Data Analytics – Lab Instruction 11

Theoretical Concepts

Suppose that $x_1, x_2, \dots, x_n$ are *iid* samples from a $N(\mu, \sigma^2 = \tau^{-1})$, where μ and τ^{-1} are unknown.

The purpose of this exercise is to use Gibbs Sampling to estimate μ and τ (and therefore σ^2).

The likelihood function equals to

$$L(\mu,\tau) = f(x \mid \mu, \tau^{-1}) = \left(\frac{\tau}{2\pi}\right)^{n/2} \ exp\left[-\frac{\tau}{2}\sum_{i=1}^{n}(x_i - \mu)^2\right] \tag{1}$$

The prior distribution for μ is assumed $N(\mu_0, \sigma_0^2)$ and the prior distribution for τ is assumed $Gamma\ (\alpha_0\ , \beta_0)$ and therefore

$$f(\mu,\tau) = f(\mu)\,.\,f(\tau) = \frac{exp(-\ (\mu-\mu_0)^2/2\sigma_0^2)}{\sqrt{2\pi\sigma_0^2}} \times \frac{(\beta_0\tau)^{\alpha_0-1}\,\beta_0\ exp(-\beta_0\tau)}{\Gamma(\alpha_0)} \tag{2}$$

Therefore, the posterior distribution would be proportional to

$$\tau^{\frac{n}{2}+\alpha_0^{-1}}\, ex\,p(-\beta_0\tau)\ exp\left(-\frac{\tau\,S}{2} - \frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right) \tag{3}$$

where $S = \sum(x_i - \mu)^2$.

We can therefore see that

$$g(\mu \mid \tau, x_i) \sim N\ \left(\ \frac{n\,\bar{x}\,\tau + \mu_0\tau_0}{n\tau + \tau_0}\ , (n\tau+\tau_0)^{-1}\right) \tag{4}$$

where $\tau_0 = \ 1/\sigma_0^2$ and

$$h(\,\tau \mid \mu\,, x_i) = \ \Gamma\ \left(\alpha_0 + \frac{n}{2}\,, \beta_0 + \ \frac{S}{2}\right) \tag{5}$$

The last two equations can be used in an iterative procedure to estimate μ and τ.

Computational Instruction

1. Use the following observations and initial values.

$x_1, x_2, \dots, x_n = 12.8, 10.5, 13.2, 13.0, 7.0, 11.0, 13.4, 13.2, 9.5, 11.0, 10.9, 4.6, 5.8, 3.2, 9.8, 0.2, 11.2,$

$7.2, 14.7, 5.9, 9.7, 17.6, 8.5, 6.8, 7.2, 12.2, 16.7, 10.4, 14.2, 5.7$

$\mu_0 = 8, \sigma_0^2 = 4, \alpha_0 = 5, \beta_0 = 1$

2. Use these figures and equation (4) to generate 1000 samples for μ. Use the estimate for μ (mean of your samples) and equation (5) to generate 1000 samples for τ. Use the estimate for τ (mean of your samples) in (5) to generate samples for μ and so on.
3. Continue the above procedure for 500 times and plot the results for 500 estimates for μ and σ^2 vs. iterations. Report the final estimates for μ and σ^2.