Differentiation

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1. $\frac{d}{dx}(ke^{f(x)}) = kf'(x)e^{f(x)}$

2. $\frac{d}{dx}(k\ln f(x)) = k\left[\frac{f'(x)}{f(x)}\right]$

3. $\frac{d}{dx}(b^x) = b^x \ln b$

4. $\frac{d}{dx}(fg) = f'g + fg'$

5. $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

5. $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$

$\frac{d}{dx}(\sec x) = (\sec x)(\tan x)$

6. $\frac{d}{dx}(\sin f(x)) = f'(x)\cos f(x)$

$\frac{d}{dx}(\operatorname{cosec} x) = -(\operatorname{cosec} x)(\cot x)$

$\frac{d}{dx}(\cos f(x)) = -f'(x)\sin f(x)$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$\frac{d}{dx}(\tan f(x)) = \frac{f'(x)}{\cos^2 f(x)}$

$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$

7. $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\operatorname{cosec}^{-1}x) = -\frac{1}{|x|\sqrt{x^2-1}}$

$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$

$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

8. $\frac{d}{dx}(\log_a(x)) = \frac{1}{x\ln a}, x>0$

Integration

$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

1. $\int(ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, n \neq 1$

2. $\int e^{ax+b}dx = \frac{e^{ax+b}}{a} + C$

3. $\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + C$

4. $\int \frac{1}{x}dx = \ln|x| + C$

6. $\int_b^a f(x)dx = -\int_a^b f(x)dx$

5. $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$

6. $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + C$

$\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + C$

$\int \tan x = \ln|\sec x| + C$

$\int \frac{1}{\cos^2 f(x)} = \tan f(x)$

$\int (\sec x)(\tan x)dx = \sec x + C$

$\int (\operatorname{cosec} x)(\cot x)dx = -\operatorname{cosec} x + C$

$\int \operatorname{cosec}^2 x\, dx = -\cot x + C$

$\int \sec x\, dx = \ln|\sec x + \tan x| + C$

$\int \operatorname{cosec} x\, dx = \ln|\operatorname{cosec} x - \cot x| + C$

$\int \cot x\, dx = \ln|\sin x| + C$

Choose u in this order
Logs
Inverse
Algebraic
Trig
Exponential

★ **Volume**

Rotate about a horizontal line

$V = \pi\int_a^b (y-h)^2 dx$

Rotate about vertical line

$V = \pi\int_a^b (x-k)^2 dy$

$\int \sec^3 x\, dx = \frac{1}{2}[(\sec x)(\tan x) + \ln|\sec x + \tan x|] + C$

$\int \operatorname{cosec}^3 x\, dx = \frac{1}{2}[-(\operatorname{cosec} x)(\cot x) + \ln|\operatorname{cosec} x - \cot x|] + C$

$\int a^x dx = \frac{a^x}{\ln a} + C$

$\int \ln x\, dx = x\ln(x) - x + C$

$\int e^{ax}\sin(bx)dx = \frac{e^{ax}}{a^2+b^2}(a\sin(bx) - b\cos(bx)) + C$

$\int e^{ax}\cos(bx)dx = \frac{e^{ax}}{a^2+b^2}(a\cos(bx) + b\sin(bx)) + C$

$\int \frac{1}{x\ln x}dx = \ln|\ln x| + C$

$\int \sin^{-1}x\, dx = x\sin^{-1}x + \sqrt{1-x^2} + C$

$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

$\int \cos^{-1}x\, dx = x\cos^{-1}x - \sqrt{1-x^2} + C$

$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

$\int \tan^{-1}x\, dx = x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C$

$\int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$

★ Integration by parts: $\int uv' = uv - \int u'v$

★ **Trig substitutions**: if integral contains these, do substitution

$\sqrt{a^2-b^2x^2} \Rightarrow x = \frac{a}{b}\sin\theta$ and $\cos^2\theta = 1 - \sin^2\theta$

$\sqrt{b^2x^2-a^2} \Rightarrow x = \frac{a}{b}\sec\theta$ and $\tan^2\theta = \sec^2\theta - 1$

$\sqrt{a^2+b^2x^2} \Rightarrow x = \frac{a}{b}\tan x$ and $\sec^2\theta = 1 + \tan^2\theta$

Trigo

1. $\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$1 + \cot^2 x = \operatorname{cosec}^2 x$

2. $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

3. $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

4. $\sin S + \sin D = 2\sin\left(\frac{S+D}{2}\right)\cos\left(\frac{S-D}{2}\right)$

$\sin S - \sin D = 2\cos\left(\frac{S+D}{2}\right)\sin\left(\frac{S-D}{2}\right)$

$\cos S + \cos D = 2\cos\left(\frac{S+D}{2}\right)\cos\left(\frac{S-D}{2}\right)$

$\cos S - \cos D = -2\sin\left(\frac{S+D}{2}\right)\sin\left(\frac{S-D}{2}\right)$

Absolute extreme: entire domain
local extreme: specified domain

5. $\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$\cos A\sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$

$\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$\sin A\sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]$

6. $\sin 2x = 2\sin x\cos x$

$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$

$\cos 2x = \cos^2 x - \sin^2 x$

$= 1 - 2\sin^2 x$

$= 2\cos^2 x - 1$

$\tan\left(\frac{x}{2}\right) = \frac{(1-\cos x)}{\sin x}$

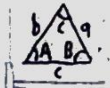
Use these when integrating $\sin^2$ or $\cos^2$ or $\tan^2 x$

7. $\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{2}}$

$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos x}{2}}$

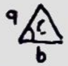
8. $a^2 = b^2 + c^2 - 2bc\cos A$

$A = \cos^{-1}\left(\frac{b^2+c^2-a^2}{2bc}\right)$



$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

9. Area of $\Delta = \frac{1}{2}ab\sin C$



Partial fraction: $\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$

$\frac{px+q}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$

$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

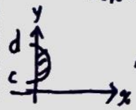
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$

$\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$

Area & volume

$A = \int_a^b f(x)dx$

Area → above x-axis → +ve
below x-axis → -ve



$A = \int_c^d g(y)dy$

Area → right y-axis → +ve
left y-axis → -ve

V about x-axis

$V = \pi\int_a^b y^2 dx$

V about y-axis

$V = \pi\int_c^d x^2 dy$

$f''(x) > 0$ concave up

$f''(x) < 0$ concave down

$f'(x) = 0$
↳ $f'(x) > 0$ local min
↳ $f'(x) < 0$ local max

★ Extreme Value

↳ $f'(x) = 0$ ①

↳ $f'(x)$ does not exist ②

↳ end points of domain of f

Critical point → the 1st two properties

L'Hôpital's Rule ($\frac{0}{0}$ or $\frac{\infty}{\infty}$)

$\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$ and $\lim_{x\to c} \frac{f'(x)}{g'(x)} = L$

or $\pm\infty$ and $\pm\infty$

$\therefore$ Then, $\lim_{x\to c} \frac{f(x)}{g(x)} = L$ $y = \frac{1}{x}$ $y = \log x$

Geometric Series

$T_n = ar^{n-1}$ $S_n = \frac{a(1-r^n)}{1-r}$ ($r \neq 1$) $S_\infty = \frac{a}{1-r}$

{r can be -ve}

if converges for all x, radius $= \infty$

if converges only at x, radius $= 0$

Ratio test

$\lim_{n\to\infty} \left| \frac{U_{n+1}}{U_n} \right| = \rho$

1) $\rho < 1$: converges
2) $\rho > 1$: diverges
3) $\rho = 1$: no conclusion

Power series about $x = a$

$\sum_{n=0}^{\infty} C_n (x-a)^n$ $n=0$ zero order, $n=1$ 1st order

Radius of convergence, h

$|x - a| < h$

* ratio test only gives you ratio, not RADIUS!

Taylor series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

Some Taylor series

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$; $e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$

$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$

★ To find Taylor Series, try to express in this form:

$\frac{1}{1-\square} = \sum_{n=0}^{\infty} \square^n$, for $|\square| < 1$

{use S_∞ of geometric series}

★ If centred at a, make sure $c(x-a)$ is in $\square$, where c is any coefficient

★ Important rps. L'Hospital's Rule

if we need to find $\lim_{x\to a} x^{f(x)}$ ①

1st calculate: $\lim_{x\to a} \ln x^{f(x)}$

$= \lim_{x\to a} f(x) \ln x$

2nd substitute it back to ① to find limit

$\lim_{x\to a} x^{f(x)} = \lim_{x\to a} e^{\ln x^{f(x)}}$

Use this technique too if you see this kind of limit $\lim_{x\to a} (g(x))^{f(x)}$

$\vec{PQ} = \vec{OQ} - \vec{OP}$

Angle between 2 vectors

$$\cos\theta = \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{\|V_1\| \|V_2\|}$$

2nd order partial derivatives (notation)

$f_{xx} = \frac{\partial^2 f}{\partial x^2}$ $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$

$f_{yy} = \frac{\partial^2 f}{\partial y^2}$ $f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$

★ $f_{xy}(a,b) = f_{yx}(a,b)$ {see whether LHS or RHS is easier to do, then use this eq to get your ans}

Chain rule $z = f(x,y)$

$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

x & y are function of 1 variables, i.e. $x(t)$ and $y(t)$

$z(t) = f(x(t), y(t))$

For 2 independent variables $z = f(x,y)$

$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$

$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$

$z(s,t) = f(x(s,t), y(s,t))$

For $f(x,y,z)$ {1 variable: t}

$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$

$w(t) = f(x(t), y(t), z(t))$

For $f(x,y,z)$ {2 variables, s,t}

$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$

$\frac{\partial w}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$

$w(s,t) = f(x(s,t), y(s,t), z(s,t))$

$y - y_1 = m(x - x_1)$

Directional derivative along the direction $u = u_1 i + u_2 j$ (must be unit vector)

$D_u f(a,b) = f_x(a,b) \cdot u_1 + f_y(a,b) \cdot u_2$

★ $D_u f(a,b) = \nabla f(a,b) \cdot u = \|\nabla f(a,b)\| \cos\theta$

Gradient of $f(x,y)$, $\nabla f = f_x i + f_y j$

max/min $D_u f(a,b) \Rightarrow \|\nabla f(a,b)\|$

★ f increases most rapidly in the direction $\nabla f(a,b)$

decreases most rapidly in the direction $-\nabla f(a,b)$

θ is the angle between $\nabla f(a,b)$ and u

Change in the value of f when we move a small distance Δt from the point (a,b) in the direction of the vector u

$\Delta f = D_u f(a,b) \cdot \Delta t$

Directional derivative along the direction $u = u_1 i + u_2 j + u_3 k$

$D_u f(a,b,c) = f_x(a,b,c) \cdot u_1 + f_y(a,b,c) \cdot u_2 + f_z(a,b,c) \cdot u_3$

Critical Points:

i) $f_x(a,b) = 0$ and $f_y(a,b) = 0$

ii) $f_x(a,b) = 0$ or $f_y(a,b)$ does not exist

Reduction to separable variable

(for equation of form $y' = g(\frac{y}{x})$) ⓐ

① let $v = \frac{y}{x}$

② $y = vx$, $y' = v + xv'$

$\therefore$ Thus ⓐ becomes $v + xv' = g(v)$ which is separable

★ 2nd Derivative Test

$D = f_{xx}(a,b) f_{yy}(a,b) - f_{xy}(a,b)^2$

a) $D > 0$; $f_{xx}(a,b) > 0 \Rightarrow$ local min

b) $D > 0$; $f_{xx}(a,b) < 0 \Rightarrow$ local max

c) $D < 0 \Rightarrow$ saddle point

d) $D = 0 \Rightarrow$ no conclusion

Linear 1st order ODEs

$\frac{dy}{dx} + P(x) y = Q(x)$

① Find $R = e^{\int P(x) dx}$

② $y = \frac{1}{R} \int R Q(x) dx$

Bernoulli Equation

$\frac{dy}{dx} + P(x) y = Q(x) y^n$

① Let $z = y^{1-n}$ ② Find $\frac{dz}{dx}$ using chain rule ③ express $\frac{dy}{dx}$ in terms of y & $\frac{dz}{dx}$

2nd order LDE {homogenous}

$y'' + ay' + by = 0$

a & b are constants

Characteristic Equation

$\lambda^2 + a\lambda + b = 0$ ①

Case 1: 2 real roots

$\therefore y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$

Case 2: Real double root ($\lambda_1 = \lambda_2$)

$\therefore y = (C_1 + C_2 x) e^{-\frac{ax}{2}}$

★ $-\frac{a}{2}$ is the double root, i.e. $\lambda_1 = \lambda_2$

Case 3: Complex roots

$\lambda_1 = \alpha + \beta i$; $\lambda_2 = \alpha - \beta i$

$\therefore y = C_1 e^{\alpha x} \cos\beta x + C_2 e^{\alpha x} \sin\beta x$

C_1 & C_2 are constants that we have to determine through initial value

Radioactive decay equation

$N = N_0 e^{-kt}$; $k = (\ln 2)/t_{1/2}$

Malthus's model

$N = N_0 e^{kt}$

$\frac{dN}{dt} = kN$, $K = B - D$ (no -ve sign!)

Logistic population model

(Note B can be >1 or <1, depending on how we express) {s is a constant which is the one in $D = sN$}

① $\frac{dN}{dt} = BN - DN = BN - (sN)N = BN - sN^2$

$$N = \frac{N_\infty}{1 + (\frac{N_\infty}{N_0} - 1) e^{-Bt}}$$; $N_\infty = B/s$

To sustain the population:

$0 < E < \frac{B^2}{4s}$, E = harvesting rate

Basic harvesting model

$\frac{dN}{dt} = (B - sN)N - E$

Max dN/dt in ① $\Rightarrow \frac{B^2}{4s}$ (N is $\frac{B}{2s}$ when $\frac{dN}{dt}$ is max)

• If cannot sustain, then T, the time the population can last for: $T = \int_0^{N_0} \frac{1}{sN^2 - BN + E} dN$

Partial DE

{if the form is $(f(x)X(x))(g(y)Y(y)) = 1$, then $f(x)X(x) = c$, $g(y)Y(y) = \frac{1}{c}$}

Consider $U_x = f(x) g(y) U_y$

① Assume solution $U(x,y) = X(x) Y(y)$ exists

② Substitute & change into $\frac{1}{f(x)} \cdot \frac{X'(x)}{X(x)} = g(y) \frac{Y'(y)}{Y(y)}$

③ Equate both sides to constant k, solve & substitute back to $U(x,y) = X(x) Y(y)$

Given angle, find unit vector

① $U = (\cos\theta) i + (\sin\theta) j$ # θ = angle between u and the x-axis

② $u = (\cos\phi)(\cos\theta) i + (\cos\phi)(\sin\theta) j + (\sin\phi) k$

ϕ = angle between u and x-y plane

θ = angle between the projection of u onto the x-y plane and the x-axis

Limits (Suppose $\lim_{x\to a} f(x) = L$; $\lim_{x\to a} g(x) = L'$)

i) $\lim_{x\to a} (f \pm g)(x) = L \pm L'$

ii) $\lim_{x\to a} (f(x))(g(x)) = LL'$

iii) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{L'}$

iv) $\lim_{x\to a} kf(x) = kL$, for any real number k

Newton's Law of Cooling

$\frac{dT}{dt} = k(T - T_s)$, k = -ve

$\Rightarrow T - T_s = Ae^{kt}$

$B = \frac{2kb}{m}$ $v = (\frac{1 + Ae^{Bt}}{1 - Ae^{Bt}}) k$

Falling

$F_{upthrust} = bv^2$

$\frac{dv}{dt} = -\frac{b}{m}(v^2 - k^2)$; $k = \sqrt{\frac{mg}{b}}$, k = terminal velocity

$x^2 - (SOR)x + (POR) = 0$

$SOR = -\frac{b}{a}$ $POR = \frac{c}{a}$

Fundamental Theorem of Calculus

$\frac{d}{dx} \int_a^x f(t) dt = f(x)$ {use chain rule if needed}