{aligned}$$

Just analyzing the R_y^n term, we get:

$$\begin{aligned}
&\frac{2}{c}\left(R_y^n\left(1 - \frac{v}{c}\cos(\gamma_y^n)\right) + R_y^n\left(1 - \frac{v}{c}\cos(\gamma_y^n)\right)\left(\frac{v}{c}\cos(\gamma_y^n)\right) + R_y^n\left(1 - \frac{v}{c}\cos(\gamma_y^n)\right)\left(\frac{v}{c}\cos(\gamma_z^n)\right)\right) \\
&\frac{2}{c}\left(R_y^n - R_y^n\frac{v}{c}\cos(\gamma_y^n) + R_y^n\left(\frac{v}{c}\cos(\gamma_y^n) - \frac{v^2}{c^2}\cos^2(\gamma_y^n)\right) + R_y^n\left(\frac{v}{c}\cos(\gamma_z^n) - \frac{v^2}{c^2}\cos(\gamma_y^n)\cos(\gamma_z^n)\right)\right) \\
&\frac{2}{c}\left(R_y^n - R_y^n\frac{v}{c}\cos(\gamma_y^n) + R_y^n\frac{v}{c}\cos(\gamma_y^n) + R_y^n\frac{v}{c}\cos(\gamma_z^n)\right) \\
&\frac{2}{c}\left(R_y^n + R_y^n\frac{v}{c}\cos(\gamma_z^n)\right) \\
&2\frac{R_y^n}{c}\left(1 + \frac{v}{c}\cos(\gamma_z^n)\right),
\end{aligned}$$

where we neglected the $\frac{v^2}{c^2}$ terms. We can easily see the extension of this analysis to the R_z^n term:

$$-2\frac{R_z^n}{c}\left(1 + \frac{v}{c}\cos(\gamma_y^n)\right).$$

Consequently,

$$
\begin{aligned}
T^n(\kappa_y^n + \kappa_z^n) &\approx 2\frac{R_y^n}{c}\left(1 + \frac{v}{c}\cos(\gamma_z^n)\right) - 2\frac{R_z^n}{c}\left(1 + \frac{v}{c}\cos(\gamma_y^n)\right) \\
&= \frac{2}{c}\left(R_y^n\left(1 + \frac{v}{c}\cos(\gamma_z^n)\right) - R_z^n\left(1 + \frac{v}{c}\cos(\gamma_y^n)\right)\right) \\
&= \frac{2}{c}\left(R_y^n - R_z^n + R_y^n\frac{v}{c}\cos(\gamma_z^n) - R_z^n\frac{v}{c}\cos(\gamma_y^n)\right) \\
&= \frac{2}{c}\left(R_y^n - R_z^n + \frac{v}{c}\left(R_y^n\cos(\gamma_z^n) - R_z^n\cos(\gamma_y^n)\right)\right) \\
&= \frac{2}{c}\left(R_y^n - R_z^n + \frac{v}{c}\left(\frac{R_y^n}{R_z^n}(z_1 - x_1^n) - \frac{R_z^n}{R_y^n}(y_1 - x_1^n)\right)\right).
\end{aligned}
$$

Where $R_{Dop} = \frac{R_y^n}{R_z^n}(z_1 - x_1^n) - \frac{R_z^n}{R_y^n}(y_1 - x_1^n)$. Using the Pythagorean expressions for the travel distances,

$$
\begin{aligned}
\left(R_y^n\right)^2 &= R'^2 + \left(x_1^n - y_1\right)^2 \\
\left(R_z^n\right)^2 &= R'^2 + \left(x_1^n - z_1\right)^2,
\end{aligned}
$$

where

$$
\begin{aligned}
R'^2 &= R^2 + l^2 - 2Rl\cos(\frac{\pi}{2} + \theta) \\
&= R^2 + l^2 + 2Rl\sin(\theta) \\
&\approx \left(R + l\sin(\theta)\right)^2
\end{aligned}
$$

is given by the cosine law. R is the slant range, slant distance from the antenna to the target. θ is the angle of incidence of the antenna when emitting the pulse. l is the difference in the range between the image and target, i.e., $y_2 - z_2$. R_{Dop}, defined above, describes the linear effect with respect to $\frac{v}{c}$ of the antenna motion on the azimuthal factor of the GAF. Additionally, introducing the following definitions:

$$
\tilde{x} = \tilde{n}\Delta x_1 \quad and \quad d = \frac{y_1 - z_1}{2},
$$

such that

$$
y_1 - x_1^n = d - \tilde{x} \quad and \quad z_1 - x_1^n = -(d + \tilde{x}).
$$

Substituting these values into R_{Dop} and Tayloring the necessary expressions,

$$
\begin{aligned}
R_{Dop} &= \frac{\sqrt{R'^2 + \left(x_1^n - y_1\right)^2}}{\sqrt{R'^2 + \left(x_1^n - z_1\right)^2}} \frac{z_1 - x_1^n}{|z - x^n|} - \frac{\sqrt{R'^2 + \left(x_1^n - z_1\right)^2}}{\sqrt{R'^2 + \left(x_1^n - y_1\right)^2}} \frac{y_1 - x_1^n}{|y - x^n|} \\
&= -\frac{d + \tilde{x}}{\sqrt{R'^2 + \left(d + \tilde{x}\right)^2}} \frac{\sqrt{R'^2 + \left(\tilde{x} - d\right)^2}}{R} - \frac{d - \tilde{x}}{\sqrt{R'^2 + \left(\tilde{x} - d\right)^2}} \frac{\sqrt{R'^2 + \left(d + \tilde{x}\right)^2}}{R} \\
&= -\frac{d + \tilde{x}}{\sqrt{R^2 + \left(d + \tilde{x}\right)^2}} \sqrt{R'^2 + \left(\tilde{x} - d\right)^2} - \frac{d - \tilde{x}}{\sqrt{R'^2 + \left(\tilde{x} - d\right)^2}} \sqrt{R^2 + \left(d + \tilde{x}\right)^2} \\
&\approx -\frac{d + \tilde{x}}{R\left(1 + \frac{1}{2}\frac{\left(d + \tilde{x}\right)^2}{R^2}\right)} R' \left(1 + \frac{1}{2}\frac{\left(\tilde{x} - d\right)^2}{R'^2}\right) - \frac{\left(d - \tilde{x}\right)^2}{R'\left(1 + \frac{1}{2}\frac{\tilde{x} - d}{R'^2}\right)} R \left(1 + \frac{1}{2}\frac{\left(d + \tilde{x}\right)^2}{R^2}\right) \\
&= -\frac{d + \tilde{x}}{R + \frac{1}{2}\frac{\left(d + \tilde{x}\right)^2}{R}} \left(R' + \frac{1}{2}\frac{\left(\tilde{x} - d\right)^2}{R'}\right) - \frac{d - \tilde{x}}{R' + \frac{1}{2}\frac{\left(\tilde{x} - d\right)^2}{R'}} \left(R + \frac{1}{2}\frac{\left(d + \tilde{x}\right)^2}{R}\right).
\end{aligned}
$$

Inspecting the first term and taking the derivative with respect to $\tilde{x}$,

$$
-\frac{d + \tilde{x}}{R + \frac{1}{2}\frac{\left(d + \tilde{x}\right)^2}{R}} \left(R' + \frac{1}{2}\frac{\left(\tilde{x} - d\right)^2}{R'}\right) = -\frac{dR'}{R + \frac{1}{2}\frac{\left(d + \tilde{x}\right)^2}{R}} + \frac{\frac{d}{2}\frac{\left(\tilde{x} - d\right)^2}{R'}}{R + \frac{1}{2}\frac{\left(d + \tilde{x}\right)^2}{R}} + \frac{\tilde{x}R'}{R + \frac{1}{2}\frac{\left(d + \tilde{x}\right)^2}{R}} + \frac{\frac{\tilde{x}}{2}\frac{\left(\tilde{x} - d\right)^2}{R'}}{R + \frac{1}{2}\frac{\left(d + \tilde{x}\right)^2}{R}}.
$$

Where we will denote:

$$
\begin{aligned}
R_{Dop_1}^{(1)} &= -\frac{dR'}{R + \frac{1}{2}\frac{\left(d + \tilde{x}\right)^2}{R}} \\
R_{Dop_1}^{(2)} &= \frac{\frac{d}{2}\frac{\left(\tilde{x} - d\right)^2}{R'}}{R + \frac{1}{2}\frac{\left(d + \tilde{x}\right)^2}{R}} \\
R_{Dop_1}^{(3)} &= \frac{\tilde{x}R'}{R + \frac{1}{2}\frac{\left(d + \tilde{x}\right)^2}{R}} \\
R_{Dop_1}^{(4)} &= \frac{\frac{\tilde{x}}{2}\frac{\left(\tilde{x} - d\right)^2}{R'}}{R + \frac{1}{2}\frac{\left(d + \tilde{x}\right)^2}{R}}.
\end{aligned}
$$

Solving for $\left.\frac{\partial R_{Dop_1}^{(i)}}{\partial \tilde{x}}\right|_{\tilde{x}=d=0}$, $i \in [1, 2, 3, 4]$ where Ξ will denote don't care values, indicating an expression that has no impact on the result due to the fact that $\tilde{x} = d = 0$ (we are also able to easily see that there will never be a divide by zero occurrence due to R^2 or R'^2 always being present in each denominator):

$$
\begin{aligned}
\left.\frac{\partial R_{Dop_1}^{(1)}}{\partial \tilde{x}}\right|_{\tilde{x}=d=0} &= \frac{\frac{d\tilde{x}}{R} + \frac{\tilde{x}}{R}\Xi}{\Xi} \\
&= \frac{0 + 0\Xi}{\Xi} \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
\left.\frac{\partial R_{Dop_1}^{(2)}}{\partial \tilde{x}}\right|_{\tilde{x}=d=0} &= \frac{\frac{d(\tilde{x}-d)}{R'}\left(R + \frac{1}{2}\frac{(d+\tilde{x})^2}{R}\right) - \frac{d}{2}\frac{(\tilde{x}-d)^2}{R'}\left(\frac{\frac{d\tilde{x}}{R} + \frac{\tilde{x}}{R}\Xi}{R + \frac{1}{2}\left(\frac{d^2+2d\tilde{x}+\tilde{x}^2}{R}\right)^2}\right)}{\Xi} \\
&= \frac{0 - 0}{\Xi} \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
\left.\frac{\partial R_{Dop_1}^{(3)}}{\partial \tilde{x}}\right|_{\tilde{x}=d=0} &= \frac{R'\left(R + \frac{1}{2}\frac{(d+\tilde{x})^2}{R}\right) - \tilde{x}R\Xi}{\left(R + \frac{1}{2}\frac{(d+\tilde{x})^2}{R}\right)^2} \\
&= \frac{R'R}{R^2 + \frac{(d+\tilde{x})^2}{R} + \frac{1}{4}\frac{(d+\tilde{x})^4}{R^2}} \\
&= \frac{R'R}{R^2} \\
&= \frac{R'}{R},
\end{aligned}
$$

$$
\begin{aligned}
\left.\frac{\partial R_{Dop_1}^{(4)}}{\partial \tilde{x}}\right|_{\tilde{x}=d=0} &= \frac{0}{\Xi} \\
&= 0.
\end{aligned}
$$

Therefore,

$$
\left.\frac{\partial R_{Dop_1}}{\partial \tilde{x}}\right|_{\tilde{x}=d=0} = \frac{R'}{R}.
$$

It is easy to see the extension of this:

$$
\left.\frac{\partial R_{Dop_2}}{\partial \tilde{x}}\right|_{\tilde{x}=d=0} = -\frac{R}{R'}.
$$

Where,

$$
\left.\frac{\partial R_{Dop}}{\partial \tilde{x}}\right|_{\tilde{x}=d=0} = \frac{R'}{R} - \frac{R}{R'} \approx -2\frac{l\sin(\theta)}{R},
$$

and, therefore,

$$R_{Dop} \approx -2d - 2\frac{l\sin(\theta)}{R}\tilde{x}.$$

1.2 Problem 2.

Derive [1, equation (6.83)] in your text. Show that, the term identified as $\mathcal{O}(1)$ on the right-hand side is indeed order one for the typical parameters that we are considering, see [1, Table 1.1].

Leveraging the analysis in problem 1, we see the range factor as:

$$W_R(y,z) = \tau^c sinc\left(\tau^c\left(\alpha T^c\left(\left(\kappa_y^c\right)^2 + \left(\kappa_z^c\right)^2\right) - \omega_0\frac{\left(\kappa_y^c - \kappa_z^c\right)}{2}\right)\right).$$

Since the maximum of a *sinc* is at zero, we can find the maximum where:

$$\begin{aligned}\tau^c\left(\alpha T^c\left(\left(\kappa_y^c\right)^2 + \left(\kappa_z^c\right)^2\right) - \omega_0\frac{\left(\kappa_y^c - \kappa_z^c\right)}{2}\right)\Bigg|_{T^c=T^c_{max}} &= 0\\ T^c_{max} &= \frac{\omega_0\left(\kappa_y^c - \kappa_z^c\right)}{2\alpha\left(\left(\kappa_y^c\right)^2 + \left(\kappa_z^c\right)^2\right)}.\end{aligned}$$

The two zeros closest near the central maximum of the *sinc* is at $\pm\pi$. To efficiently determine this we must find τ^c, which we will via the bounds determined by the indicator functions, $\chi_\tau\left(\left(t-t_y^n\right)\kappa_y^n\right)$ & $\chi_\tau\left(\left(t-t_z^n\right)\kappa_z^n\right)$, where we analyzed the case corresponding to $t_z^n < t_y^n$ corresponding to the following interval determined in problem 1:

$$t \in \left[t_y^n - \frac{\tau}{2\kappa_y^n}, t_z^n + \frac{\tau}{2\kappa_z^n}\right] \approx \left[t_y^n - \frac{\tau}{2}\left(1 - 2\frac{v}{c}\cos(\gamma_y^n)\right), t_z^n + \frac{\tau}{2}\left(1 - 2\frac{v}{c}\cos(\gamma_z^n)\right)\right].$$

Previously, we also introduced the notation $T^n = \frac{t_y^n - t_z^n}{2}$ and, now, we will also introduce:

$$\tilde{t} = t - \frac{t_y^n + t_z^n}{2}.$$

Thus, the new interval of integration will be:

$$\begin{aligned}\tilde{t} &\in \left[t_y^n - \frac{\tau}{2}\left(1 - 2\frac{v}{c}\cos(\gamma_y^n)\right) - \frac{t_y^n + t_z^n}{2}, t_z^n + \frac{\tau}{2}\left(1 - 2\frac{v}{c}\cos(\gamma_z^n)\right) - \frac{t_y^n + t_z^n}{2}\right]\\ &\in \left[-\frac{\tau}{2} + \frac{t_y^n - t_z^n}{2} + \tau\frac{v}{c}\cos(\gamma_y^n), \frac{\tau}{2} - \frac{t_y^n - t_z^n}{2} - \tau\frac{v}{c}\cos(\gamma_z^n)\right]\\ &\in \left[-\frac{\tau}{2} + T^n + \tau\frac{v}{c}\cos(\gamma_y^n), \frac{\tau}{2} - T^n - \tau\frac{v}{c}\cos(\gamma_z^n)\right]\end{aligned}.$$

We can clearly see that the integration interval is asymmetric when $\tilde{t} = 0$ as the intersecting intervals are of different lengths, i.e., the times it takes between the target and image points to the moving antenna are different. In changing the integration variable of $W(y, z)$ we will notice that the quantities of some of these variables will be $\ll 1$, when looking at values from Table 1.1, where $\frac{|y_1 - z_1|}{R} \sim \frac{\Delta_A}{R} \sim 10^{-5}$, $|T^n| \lesssim \frac{1}{B}$ and $\alpha|\tilde{t}||T^n| \lesssim \alpha\tau|T^n| \sim B|T^n| = 0(1)$. Thus, it is beneficial to change the integration variable an additional time where:

$$\tilde{t}' = \tilde{t} - \frac{\tau}{2}\frac{v}{c}\left(\cos(\gamma_y^n) - \cos(\gamma_z^n)\right) \Leftrightarrow \tilde{t} = \tilde{t}' + \frac{\tau}{2}\frac{v}{c}\left(\cos(\gamma_y^n) - \cos(\gamma_z^n)\right).$$

Then,

$$\begin{aligned}
\tilde{t}' &\in \left[-\frac{\tau}{2} + T^n + \tau\frac{v}{c}\cos(\gamma_y^n) - \frac{\tau}{2}\frac{v}{c}\left(\cos(\gamma_y^n) - \cos(\gamma_z^n)\right), \frac{\tau}{2} - T^n + \tau\frac{v}{c}\cos(\gamma_z^n) - \frac{\tau}{2}\frac{v}{c}\left(\cos(\gamma_y^n) - \cos(\gamma_z^n)\right)\right] \\
&\in \left[-\frac{\tau}{2} + T^n + \frac{\tau}{2}\frac{v}{c}\left(\cos(\gamma_y^n) + \cos(\gamma_z^n)\right), \frac{\tau}{2} - T^n - \frac{\tau}{2}\frac{v}{c}\left(\cos(\gamma_y^n) + \cos(\gamma_z^n)\right)\right].
\end{aligned}$$

Using these new bounds to solve for τ^c, where $n = n_c$:

$$\begin{aligned}
\tau^c &= \frac{\tau}{2} - T^c - \frac{\tau}{2}\frac{v}{c}\left(\cos(\gamma_y^n + \cos(\gamma_z^n))\right) - \left(-\frac{\tau}{2} + T^c + \frac{\tau}{2}\left(\cos(\gamma_y^n) + \cos(\gamma_z^n)\right)\right) \\
&= \tau - 2T^c - \tau\frac{v}{c}\left(\cos(\gamma_y^n) + \cos(\gamma_z^n)\right).
\end{aligned}$$

We will also denote $\delta T^c = T^c - T^c_{max}$. We will now try to find the zeros closest to the center of the *sinc*, which we can find using this equation determined in our first homework $B(1 - \frac{2T^c}{\tau})T^c = \pi$, where B is the bandwidth:

$$B\left(1 - \frac{2\delta T^c}{\tau} - \frac{2T^c_{max}}{\tau} - \frac{v}{c}\left(\cos(\gamma_y^c) + \cos(\gamma_z^c)\right)\right)\frac{\delta T^c}{2}\left(\left(\kappa_y^c\right)^2 + \left(\kappa_z^c\right)^2\right) = \pm\pi.$$

We know that $|\cos(\gamma_z^c)| \lesssim \frac{L_{SA}}{R}$ & $\left|\cos(\gamma_y^c)\right| \lesssim \frac{L_{SA}}{R}$, therefore,

$$|T^c_{max}| = \frac{\omega_0\frac{v}{c}\left|\cos(\gamma_y^c) - \cos(\gamma_z^c)\right|}{\frac{B}{\tau}\left(2 + 4\frac{v}{c}\left(\cos(\gamma_y^n) + \cos(\gamma_z^n)\right)\right)} \lesssim \tau\frac{\omega_0}{B}\frac{v}{c}\frac{L_{SA}}{R} \ll \tau.$$

Where $\frac{\omega_0}{B}\frac{v}{c}\frac{L_{SA}}{R}$ is typically very small according to Table 1.1 and 1.2, e.g., $\frac{\omega_0}{B} \sim 3.33 \cdot 10^1$, $\frac{v}{c} \sim 2.53 \cdot 10^{-5}$, & $\frac{L_{SA}}{R} \sim 5 \cdot 10^{-2}$. Thus, $\frac{\omega_0}{B}\frac{v}{c}\frac{L_{SA}}{R} \sim 10^{-5}$.

To obtain our estimate of $|T^c_{max}|$, we dropped the quadratic terms $\frac{v^2}{c^2}$, implying:

$$\left(\kappa_y^c\right)^2 + \left(\kappa_z^c\right)^2 \approx 2 + 4\frac{v}{c}\left(\cos(\gamma_y^c) + \cos(\gamma_z^c)\right).$$

Thus, finding the roots of δT^c,

$$-\left(\delta T^c\right)^2\left(\frac{4B}{\tau}\left(\left(\kappa_y^c\right)^2 + \left(\kappa_z^c\right)^2\right)\right) + \delta T^c\left(B\left(2 - \frac{4T^c_{max}}{\tau} - \frac{v}{c}\left(\cos(\gamma_y^c) + \cos(\gamma_z^c)\right)\right)\left(\left(\left(\kappa_y^c\right)^2 + \left(\kappa_z^c\right)^2\right)\right)\right) \mp \pi = 0.$$

Thus,

$$\delta T^c = \pm \frac{2\pi c}{B\left(\left(\kappa_y^c\right)^2+\left(\kappa_z^c\right)^2\right)} := \pm\Delta_R \Leftrightarrow T^c = T^c_{max}\pm\Delta_R.$$

For $\Delta_R \leq T^c - T^c_{max} \leq \Delta_R$ we can replace τ^c on the right-hand side of $W_R(y,z)$ with τ following from $|T^c_{max}| \lesssim \tau\frac{\omega_0}{B}\frac{v}{c}\frac{L_{SA}}{R} \ll \tau$ and the estimate that $\Delta_R \ll \tau$ and making the appropriate substitutions into $W_R(y,z)$ and where $\alpha = \frac{B}{2\tau}$.:

$$\begin{aligned}
W_R(y,z) &= \tau^c sinc\left(\tau^c \frac{t_y^c - t_z^c}{2}\frac{B}{2\tau}\left(\left(\kappa_y^c\right)^2+\left(\kappa_z^c\right)^2\right) - \tau^c\omega_0\frac{\kappa_y^c-\kappa_z^c}{2}\right)\\
&\approx \tau sinc\left(\tau\frac{\frac{2R_y^c}{c}\left(1-\frac{v}{c}\cos(\gamma_y^c)\right) - \frac{2R_z^c}{c}\left(1-\frac{v}{c}\cos(\gamma_z^c)\right)}{2}\frac{B}{2\tau}\left(\left(\kappa_y^c\right)^2+\left(\kappa_z^c\right)^2\right) - \tau\omega_0\frac{c}{B}\frac{B}{c}\frac{\kappa_y^c-\kappa_z^c}{2}\right)\\
&= \tau sinc\left(\frac{B}{2c}\left(\left(\kappa_y^c\right)^2+\left(\kappa_z^c\right)^2\right)\left(R_y^c\left(1-\frac{v}{c}\cos(\gamma_y^c)\right) - R_z^c\left(1-\frac{v}{c}\cos(\gamma_z^c)\right)\right) - \tau c\frac{\omega_0}{B}\frac{B}{c}\frac{\kappa_y^c-\kappa_z^c}{2}\right)\\
&= \tau sinc\left(\frac{B}{c}\left(\frac{\left(\kappa_y^c\right)^2+\left(\kappa_z^c\right)^2}{2}\left(R_y^c\left(1-\frac{v}{c}\cos(\gamma_y^c)\right) - R_z^c\left(1-\frac{v}{c}\cos(\gamma_z^c)\right)\right) - \tau c\frac{\omega_0}{B}\frac{\kappa_y^c-\kappa_z^c}{2}\right)\right).
\end{aligned}$$

Continuing to the factorization error of the GAF, we take the difference of the non-factorized GAF, W with that of the factorized GAF, $W_{(R,\sum)}$:

$$W = \sum_n \chi_{L_{SA}}\left(y_1 - x_1^n\right)\chi_{L_{SA}}\left(z_1 - x_1^n\right)e^{-i\omega_0 T^n\left(\kappa_y^n+\kappa_z^n\right)}\cdot\tau^n sinc\left(\tau^n\left(\alpha T^n\left(\left(\kappa_y^n\right)^2+\left(\kappa_z^n\right)^2\right)-\omega_0\frac{\kappa_y^n-\kappa_z^n}{2}\right.\right.$$

$$W_{(R,\sum)} = \tau^c sinc\left(\tau^c\left(\alpha T^c\left(\left(\kappa_y^c\right)^2+\left(\kappa_z^c\right)^2\right)-\omega_0\frac{\kappa_y^c-\kappa_z^c}{2}\right)\right)\cdot\sum_n \chi_{L_{SA}}\left(y_1 - x_1^n\right)\chi_{L_{SA}}\left(z_1 - x_1^n\right)e^{-i\omega_0 T^n\left(\kappa_y^n+\kappa\right.}$$

$$W - W_{(R,\sum)} = \sum_n \chi_{L_{SA}}\left(y_1 - x_1^n\right)\chi_{L_{SA}}\left(z_1 - x_1^n\right)e^{-i\omega_0 T^n\left(\kappa_y^n+\kappa_z^n\right)}\cdot\left[\tau^n sinc\left(\tau^n\left(\alpha T^n\left(\left(\kappa_y^n\right)^2+\left(\kappa_z^n\right)^2\right)-\omega_0\frac{\kappa_y^n-\kappa}{2}\right.\right.\right.$$

Analyzing $T^n\left(\left(\kappa_y^n\right)^2+\left(\kappa_z^n\right)^2\right)$ using the definition from earlier that $\left(\kappa_y^n\right)^2+\left(\kappa_z^n\right)^2 \approx 2 + 4\frac{v}{c}\left(\cos(\gamma_y^n)+\cos(\gamma_z^n)\right)$:

$$
\begin{aligned}
T^n \Big(\left(\kappa_y^n\right)^2 + \left(\kappa_z^n\right)^2 \Big) &= \frac{t_y^n - t_z^n}{2} \Big(2 + 4\frac{v}{c} \left(\cos(\gamma_y^n) + \cos(\gamma_z^n) \right) \Big) \\
&= \frac{\frac{2}{c} \left(R_y^n \left(1 - \frac{v}{c}\cos(\gamma_y^n) \right) - R_z^n \left(1 - \frac{v}{c}\cos(\gamma_z^n) \right) \right)}{2} \Big(2 + 4\frac{v}{c} \left(\cos(\gamma_y^n) + \cos(\gamma_z^n) \right) \Big) \\
&= \frac{2}{c} \Big(R_y^n \Big(1 - \frac{v}{c}\cos(\gamma_y^n) \Big) \Big(1 + 2\frac{v}{c} \left(\cos(\gamma_y^n) + \cos(\gamma_z^n) \right) \Big) - R_z^n \Big(1 - \frac{v}{c}\cos(\gamma_z^n) \Big) \Big(1 + 2\frac{v}{c} \big(\cos(\gamma \\
&= \frac{2}{c} \Bigg(R_y^n \Bigg(1 + 2\frac{v}{c}\cos(\gamma_y^n) + 2\frac{v}{c}\cos(\gamma_z^n) - \frac{v}{c}\cos(\gamma_y^n) - o(\frac{v^2}{c^2}) \Bigg) - R_z^n \Bigg(1 + 2\frac{v}{c}\cos(\gamma_y^n) + 2\frac{v}{c}\cos \\
&\approx \frac{2}{c} \Big(R_y^n \Big(1 + \frac{v}{c}\cos(\gamma_y^n) + 2\frac{v}{c}\cos(\gamma_z^n) \Big) - R_z^n \Big(1 + \frac{v}{c}\cos(\gamma_y^n) + 2\frac{v}{c}\cos(\gamma_z^n) \Big) \Big)
\end{aligned}
$$

$$
\begin{aligned}
\cos(\gamma_y^n) &= \frac{y_1 - x_1^n}{R_y^n} \\
\cos(\gamma_z^n) &= \frac{z_1 - x_1^n}{R_z^n}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2}{c} \left(R_y^n \left(1 + \frac{v}{c}\frac{y_1 - x_1^n}{R_y^n} + 2\frac{v}{c}\frac{z_1 - x_1^n}{R_z^n} \right) - R_z^n \left(1 + \frac{v}{c}\frac{z_1 - x_1^n}{R_z^n} + 2\frac{v}{c}\frac{y_1 - x_1^n}{R_y^n} \right) \right) \\
&= \frac{2}{c} \left(\left(R_y^n - R_z^n \right) + \frac{v}{c} \left[R_y^n \left(\frac{y_1 - x_1^n}{R_y^n} + 2\frac{z_1 - x_1^n}{R_z^n} \right) - R_z^n \left(\frac{z_1 - x_1^n}{R_z^n} + 2\frac{y_1 - x_1^n}{R_y^n} \right) \right] \right) \\
&= \frac{2}{c} \left(\left(R_y^n - R_z^n \right) + \frac{v}{c} \left[y_1 - x_1^n - (z_1 - x_1^n) + 2 \left(\frac{R_y^n}{R_z^n} \left(z_1 - x_1^n \right) - \frac{R_z^n}{R_y^n} \left(y_1 - x_1^n \right) \right) \right] \right) \\
&= \frac{2}{c} \left(\left(R_y^n - R_z^n \right) + \frac{v}{c} \left[y_1 - z_1 + 2 \left(\frac{R_y^n}{R_z^n} \left(z_1 - x_1^n \right) - \frac{R_z^n}{R_y^n} \left(y_1 - x_1^n \right) \right) \right] \right).
\end{aligned}
$$

Where

$$
R'_{Dop} = y_1 - z_1 + 2 \left(\frac{R_y^n}{R_z^n} \left(z_1 - x_1^n \right) - \frac{R_z^n}{R_y^n} \left(y_1 - x_1^n \right) \right).
$$

Since, we are only interested in the portion of the factorization error corresponding to the Doppler effect, with relation to the Start-Stop approximation, we are particularly interested in the terms $\propto \frac{v}{c}$ and according to previous analysis of R_{Dop}:

$$
R'_{Dop} = 2d + 2R_{Dop} \approx -2d - 4\frac{l\sin(\theta)}{R}\tilde{x}.
$$

Therefore, we finally get:

$$
T^n\left(\left(\kappa_y^n\right)^2+\left(\kappa_z^n\right)^2\right)\approx\frac{2}{c}\left(\left(R_y^n-R_z^n\right)+\frac{v}{c}\left[-2d-4\frac{l\sin(\theta)}{R}\tilde{x}\right]\right)
$$

$$
d=\frac{y_1-z_1}{2}
$$

$$
\begin{aligned}
&=\frac{2}{c}\left(\left(R_y^n-R_z^n\right)+\frac{v}{c}\left[-\left(y_1-z_1\right)-4\frac{l\sin(\theta)}{R}\tilde{x}\right]\right)\\
&=\frac{2}{c}\left(\left(R_y^n-R_z^n\right)-\frac{v}{c}\left(y_1-z_1\right)-4\frac{v}{c}\frac{l\sin(\theta)}{R}\tilde{x}\right)
\end{aligned}
$$

Additionally, the term proportional to ω_0 in $sinc(\cdot)$, $\frac{\kappa_y^n-\kappa_z^n}{2}$, can be expanded using previous definitions for κ_y^n, κ_z^n, $\cos(\gamma_y^n)$, and $\cos(\gamma_y^n)$, as well as utilizing the Taylor Series of the appropriate terms:

$$
\begin{aligned}
\frac{\kappa_y^n-\kappa_z^n}{2}&=\frac{\left(1+2\frac{v}{c}\cos(\gamma_y^n)\right)-\left(1+2\frac{v}{c}\cos(\gamma_z^n)\right)}{2}\\
&=\frac{2\frac{v}{c}\left(\cos(\gamma_y^n)-\cos(\gamma_z^n)\right)}{2}\\
&=\frac{v}{c}\left(\cos(\gamma_y^n)-\cos(\gamma_z^n)\right)\\
&=\frac{v}{c}\left(\frac{d-\tilde{x}}{\sqrt{R'^2+\left(d-\tilde{x}\right)^2}}-\frac{-\left(d+\tilde{x}\right)}{\sqrt{R^2+\left(d+\tilde{x}\right)^2}}\right)\\
&\approx\frac{v}{c}\left(d\left(\frac{1}{R}+\frac{1}{R'}\right)+\frac{R'-R}{RR'}\tilde{x}\right)\\
&\approx\frac{v}{c}\left(\frac{y_1-z_1}{R}+\frac{l\sin(\theta)}{R^2}\tilde{x}\right).
\end{aligned}
$$

Finally, solving for:

$$
\begin{aligned}
T^n\left(\left(\kappa_y^n\right)^2+\left(\kappa_z^n\right)^2\right)-\omega_0\frac{\kappa_y^n-\kappa_z^n}{2\alpha} &\approx \frac{2}{c}\left(\left(R_y^n-R_z^n\right)-\frac{v}{c}\left(y_1-z_1\right)-4\frac{v}{c}\frac{l\sin(\theta)}{R}\tilde{x}\right)-\frac{\omega_0}{\alpha}\frac{v}{c}\left(\frac{y_1-z_1}{R}+\tilde{x}\frac{l\sin(\theta)}{R^2}\right)\\
&= \frac{2}{c}\left(\left(R_y^n-R_z^n\right)-\frac{v}{c}\left(y_1-z_1\right)\right)-\frac{8v}{c^2}\frac{l\sin(\theta)}{R}\tilde{x}-\frac{\omega_0}{\alpha}\frac{v}{c}\frac{l\sin(\theta)}{R^2}\tilde{x}-\frac{\omega_0}{\alpha}\frac{v}{c}\frac{y_1-z_1}{R}\\
&= \frac{2}{c}\left(\left(R_y^n-R_z^n\right)-\frac{v}{c}\left(y_1-z_1\right)\right)-\frac{2v}{c^2}\frac{l\sin(\theta)}{R}\tilde{x}\left(4+\frac{\omega_0}{\alpha}\frac{c}{2R}\right)-\frac{\omega_0}{\alpha}\frac{v}{c}\frac{y_1-z_1}{R}
\end{aligned}
$$

$$
\cos^2(\theta)=1-\frac{L^2}{R^2}, \quad \therefore \quad \sin(\theta)=\frac{L}{R}
$$

$$
\begin{aligned}
&= \frac{2}{c}\left(\left(R_y^n-R_z^n\right)-\frac{v}{c}\left(y_1-z_1\right)\right)-\frac{2v}{c^2}\frac{Ll}{R^2}\tilde{x}\left(4+\frac{\omega_0}{\alpha}\frac{c}{2R}\right)-\frac{\omega_0}{\alpha}\frac{v}{c}\frac{y_1-z_1}{R}\\
&= \frac{2}{c}\left(\left(R_y^n-R_z^n\right)-\frac{v}{c}\left(y_1-z_1\right)-\frac{v}{c}\frac{Ll}{R^2}\tilde{x}\left(4+\frac{\omega_0}{\alpha}\frac{c}{2R}\right)\right)-\frac{\omega_0}{\alpha}\frac{v}{c}\frac{y_1-z_1}{R}\\
&= \frac{2}{c}\left(\frac{Ll}{R}-\frac{v}{c}\left(y_1-z_1\right)-\tilde{x}\left[\frac{y_1-z_1}{R}+\frac{v}{c}\frac{Ll}{R^2}\left(4+\frac{\omega_0}{\alpha}\frac{c}{2R}\right)\right]\right)-\frac{\omega_0}{\alpha}\frac{v}{c}\frac{y_1-z_1}{R}.
\end{aligned}
$$

Substituting this into our original expression for W_R at the beginning of problem 2 and taking $\tau^c=\tau$ for the right-hand side and evaluating at $\tilde{x}=0$, i.e., when the antenna is not moving and the Start-Stop approximation is being used:

$$
W_R \approx \tau sinc\left(\tau\left(\alpha\left[\frac{2}{c}\left(\frac{Ll}{R}-\frac{v}{c}\left(y_1-z_1\right)\right)-\frac{\omega_0}{\alpha}\frac{v}{c}\frac{y_1-z_1}{R}\right]\right)\right)
$$

$$
\alpha=\frac{B}{2\tau}
$$

$$
= \tau sinc\left(\frac{B}{c}\left(\frac{Ll}{R}-\frac{v}{c}\left(y_1-z_1\right)-\frac{\omega_0}{\alpha}\frac{v}{c}c\frac{y_1-z_1}{R}\right)\right)
$$

$$
\frac{L}{R}=\sin(\theta), \quad l=y_2-z_2
$$

$$
= \tau sinc\left(\frac{B}{c}\left(\left(y_2-z_2\right)\sin(\theta)-\left(y_1-z_1\right)\frac{v}{c}\left(1+\frac{\omega_0 c}{2\alpha R}\right)\right)\right)
$$

$$
\Delta_R=\frac{\pi c}{B}
$$

$$
= \tau sinc\left(\frac{\pi}{\Delta_R}\left(\left(y_2-z_1\right)\sin(\theta)-\left(y_1-z_1\right)\frac{v}{c}\left(1+\frac{\omega_0 c}{2\alpha R}\right)\right)\right).
$$

Now, we can show that $1+\frac{\omega_0 c}{2\alpha R}$ is $\mathcal{O}(1)$. Using typical values obtained from [1, Table 1.1]: $\omega_0 \sim 300\cdot 2\pi MHz \sim 5.3\cdot 10^{-10}s$, $c\sim 3\cdot 10^8 m/s$, $R\sim 1\cdot 10^6 m$ & $\alpha \ll \omega_0 \sim 5.3\cdot 10^{-10}s$. If we substitute these approximate values into the expression, then: $\sim 1 + 7.5 \cdot 10^{-5} Hz \sim 1$. Thus, we can see that this term is $\mathcal{O}(1)$.

Continuing from the analysis of problem one where,

$$
\begin{aligned}
T^n \left(\kappa_y^n + \kappa_z^n\right) &= \frac{2}{c}\left(R_y^n - R_z^n + \frac{v}{c}\left(\frac{R_y^n}{R_z^n}\left(z_1 - x_1^n\right) - \frac{R_z^n}{R_y^n}\left(y_1 - x_1\right)\right)\right) \\
R_{Dop} &= \frac{R_y^n}{R_z^n}\left(z_1 - x_1^n\right) - \frac{R_z^n}{R_y^n}\left(y_1 - x_1\right) \\
R_{Dop} &\approx -2d - 2\frac{l\sin(\theta)}{R}\tilde{x}.
\end{aligned}
$$

When substituting this approximation of R_{Dop} into $T^n\left(\kappa_y^n + \kappa_z^n\right)$, we see that the first term, $-2d$, will be multiplied with $\frac{v}{c}$, which will be a small dimensionless quantity, and d will be a small quantity as it is defined as $d = \frac{y_1 - z_1}{2}$ and we are only interested when $|y_1 - z_1| \ll 1$. Therefore, we can omit the $-2d$ term and get:

$$
\begin{aligned}
T^n\left(\kappa_y^n + \kappa_z^n\right) &= \frac{2}{c}\left(R_y^n - R_z^n - 2\frac{v}{c}\frac{l\sin(\theta)}{R}\tilde{x}\right) \\
&= \frac{2}{c}\left(R_y^n - R_z^n - 2\frac{v}{c}\frac{l\sin(\theta)}{R}\Delta x_1 \tilde{n}\right), \quad \text{where} \quad \tilde{x} = \Delta x_1 \tilde{n}.
\end{aligned}
$$

Solving for R_z^n & R_y^n, where $z_2 = 0$ & $y_2 - z_2 = y_2 = l$:

$$
\begin{aligned}
R_z^n &= \left(H^2 + L^2 + \left(x_1^n - z_1\right)^2\right)^{\frac{1}{2}}, \quad \text{where} \quad R^2 = H^2 + L^2 \\
&= \left(R^2 + \left(x_1^n - z_1\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&= R\left(1 + \frac{\left(x_1^n - z_1\right)^2}{R^2}\right)^{\frac{1}{2}} \\
&\approx R\left(1 + \frac{1}{2}\frac{\left(x_1^n - z_1\right)^2}{R^2}\right) \\
&= R + \frac{1}{2}\frac{\left(x_1^n - z_1\right)^2}{R}
\end{aligned}
$$

$$
\begin{aligned}
R_y^n &= \left(H^2 + \left(L + l\right)^2 + \left(x_1^n - y_1\right)^2\right)^{\frac{1}{2}} \\
&= \left(H^2 + L^2 + 2Ll + l^2 + \left(x_1^n - y_1\right)^2\right)^{\frac{1}{2}} \\
&= \left(R^2 + 2Ll + l^2 + \left(x_1^n - y_1\right)^2\right)^{\frac{1}{2}} \\
&= R\left(1 + \frac{2Ll + l^2 + \left(x_1^n - y_1\right)^2}{R^2}\right)^{\frac{1}{2}} \\
&\approx R\left(1 + \frac{1}{2}\frac{2Ll + l^2 + \left(x_1^n - y_1\right)^2}{R^2} - \frac{1}{8}\left(\frac{2Ll + l^2 + \left(x_1^n - y_1\right)^2}{R^2}\right)^2\right).
\end{aligned}
$$

Further reducing R_y^n by analyzing the quadratic term in the Taylor expansion expression:

$$
\begin{aligned}
\left(\frac{2Ll + l^2 + \left(x_1^n - y_1\right)^2}{R^2}\right)^2 &= \frac{1}{R^4}\left(4L^2l^2 + 2Ll^3 + 2Ll\left(x_1^n - y_1\right)^2 + 2Ll^3 + l^4 + l^2\left(x_1^n - y_1\right)^2 + 2Ll\left(x_1^n - y_1\right)^2 + l^2\left(\right.\right. \\
&= \frac{1}{R^4}\left(4L^2l^2 + 4Ll^3 + 4Ll\left(x_1^n - y_1\right)^2 + l^4 + 2l^2\left(x_1^n - y_1\right)^2 + \left(x_1^n - y_1\right)^4\right) \\
&\approx \frac{4L^2l^2}{R^4}.
\end{aligned}
$$

Where the other terms are inconsequential magnitudes as they are not very large numerators to begin with and become dimensionless when divided by R^4. Hence,

$$
\begin{aligned}
R_y^n &\approx R\left(1+\frac{1}{2}\frac{2Ll+l^2+\left(x_1^n-y_1\right)^2}{R^2}-\frac{1}{8}\frac{4L^2l^2}{R^4}\right)\\
&= R+\frac{1}{2}\frac{2Ll+l^2+\left(x_1^n-y_1\right)^2}{R}-\frac{1}{8}\frac{4L^2l^2}{R^3}\\
&= R+\frac{1}{2}\frac{2Ll+l^2\left(1-\frac{L^2}{R^2}\right)+\left(x_1^n-y_1\right)^2}{R}, \quad \text{where} \quad 1-\frac{L}{R}=\cos(\theta)\\
&= R+\frac{1}{2}\frac{2Ll+l^2\cos^2(\theta)+\left(x_1^n-y_1\right)^2}{R}.
\end{aligned}
$$

Henceforth,

$$
\begin{aligned}
R_y^n-R_z^n &= R+\frac{1}{2}\frac{2Ll+l^2\cos^2(\theta)+\left(x_1^n-y_1\right)^2}{R}-R-\frac{1}{2}\frac{\left(x_1^n-z_1\right)^2}{R}\\
&=\frac{1}{2}\frac{2Ll+l^2\cos^2(\theta)+\left(x_1^n-y_1\right)^2-\left(x_1^n-z_1\right)^2}{R}\\
&=\frac{Ll}{R}+\frac{1}{2}\frac{l^2\cos^2(\theta)}{R}+\frac{y_1^2-z_1^2}{R}+\frac{2\left(y_1-z_1\right)x_1^n}{R}.
\end{aligned}
$$

Using the trig identity $\sin^2(\theta)+\cos^2(\theta)=1$ & and the previous definition, $\cos^2(\theta)=1-\frac{L^2}{R^2}$, we know $\sin(\theta)=\frac{L}{R}$. Substituting these values into $T^n\left(\kappa_y^n+\kappa_z^n\right)$:

$$
T^n\left(\kappa_y^n+\kappa_z^n\right)=\frac{2}{c}\left(\frac{Ll}{R}+\frac{1}{2}\frac{l^2\cos^2(\theta)}{R}+\frac{y_1^2-z_1^2}{R}+\frac{2\left(z_1-y_1\right)x_1^n}{R}-2\frac{Ll}{R^2}\frac{v}{c}\Delta x_1\tilde{n}\right)
$$

$$
\begin{aligned}
x_1^n &= y_1+\tilde{x}=y_1+\Delta x_1\tilde{n}\\
&= z_1+\tilde{x}=z_1+\Delta x_1\tilde{n}
\end{aligned}
$$

$$
=\frac{2}{c}\left(\frac{Ll}{R}+\frac{l^2\cos^2(\theta)}{2R}\right)-\frac{2}{c}\left(\frac{y_1-z_1}{R}+2\frac{Ll}{R^2}\frac{v}{c}\right)\Delta x_1\tilde{n}.
$$

1.3 Problem 3.

Expand on the analysis of [1, Appendix 6.A] in your text. In particular, derive [1, equation (6.111)] carefully, i.e., show how the change of variable $\mu(t')=|z-x(t')|+ct'$ leads to the last line of the equation. Analyze the roots of [1, equation (6.113)] (a quadratic equation) and explain why it is the root given by [1, equation (6.114)] that is deemed appropriate. What happens to the other root? Finally, substitute [1, equation (6.114)] and [1, equation (6.115)] into [1, equation (6.112)] and show that the resulting $u(t,z)$ indeed reduces to [1, equation (6.4)].

We will begin by looking at the wave equation that governs the radiation of waves by a moving source:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial z_1^2}-\frac{\partial^2 u}{\partial z_2^2}-\frac{\partial^2 u}{\partial z_3^2}=P(t)\delta(z-x(t))\equiv P(t)\delta(z_1-vt)\delta(z_2+L)\delta(z_3-H),$$

where the point source $P(t)$ is located at $x(t)=(vt,-L,H)$. Using the Lorentz transform, this becomes:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial \sigma^2}-\frac{\partial^2 u}{\partial \zeta^2}-\frac{\partial^2 u}{\partial z_2^2}-\frac{\partial^2 u}{\partial z_3^2}=P(\frac{\sigma}{\beta}+\frac{\frac{v}{c}}{\beta}\frac{\zeta_1}{c})\delta(\beta\zeta_1)\delta(z_2+L)\delta(z_3-H):=f(\sigma,\zeta_1,z_2,z_3).$$

This above solution is the incident field. Then using Kirchhoff's integral, the solution in the new coordinates would be:

$$u^{(0)}\left(\sigma,\zeta_1,z_2,z_3\right)=\frac{1}{4\pi}\int\int\int\frac{f\left(\sigma-\frac{\rho'}{c},\zeta_1',z_2',z_3'\right)}{\rho'}d\zeta_1'dz_2'dz_3',$$

where

$$\rho'=\left(\left(\zeta_1-\zeta_1'\right)^2+\left(z_2-z_2'\right)^2+\left(z_3-z_3'\right)^2\right)^{\frac{1}{2}},$$

which reduces to

$$u^{(0)}(t,z)=\frac{1}{4\beta\pi}\frac{P\left(\frac{\sigma-\frac{\rho}{c}}{\beta}\right)}{\rho},$$

where

$$\rho=\sqrt{\zeta_1^2+\left(z_2+L\right)^2+\left(z_3-H\right)^2}.$$

This is the resultant expression we wish to reduce our following derivations to. The fundamental solution of the d'Alembert operator on the left-hand side of the wave equation above is given by:

$$\varepsilon(t,z)=\frac{\mathcal{H}(t)}{4\pi}\frac{\delta\left(|z|-ct\right)}{t},$$

where

$$\mathcal{H}(t):=\begin{cases}1, & t\geq 0\\ 0, & t<0.\end{cases}$$

Which is the Heaviside function, and $\delta\left(|z|-ct\right)$ is a single layer of unit magnitude on the expanding sphere of radius ct centered at the origin. Then the solution of the wave equation is obtained via convolution:

$$u^{(0)}(t,z) = \frac{1}{4\pi}\int_{-\infty}^{t} dt' \iiint\limits_{\mathbb{R}^3} \frac{\delta\left(|z-z'| - c\left(t-t'\right)\right)}{t-t'} \cdot P(t')\delta(z_1' - vt')\delta(z_2' + L)\delta(z_3' - H)dz'.$$

Where $\int_{-\infty}^{t}$ comes from the Heaviside function. Integrating only over the bounds where $\delta(z_1' - vt')\delta(z_2' + L)\delta(z_3' - H)$ is non-zero, we get:

$$u^{(0)}(t,z) = \frac{1}{4\pi}\int_{-\infty}^{t} \frac{\delta\left(|z-x(t')| - c\left(t-t'\right)\right)}{t-t'}P(t')dt',$$

where $x(t') \equiv z'$ as z' is the antenna position upon receiving the scattered signal and $x(t')$ is a function of the original position of the antenna and the time between the emission of the signal and the reception of the scattered signal at the antenna. Performing a change of variable with $\mu = \mu(t') = |z - x(t)| + ct' \equiv \sqrt{\left(z_1 - vt'\right)^2 + \left(z_2 + L\right)^2 + \left(z_3 - H\right)^2} + ct'$:

$$d\mu = dt'\left[\left(\left(z_1 - vt'\right)^2 + \left(z_2 + L\right)^2 + \left(z_3 - H\right)^2\right)^{\frac{1}{2}} + ct'\right]$$

$$dt'\left[\left(\left(z_1 - vt'\right)^2 + \left(z_2 + L\right)^2 + \left(z_3 - H\right)^2\right)^{\frac{1}{2}}\right] = \frac{1}{2}\left(\left(z_1 - vt'\right)^2 + \left(z_2 + L\right)^2 + \left(z_3 - H\right)^2\right)^{-\frac{1}{2}} \cdot \frac{d}{dt'}\left[\left|z - x(t')\right|^2\right]$$

$$\begin{aligned}\frac{d}{dt'}\left[\left|z - x(t')\right|^2\right] &= \frac{d}{dt'}\left[\left(z_1 - vt'\right)^2\right] + \frac{d}{dt'}\left[\left(z_2 + L\right)^2\right] + \frac{d}{dt'}\left[\left(z_3 - H\right)^2\right] \\ &= 2\left(z_1 - vt'\right)\cdot(-v) \\ &= -2v\left(z_1 - vt'\right)\end{aligned}$$

$$dt'\left[\left(\left(z_1 - vt'\right)^2 + \left(z_2 + L\right)^2 + \left(z_3 - H\right)^2\right)^{\frac{1}{2}}\right] = -frac2v\left(z_1 - vt'\right)2\left|z - x(t')\right|$$

$$\begin{aligned}dt'\left[\left(\left(z_1 - vt'\right)^2 + \left(z_2 + L\right)^2 + \left(z_3 - H\right)^2\right)^{\frac{1}{2}} + ct'\right] &= dt'\left(-\frac{v\left(z_1 - vt'\right)}{\left|z - x(t')\right|} + c\right) \\ &= dt'\left(\frac{-v\left(z_1 - vt'\right) + c\left|z - x(t')\right|}{\left|z - x(t')\right|}\right)\end{aligned}$$

$$dt' = d\mu\left(\frac{\left|z - x(t')\right|}{c\left|z - x(t')\right| - v\left(z_1 - vt'\right)}\right)$$

$$\begin{aligned}u(t,z) &= \frac{1}{4\pi}\int_{-\infty}^{\mu(t)} \frac{\delta\left(\mu - ct\right)}{t-t'}P(t')d\mu\left(\frac{\left|z - x(t')\right|}{c\left|z - x(t')\right| - v\left(z_1 - vt'\right)}\right) \\ &= \frac{1}{4\pi}\int_{-\infty}^{\mu(t)} \frac{\left|z - x(t')\right|\delta\left(\mu - ct\right)}{\left(c\left|z - x(t')\right| - v\left(z_1 - vt'\right)\right)\left(t-t'\right)}P(t')d\mu.\end{aligned}$$

In order to evaluate this integral we must find $\mu(t')$ which satisfies $\delta\left(\mu - ct\right)$ to be evaluated as non-zero, i.e., where $\mu(t') = ct$:

$$|z - x(t')| = c(t - t').$$

Substituting this value into our solution for $u(t, z)$:

$$\begin{aligned} u(t,z) &= \frac{1}{4\pi}\left(\frac{c\left(t-t'\right)P(t')}{\left(c\left|z-x(t')\right| - v\left(z - vt'\right)\right)\left(t-t'\right)}\right)\Bigg|_{\mu(t')=ct} \\ &= \frac{1}{4\pi}\frac{cP(t')}{c\left|z-x(t')\right| - v\left(z-vt'\right)}. \end{aligned}$$

Consequently, the equation $\mu(t') = ct$ can be written as:

$$\sqrt{\left(z_1 - vt'\right)^2 + \left(\right)^2 + \left(z_3 - H\right)^2} + ct' = ct.$$

Solving for this equation with respect to t':

$$\begin{aligned} z_1^2 - 2t'vz_1 + v^2(t')^2 + (z_2+L)^2 + (z_3-H)^2 + 2c^2tt' - (ct)^2 - c^2(t')^2 &= 0 \\ (t')^2(v^2-c^2) + 2t'(c^2t - vz_1) + z_1^2 + (z_2+L)^2 + (z_3-H)^2 - (ct)^2 &= 0 \end{aligned}$$

$$a = v^2 - c^2, \quad b = 2(c^2t - vz_1), \quad c = z_1^2 + (z_2+L)^2 + (z_3-H)^2 - (ct)^2.$$

$$\begin{aligned} t' &= \frac{2\left(vz_1 - c^2t\right) \pm \sqrt{4\left(c^2t - vz_1\right)^2 - 4\left(v^2-c^2\right)\left(z_1^2 + \left(z_2+L\right)^2 + \left(z_3-H\right)^2 - \left(ct\right)^2\right)}}{2\left(v^2-c^2\right)} \\ &= \frac{\left(vz_1 - c^2t\right) \pm \sqrt{\left(c^2t - vz_1\right)^2 - \left(v^2-c^2\right)\left(z_1^2 + \left(z_2+L\right)^2 + \left(z_3-H\right)^2 - \left(ct\right)^2\right)}}{\left(v^2-c^2\right)}. \end{aligned}$$

Analyzing the second term under the square root:

$$\begin{aligned} \left(v^2-c^2\right)\left(z_1^2 + \left(z_2+L\right)^2 + \left(z_3-H\right)^2 - \left(ct\right)^2\right) &= z_1^2v^2 + v^2\left(z_2+L\right)^2 + v^2\left(z_3-H\right)^2 - v^2c^2t^2 - c^2z_1^2 - c^2\left(z_2+L\right)^2 \\ &= v^2\left(z_1^2 + \left(z_2+L\right)^2 + \left(z_3-H\right)^2 - c^2t^2\right) - c^2\left(z_1^2 + \left(z_2+L\right)^2 + (z_3 \right. \\ &= c^2v^2\left(\frac{z_1^2 + \left(z_2+L\right)^2 + \left(z_3-H\right)^2 - c^2t^2}{c^2} - \frac{z_1^2 + \left(z_2+L\right)^2 + (z_3}{v^2}\right. \end{aligned}$$

Analyzing the first term under the square root:

$$\left(c^2t - vz_1\right)^2 = c^2\left(t - \frac{vz_1}{c^2}\right).$$

Analyzing the square root expression:

$$
\begin{aligned}
&= \sqrt{c^4 \left(t - \frac{vz_1}{c^2}\right)^2 - c^2v^2 \left(\frac{z_1^2 + (z_2+L)^2 + (z_3-H)^2 - c^2t^2}{c^2} - \frac{z_1^2 + (z_2+L)^2 + (z_3-H)^2 - c^2t^2}{v^2}\right)} \\
&= c^2 \sqrt{\left(t - \frac{vz_1}{c^2}\right)^2 - \frac{v^2}{c^2} \left(\frac{z_1^2 + (z_2+L)^2 + (z_3-H)^2}{c^2} - t^2 - \frac{z_1^2 + (z_2+L)^2 + (z_3-H)^2 - c^2t^2}{v^2}\right)} \\
&= c^2 \sqrt{\left(t - \frac{vz_1}{c^2}\right)^2 - \left(\frac{v^2}{c^2} - 1\right) \left(\frac{z_1^2 + (z_2+L)^2 + (z_3-H)^2}{c^2} - t^2\right)}
\end{aligned}
$$

$$
\begin{aligned}
\beta &= \sqrt{1 - \frac{v^2}{c^2}} \\
\beta^2 &= 1 - \frac{v^2}{c^2}
\end{aligned}
$$

$$
= c^2 \sqrt{\left(t - \frac{vz_1}{c^2}\right)^2 - \beta^2 \left(t^2 - \frac{z_1^2 + (z_2+L)^2 + (z_3-H)^2}{c^2}\right)}.
$$

Substituting this back into the solution for t':

$$
\begin{aligned}
t' &= \frac{(vz_1 - c^2 t) \pm c^2 \sqrt{\left(t - \frac{vz_1}{c^2}\right)^2 - \beta^2 \left(t^2 - \frac{z_1^2 + (z_2+L)^2 + (z_3-H)^2}{c^2}\right)}}{v^2 - c^2} \\
&= \frac{-c^2 \left(t - \frac{vz_1}{c^2}\right) \pm c^2 \sqrt{\left(t - \frac{vz_1}{c^2}\right)^2 - \beta^2 \left(t^2 - \frac{z_1^2 + (z_2+L)^2 + (z_3-H)^2}{c^2}\right)}}{v^2 - c^2} \\
&= -\frac{c^2}{v^2} \left(\frac{t - \frac{vz_1}{c^2} \mp \sqrt{\left(t - \frac{vz_1}{c^2}\right)^2 - \beta^2 \left(t^2 - \frac{z_1^2 + (z_2+L)^2 + (z_3-H)^2}{c^2}\right)}}{\frac{c^2}{v^2} - 1} \right) \\
&= \frac{1}{\beta^2} \left(t - \frac{vz_1}{c^2} \mp \sqrt{\left(t - \frac{vz_1}{c^2}\right)^2 - \beta^2 \left(t^2 - \frac{z_1^2 + (z_2+L)^2 + (z_3-H)^2}{c^2}\right)} \right)
\end{aligned}
$$

$$
\begin{aligned}
\sigma &= \frac{1}{\beta}\left(t - \frac{vz_1}{c^2}\right) \\
&= \frac{\sigma}{\beta} \pm \frac{1}{\beta c} \sqrt{\frac{(z_1 - vt)^2 + (z_2 + L)^2 + (z_3 - H)^2}{\beta^2}} \\
\zeta_1 &= \frac{1}{\beta}(-vt + z_1) \\
&= \frac{\sigma}{\beta} \pm \frac{\zeta_1^2 + (z_2 + L)^2 + (z_3 - H)^2}{\beta c} \\
\rho &= \sqrt{\zeta_1^2 + (z_2 + L)^2 + (z_3 - H)^2} \\
&= \frac{\sigma}{\beta} \pm \frac{\rho}{\beta c}.
\end{aligned}
$$

Where we then choose the solution corresponding to $t' = \frac{\sigma}{\beta} - \frac{\rho}{\beta c}$. We choose this solution, because we are solving for the time at which the scattered signal will be received at the antenna. If we were to choose the + root, we would be looking at some time that would not align with the travel time of the signal and thus only exist at a negative time in order to satisfy $\mu(t') = ct$. Now to solve for the denominator:

$$
\begin{aligned}
c\left|z-x(t')\right|-v\left(z_1-vt'\right) &= c^2\left(t-t'\right)-v\left(z_1-vt'\right) \\
&= c^2 t - z_1 v - t'\left(c^2-v^2\right) \\
&= c^2\left(t-\frac{z_1 v}{c^2}-t'\left(1-\frac{v^2}{c^2}\right)\right) \\
&= c^2\left(t-\frac{z_1 v}{c^2}-t'\beta^2\right)
\end{aligned}
$$

using the solution for t'

$$
\begin{aligned}
&= c^2\sqrt{\left(t-\frac{vz_1}{c^2}\right)^2-\beta^2\left(t^2-\frac{z_1^2+\left(z_2+L\right)^2+\left(z_3-H\right)^2}{c^2}\right)} \\
&= \beta c\rho.
\end{aligned}
$$

Substituting our solutions for t' and the denominator back into $u(t,z)$:

$$
\begin{aligned}
u(t,z) &= \frac{1}{4\pi}\frac{cP\left(t'\right)}{c\left|z-x(t')\right|-v\left(z_1-vt'\right)}\Bigg|_{\mu(t')=ct} \\
&= \frac{1}{4\pi}\frac{cP\left(\frac{\sigma}{\beta}-\frac{\rho}{\beta c}\right)}{\beta c\rho} \\
&= \frac{1}{4\pi\beta}\frac{P\left(\frac{1}{\beta}\left(\sigma-\frac{\rho}{c}\right)\right)}{\rho}.
\end{aligned}
$$

Where we can see that $u(t,z)$ does indeed reduce down to our original statement at the beginning of this problem.

References

[1] Mikhail Gilman, Erick Smith, and Semyon Tsynkov. *Transionospheric synthetic aperture imaging.* Applied and Numerical Harmonic Analysis. Birkhauser/Spring, Cham, Switzerland, 2017.