638 : Hw 1

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Spring 2022

2.2.x

Analyze the following equations graphically. In each case, sketch the vector field on the real line, find all the fixed points, classify their stability, and sketch the graph of $x(t)$ for different initial conditions. Then try for a few minutes to obtain the analytical solution for $x(t)$; if you get stuck, don't try for too long since in several cases it's impossible to solve the equation in closed form!

2.2.3

$$\dot{x} = x - x^3 \tag{1}$$

Fixed Points x^*

We set $f(x) = \dot{x} = 0$:

$$\begin{aligned} 0 &= x - x^3 \\ x &= x^3 \implies x = 0, 1 \end{aligned}$$

Collecting our fixed points,

$$\begin{cases} x_1^* = 0 \\ x_2^* = 1 \\ x_3^* = -1 \end{cases}$$

Stability

We find $f'(x)$, and observe the sign of $f'(x^*)$ for each fixed point above:

$$\begin{aligned} f'(x) &= \frac{d}{dx}[x - x^3] \\ &= -3x^2 + 1 \end{aligned}$$

Thus for $x_1^* = 0$,

$$f'(x_1^*) = 1 > 0$$

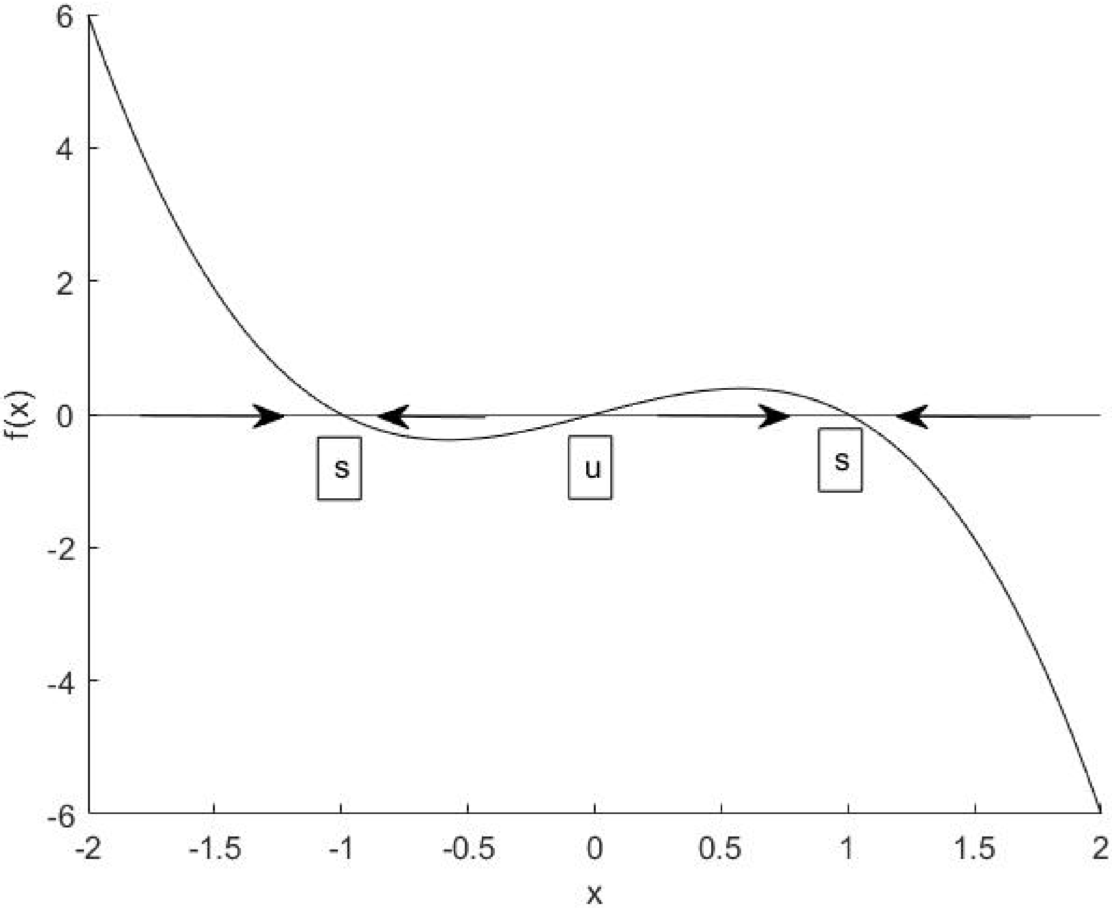
and so $x_1^* = 0$ is an **unstable** fixed point. Performing the same analysis for $x_2^* = 1$,

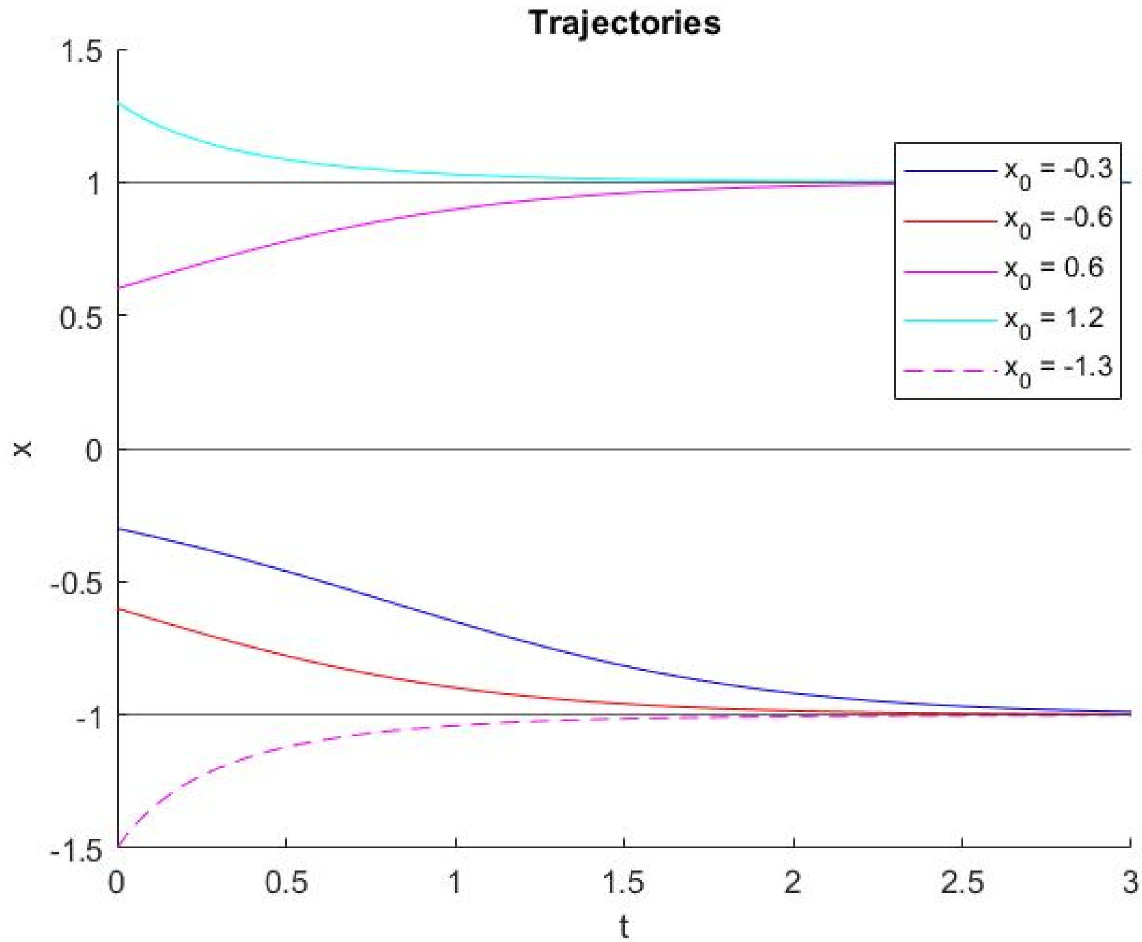
$$f'(x_2^*) = -2 < 0$$

and so $x_2^* = 1$ is a **stable** fixed point. Clearly, $x_3^* = -1$ is also a **stable** fixed point.

Vector Field and Trajectories

Below is a plot of $f(x)$ with arrows indicating flow, and plots of various solutions of (1) with different initial conditions.





Analytical Solution

This is a first Bernoulli ODE with a solution:

$$x(t) = \frac{e^t}{\sqrt{c + e^{2t}}}$$

2.2.4

$$f(x) = \dot{x} = e^{-x}\sin(x) \tag{2}$$

Fixed Points x^*

We set $f(x) = \dot{x} = 0$ and solve:

$$0 = e^{-x}\sin(x) \implies \sin(x) = 0 \implies x = n \cdot \pi, n \in \mathbb{Z}$$

since the exponential term is clearly never zero. Collecting our fixed points,

$$\left\{ x^* = n \cdot \pi, n \in \mathbb{Z} \right.$$

Stability

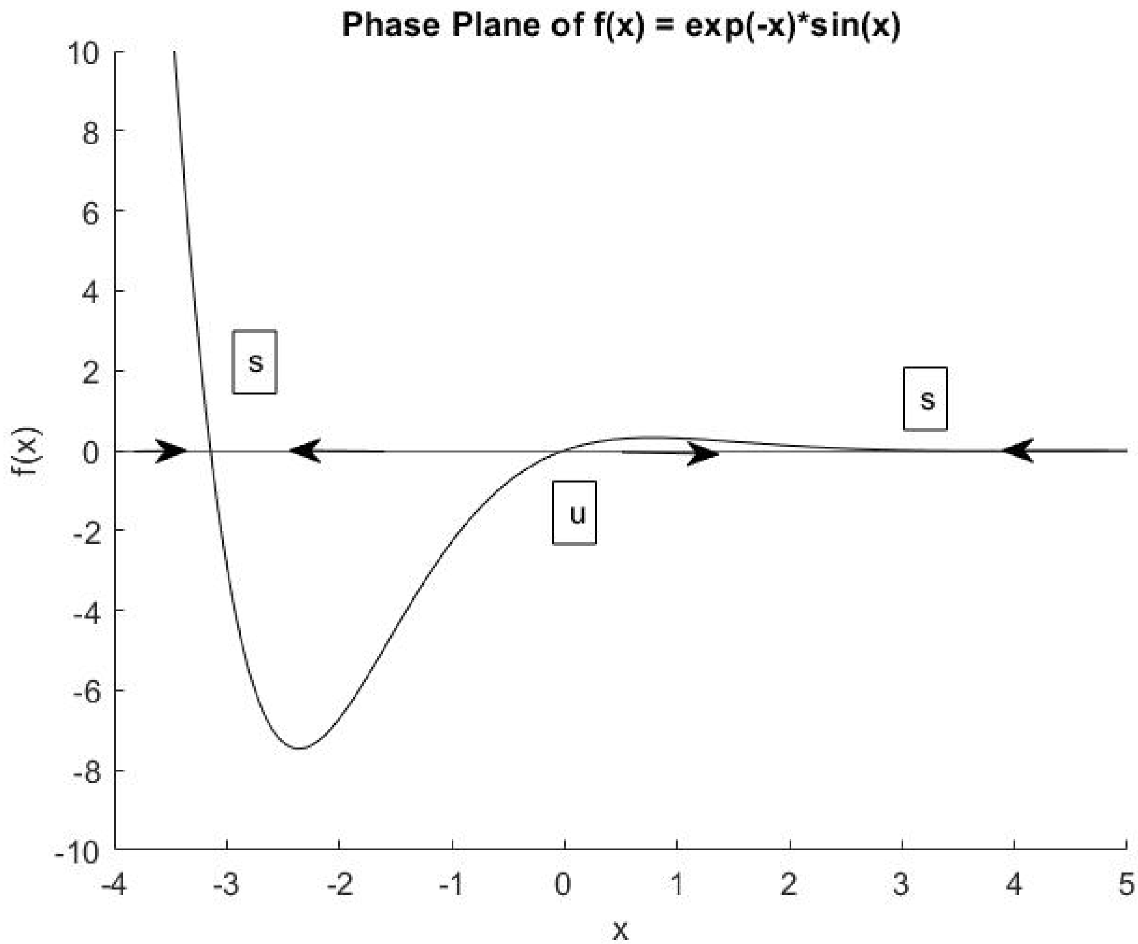
We find $f'(x)$, and observe the sign of $f'(x^*)$:

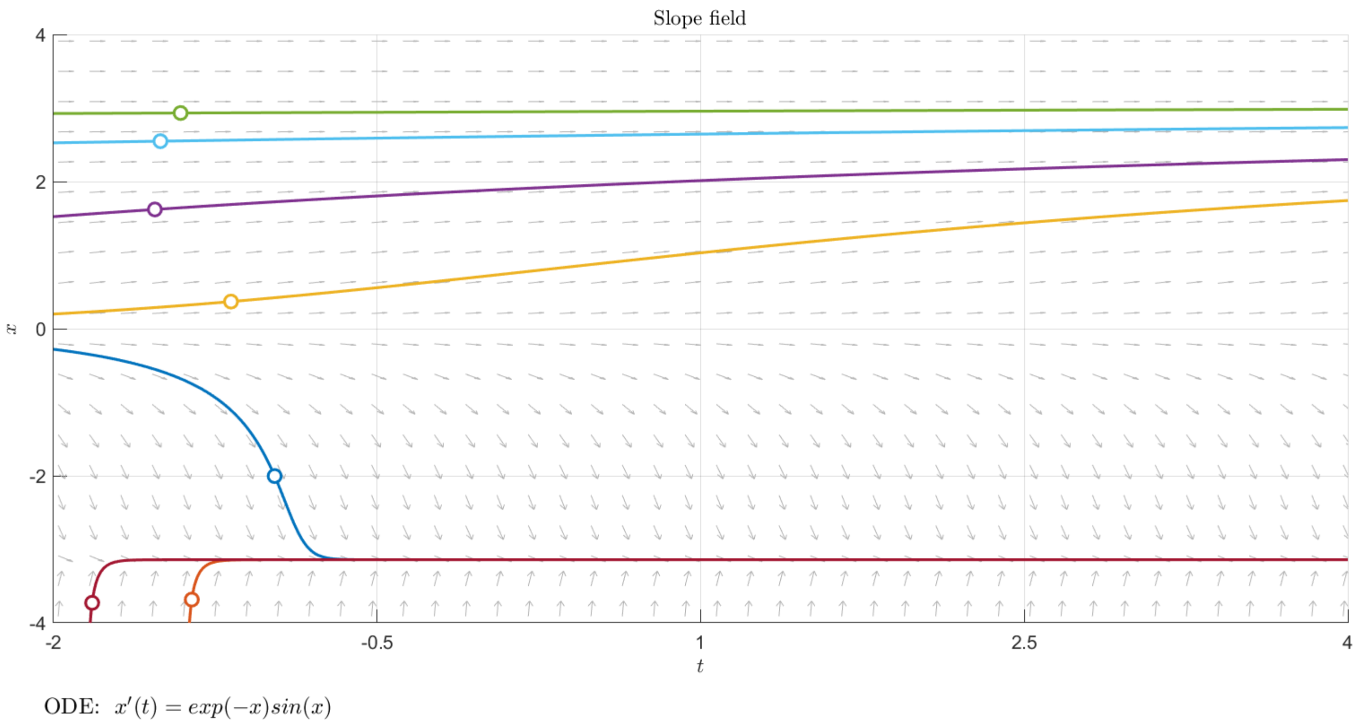
$$f'(x) = e^{-x}\cos(x) - \sin(x)e^{-x}$$

Thus for $x^* = k \cdot \pi$ where $k = 2n+1$, we have that $f'(x^*) = -e^{-x} < 0$, yielding a **stable** fixed point. Similarly, for $x^* = k' \cdot \pi$ where $k' = 2n$, we have that $f'(x^*) = e^{-x} > 0$, yielding an **unstable** fixed point.

Vector Field and Trajectories

Below is a plot of $f(x)$ with arrows indicating flow, and plots of various solutions of (2) with different initial conditions. This ODE has no analytical solution.





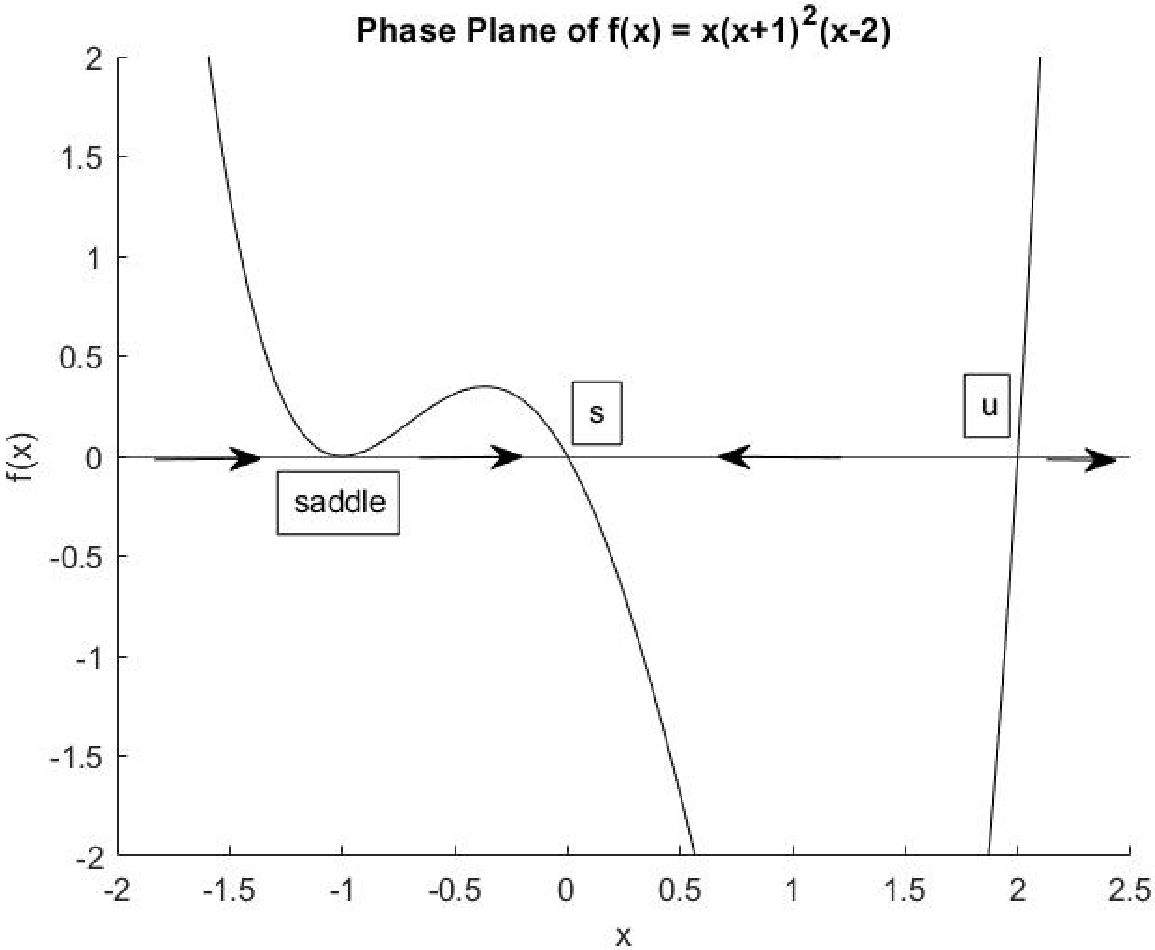
2.2.8

For the phase portrait shown below, find an equation that is consistent with it.

Clearly, we need an equation $f(x) = \dot{x}$ such that there is a root of even multiplicity at $x = -1$ and odd multiplicities at $x = 0$ and $x = 2$. Such a function is the following:

$$f(x) = \dot{x} = x(x+1)^2(x-2)$$

Below is the phase plane of $f(x)$.



2.3.x

2.3.3

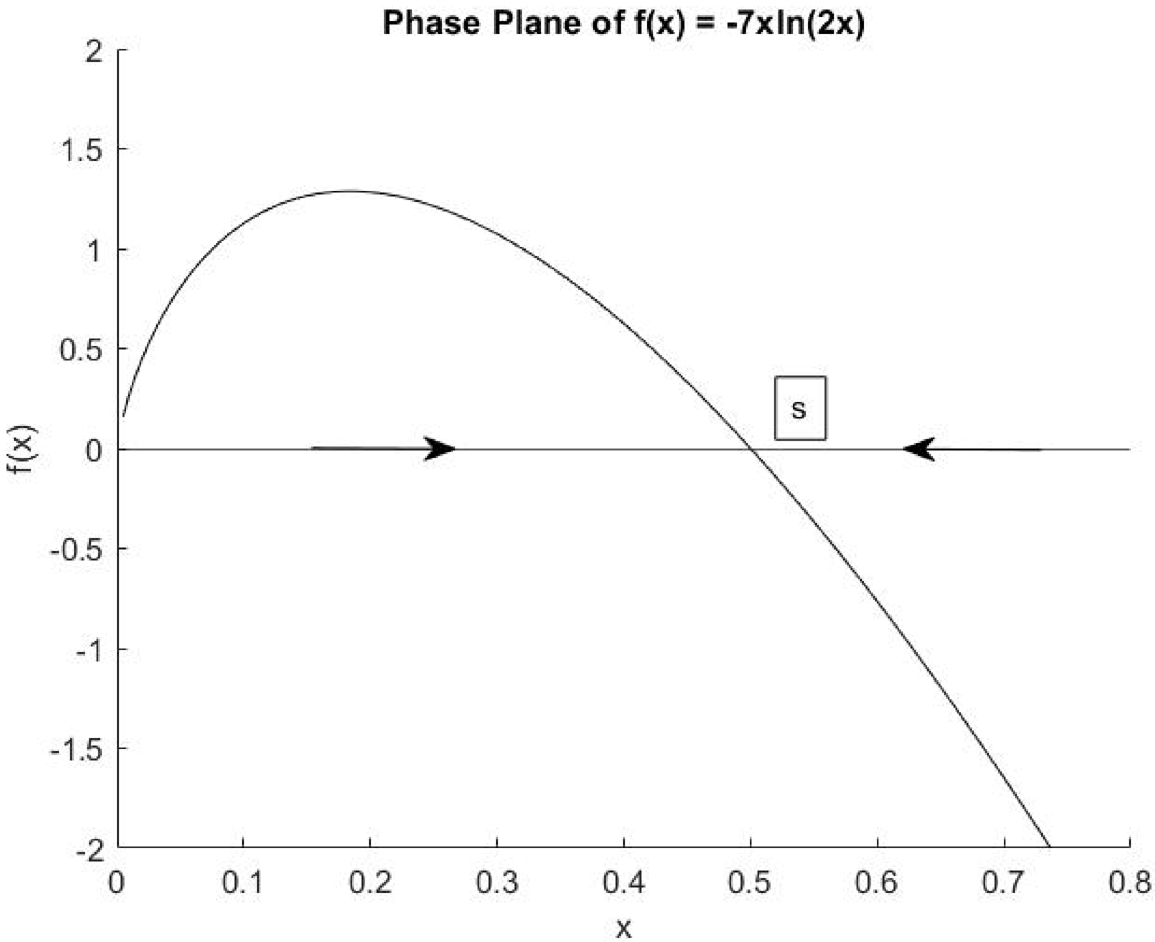
The growth of cancerous tumors can be modeled by the Gompertz law $\dot{N} = -aN\ln(bN)$, where $N(t)$ is proportional to the number of cells in the tumor, and $a, b > 0$ are parameters.

(a)

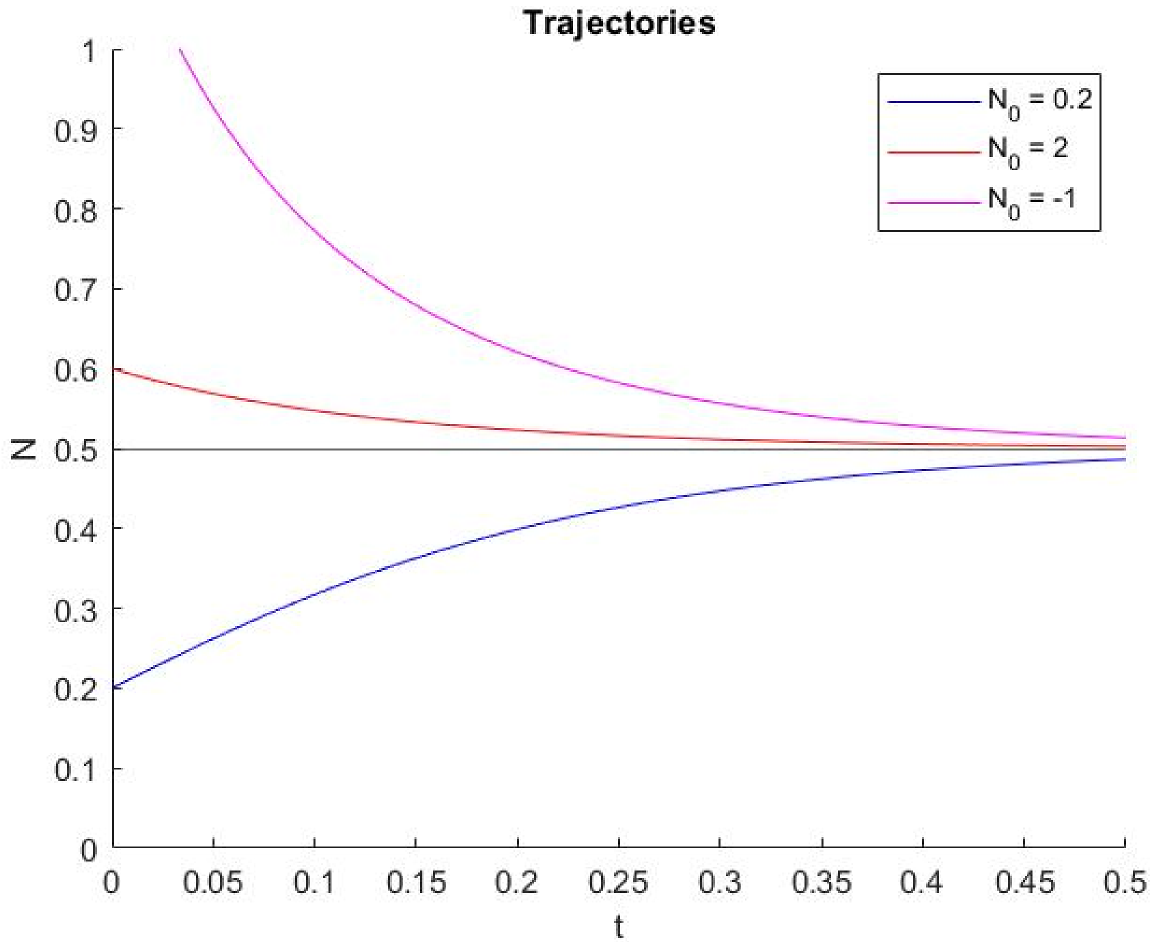
The parameter a may be a scaling factor for the growth of the number of tumors, directly affecting the maximum rate. It may be related to the number of tumors at a time t. The parameter b may be how quickly any individual tumor duplicates.

(b)

Below is a graph of the vector field, and various solutions $N(t)$ for different initial values, given $a = 7$ and $b = 2$.



Phase Plane of f(x) = -7xln(2x)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
f(x)
s
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
x



2.3.4

For certain species of organisms, the effective growth rate $\dot{N}/N$ is highest at intermediate N. This is the called the Allee effect (Edelstein-Keshet 1988). For example, imagine that it is too hard to find mates when N is very small, and there is too much competition for food and other resources when N is large.

(a)

As will be shown, in order for the system to exhibit the Allee effect, the following conditions must be satisfied regarding the parameters:

$$\begin{cases} a, b, r > 0 \\ b > \sqrt{\frac{r}{a}} \end{cases}$$

(b)

We find fixed points of (3) by setting $\dot{N} = 0$:

$$0 = N^*[r - a(N^* - b)^2]$$

$$N^* = 0 \qquad \text{or} \qquad r - a(N^* - b)^2 = 0$$

$$(N^* - b)^2 = \frac{r}{a}$$

$$N^* = \pm\sqrt{\frac{r}{a}} + b$$

Collecting these fixed points,

$$\begin{cases} N_1^* = 0 \\ N_2^* = \sqrt{\frac{r}{a}} + b \\ N_3^* = -\sqrt{\frac{r}{a}} + b \end{cases}$$

We note that N_3^* is not physically possible in the case that $\sqrt{\frac{r}{a}} > b$. Further, in answer to part (a), our equation produces the Allee effect only if all three fixed points occur and are positive. Thus, we will only be considering the case that $\sqrt{\frac{r}{a}} < b \iff r < ab^2$. This restricts b so that $b > 0$. Regarding r and a, we have two possibilities to consider:

(i) $r, a > 0$

To analyze the stability of the above fixed points, we find $\ddot{N}$:

$$\ddot{N} = -2aN(N - b) + r - a(N - b)^2$$

We now consider the following parameter values according to conditions (i):

$$\begin{cases} r = 2 \\ a = 8 \\ b = 1 \end{cases}$$

So that all three fixed points are greater than zero, and $f'(N)$ becomes

$$\ddot{N} = -16N(N - 1) + 2 - 5(N - 1)^2$$

For $N_1^* = 0$, we have that

$$\ddot{N} = 2 - 5 < 0$$

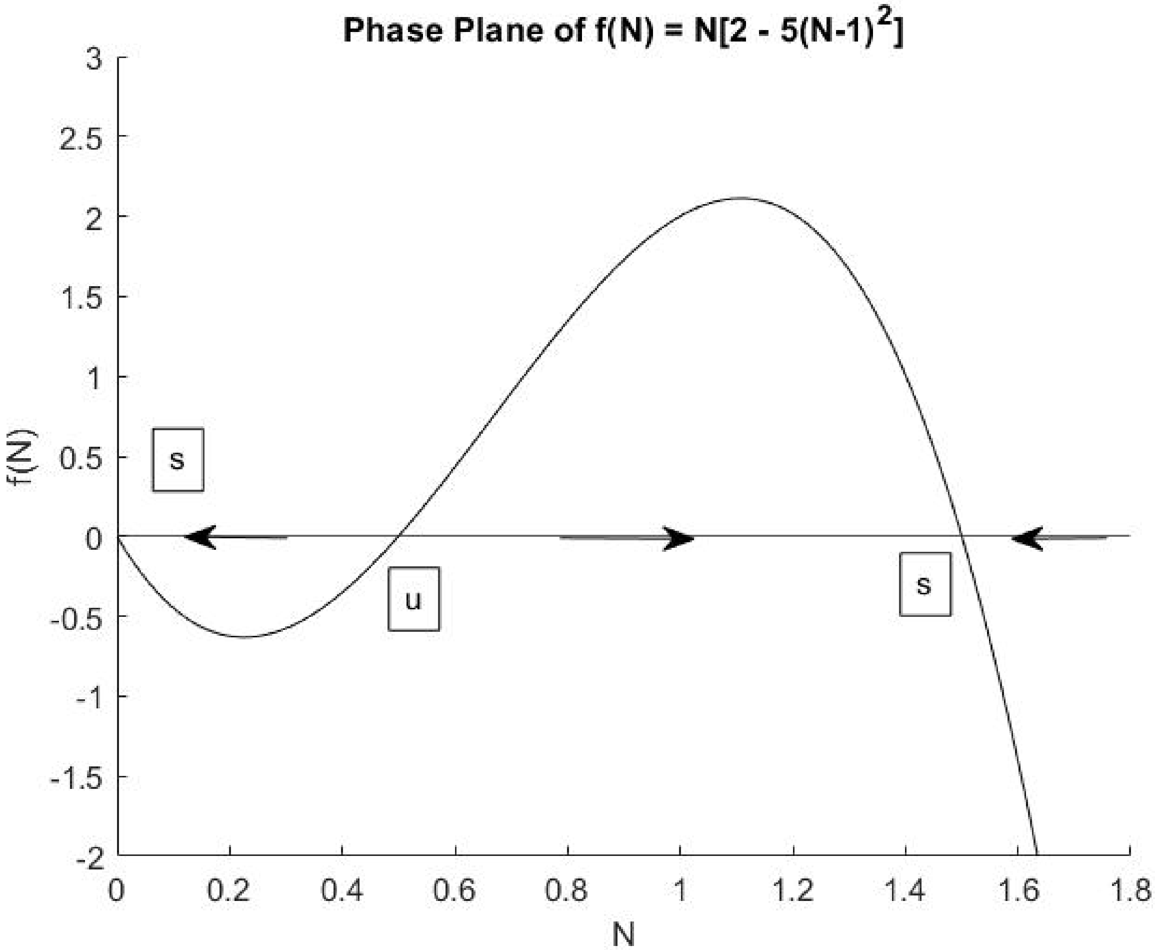
implying that N_1^* is a **stable** fixed point.

With the chosen parameter values, $N_2^* = \frac{1}{2} + 1 = 1.5$. Subbing this into $\ddot{N}$:

$$\ddot{N} = -10(1.5)(0.5) + 2 - 8(.25) < 0$$

indicating that $N_2^* = 1.5$ is a **stable** fixed point.

This must mean that $N_3^* = 0.5$ is an **unstable** fixed point, which is clear from the sketch and flow of the phase line, shown below. Further, condition (i) imposes the Allee effect on the system.



(ii) $r, a < 0$

We now consider the case that r and a are both negative. Let us observe $\ddot{N}$ by substituting $(-a)$ and $(-r)$:

$$\ddot{N} = 2aN(N-b) - r + a(N-b)^2$$

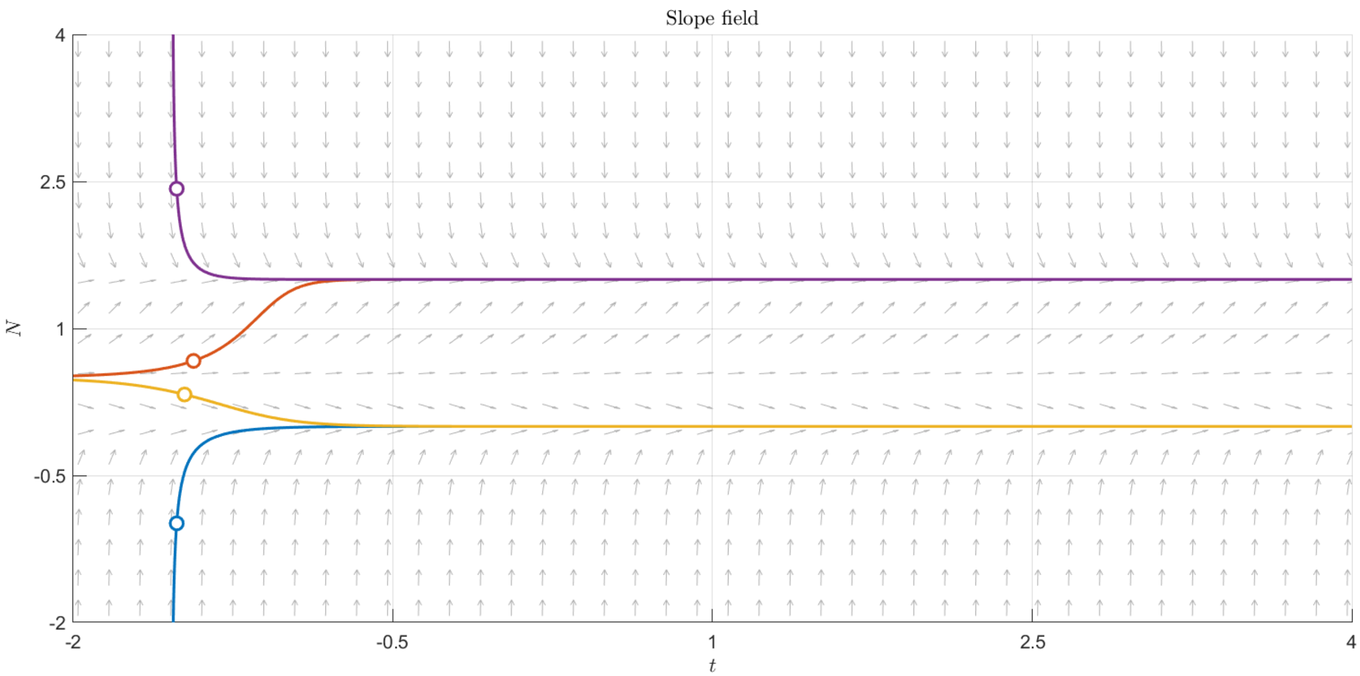
At $N_1^* = 0$, we have that

$$\ddot{N} = -r + ab^2$$

By our condition that $r < ab^2$, this must mean that $\ddot{N} > 0$. However, in order to produce the Allee effect, we must have that $\ddot{N}|_{N_1^*=0} < 0$ so that $N_1^* = 0$ is a stable fixed point, N_3^* is unstable, and N_2^* is stable. Thus, condition (ii) ensures that the system does not produce the Allee effect, and we disregard further analysis.

(c)

Below is a graph of several solutions $N(t)$ for various initial conditions, with the parameter values stated in case (i) above.



ODE: $N'(t) = N(r - a(N - b)(N - b))$

(d)

As we know, the logistic equation has only two fixed points, while our system has 3. They both share the fixed point at 0, however, ours is stable, while that of the logistic equation is unstable so that solutions trend towards 0 for small N in our system, but away from 0 in context of the logistic equation. However, they both have a stable fixed point greater than zero towards which solutions trend.

2.4.x

Use linear stability analysis to classify the fixed points of the following systems. If linear stability analysis fails because $f'(x*) = 0$, use a graphical argument to decide the stability.

2.4.4

$$\dot{x} = x^2(6 - x) \tag{3}$$

Fixed Points x^*

Setting (3) equal to zero:

$$0 = x^2(6 - x)$$
$$x = 0 \qquad\qquad x = 6$$

Collecting these fixed points:

$$\begin{cases} x_1^* = 0 \\ x_2^* = 6 \end{cases}$$

Stability

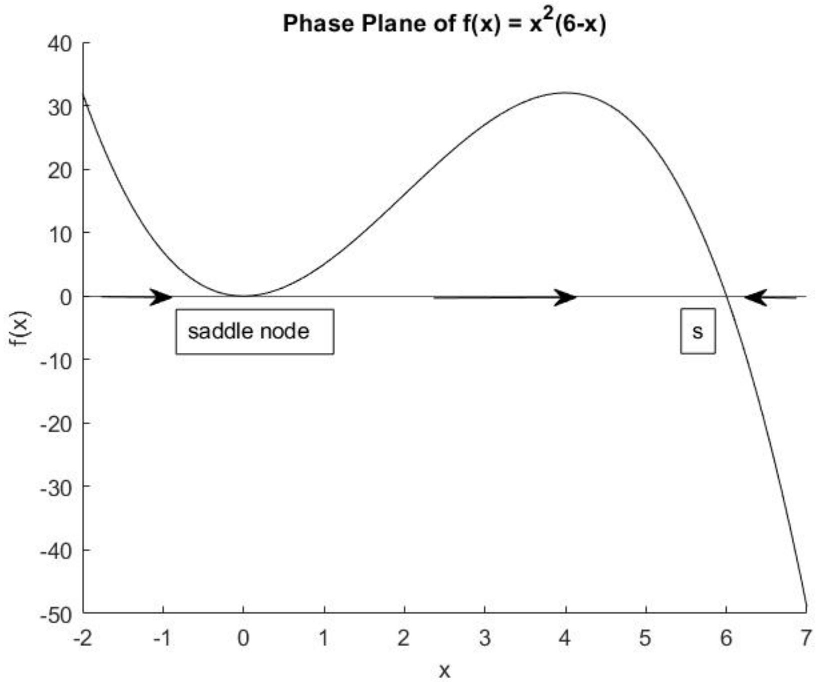
We find $\ddot{x}$

$$\ddot{x} = -3x^2 + 12x$$

Then for each of our fixed points,

$$\begin{cases} \ddot{x}(x_1^*) = 0 \implies & \text{linear stability analysis inconclusive} \\ \ddot{x}(x_2^*) = -36 < 0 \implies & x_2^* \text{ is } \textbf{stable} \end{cases}$$

For x_1^*, we refer to graphical analysis, which indicates that it is a **saddle node**:



2.4.8

Using linear stability analysis, classify the fixed points of the Gompertz model of tumor growth $\dot{N} = -aN\ln(bN)$. (As in Exercise 2.3.3, $N(t)$ is proportional to the number of cells in the tumor and a, b > 0 are parameters.)

Fixed Points x^*

Setting $\dot{N} = 0$, we have that

$$-aN\ln(bN) = 0$$

$$N = 0 \qquad\qquad \ln(bN) = 0$$

$$N = \frac{1}{b}$$

Clearly, we cannot have that $N = 0$, thus our only fixed point is:

$$\left\{ N = 1/b \right.$$

Stability

In the spirit of linear stability analysis, we find $\ddot{N}$:

$$\ddot{N} = -a[N \cdot \frac{b}{bN} + \ln(bN)]$$
$$\ddot{N} = -a[1 + \ln(bN)]$$

Subbing in our only fixed point,

$$\ddot{N} = -a[1 + 0]$$
$$= -a$$

Since $a > 0$, we have $\ddot{N} < 0$, thus our only fixed point is **stable**.

2.5.x

2.5.1

A particle travels on the half-line $x \geq 0$ with a velocity given by $\dot{x} = -x^c$, where c is real and constant.

(a)

We first find the derivative of $\dot{x}$:

$$\ddot{x} = -cx^{c-1}$$

Clearly, subbing the only fixed point $x^* = 0$ would yield $\ddot{x} = 0$ and thus our linear stability analysis fails. However since we are only considering $x \geq 0$, in order for x^* to be stable, we only need that $\dot{x} < 0$. This is true of all values of c. However, for $c < 0$, our ODE becomes

$$\dot{x} = -\frac{1}{x^{|c|}}$$

in which case x^* is not defined and is thus not a fixed point. Further, for $c = 0$, our ODE becomes

$$\dot{x} = -1$$

which clearly has no fixed points. Thus, x^* is a stable fixed point of our ODE if and only if $c > 0$.

Part (b)

We start by solving our ODE so that we can solve for t given x. When $c \neq 1$ (we will get to that boring case later),

$$\dot{x} = -x^c$$
$$\int \frac{1}{x^c} dx = -\int 1 dt$$
$$\frac{x^{1-c}}{c-1} = t + K$$

Considering the initial condition $x(0) = 1$,

$$\frac{1}{c-1} = K$$

our implicit solution becomes

$$\frac{x^{1-c}}{c-1} = t + \frac{1}{c-1}$$
$$\frac{x^{1-c}-1}{c-1} = t$$

To find the time that it takes for the particle to travel to the origin, we substitute $x = 0$ which yields

$$\frac{1}{1-c} = t \tag{4}$$

Since we are assuming that $c = 1$, t is clearly defined. Thus, (4) gives the finite time in which the particle reaches the origin. In the case that c is equal to 1, our ODE becomes

$$\dot{x} = -x$$

which has the obvious solution

$$x = Ke^{-t}$$

With the initial condition $x(0) = 1$, we have that

$$1 = K$$

which implies

$$x(t) = e^{-t}$$

But this solution will clearly never reach $x = 0$ for any t. Therefore we can say that the particle will reach the origin with the given initial condition, if $c \in (0, 1) \cup (0, \infty)$.

2.5.3

Consider the equation $\dot{x} = rx + x^3$, where $r > 0$ is fixed. Show that $x(t) \to \pm\infty$ in finite time, starting from any initial condition $x_0 \neq 0$.

First we solve the separable first-order ODE to arrive at the solution

$$x(t) = \frac{\sqrt{cr^2e^{-2rt} - r}}{cre^{-2rt} - 1}$$

Consider $x(0)$:

$$x(0) = \frac{\sqrt{cr^2 - r}}{cre^{-2rt} - 1}$$

Imposing the condition that $x(0) \neq 0$, we therefore have that

$$\begin{cases} c > 0 \\ cr^2 > r \iff cr > 1 \end{cases}$$

With this in mind, we can solve for the time that $x(t) \to \pm\infty$ by solving, for t:

$$cre^{-2rt} - 1 = 0$$
$$e^{-2rt} = \frac{1}{cr}$$
$$-2rt = \ln(\frac{1}{cr})$$
$$t = -\frac{\ln(\frac{1}{cr})}{2r}$$

which is positive since $cr > 1$.

2.7.x

2.7.1

$$\dot{x} = x - x^2 \tag{5}$$

We are interested in $V(x)$ such that

$$-\frac{dV}{dx} = x - x^2$$
$$\frac{dV}{dx} = x^2 - x$$
$$\int dV = \int (x^2 - x)dx$$
$$V(x) = \frac{x^3}{3} - \frac{x^2}{2} + c$$

setting $c = 0$, our potential function becomes

$$V(x) = \frac{x^3}{3} - \frac{x^2}{2}$$

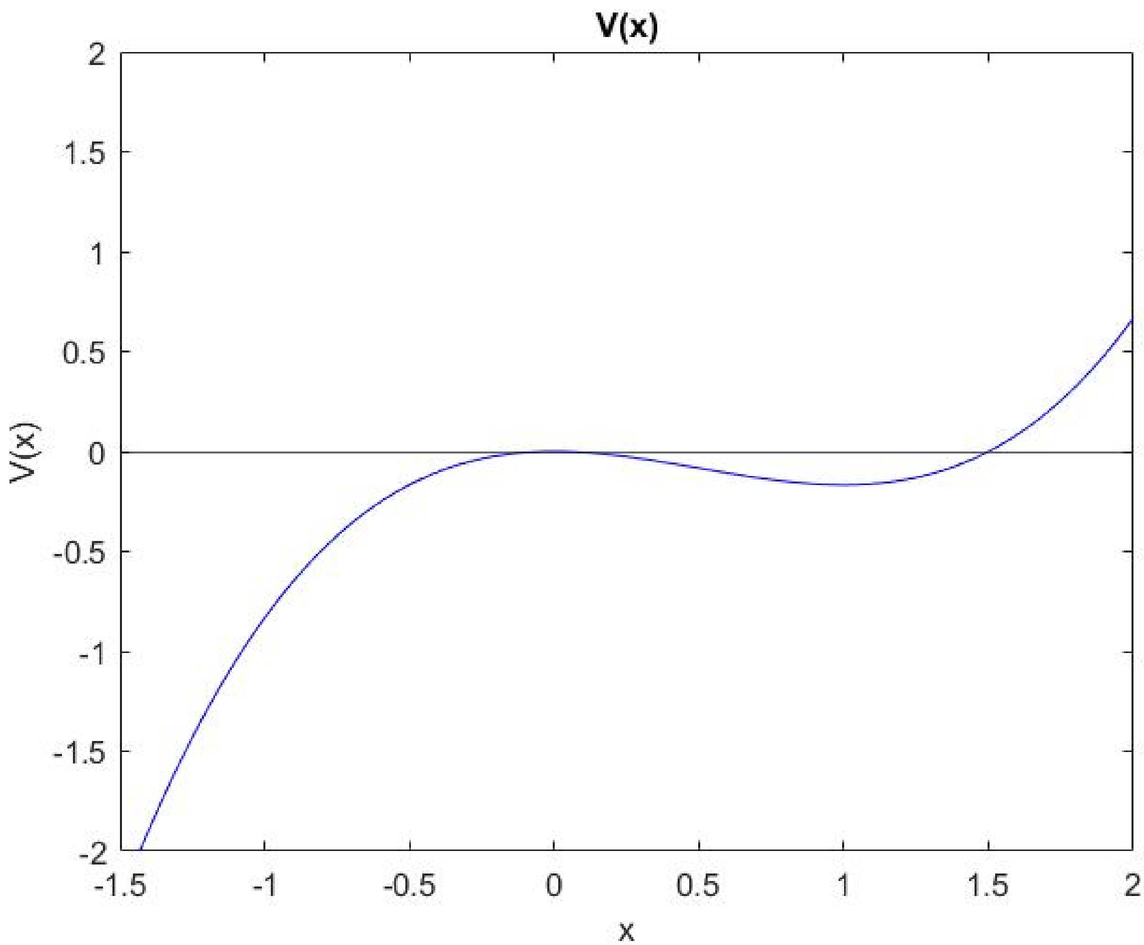
Setting $\frac{V(x)}{dx} = 0$ yields our equilibria:

$$\begin{cases} x_1^* = 0 \\ x_2^* = 1 \end{cases}$$

To analytically determine if these extrema of our potential function are local minimums or maximums, we find $\frac{d^2V}{dx^2}$:

$$\ddot{V}(x) = 2x - 1$$

then $\ddot{V}(x_1^* = 0) = -1$ which indicates that x_1^* is an **unstable** equilibrium of our potential function. On the other hand, $\ddot{V}(x_2^* = 1) = 1$, so that x_2^* is a **stable** equilibrium.


V(x)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
V(x)
-1.5
-1
-0.5
0
0.5
1
1.5
2
x

2.7.6

$$\dot{x} = r + x - x^3 \tag{6}$$

We are interested in $V(x)$ such that

$$-\frac{dV}{dx} = r + x - x^3$$
$$\int dV = \int (x^3 - x - r)dx$$
$$V(x) = \frac{x^4}{4} - \frac{x^2}{2} - rx$$

so that $V(x)$ is our potential function in the case that $c = 0$. To find the equilibria of our potential function, we solve

$$r + x - x^3 = 0$$
$$x^3 - x - r = 0$$

Clearly, depending on the value of r, we will have 1, 2 or 3 real equilibria. To determine the interval of values such that we have 3 real equilibria, we consider when

$$\left(\frac{-1}{3}\right)^3 + \left(\frac{-r}{2}\right)^2 < 0$$
$$\frac{r^2}{4} < \frac{1}{27}$$
$$r^2 < \frac{4}{27}$$
$$\implies r \in \left(-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right)$$

We can now collect the cases of real equilibria:

$$\begin{cases} r \in (-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}) & \text{3 real} \\ r = \frac{2}{3\sqrt{3}} & \text{2 equilibria} \\ r \notin [-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}] & \text{1 real} \end{cases}$$

For stability, we turn our attention to

$$\ddot{V}(x) = 3x^2 - 1$$

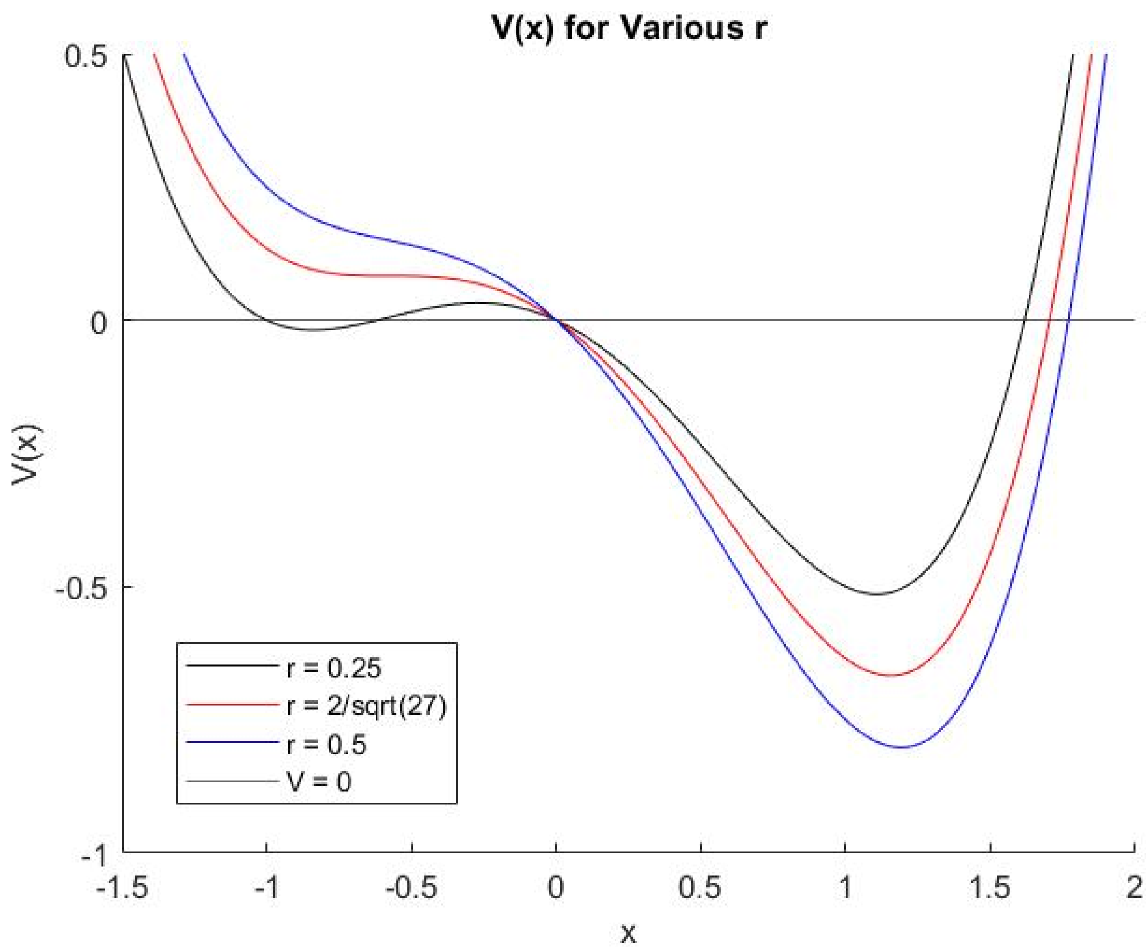
It is intuitive that the equilibria within the defined intervals of r will exhibit the same stability properties. For example, in general for $r \in (-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}})$, say that our equilibria are x_1, x_2, x_3 where $x_1 < x_2 < x_3$. Then,

$$\begin{cases} \ddot{V}(x_1) > 0 \implies x_1 \text{ is } \textbf{stable} \\ \ddot{V}(x_2) < 0 \implies x_2 \text{ is } \textbf{unstable} \\ \ddot{V}(x_3) > 0 \implies x_3 \text{ is } \textbf{stable} \end{cases}$$

Then, for $r = \frac{2}{3\sqrt{3}}$, our equilibria are $x_3 = -0.577$ and $x_4 = 0.577$.

$$\begin{cases} \ddot{V}(x_3)00 \implies x_3 \text{ is a } \textbf{saddle node} \\ \ddot{V}(x_4) < 0 \implies x_4 \text{ is } \textbf{unstable} \end{cases}$$

Finally, we consider the case where $r \notin [-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}]$. The single equilibria x_5 in this case is **stable** since $\ddot{V}(x_5) > 0$.
Below are some graphs of $V(x)$ for various r values.


V(x) for Various r
0.5
0
-0.5
-1
V(x)
-1.5
-1
-0.5
0
0.5
1
1.5
2
x
r = 0.25
r = 2/sqrt(27)
r = 0.5
V = 0