# Continuous Dynamical Systems Assignment - Hw 6

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## Problem 7.2.12

**7.2.12** Show that $\dot{x} = -x + 2y^3 - 2y^4$, $\dot{y} = -x - y + xy$ has no periodic solutions. (Hint: Choose $a$, $m$, and $n$ such that $V = x^m + ay^n$ is a Liapunov function.)

We are working with the system

$$\begin{cases} \dot{x} = -x + 2y^3 - 2y^4 \\ \dot{y} = -x - y + xy \end{cases}$$

We want to find a function $V(\boldsymbol{x})$, where $\boldsymbol{x} = (xy)$, such that

$$\begin{cases} V(\boldsymbol{x}) > 0 \text{ for } \boldsymbol{x} \neq \boldsymbol{x}^* \\ \dot{V} < 0 \text{ for } \boldsymbol{x} \neq \boldsymbol{x}^* \end{cases}$$

We will be looking at even $m, n$ to satisfy the first condition. Specifically, let's consider

$$V = x^2 + ay^4$$

Then

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} \\ &= (2x)(-x + 2y^3 - 2y^4) + (4ay^3)(-x - y + xy) \\ &= -2x^2 + 4xy^3 - 4xy^4 - 4axy^3 - 4ay^4 + 4axy^4 \\ &= -2x^2 + 4xy^3(1-a) + 4y^4(-x - a + ax) \end{aligned}$$

If we have that $a = 1$, then

$$\dot{V} = -2x^2 - 4y^4 < 0 \text{ for all } x, y$$

Then $V$ is a Liapunov function, and all trajectories must move monotonically towards $x^* = (0,0)$, ruling out the existence of closed orbits in our system.

# Problem 7.3.1

**7.3.1** Consider $\dot{x} = x - y - x(x^2 + 5y^2)$, $\dot{y} = x + y - y(x^2 + y^2)$.

a) Classify the fixed point at the origin.

b) Rewrite the system in polar coordinates, using $r\dot{r} = x\dot{x} + y\dot{y}$ and $\dot{\theta} = (x\dot{y} - y\dot{x})/r^2$.

c) Determine the circle of maximum radius, $r_1$, centered on the origin such that all trajectories have a radially *outward* component on it.

d) Determine the circle of minimum radius, $r_2$, centered on the origin such that all trajectories have a radially *inward* component on it.

e) Prove that the system has a limit cycle somewhere in the trapping region $r_1 \leq r \leq r_2$.

## Part (a)

We have a nonlinear system, so let's begin by computing the Jacobian:

$$J\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 - 3x^2 - 5y^2 & -1 - 10xy \\ 1 - 2xy & 1 - x^2 - 3y^2 \end{bmatrix}$$

Then evaluating this at the origin,

$$J\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

which has the eigenvalues

$$\begin{cases} \lambda_- = 1 + i \\ \lambda_+ = 1 - i \end{cases}$$

Thus the origin is an **unstable spiral**.

## Part (b)

From the transformation

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ x^2 + y^2 = r^2 \end{cases}$$

we can rewrite our system in polar coordinates using:

$$\begin{cases} r\dot{r} = x\dot{x} + y\dot{y} \\ \dot{\theta} = (x\dot{y} - y\dot{x})/r^2 \end{cases}$$

via substitution of our system:

$$
\begin{aligned}
r\dot{r} &= x(x - y - x(x^2 + 5y^2)) + y(x + y - y(x^2 + y^2)) \\
&= x^2 - xy - x^2(x^2 + 5y^2) + xy + y^2 - y^2 r^2 \\
&= r^2 - r^4 \sin^2\theta - r^4 \cos^2\theta[\cos^2\theta + 5\sin^2\theta] \\
&= r^2 - r^4 \sin^2\theta - r^4\cos^4\theta - 5r^4\cos^2\theta\sin^2\theta \\
&= r^2 - r^4[\cos^4\theta + 6\cos^2\sin^2\theta + \sin^4\theta]
\end{aligned}
$$

along with

$$
\begin{aligned}
\dot{\theta} &= \frac{1}{r^2}[x(x + y - y(x^2 + y^2)) - y(x - y - x(x^2 + 5y^2))] \\
&= \frac{1}{r^2}[x^2 + xy - xyr^2 - yx + y^2 + xy(x^2 + 5y^2))] \\
&= \frac{1}{r^2}[r^2 - r^4\cos\theta\sin\theta + r^4\cos\theta\sin\theta(\cos^2\theta + 5\sin^2\theta)] \\
&= 1 - r^2\cos\theta\sin\theta + r^2\cos\theta\sin\theta(\cos^2\theta + 5\sin^2\theta) \\
&= 1 + r^2\cos\theta\sin\theta[-1 + \cos^2\theta + 5\sin^2\theta] \\
&= 1 + 4r^2\cos\theta\sin^3\theta
\end{aligned}
$$

And so our system becomes

$$
\begin{cases}
\dot{r} = r - r^3[\cos^4\theta + 6\cos^2\sin^2\theta + \sin^4\theta] \\
\dot{\theta} = 1 + 4r^2\cos\theta\sin^3\theta
\end{cases}
$$

## Part (c)

We are interested in the circumstance when $\dot{r} > 0$. From our definition of $\dot{r}$, let us define

$$g(\theta) = \cos^4\theta + 6\cos^2\sin^2\theta + \sin^4\theta$$

which, since $g$ is periodic with period $2\pi$, is bounded such that

$$1 \leq g(\theta) \leq 2$$

for $0 \leq \theta \leq 2\pi$. Thus, for any $\theta$,

$$r(1 - 2r^2) \leq \dot{r} \leq r(1 - r^2) \tag{1}$$

This tells us that $\dot{r} \leq 0$ corresponds to

$$r(1 - r^2) \leq 0 \implies r \geq 1$$

Clearly, then for $r > 1$, $\dot{r} < 0$, which tells us that the desired maximum radius, below which all trajectories have a radially outward component, is $r_2 = 1$.

## Part (d)

Again appealing to (1), we know that $\dot{r} \geq 0$ corresponds to

$$r(1-2r^2) \geq 0 \implies r \leq \frac{1}{\sqrt{2}}$$

Clearly, then for $r < \frac{1}{\sqrt{2}}$, trajectories have radially outward components. Thus $r_1 = \frac{1}{\sqrt{2}}$ is the minimum value for which all trajectories have radially inward components.

## Part (e)

We know that the only fixed point of our system, in cartesian, is at (0,0), which corresponds to (0,$\theta$) in polar coordinates. Thus in polar, our only fixed point is at the origin. Since $r_1 > 0$, we know that there are no fixed points in the trapping region $R$ described by $r_1 \leq r \leq r_2$, which is closed and bounded. Further, we know that a trajectory $C$ is confined in $R$. Thus by the Poincare-Bendixson theorem, $C$ is either a closed orbit or asympotically approaches it, and thus there exists a limit cycle in $R$.

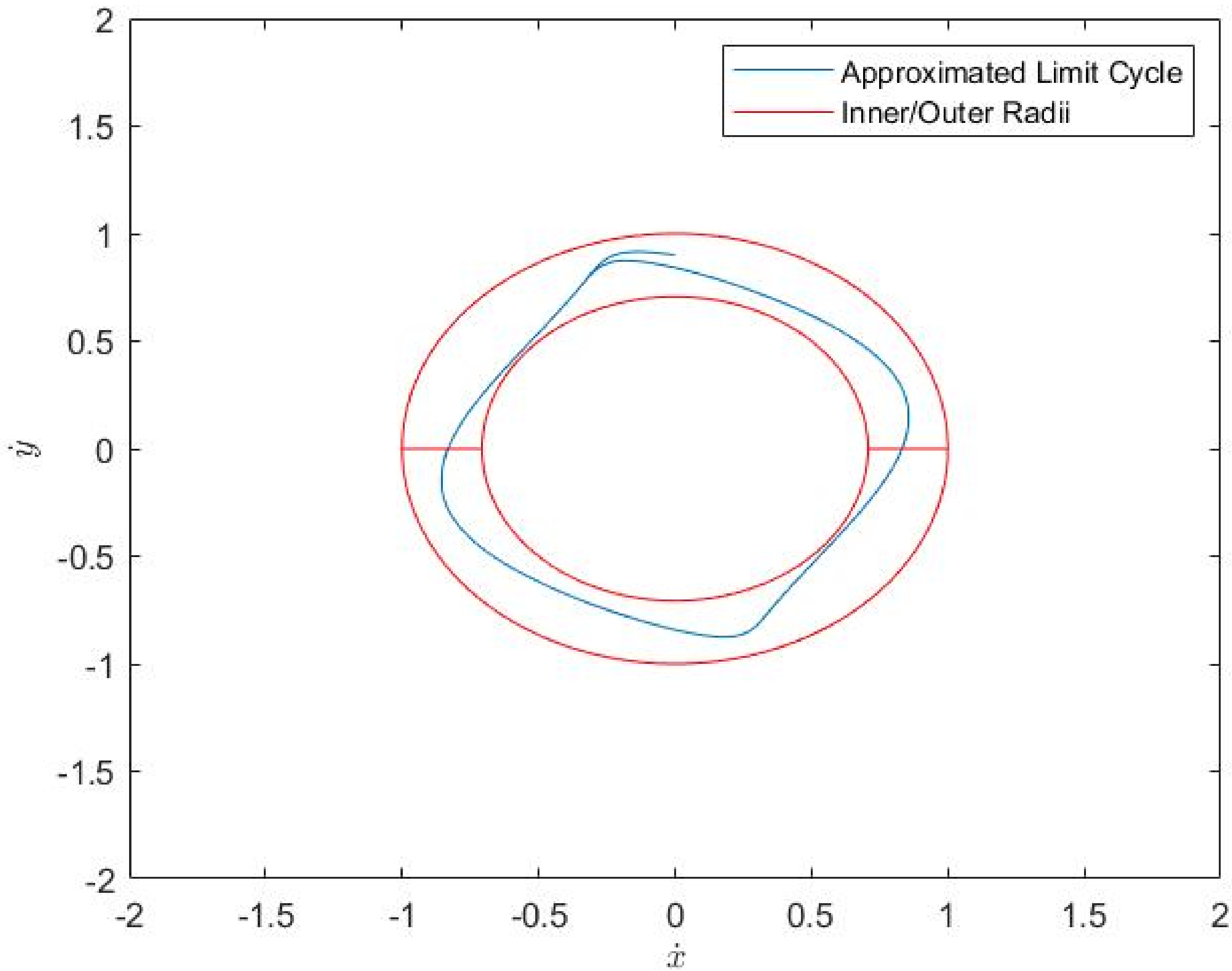
# Problem 7.3.2

**7.3.2** Using numerical integration, compute the limit cycle of Exercise 7.3.1 and verify that it lies in the trapping region you constructed.

Code:

```
x1p = linspace(-1,1,1001);
a=0;
b=15000;
h = 0.01;
nsteps = (b-a)/h + 1;
x=zeros(1,nsteps);
y=zeros(1,nsteps);
x(1)=0;
y(1)=0.9;
dxdt = @(x,y) x - y - x*(x^2 + 5*y^2);
dydt = @(x,y) x + y - y*(x^2 + y^2);
for n=1:nsteps
    x(n+1)=x(n)+h*dxdt(x(n),y(n));
    y(n+1)=y(n)+h*dydt(x(n),y(n));
end
plot(x,y)
hold on
y1 = sqrt(1-x1p.^2);
y2 = - sqrt(1-x1p.^2);
y3 = sqrt(1/2-x1p.^2);
y4 = -sqrt(1/2-x1p.^2);
plot(x1p,y1,'r',x1p,y2,'r',x1p,y3,'r',x1p,y4,'r')
legend('Approximated Limit Cycle','Inner/Outer Radii')
xlabel('$\dot{x}$','Interpreter','latex')
ylabel('$\dot{y}$','Interpreter','latex')
ylim([-2 2])
xlim([-2 2])
```

Figure:



Approximated Limit Cycle
Inner/Outer Radii
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
$\dot{y}$
$\dot{x}$

# Problem 7.3.4

**7.3.4** Consider the system

$$\dot{x} = x(1-4x^2-y^2) - \tfrac{1}{2}y(1+x), \qquad \dot{y} = y(1-4x^2-y^2) + 2x(1+x).$$

a) Show that the origin is an unstable fixed point.

b) By considering $\dot{V}$, where $V = (1-4x^2-y^2)^2$, show that all trajectories approach the ellipse $4x^2+y^2=1$ as $t \to \infty$.

## Part (a)

Let us begin by finding the Jacobian of our system:

$$J\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1-12x^2-y^2-\frac{1}{2}y & -2xy-\frac{1}{2}(1+x) \\ -8yx+2+4x^2 & 1-4x^2-3y^2 \end{bmatrix}$$

which evaluated at the origin becomes

$$J\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 2 & 1 \end{bmatrix}$$

which has the eigenvalues

$$\begin{cases} \lambda_+ = 1+i \\ \lambda_- = 1-i \end{cases}$$

which is an unstable spiral.

## Part (b)

If

$$V = (1-4x^2-y^2)^2$$

then by multi-variable chain rule,

$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt}$$

We can easily see that

$$\begin{aligned} \frac{\partial V}{\partial x} &= 2(1-4x^2-y^2)(-8x) \\ &= -16x(1-4x^2-y^2) \\ \frac{\partial V}{\partial y} &= 2(1-4x^2-y^2)(-2y) \\ &= -4y(1-4x^2-y^2) \end{aligned}$$

Then, for $\{(x,y) : (x \neq 0 \wedge y \neq 0) \vee (4x^2 + y^2 \neq 1)\}$, we have that

$$\begin{aligned}
\frac{\partial V}{\partial t} &= -16x(1-4x^2-y^2)[x(1-4x^2-y^2) - \frac{1}{2}y(1+x)] - 4y(1-4x^2-y^2)[y(1-4x^2-y^2) + 2x(1+x)] \\
&= -16x^2(1-4x^2-y^2)^2 + 8xy(1+x)(1-4x^2-y^2) - 4y^2(1-4x^2-y^2)^2 - 8xy(1+x)(1-4x^2-y^2) \\
&= -4\underbrace{[1-(4x^2+y^2)]^2}_{>0}\underbrace{(4x^2+y^2)}_{>0}
\end{aligned}$$

It is clear, then, that for any initial condition not at the origin or on the ellipse, $\dot{V} < 0$ such that $V \to 0$ as $t \to \infty$. This clearly means that, as $t \to \infty$,

$$(1 - (4x^2 + y^2)) \to 0 \implies 4x^2 + y^2 \to 1$$

If we start on the ellipse, then $V = \dot{V} = 0$, thus we stay on the ellipse. Thus all trajectories approach the ellipse $4x^2 + y^2 = 1$ as $t \to \infty$.

## Problem 7.4.2

**7.4.2** Consider the equation $\ddot{x}+\mu(x^4-1)\dot{x}+x=0$.

a) Prove that the system has a unique stable limit cycle if $\mu>0$.

b) Using a computer, plot the phase portrait for the case $\mu=1$.

c) If $\mu<0$, does the system still have a limit cycle? If so, is it stable or unstable?

### Part (a)

First let us rewrite this second order ODE into a system of first order ODEs:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x-\mu(x^4-1)y \end{cases}$$

Now let us define

$$\begin{aligned} f(x) &:= \mu(x^4-1) \\ g(x) &:= x \end{aligned}$$

We wish to invoke Lienard's Theorem, which requires the following conditions to be satisfied:

1. **$f(x)$ and $g(x)$ are continuously differentiable for all $x$.**
   Clearly, this conditions is satisfied since $f(x)$ and $g(x)$ are both polynomials.

2. **$g(-x)=-g(x)$ for all $x$.**
   Consider:

   $$g(-x)=-x=-g(x)$$

   Thus $g$ is an odd function.

3. **$g(x)>0$ for $x>0$.**
   This is clearly true.

4. **$f(-x)=f(x)$ for all $x$.** Consider:

   $$f(-x)=\mu((-x)^4-1)=\mu((x)^4-1)=f(x)$$

   Thus $f$ is an even function.

5. **The odd function $F(x)=\int_0^x f(u)du$ has exactly one positive zero at $x=a$, is negative for $0<x<a$, is positive and nondecreasing for $x>a$, and $F(x)\to\infty$ as $x\to\infty$.**
   Let us find $F(x)$ as defined above:

   $$\begin{aligned} F(x) = \int_0^x \mu(u^4-1)du &= \mu\left(\frac{x^5}{5}-x\right) \\ &= \mu x\left(\frac{x^4}{5}-1\right) \end{aligned}$$

It is easy to see that

$$F(x) = 0 \iff \begin{cases} x = 0 \\ x = \pm(5)^{1/4} \end{cases}$$

Let us define $(5)^{1/4} := a$. Then $F$ has only one positive root at $a$. It is also easy to see that

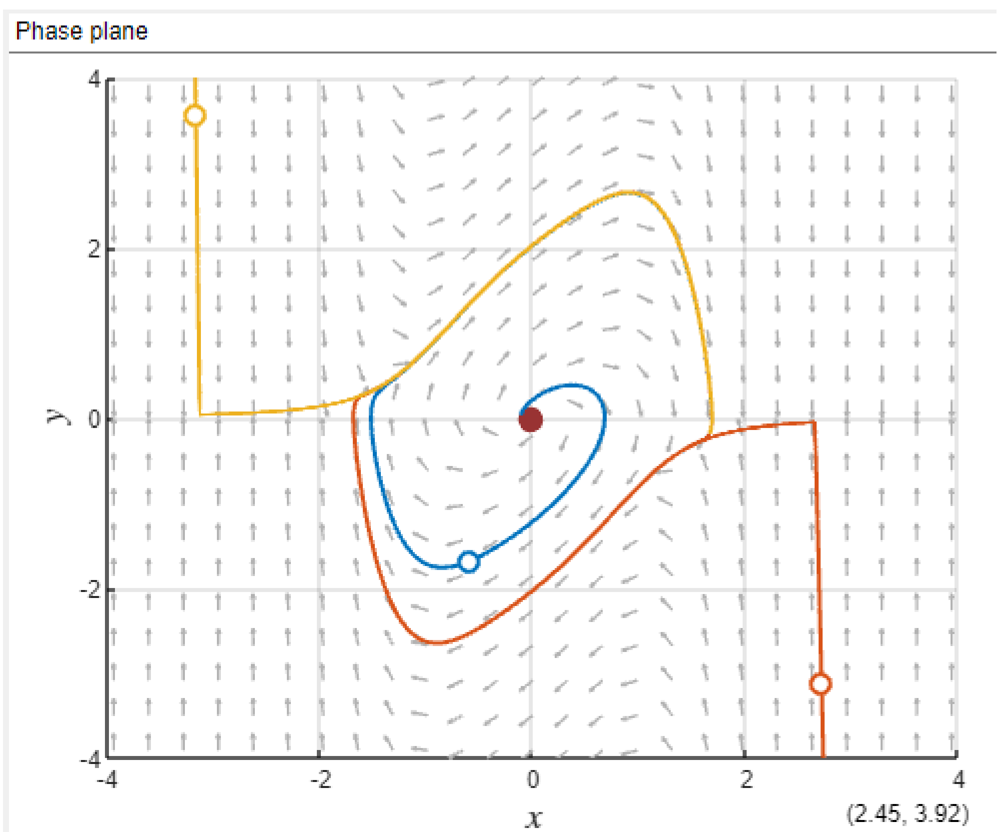
$$0 < x < a \implies \frac{x^4}{5} - 1 < 0 \implies F(x) < 0$$

for positive $\mu$. Then, since $f(x) > 0$ for $x > a$, we know that $F$ is nondecreasing for $x > a$. Lastly,

$$x \to \infty \implies F(x) \to \infty$$

We have shown that all conditions of Lienar's Theorem are satisfied for the system in question. Thus the system has a unique, stable limit cycle surrounding the origin in the phase plane.

## Part (b)



## Part (c)

For $\mu < 0$, it is easy to see that the first four conditions outlined above for Lienard's theorem are still satisfied. Regarding $F(x)$, though,

$$F(x) = -\int_0^u \mu(u^4 - 1)du = -\mu x \left(\frac{x^4}{5} - 1\right)$$

Clearly $F(x) \to -\infty$ for $x > a := (5)^{1/4}$. This tells us that the damping from $f(x)$ is positive for small $|x|$, and negative for large $|x|$. This follows with the fact that $(0,0)$ becomes a **stable** fixed point. Thus while a limit cycle still exists, it is unstable, as initial conditions inside the limit cycle converge towards the fixed point, and initial conditions outside the limit cycle diverge away from it.