Continuous Dynamical Systems Assignment - Hw 6

Christopher Mack

Spring 2022

Problem 7.5.6

7.5.6 (Biased van der Pol) Suppose the van der Pol oscillator is biased by a constant force: $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = a$, where a can be positive, negative, or zero. (Assume $\mu > 0$ as usual.)

a) Find and classify all the fixed points.

b) Plot the nullclines in the Liénard plane. Show that if they intersect on the *middle* branch of the cubic nullcline, the corresponding fixed point is unstable.

c) For $\mu >> 1$, show that the system has a stable limit cycle if and only if $|a| < a_c$, where a_c is to be determined. (Hint: Use the Liénard plane.)

d) Sketch the phase portrait for a slightly greater than a_c. Show that the system is *excitable* (it has a globally attracting fixed point, but certain disturbances can send the system on a long excursion through phase space before returning to the fixed point; compare Exercise 4.5.3.)

This system is closely related to the Fitzhugh–Nagumo model of neural activity; see Murray (1989) or Edelstein–Keshet (1988) for an introduction.

Part (a)

We want to go to the Lienard plane obviously so let's define

$$F(x) := \frac{1}{3}x^3 - x$$
$$w := \dot{x} + \mu F(x)$$
$$\implies \dot{w} = \ddot{x} + \mu\dot{F}(x) = \ddot{x} + \mu\dot{x}(x^2 - 1) = a - x$$

And so we can translate our system into

$$\begin{cases} \dot{x} = w - \mu F(x) \\ \dot{w} = a - x \end{cases}$$

Lets also use

$$y = \frac{w}{\mu}$$
$$\implies \dot{w} = \mu\dot{y}$$

which allows us to write

$$\begin{cases} \dot{x} = \mu[y - F(x)] \\ \dot{y} = \frac{1}{\mu}(a - x) \end{cases}$$

where $a \in \mathbb{R}$ and $\mu > 0$. We search for fixed points by observing the system

$$0 = \mu[y - F(x)] \tag{1}$$
$$0 = \frac{1}{\mu}(a - x) \tag{2}$$

Clearly, (2) is satisfied for $x = a$, and thus we have a critical point when:

$$y = F(a) = \frac{1}{3}a^3 - a$$

So our critical points are:

$$\boxed{(a, \frac{1}{3}a^3 - a)}$$

For classification, we refer to the Jacobian of our system:

$$J\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\mu(x^2 - 1) & \mu \\ -\frac{1}{\mu} & 0 \end{bmatrix}$$

Then evaluating this at the origin,

$$J\begin{bmatrix} a \\ \frac{1}{3}a^3 - a \end{bmatrix} = \begin{bmatrix} -\mu(a^2 - 1) & \mu \\ -\frac{1}{\mu} & 0 \end{bmatrix}$$

we then refer to the determinant of:

$$J\begin{bmatrix} a \\ \frac{1}{3}a^3 - a \end{bmatrix} = \begin{bmatrix} -\mu(a^2 - 1) - \lambda & \mu \\ -\frac{1}{\mu} & -\lambda \end{bmatrix}$$

which, setting it to zero, yields the equation

$$\lambda[\mu(a^2 - 1) + \lambda] + 1 = 0$$
$$\lambda^2 + \lambda\mu(a^2 - 1) + 1 = 0$$

Which from the quadratic formula, yields

$$\lambda_{\pm} = \frac{-\mu(a^2 - 1) \pm \sqrt{\mu^2(a^2 - 1)^2 - 4}}{2}$$

Now, when do we have oscillatory vs. nodal fixed points? This is of course determined by the sign of our discriminant:

$$\begin{cases} Im(\lambda_{\pm}) = 0 \quad \mu^2(a^2 - 1)^2 \geq 4 \implies \textbf{nodal} \text{ fixed points} \\ Im(\lambda_{\pm}) \neq 0 \quad 0 \leq \mu^2(a^2 - 1)^2 < 4 \implies \textbf{spiral} \text{ fixed points} \end{cases}$$

Lets assume the **second case, where we have a spiral fixed point**, such that $0 \leq \mu^2(a^2 - 1)^2 < 4$. Then we can easily make cases regarding the real part of our eigenvalues:

$$\begin{cases} Re(\lambda_{\pm}) > 0 \quad -1 < a < 1 \implies \textbf{unstable} \text{ spirals} \\ Re(\lambda_{\pm}) < 0 \quad |a| > 1 \implies \textbf{stable} \text{ spirals} \end{cases}$$

We note that these two cases of spirals is always possible, since no matter how small/large a is, there exists a μ such that the condition for $Im(\lambda) \neq 0$ is satisfied.
We then refer to the **case of a nodal fixed point**.

$$\begin{cases} \lambda_\pm > 0 & -\mu(a^2-1) \pm \sqrt{\mu^2(a^2-1)^2 - 4} > 0 \implies \textbf{unstable nodes} \\ \lambda_\pm < 0 & -\mu(a^2-1) \pm \sqrt{\mu^2(a^2-1)^2 - 4} < 0 \implies \textbf{stable nodes} \end{cases}$$

To get more detailed about these conditions of stability, suppose that, for $|a| < 1$, we have $\mu > 0$ such that $\mu^2(a^2-1)^2 \geq 4$. Then

$$-\mu(a^2-1) > 0$$

Further, we can see clearly that

$$\begin{aligned} \mu(a^2-1) &> \sqrt{\mu^2(a^2-1) - 4} \\ \implies -\mu(a^2-1) \pm \sqrt{\mu^2(a^2-1) - 4} &> 0 \end{aligned}$$

thus, $Re(\lambda_\pm) = \lambda_\pm > 0$. So we can make the following generalizations about the stability of nodal fixed points, i.e., fixed points under the condition that $\mu^2(a^2-1)^2 \geq 4$:

$$\begin{cases} \lambda_\pm > 0 & |a| < 1 \implies \textbf{unstable nodes} \\ \lambda_\pm < 0 & |a| > 1 \implies \textbf{stable nodes} \end{cases}$$

We note that we never had saddle nodes. Lastly, we have **centers** when

$$\begin{cases} a = \pm 1 \end{cases}$$

Part (b)

Our nullclines follow from:

$$\begin{aligned} \dot{x} = 0 &\implies y = F(x) = \frac{1}{3}x^3 - x \\ \dot{y} = 0 &\implies x = a \end{aligned}$$

It is easy to see that the local extrema of $F(x)$, the cubic nullcline, occur at $x = \pm 1$. Thus, for the nullclines to intersect in the middle crest, we must have that $|a| < 1$. From part (a), this translates to either an unstable spiral or an unstable node, depending on our discriminant.

Part (c)

Let's recall our system in the Lienard plane:

$$\begin{cases} \dot{x} = \mu[y - F(x)] \\ \dot{y} = \frac{1}{\mu}(a - x) \end{cases}$$

Let's assume that $\mu \gg 1$, and suppose that a initial condition for our system starts far away from the cubic nullcline such that

$$y - F(x) \sim \mathcal{O}(1)$$

This tells us, from our system in the Lienard plane, that

$$|\dot{x}| \sim \mathcal{O}(\mu) \gg 1$$

while, assuming that $x \approx \mathcal{O}(1)$

$$|\dot{y}| \sim \frac{\mathcal{O}(a) - \mathcal{O}(1)}{\mu} \tag{3}$$

Recall from parts (a) and (b) that we only have an unstable point for $|a| < 1$. This is necessary for a limit cycle. Further, we want $|\dot{y}| \ll \mathcal{O}(\mu)$ so that horizontal movement occurs much more quickly than vertical movement, which is achieved by the assumption that $\mu \gg 1$, and that $|a| < 1$. Thus from (3),

$$|\dot{y}| \ll \mathcal{O}(\mu^{-1})$$

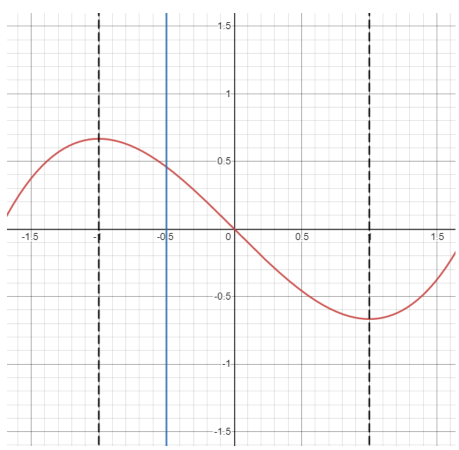
causing the trajectory to move towards the nullcline. Then, at the nullcline, such that

$$y - F(x) \sim \mathcal{O}(\mu^{-2})$$

we have that

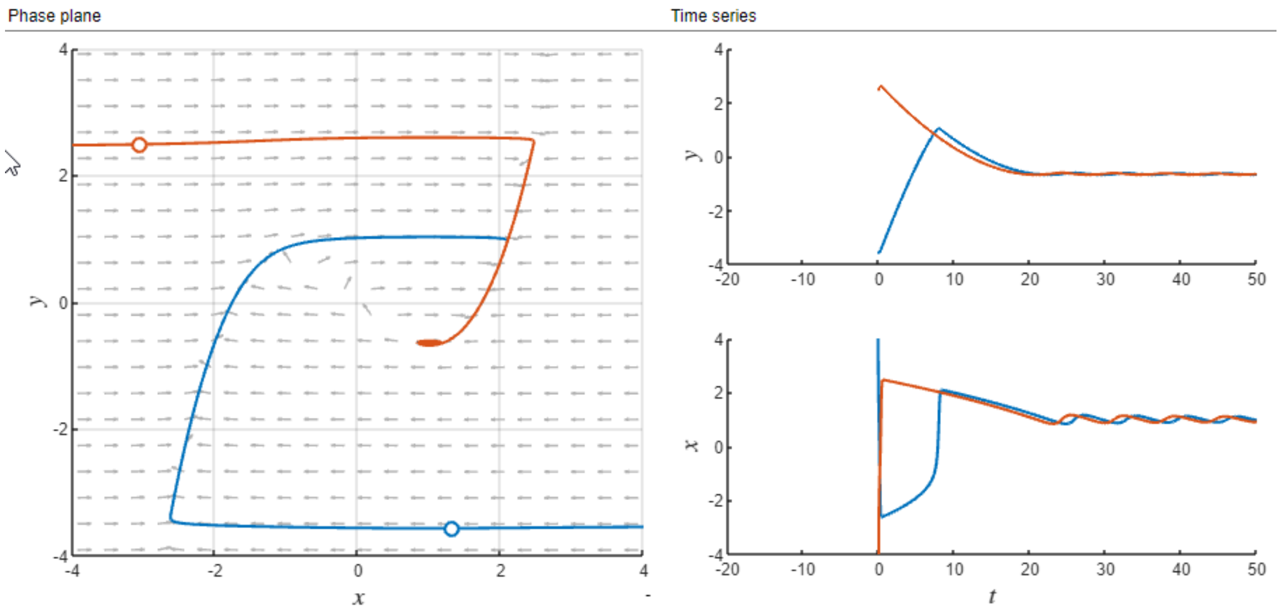
$$|\dot{x}| \sim \mathcal{O}(\mu^{-1}) \ll 1$$

which is on the order of $|\dot{y}|$ as shown. Thus the trajectory moves slowly along the nullcine, until it gets to the knee–at which point it once again becomes far enough away to exhibit quick horizontal movement relative to its vertical. Thus we have a limit cycle.



7.5.6 Part (b) : Nullclines (in blue and red) for some a

Part (d)



7.5.6 Part (d) : Excited Trajectory for $a = 1.001$

Problem 7.5.7

7.5.7 (Cell cycle) Tyson (1991) proposed an elegant model of the cell division cycle, based on interactions between the proteins cdc2 and cyclin. He showed that the model's mathematical essence is contained in the following set of dimensionless equations:

$$\dot{u} = b(v-u)(\alpha+u^2)-u, \qquad \dot{v} = c-u,$$

where u is proportional to the concentration of the active form of a cdc2-cyclin complex, and v is proportional to the total cyclin concentration (monomers and dimers). The parameters $b >> 1$ and $\alpha << 1$ are fixed and satisfy $8\alpha b < 1$, and c is adjustable.

a) Sketch the nullclines.

b) Show that the system exhibits relaxation oscillations for $c_1 < c < c_2$, where c_1 and c_2 are to be determined approximately. (It is too hard to find c_1 and c_2 exactly, but a good approximation can be achieved if you assume $8\alpha b << 1$.)

c) Show that the system is excitable if c is slightly less than c_1.

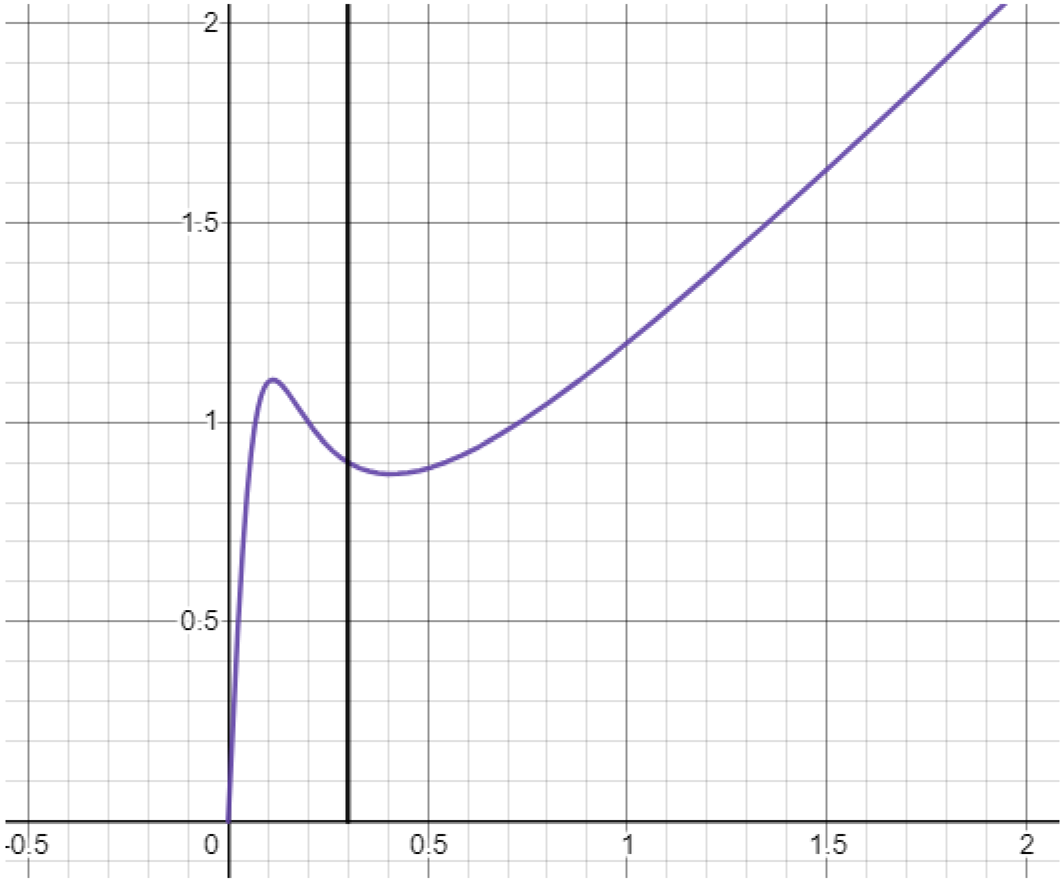
We are working with the system

$$\dot{u} = b(v-u)(\alpha+u^2)-u \tag{4}$$
$$\dot{v} = c-u \tag{5}$$

Part (a)

We find the nullclines by observing

$$\dot{u} = 0 \implies v = \frac{u}{b(\alpha+u^2)} + u$$
$$\dot{v} = 0 \implies u = c$$



7.5.7 (a) : Nullclines for $c = 0.3$, $b = 5$ and $\alpha = 0.01$

Part (b)

Let's start by computing the derivative our "cubic" nullcline:

$$\begin{aligned}
\frac{dv}{du} &= \frac{b(\alpha+u^2)(1)-u(2bu)}{b^2(\alpha+u^2)^2}+1 \\
&= \frac{b(\alpha+u^2)-2bu^2}{b^2(\alpha+u^2)^2}+\frac{b^2(\alpha+u^2)^2}{b^2(\alpha+u^2)^2} \\
&= \frac{b\alpha+bu^2-2bu^2+b^2\alpha^2+2b^2\alpha u^2+b^2u^4}{b^2(\alpha+u^2)^2} \\
&= \frac{b^2u^4+(-b+2b^2\alpha)u^2+(b\alpha+b^2\alpha^2)}{b^2(\alpha+u^2)^2}
\end{aligned}$$

The zeros of this derivative are the critical points of our cubic nullcline, so let's look at:

$$\begin{aligned}
0 &= b^2u^4+(-b+2b^2\alpha)u^2+(b\alpha+b^2\alpha^2) \\
&= b^2x^2+(-b+2b^2\alpha)x+(b\alpha+b^2\alpha^2)
\end{aligned}$$

Letting $x := u^2$. This yields two critical points:

$$\begin{aligned}
x_\pm &= \frac{(b-2b^2\alpha)\pm\sqrt{(-b+2b^2\alpha)^2-4(b\alpha+b^2\alpha^2)(b^2)}}{2b^2} \\
&= \frac{(b-2b^2\alpha)\pm\sqrt{b^2-4b^3\alpha+4b^4\alpha^2-4b^3\alpha-4b^4\alpha^2}}{2b^2} \\
&= \frac{(b-2b^2\alpha)\pm\sqrt{b^2-8b^3\alpha}}{2b^2} \\
&= \frac{(b-2b^2\alpha)\pm\sqrt{b^2(1-8b\alpha)}}{2b^2} \\
&= \frac{(b-2b^2\alpha)\pm b\sqrt{1-8b\alpha}}{2b^2}
\end{aligned}$$

Assuming that $8\alpha b \ll 1$, our critical points are simplified:

$$x_\pm \approx \frac{(b-2b^2\alpha)\pm b}{2b^2}$$

Which tells us that

$$u_\pm \approx \sqrt{\frac{(b-2b^2\alpha)\pm b}{2b^2}}$$

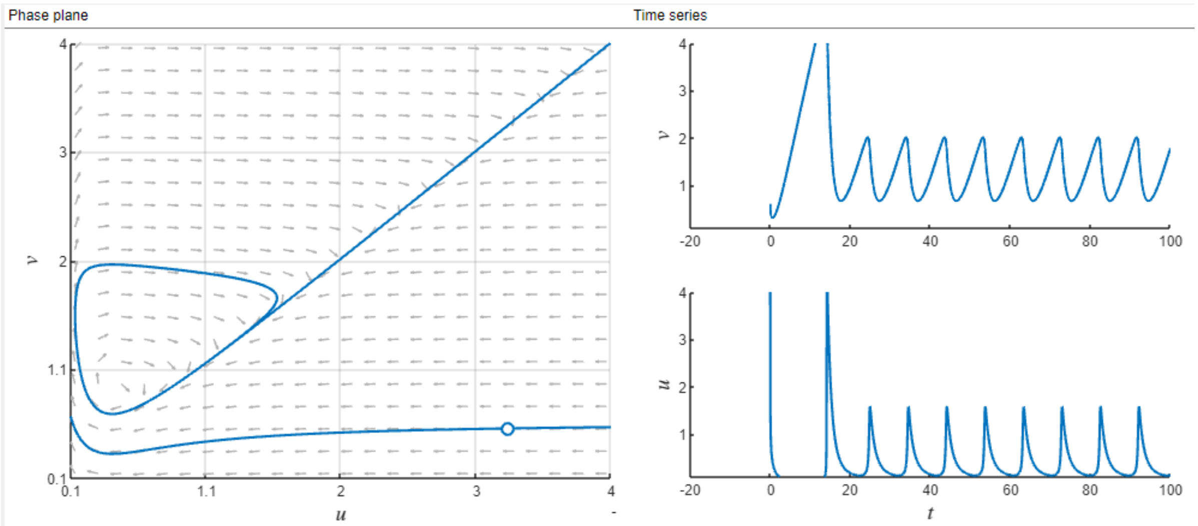
since we are only interested $u > 0$. Now, let's consider our other nullcine

$$u = c$$

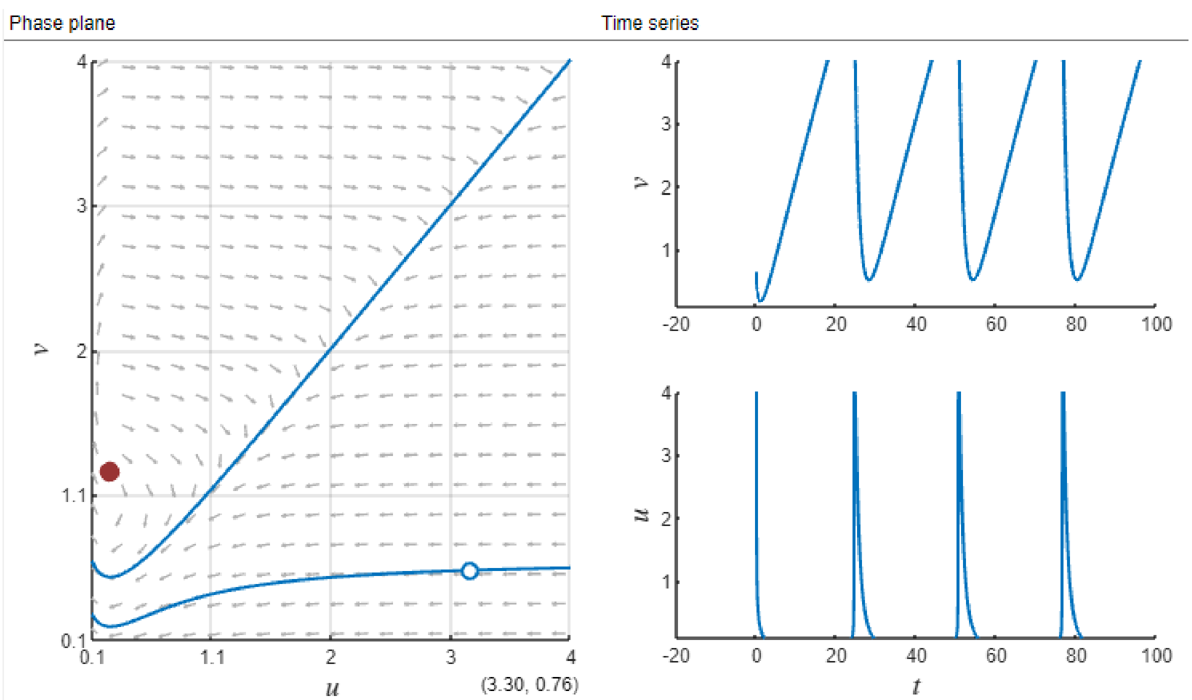
We know that we achieve relaxation oscillations when the vertical nullcine is intersecting the middle section of the cubic nullcline, i.e., for

$$\boxed{c_1 \approx \sqrt{\alpha} < c < \sqrt{\frac{1-b\alpha}{b}} \approx c_2}$$

Part (c)



7.5.7 (c): An excited trajectory for $c = 0.4$ slightly less than $c_2 = 0.446$



7.5.7 (c) : An excited trajectory for $c = 0.025$ slightly less than $c_1 = 0.031$

Problem 7.6.14

7.6.14 (Computer test of two-timing) Consider the equation $\ddot{x} + \varepsilon \dot{x}^3 + x = 0$.

a) Derive the averaged equations.

b) Given the initial conditions $x(0) = a$, $\dot{x}(0) = 0$, solve the averaged equations and thereby find an approximate formula for $x(t, \varepsilon)$.

c) Solve $\ddot{x} + \varepsilon \dot{x}^3 + x = 0$ numerically for $a = 1$, $\varepsilon = 2$, $0 \le t \le 50$, and plot the result on the same graph as your answer to part (b). Notice the impressive agreement, even though ε is not small!

Part (a)

It is taken for granted that, for some $h(\theta)$ related to the differential equation

$$\ddot{x} + x + \epsilon h(x, \dot{x}) = 0$$

the averaged equations are defined

$$\begin{aligned} r' &= \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\sin\theta d\theta := \langle h, \sin\theta\rangle \\ r\phi' &= \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\cos\theta d\theta := \langle h, \cos\theta\rangle \end{aligned}$$

Thus it has become our mission to find $h(\theta)$. We have that, where $\theta = \tau + \phi$

$$\begin{aligned} &h(x, \dot{x}) = \dot{x}^3 \\ \implies &h(\theta) = (-r\sin\theta)^3 = -r^3\sin^3\theta \end{aligned}$$

Thus we have

$$\begin{aligned} r' &= -\frac{r^3}{2\pi}\int_0^{2\pi}\sin^4\theta d\theta \\ &= -\frac{r^3}{2\pi}\left[\frac{\left(2\,(\cos(x))^3 - 5\,\cos(x)\right)\sin(x)}{8} + \frac{3\,x}{8}\right]_0^{2\pi} \\ &= -\frac{r^3}{2\pi}\left[\left(\frac{(2(1-5)\cdot 0)}{8} + \frac{6\pi}{8}\right) - \left(\frac{(2(1-5)\cdot 0)}{8} + 0\right)\right] \\ &= -\frac{r^3}{2\pi}\cdot\frac{6\pi}{8} \\ &= -\frac{3r^3}{8} \end{aligned}$$

Similarly,

$$\begin{aligned} h\phi' &= \frac{-r^3}{2\pi}\int_0^{2\pi}\sin^3\theta\cos\theta d\theta \\ &= \frac{-r^3}{8\pi}[\sin^4\theta]_0^{2\pi} = 0 \end{aligned}$$

Thus our averaged equations are

$$\boxed{\begin{cases} r' = -\frac{3r^3}{8} \\ r\phi' = 0 \end{cases}}$$

Part (b)

We are now considering the initial conditions

$$\begin{cases} x(0) = a \\ \dot{x}(0) = 0 \end{cases}$$

To understand these conditions in the context of our averaged equations, we recall that

$$x_0(t) = r(T)\cos(\tau + \phi(T))$$

which tells us that

$$\begin{aligned} \dot{x}(0) = -r(0)\sin(\phi(0)) = 0 &\implies \dot{\phi}(0) = 0 \\ x(0) = r(0)\cos(\phi(0)) = a &\implies r(0) = a \end{aligned}$$

Now we can solve for solutions for r and ϕ. Starting with r,

$$\begin{aligned} \frac{dr}{dT} &= -\frac{3r^3}{8} \\ -\frac{8}{3}r^{-3}dr &= dT \\ \frac{4}{3}r^{-2} &= T + C \\ \frac{1}{r^2} &= \frac{3}{4}(T + C) \\ r^2 &= \frac{4}{3}\frac{1}{T + C} \\ r &= \sqrt{\frac{4}{3}\frac{1}{T + C}} \end{aligned}$$

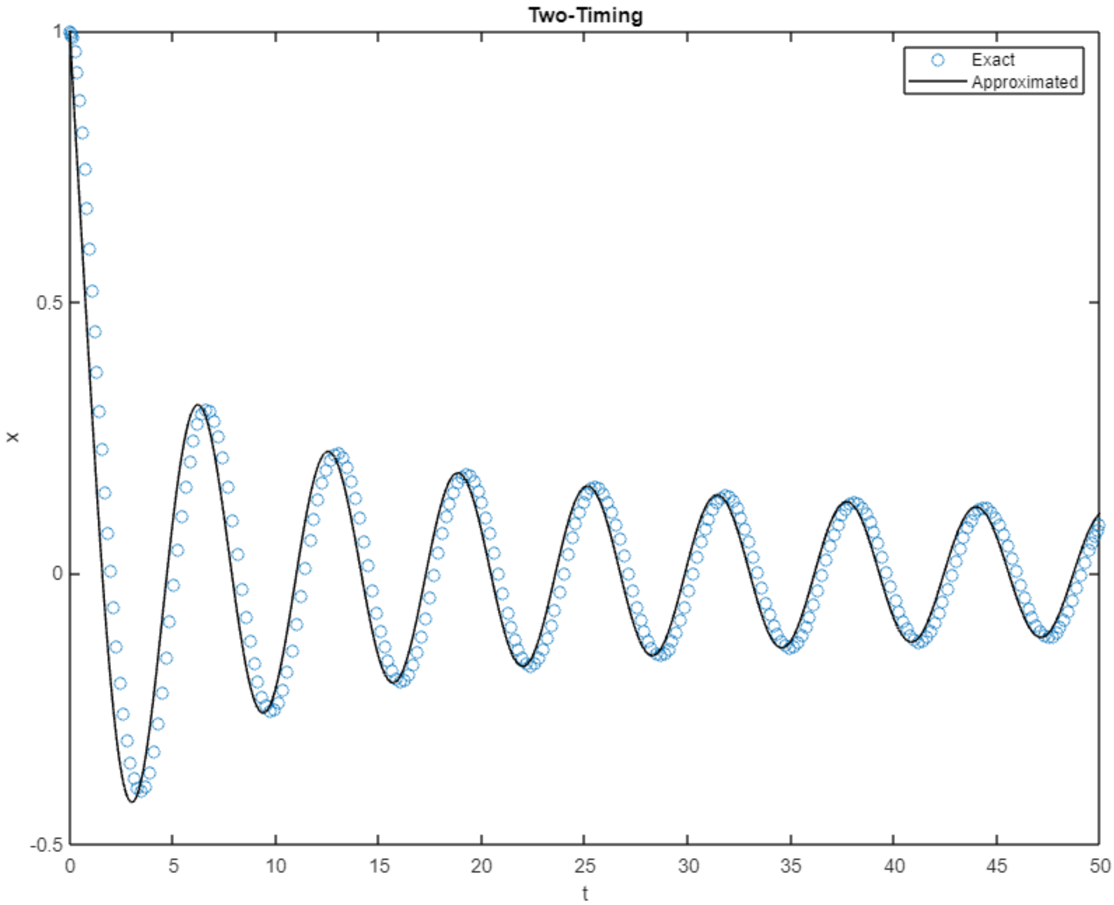
By $r(0) = a$,

$$\begin{aligned} a &= \sqrt{\frac{4}{3}\frac{1}{C}} \\ \frac{3}{4}a^2 &= C^{-1} \end{aligned}$$

Thus

$$\boxed{\begin{cases} r(\epsilon t) = \sqrt{\frac{4}{3}\frac{1}{\epsilon t + \frac{4}{3}a^{-2}}} \\ \phi(T) = 0 \end{cases}}$$

Part (c)



7.6.14 (c) Two Timer

Code:

```
TIME = linspace(0,50,1000);
a = 1;
y0 = [a 0];
e = 2;

[t,y] = ode45(@(t,y) odefcn(y,e), tspan, y0);

plot(t,y(:,1),'o')
hold on
x = sqrt((4/3).*(1./(e.*TIME + (4/3)*(1/a^2)))).*cos(TIME);
plot(TIME,x,'k')
xlabel('t')
ylabel('x')
legend('Exact','Approximated')
title('Two-Timing')


function dydt = odefcn(y,e)
dydt = zeros(2,1);
dydt(1) = y(2);
dydt(2) = -e.*(y(2)).^3 - y(1);
end
```

Problem 7.6.15

7.6.15 (Pendulum) Consider the pendulum equation $\ddot{x} + \sin x = 0$.

a) Using the method of Example 7.6.4, show that the frequency of small oscillations of amplitude $a << 1$ is given by $\omega \approx 1 - \frac{1}{16}a^2$. (Hint: $\sin x \approx x - \frac{1}{6}x^3$, where $\frac{1}{6}x^3$ is a "small" perturbation.)

b) Is this formula for ω consistent with the exact results obtained in Exercise 6.7.4?

$$\ddot{x} + \sin x = 0 \tag{6}$$

Part (a)

First let's utilize the hint given that $\sin x = x - \frac{1}{6}x^3$ so that our ODE becomes

$$\ddot{x} + x - \frac{1}{6}x^3 = 0$$

Let's let $\epsilon = -1/6$. To find our averaged equations, we consider

$$\begin{aligned} h(x, \dot{x}) &= x^3 \\ \implies h(\theta) &= r^3 \cos^3 \theta \end{aligned}$$

Following a similar process as in the previous problem, we find that

$$\begin{aligned} r' &= r^3 \langle \cos^3 \theta, \sin \theta \rangle = 0 \\ r\phi' &= r^3 \langle \cos^3 \theta, \cos \theta \rangle = \frac{3}{8} r^3 \end{aligned}$$

This tells us that

$$\begin{aligned} r(T) &= a \in \mathbb{R} \\ \implies \phi'(T) &= \frac{3}{8} a^2 \end{aligned}$$

And we know that the angular frequency ω is given by

$$\omega = 1 + \epsilon \phi'$$

Thus in our case, we have that

$$\boxed{\omega \approx 1 - \frac{1}{6}\frac{3}{8} a^2 = 1 - \frac{1}{16} a^2}$$

Part (b)

We recall the results of problem 6.7.4 where

$$T(\alpha) = 2\pi[1 + \frac{1}{16}\alpha^2 + \mathcal{O}(\alpha^4)]$$

and we know the relationship between period and frequency to be:

$$T = \frac{2\pi}{\omega}$$

Subbing the exact results from 6.7.4 into this relationship, we have that

$$\begin{aligned}\omega &= \frac{2\pi}{2\pi[1 + \frac{1}{16}\alpha^2 + \mathcal{O}(\alpha^4)]} \\ &= \frac{1}{1 + \frac{1}{16}\alpha^2}\end{aligned}$$

negating the higher order terms. We know that, from power series expansion,
$\dfrac{1}{1-x} = \dfrac{1}{1+(-x)} = 1 - x + x^2 - \dots$
Thus

$$\frac{1}{1 + \frac{1}{16}\alpha^2} = 1 - \frac{1}{16}\alpha^2 + \dots$$

Thus these results are consistent with each other.

Problem 7.6.19

7.6.19 (Poincaré–Lindstedt method) This exercise guides you through an improved version of perturbation theory known as the ***Poincaré–Lindstedt method***. Consider the Duffing equation $\ddot{x} + x + \varepsilon x^3 = 0$, where $0 < \varepsilon << 1$, $x(0) = a$, and $\dot{x}(0) = 0$. We know from phase plane analysis that the true solution $x(t, \varepsilon)$ is periodic; our goal is to find an approximate formula for $x(t, \varepsilon)$ that is valid for all t. The key idea is to regard the frequency ω as *unknown* in advance, and to solve for it by demanding that $x(t, \varepsilon)$ contains no secular terms.

a) Define a new time $\tau = \omega t$ such that the solution has period 2π with respect to τ. Show that the equation transforms to $\omega^2 x'' + x + \varepsilon x^3 = 0$.

b) Let $x(\tau, \varepsilon) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + O(\varepsilon^3)$ and $\omega = 1 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + O(\varepsilon^3)$. (We know already that $\omega_0 = 1$ since the solution has frequency $\omega = 1$ when $\varepsilon = 0$.) Substitute these series into the differential equation and collect powers of ε. Show that

$$O(1):\ x_0'' + x_0 = 0$$
$$O(\varepsilon):\ x_1'' + x_1 = -2\omega_1 x_0'' - x_0^3.$$

c) Show that the initial conditions become $x_0(0) = a$, $\dot{x}_0(0) = 0$; $x_k(0) = \dot{x}_k(0) = 0$ for all $k > 0$.

d) Solve the $O(1)$ equation for x_0.

e) Show that after substitution of x_0 and the use of a trigonometric identity, the $O(\varepsilon)$ equation becomes $x_1'' + x_1 = (2\omega_1 a - \frac{3}{4}a^3)\cos\tau - \frac{1}{4}a^3\cos 3\tau$. Hence, *to avoid secular terms*, we need $\omega_1 = \frac{3}{8}a^2$.

f) Solve for x_1.

Part (a)

If we define

$$\tau = \omega t$$

Then we have that, by chain rule,

$$\frac{d\tau}{dt} = \omega$$

So that our ODE becomes

$$\omega^2 \ddot{x} + x + \epsilon x^3 = 0 \tag{7}$$

now in terms of τ.

Part (b)

Now we perturb our variables

$$\begin{aligned} x(\tau, \epsilon) &= x_0(\tau) + \epsilon x_1(\tau) + \epsilon^2 x_2(\tau) + \mathcal{O}(\epsilon^3) \\ \omega &= 1 + \epsilon\omega_1 + \epsilon^2 \omega_2 + \mathcal{O}(\epsilon^3) \end{aligned}$$

Substituting this into (7) yields, omitting the higher order terms,

$$\begin{aligned} 0 &= [1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \mathcal{O}(\epsilon^3)]^2[\ddot{x}_o + \epsilon\ddot{x}_1 + \epsilon^2\ddot{x}_2 + \mathcal{O}(\epsilon^3)] + [x_o + \epsilon x_1 + \epsilon^2 x_2 + \mathcal{O}(\epsilon^3)] + \epsilon[x_0 + \epsilon x_1 + \epsilon^2 x_2 + \mathcal{O}(\epsilon^3)]^2 \\ &= [1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \epsilon\omega_1 + \epsilon^2\omega_1^2 + \epsilon^2\omega_1^2][\ddot{x}_0 + \epsilon\ddot{x}_1 + \epsilon^2\ddot{x}_2] + [x_0 + \epsilon x_1 + \epsilon^2 x_2] + \epsilon[(x_0^2 + \epsilon x_0 x_1 + \epsilon^2 x_1 x_2) + \epsilon^2 x_2] = 0 \\ &\vdots \end{aligned}$$

utilizing Maple instead......we arrive at

$$\mathcal{O}(1) : \ddot{x}_0 + x_0 = 0 \tag{8}$$

$$\mathcal{O}(\epsilon) : \ddot{x}_1 + x_1 = -2\omega_1\ddot{x}_0 - x_0^3 \tag{9}$$

Part (c)

We have that $x(0) = a$. By our perturbation,

$$x(0) = x_0(0) + \epsilon x_1(0) + \epsilon^2 x_2(0) = a$$

Which must mean that $x_0(0) = a$, as there are no ϵ terms on the right hand side. Similarly, it is clear that $x_0(0) = 0$.

Part (d)

Let us solve (8):

$$\begin{aligned} \ddot{x}_0 &= -x_0 \\ \implies x_0(\tau) &= c_1 \cos(\tau) + c_2 \sin(\tau) \end{aligned}$$

By one initial condition

$$x_0(1) = c_1 = a$$

and by the other,

$$\dot{x}_0(0) = -a\sin(0) + c_2\cos(0) = c_2 = 0$$

Thus

$$\boxed{x_0(\tau) = a\cos(\tau)}$$

Part (e)

$$\begin{aligned} \ddot{x}_1 + x_1 &= 2a\omega_1 \cos\tau - a^3\cos^3\tau \\ &= 2a\omega_1\cos\tau - a^3\frac{1}{4}\left(\cos 3\tau + 3\cos\tau\right) \\ &= \left(2a\omega_1 - \frac{3}{4}a^3\right)\cos\tau - \frac{1}{4}a^3\cos 3\tau \end{aligned}$$

To avoid secular terms, we must have that

$$\begin{aligned} 2a\omega_1 - \frac{3}{4}a^3 &= 0 \\ \implies \omega_1 &= \frac{3}{8}a^3 \end{aligned}$$

Part (f)

Let's recall our ODE for x_1, under the assumption $\omega_1 = \frac{3}{8}a^3$:

$$\ddot{x}_1 + x_1 = -\frac{1}{4}a^3\cos 3\tau \qquad \text{subject to} \qquad \begin{cases} x_1(0) = 0 \\ \dot{x}_1(0) = 0 \end{cases}$$

With the help of Maple,

$$\boxed{x\left(\tau\right) = -\frac{a^3\cos\left(\tau\right)}{32} + \frac{a^3\cos\left(3\,\tau\right)}{32}}$$

where

$$\tau = t + \frac{3\epsilon}{8}t + \ldots$$