537 : Hw 6

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Problem 1

We are working with the equation

$$\epsilon x^3 + x - 2 = 0 \tag{1}$$

Part (a)

Use a dominate balancing method to determine the leading order behavior of the roots (1). Find a first-order leading behavior. Compare answers for $\epsilon = 0.01$ and 0.0001.

First we consider the possibility that ϵx^3 and -2 are the same order, such that:

$$\epsilon x^3 \sim -2$$
$$x = \mathcal{O}(\frac{1}{\sqrt[3]{\epsilon}})$$

In this case, $x >> 1$, as ϵ decreases which contradicts (1). Thus we must have that

$$\epsilon x^3 \sim x$$
$$x = \mathcal{O}(\frac{1}{\sqrt[2]{\epsilon}})$$

This inspires a new scaled variable y defined by

$$y = \frac{x}{\frac{1}{\sqrt{\epsilon}}} = \sqrt{\epsilon}x$$
$$\implies x = \frac{y}{\sqrt{\epsilon}}$$

Subbing this into (1),

$$\epsilon(\frac{y}{\sqrt{\epsilon}})^3 + (\frac{y}{\sqrt{\epsilon}}) - 2 = 0$$
$$\epsilon^{-1/2}(y^3 + y) - 2 = 0$$
$$y^3 + y - 2\epsilon^{1/2} = 0$$

To find a first-order approximation, let

$$y = y_0 + \sqrt{\epsilon}y_1 + \epsilon y_2 + \epsilon^{3/2}y_3 + \mathcal{O}(\epsilon^2)$$

So that (1) becomes

$$(y_0+\sqrt{\epsilon}y_1+\epsilon y_2+\epsilon^{3/2}y_3)^3+y_0+\sqrt{\epsilon}y_1+\epsilon y_2+\epsilon^{3/2}y_3-2\epsilon^{1/2}+\cdots=0$$
$$y_0^3+y_0+\sqrt{\epsilon}(3y_0^2y_1+y_1)+\epsilon(3y_0^2y_2+3y_0y_1^2+y_2)+\epsilon^{3/2}(3y_0^2y_3+6y_0y_1y_2+y_1^3+y_3)-2\epsilon^{1/2}+\cdots=0$$

Yielding the equations:

$$\epsilon^0: \quad y_0^3+y_0=0 \tag{2}$$
$$\sqrt{\epsilon}: \quad 3y_0^2y_1+y_1-2=0 \tag{3}$$
$$\epsilon: \quad 3y_0^2y_2+3y_0y_1^2+y_2=0 \tag{4}$$
$$\epsilon^{3/2}: \quad 3y_0^2y_3+6y_0y_1y_2+y_1^3+y_3=0 \tag{5}$$

From (2):

$$y_0=\pm i,0$$

Thus the leading order roots of (1) are, by the defined relationship between y and x,

$$x=\pm i\epsilon^{-1/2},2$$

We now consider the following cases for the higher-order approximations:

(i) $y_0=0$

Then by (3),

$$y_1=2$$

And by (4),

$$y_2=0$$

Finally from (5),

$$y_3=-8$$

Thus one approximated solution for y is

$$y_z=\sqrt{\epsilon}2-\epsilon^{3/2}8 \tag{6}$$

(ii) $y_0=i$

By (3),

$$\begin{aligned}-3y_1+y_1-2&=0\\ y_1&=-1\end{aligned}$$

Then by (4),

$$\begin{aligned}-3y_2+3i(1)+y_2&=0\\ -2y_2&=-3i\\ y_2&=\frac{3}{2}i\end{aligned}$$

Finally by (5),

$$\begin{aligned}-3y_3 + 6i(-1)(\frac{3}{2}i) + (-1) + y_3 &= 0\\ -2y_3 + 8 &= 0\\ y_3 &= 4\end{aligned}$$

Thus the second approximated solution for y is

$$y_{(+i)} = i - \sqrt{\epsilon} + \epsilon\frac{3}{2}i + \epsilon^{3/2}4 \tag{7}$$

(iii) $y_0 = -i$

Then by (3),

$$\begin{aligned}-3y_1 + y_1 - 2 &= 0\\ y_1 &= -1\end{aligned}$$

And by (4),

$$\begin{aligned}-3y_2 - 3i + y_2 &= 0\\ -2y_2 &= 3i\\ y_2 &= -\frac{3}{2}i\end{aligned}$$

Finally from (5),

$$y_3 = 4$$

Thus our final approximated solution for y is

$$y_{(-i)} = -i - \sqrt{\epsilon} + \epsilon\frac{3}{2}i + \epsilon^{3/2}4 \tag{8}$$

We can now estimate the roots with our first order approximations, as they are written below in part **(b)**:

(i) Let $\epsilon = 0.01$. Then

$$\begin{aligned}x_{1f} &= -1.00 + 10.00i\\ x_{exact} &= -0.9641 + 10.1385i\\ \hline x_{2f} &= -1.00 + 10.00i\\ x_{exact} &= -0.9641 - 10.1385i\\ \hline x_{3f} &= 2\\ x_{exact} &= 1.9283\end{aligned}$$

(ii) Let $\epsilon = 0.0001$. Then

$$\begin{aligned} x_{1f} &= -1.00 + 100.00i \\ x_{exact} &= -0.999600479 + 100.014987i \\ \hline x_{2f} &= -1.00 + 100.00i \\ x_{exact} &= -0.999600479 + 100.014987i \\ \hline x_{3f} &= 1.99920 \\ x_{exact} &= 2 \end{aligned}$$

Interestingly, even though x_{3f} is a constant, its approximation still gets more accurate as ϵ gets smaller, which is also true for the real parts of the complex approximations.

Part (b)

And so (6), (7) and (8) include the second and third order approximations for y. Using these, we can write our approximated roots for (1) as:

$$x_1 = \underbrace{i\epsilon^{-1/2} - 1}_{\text{First-Order Approximation}} + \sqrt{\epsilon}\frac{3}{2}i + \epsilon 4 \tag{9}$$

$$x_2 = \underbrace{-i\epsilon^{-1/2} - 1}_{\text{First Order Approximation}} - \sqrt{\epsilon}\frac{3}{2}i + \epsilon 4 \tag{10}$$

$$x_3 = \underbrace{2}_{\text{First Order Approximation}} - \epsilon 8 \tag{11}$$

We now compare these approximations with the exact roots, found from Matlab, for two different epsilon values:

(i) Let $\epsilon = 0.01$. Then

$$\begin{aligned} x_1 &= -0.9600 + 10.1500i \\ x_{exact} &= -0.9641 + 10.1385i \\ \hline x_2 &= -0.9600 - 10.1500i \\ x_{exact} &= -0.9641 - 10.1385i \\ \hline x_3 &= 1.9200 \\ x_{exact} &= 1.9283 \end{aligned}$$

(ii) Let $\epsilon = 0.0001$

$$\begin{aligned} x_1 &= -0.999600 + 100.015i \\ x_{exact} &= -0.999600479 + 100.014987i \\ \hline x_2 &= -0.99600 - 100.015i \\ x_{exact} &= -0.999600479 + 100.014987i \\ \hline x_3 &= 1.99920 \\ x_{exact} &= 1.999200958 \end{aligned}$$

Clearly these provide better approximations than the first-order ones in part **(a)**.

Problem 2

Use singular perturbation methods to obtain a uniform approximation to the solution of the BVP:

$$\epsilon y'' + 2y' + y = 0, \ \ y(0) = 0, \ \ y(1) = 1, \ \ 0 < \epsilon << 1 \tag{12}$$

State the inner and outer solutions that you derived. Find the exact solution to this BVP problem.

Part (a)

Here we define

$$\begin{aligned} p(x) &= 2 \\ q(x) &= 1 \\ a &= 0 \\ b &= 1 \end{aligned}$$

With these definitions, we make use of a nice theorem which states that the inner and outer approximations of (12) are given by

$$\begin{aligned} y_i(x) &= C_1 + (a - C_1)e^{-p(0)x/\epsilon} \\ y_o(x) &= b\exp\Big(\int_x^1 \frac{q(s)}{p(s)} ds\Big) \end{aligned}$$

where

$$C_1 = b\exp\Big(\int_0^1 \frac{q(s)}{p(s)} ds\Big)$$

Further, the uniform approximations are defined

$$y_u(x) = y_o(x) + y_i(x) - C_1$$

First, we will find C_1:

$$C_1 = \exp(\int_0^1 \frac{1}{2} ds) = \exp(\frac{s}{2}|_0^1) = e^{\frac{1}{2}}$$

Then the inner solution is

$$y_i(x) = e^{\frac{1}{2}} - e^{\frac{1}{2}}e^{-\frac{2x}{\epsilon}} = e^{\frac{1}{2}} - e^{-\frac{2x}{\epsilon}+\frac{1}{2}} \tag{13}$$

For the outer solution, we find

$$\int_x^1 \frac{1}{2} ds = \frac{s}{2}|_x^1 = \frac{1}{2} - \frac{x}{2}$$

then

$$y_o(x) = e^{\frac{1}{2}-\frac{x}{2}} \tag{14}$$

Lastly, the uniform approximation is given by

$$y_u(x) = e^{\frac{1}{2}-\frac{x}{2}} - e^{-\frac{2x}{\epsilon}+\frac{1}{2}}$$

We now find the exact solution, since (12) has the characteristic equation

$$\epsilon r^2 + 2r + 1 = 0$$

Yielding two values of r:

$$r_1 = \frac{-2+\sqrt{4-4\epsilon}}{2\epsilon} = \frac{-1+\sqrt{1-\epsilon}}{\epsilon}$$
$$r_2 = \frac{-1-\sqrt{1-\epsilon}}{\epsilon}$$

So that our exact (general) solution is

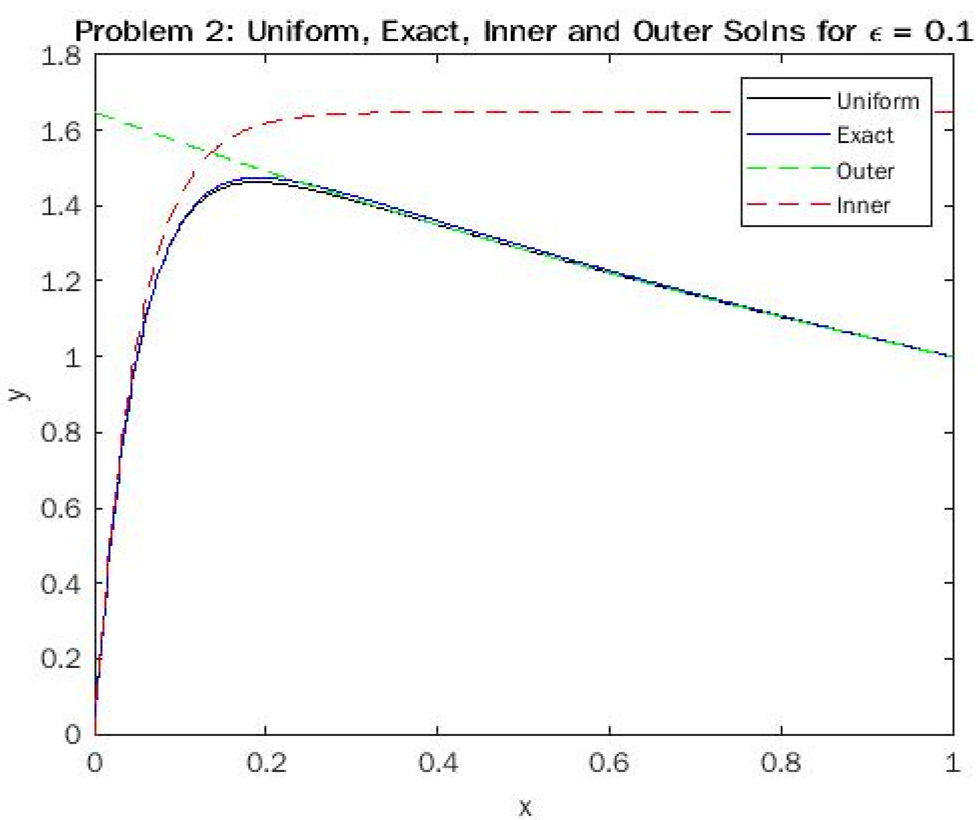
$$y(t) = A\exp\left(\frac{-1+\sqrt{1-\epsilon}}{\epsilon}x\right) + B\exp\left(\frac{-1-\sqrt{1-\epsilon}}{\epsilon}x\right) \tag{15}$$

where it can be shown that, by the conditions given,

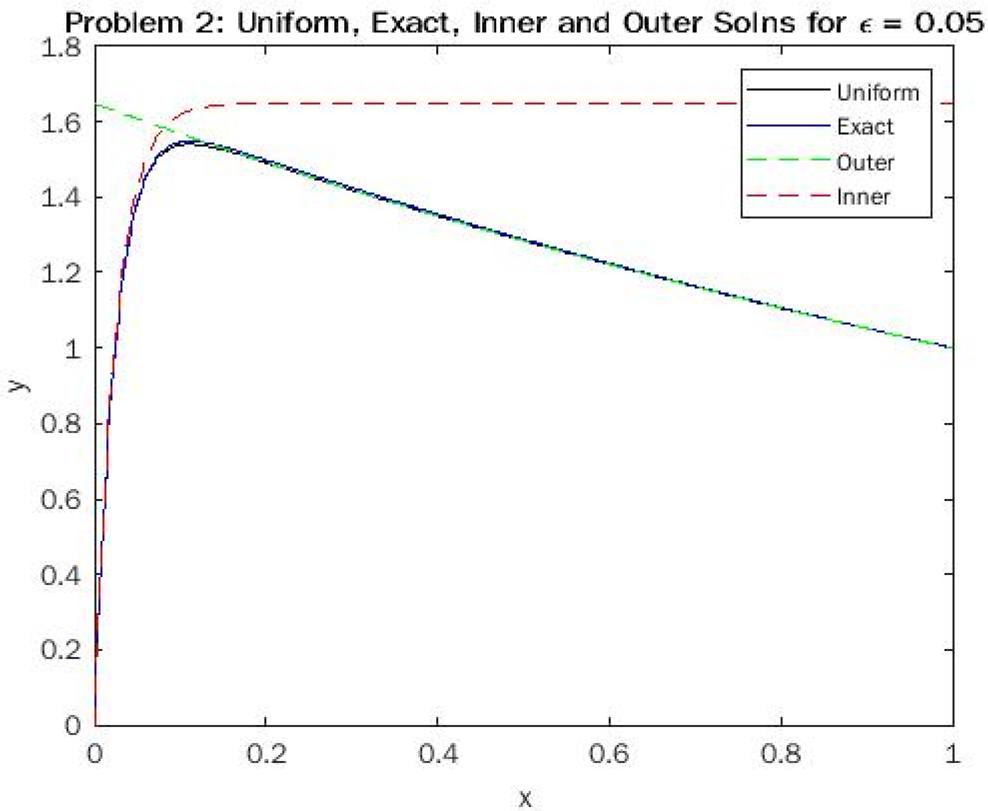
$$A = \exp\left(\frac{-1-\sqrt{1-\epsilon}}{\epsilon}\right) - \exp\left(\frac{-1+\sqrt{1-\epsilon}}{\epsilon}\right)$$
$$B = -A$$

Part (b)

Below is a plot of the exact, uniform, outer and inner solutions for $\epsilon = 0.1$.



Below are the outer, inner, and uniform solutions for $\epsilon = 0.05$.



We see that the inner solution follows the exact solution nicely for the inner inner layer, but starts to diverge around x = 0.15, which is the start of the overlap domain. Around x = 0.2, the start of the outer layer, the outer solution follows the exact solution nicely. Also, the uniform solution traces the exact solution quite well. Of course, these approximations become more accurate with smaller epsilon.

Problem 3

Use singular perturbation methods to obtain a uniform approximation to the solution of the BVP:

$$\epsilon y'' + y' + y^2 = 0, \;\; y(0) = \frac{1}{4}, \;\; y(1) = \frac{1}{2}, \;\; 0 < \epsilon << 1 \tag{16}$$

State clearly both the inner and outer solutions that you derived.

Part (a)

We can find our outer approximation by setting $\epsilon = 0$ so that (16) becomes

$$\begin{aligned} y_o' + y_0^2 &= 0 \\ y_0(x) &= \frac{1}{x - C} \end{aligned}$$

From the boundary condition (corresponding to the outer layer) $y(1) = \frac{1}{2}$,

$$\begin{aligned} y_o(x) = \frac{1}{x - C} &= \frac{1}{2} \\ C &= -1 \end{aligned}$$

So that our outer solution becomes

$$y_o(x) = \frac{1}{x+1} \tag{17}$$

For the inner solution, we have to consider changes in the boundary layer by making a change of variables

$$\xi = \frac{x}{\delta(\epsilon)} \quad \text{and} \quad Y(\xi) = y(\delta(\epsilon)\xi)$$

So that (16) becomes

$$\frac{\epsilon}{\delta(\epsilon)^2} Y''(\xi) + \frac{Y'(\xi)}{\delta(\epsilon)} + Y(\xi)^2 = 0$$

We want to ensure that no terms reach asymptotic behavior in relation to the others. So we compare the coefficients $\frac{\epsilon}{\delta(\epsilon)^2}$, $\frac{1}{\delta(\epsilon)}$, and 1. This leaves us with the cases

1. $\dfrac{\epsilon}{\delta(\epsilon)^2}$ and $\dfrac{1}{\delta(\epsilon)}$ have the same order.
2. $\dfrac{\epsilon}{\delta(\epsilon)^2}$ and 1 have the same order.

Case (1) is desirable so that $\delta(\epsilon) = \mathcal{O}(\epsilon)$, otherwise the second and first order terms would not be comparatively small. So we take $\delta(\epsilon) = \epsilon$ which leads to the scaled ODE

$$Y'' + Y' + \epsilon Y = 0$$

which is amenable to regular perturbation. So we observe the leading-order approximation, where $\epsilon = 0$, which yields the new ODE

$$Y'' + Y' = 0, \qquad Y(0) = 0.25$$

whose general solution is

$$Y = 0.25 + C(-1 + e^{-\xi})$$

which provides our inner approximation

$$y_i(x) = 0.25 + C(-1 + e^{-x/\epsilon}) \tag{18}$$

Collecting out outer and inner solutions,

$$\begin{cases} y_o(x) = \frac{1}{x+1} & x = \mathcal{O}(1) \\ y_i(x) = 0.25 + C(-1 + e^{-x/\epsilon}) & x = \mathcal{O}(\epsilon) \end{cases}$$

for $x \in [0, 1]$. To study the overlap region, where, say, $x = \mathcal{O}(\epsilon)$ we create the intermediate variable

$$\eta = \frac{x}{\sqrt{\epsilon}}$$

We should have that the outer and inner approximations converge to the same value as $e \to 0^+$ within this overlap region, i.e.

$$\lim_{\epsilon \to 0+} y_o(\sqrt{\epsilon}\eta) = \lim_{\epsilon \to 0+} y_i(\sqrt{\epsilon}\eta)$$

Computing the LHS,

$$\lim_{\epsilon \to 0+} y_o(\sqrt{\epsilon}\eta) = \lim_{\epsilon \to 0+} \frac{1}{\sqrt{\epsilon}\eta + 1} = 1$$

since these must match,

$$\lim_{\epsilon \to 0+} y_i(\sqrt{\epsilon}\eta) = \lim_{\epsilon \to 0+} 0.25 + C(-1 + e^{-\eta/\sqrt{\epsilon}}) = 0.25 - C = 1 \implies C = -\frac{3}{4}$$

And so our updated list of solutions becomes

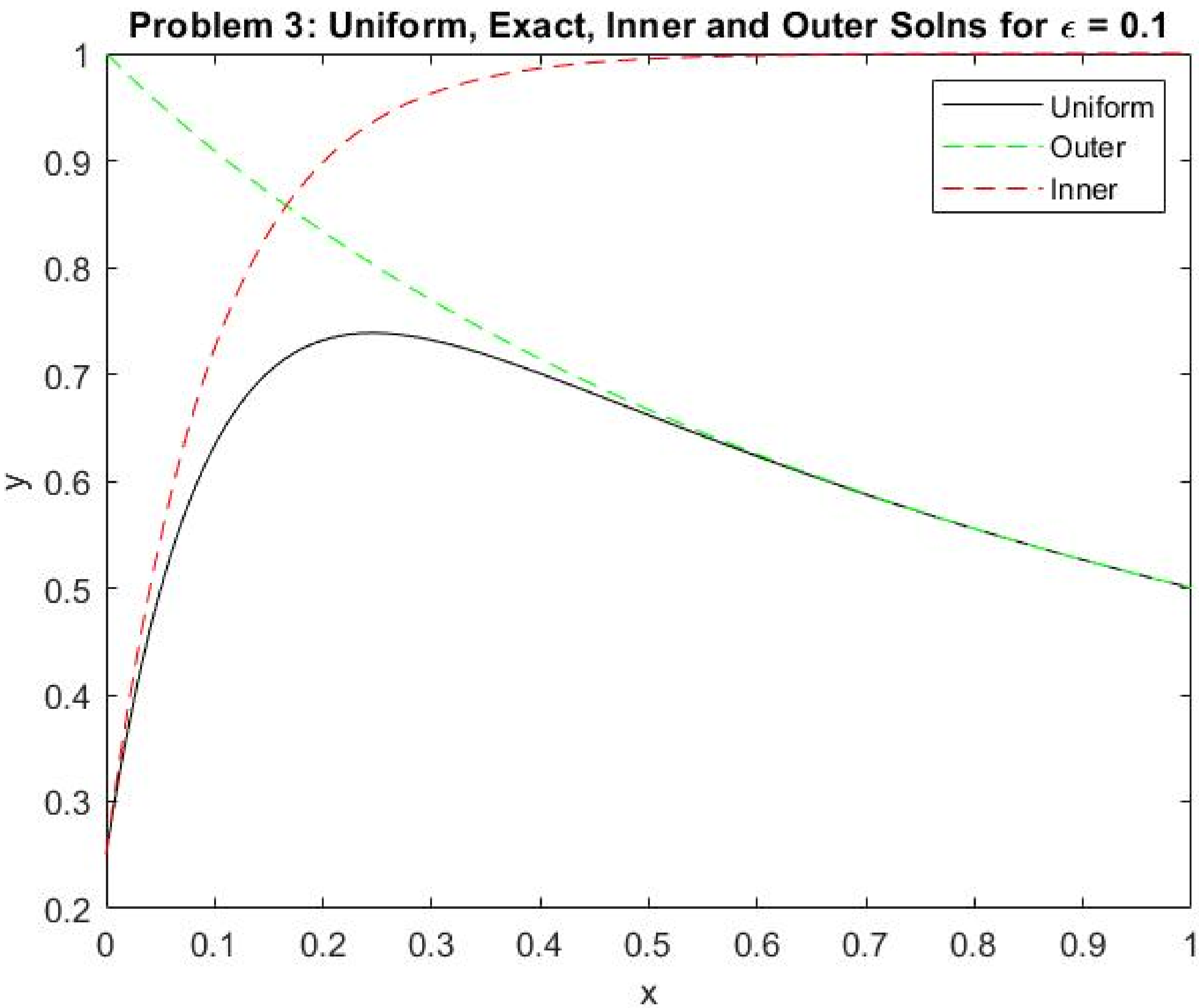
$$\begin{cases} y_o(x) = \frac{1}{x+1} & x = \mathcal{O}(1) \\ y_i(x) = 0.25 - \frac{3}{4}(-1 + e^{-x/\epsilon}) & x = \mathcal{O}(\epsilon) \end{cases}$$

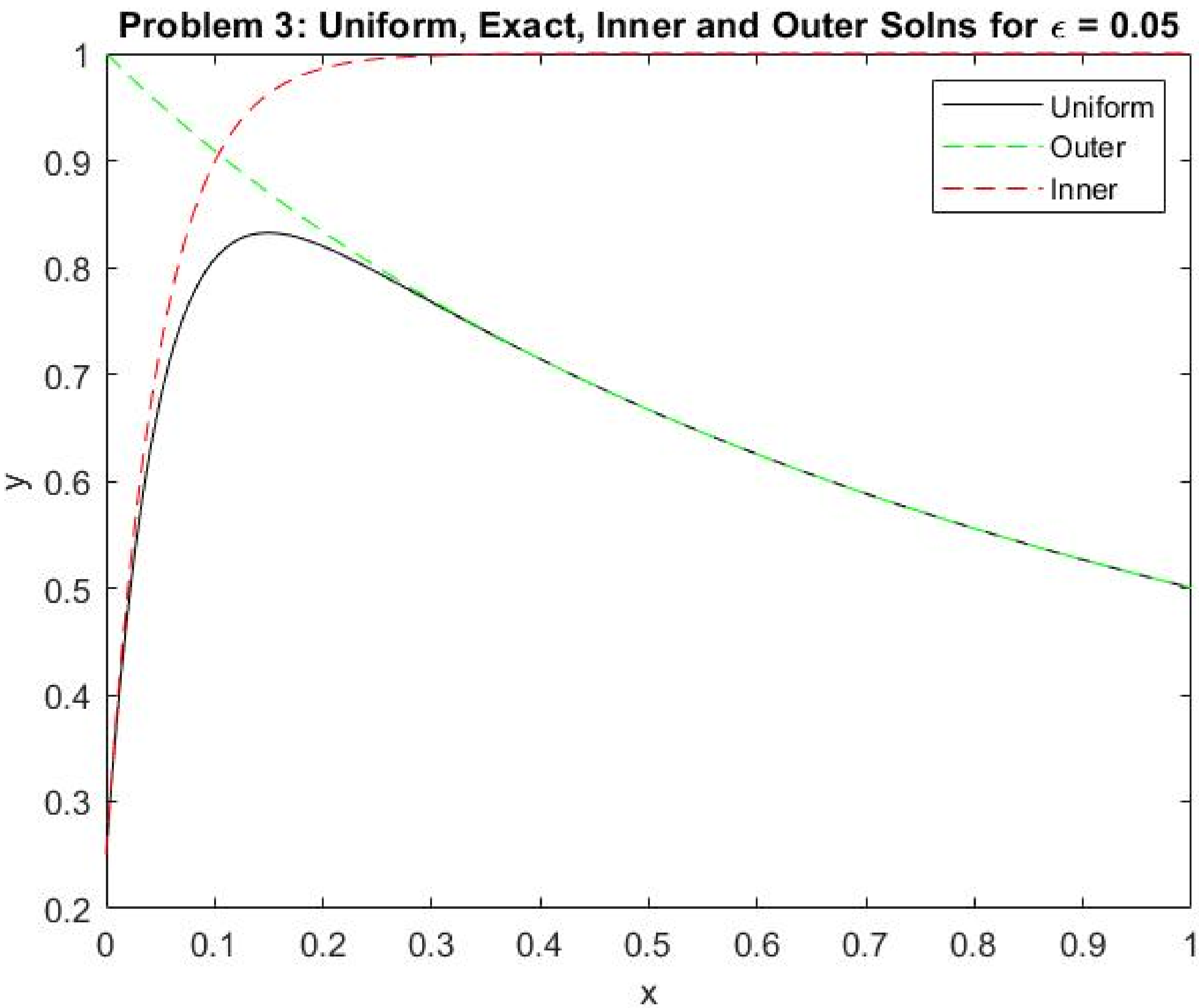
We now take the sum of the inner and outer approximations, and subtract the matching condition, to find our uniform approximation:

$$\begin{aligned} y_u(x) &= y_o(x) + y_i(x) - 1 \\ &= \frac{1}{x+1} + 0.25 - \frac{3}{4}(-1 + e^{-x/\epsilon}) - 1 \end{aligned}$$

Part (b)

Below are the plots of the above solutions, for $\epsilon = 0.1$ and $\epsilon = 0.05$. It appears that the overlap region, where neither the inner nor outer approximations match the uniform solution that accurately, is quite large. That being said, the approximation gets better with lower epsilon.


Problem 3: Uniform, Exact, Inner and Outer Solns for ϵ = 0.1
Uniform
Outer
Inner
y
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



Problem 3: Uniform, Exact, Inner and Outer Solns for ϵ = 0.05
Uniform
Outer
Inner
y
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Problem 4

Use singular perturbation methods to obtain a uniform approximation to the solution of the BVP:

$$\epsilon y'' + (1+x)y' = 1,\ \ y(0) = 0,\ \ y(1) = 1 + \ln(2),\ \ 0 < \epsilon << 1 \tag{19}$$

State clearly both the inner and outer solutions that you derived.

Part (a)

We can find our outer approximation by setting $\epsilon = 0$ so that (16) becomes

$$\begin{aligned}(1+x)y_0'(x) &= 1\\ \implies y_o(x) &= \ln(1+x) + C\end{aligned}$$

From the boundary condition $y(1) = 1 + \ln(2)$,

$$\begin{aligned}y_0(1) = \ln(2) + C &= 1 + \ln(2)\\ C &= 1\end{aligned}$$

So that our outer solution becomes

$$y_o(x) = \ln(1+x) + 1 \tag{20}$$

For the inner solution, we have to consider changes in the boundary layer by making a change of variables

$$\xi = \frac{x}{\delta(\epsilon)} \quad \text{and} \quad Y(\xi) = y(\delta(\epsilon)\xi)$$

So that (19) becomes

$$\begin{aligned}\frac{\epsilon}{\delta(\epsilon)^2}Y''(\xi) + (1+\xi\delta(\epsilon))\frac{Y'(\xi)}{\delta(\epsilon)} &= 1\\ \frac{\epsilon}{\delta(\epsilon)^2}Y''(\xi) + \frac{Y'(\xi)}{\delta(\epsilon)} + \xi\delta(\epsilon)\frac{Y'(\xi)}{\delta(\epsilon)} &= 1\\ \frac{\epsilon}{\delta(\epsilon)^2}Y''(\xi) + \frac{Y'(\xi)}{\delta(\epsilon)} + \xi Y'(\xi) &= 1\end{aligned}$$

We want to ensure that no (non-zeroth order) terms reach asymptotic behavior in relation to the others. So we compare the coefficients $\frac{\epsilon}{\delta(\epsilon)^2}$, $\frac{1}{\delta(\epsilon)}$, and ξ. This leaves us with the cases

1. $\dfrac{\epsilon}{\delta(\epsilon)^2}$ and $\dfrac{1}{\delta(\epsilon)}$ have the same order.
2. $\dfrac{\epsilon}{\delta(\epsilon)^2}$ and ξ have the same order.

Case (1) is again desirable, where $\delta(\epsilon) = \mathcal{O}(\epsilon)$, so that ξ and 1 are comparatively small. This leads to the scaled ODE

$$Y'' + Y' + \epsilon Y' = \epsilon$$

which is amenable to regular perturbation. So we observe the leading-order approximation, where $\epsilon = 0$, which yields the new ODE

$$Y'' + Y' = 0, \quad Y(0) = 0$$

whose general solution is

$$Y(\xi) = C_1(-1 + e^{-\xi})$$

which provides our inner approximation by replacing ξ with $\frac{x}{\epsilon}$

$$y_i(x) = C_1(-1 + e^{-\frac{x}{\epsilon}}) \tag{21}$$

Collecting out outer and inner solutions,

$$\begin{cases} y_o(x) = \ln(1+x) + 1 & x = \mathcal{O}(1) \\ y_i(x) = C_1(-1 + e^{-\frac{x}{\epsilon}}) & x = \mathcal{O}(\epsilon) \end{cases}$$

for $x \in [0, 1]$. To study the overlap region, we create the intermediate variable

$$\eta = \frac{x}{\sqrt{\epsilon}}$$

We should have that, as a matching condition:

$$\lim_{\epsilon \to 0+} y_o(\sqrt{\epsilon}\eta) = \lim_{\epsilon \to 0+} y_i(\sqrt{\epsilon}\eta)$$

Computing $\lim_{\epsilon \to 0+} y_o(\sqrt{\epsilon}\eta)$:

$$\lim_{\epsilon \to 0+} y_o(\sqrt{\epsilon}\eta) = \lim_{\epsilon \to 0+} \ln(1 + \sqrt{\epsilon}\eta) + 1 = 1$$

since these must match,

$$\lim_{\epsilon \to 0+} y_i(\sqrt{\epsilon}\eta) = \lim_{\epsilon \to 0+} C_1(-1 + e^{-\frac{\sqrt{\epsilon}\eta}{\epsilon}}) = -C = 1 \implies C = -1$$

And so our updated list of solutions becomes

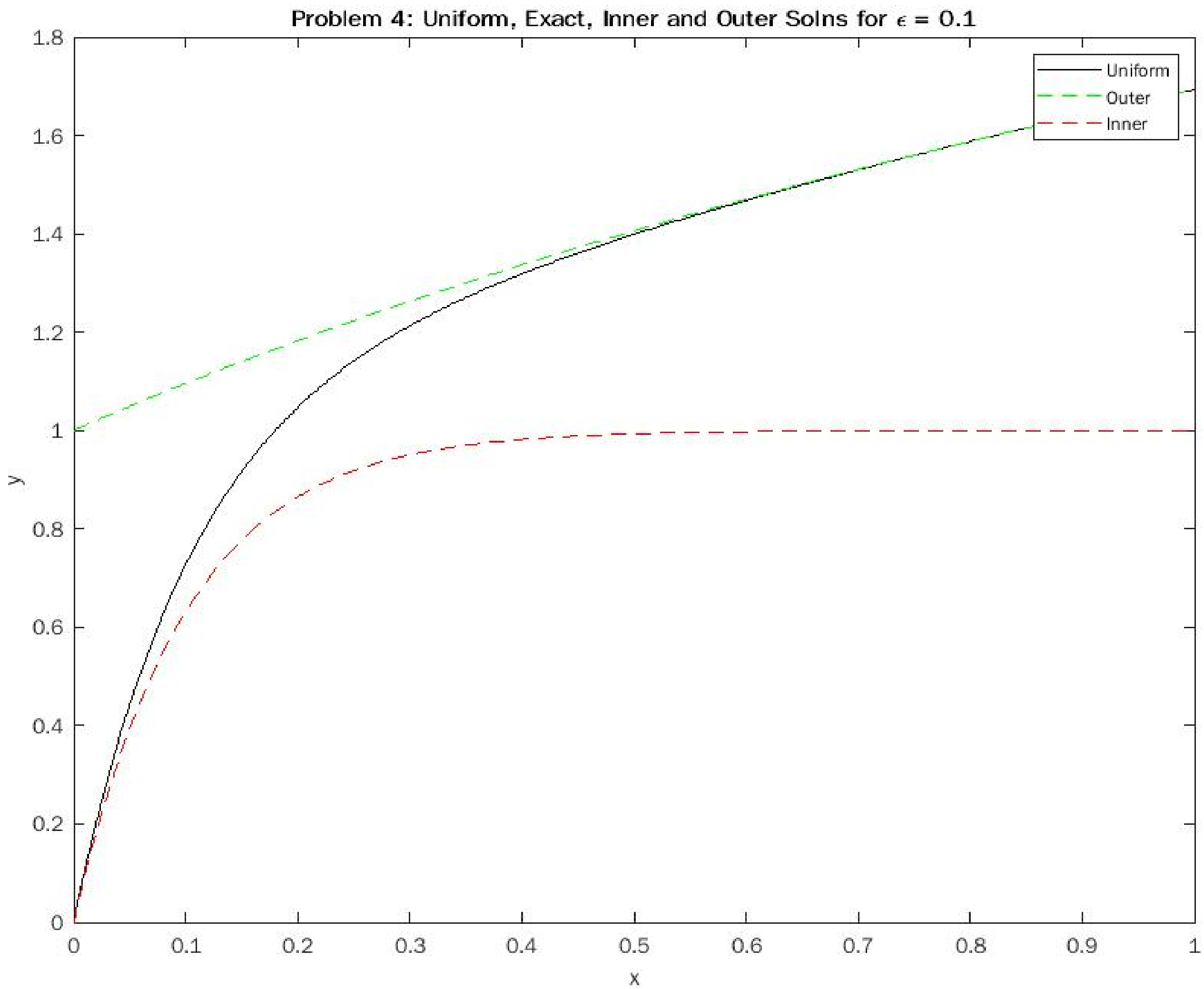
$$\begin{cases} y_o(x) = \ln(1+x) + 1 & x = \mathcal{O}(1) \\ y_i(x) = 1 - e^{-\frac{x}{\epsilon}} & x = \mathcal{O}(\epsilon) \end{cases}$$

We now take the sum of the inner and outer approximations, and subtract the matching condition, to find our uniform approximation:

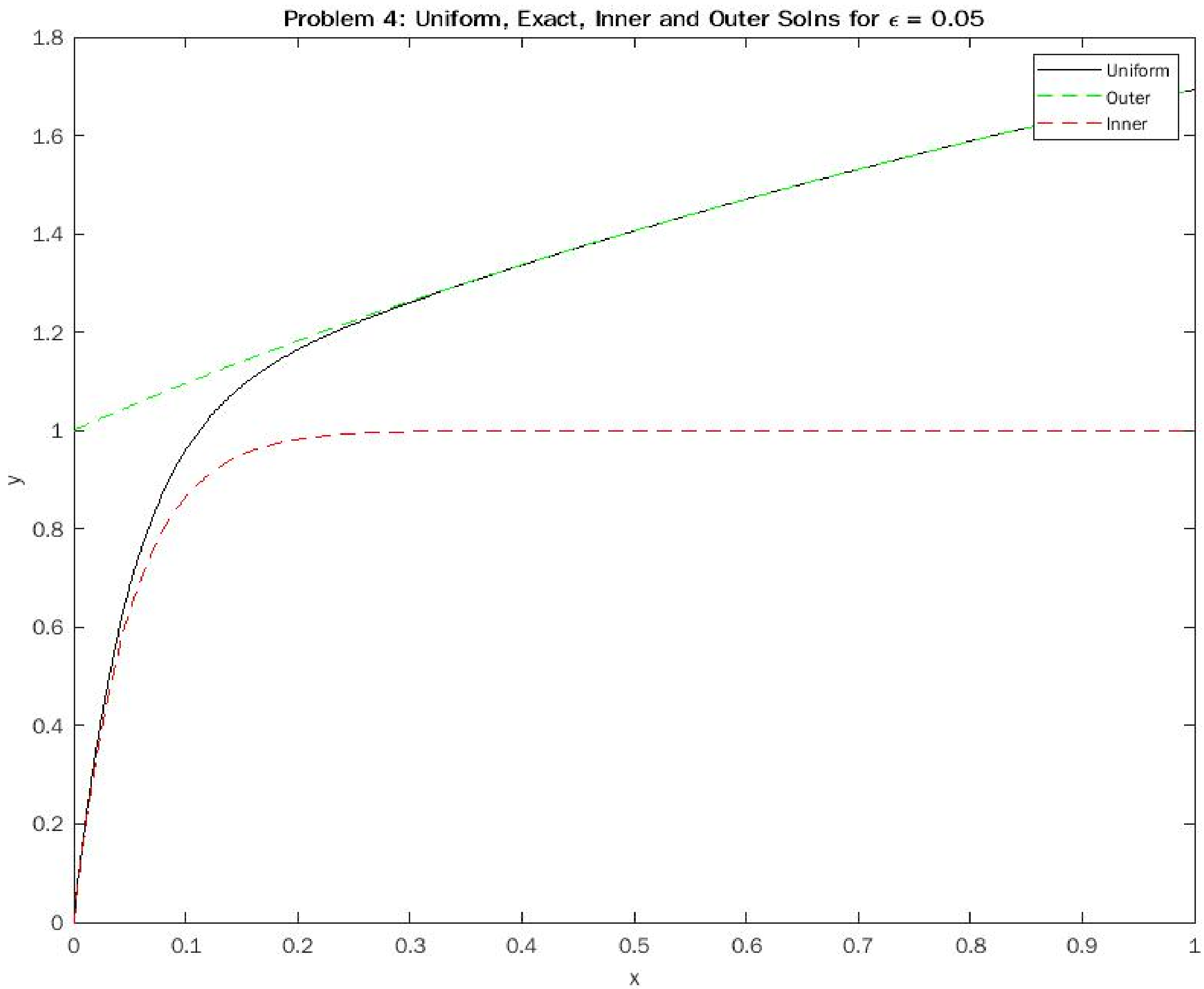
$$\begin{aligned} y_u(x) &= y_o(x) + y_i(x) - 1 \\ &= \ln(1+x) + 1 + 1 - e^{-\frac{x}{\epsilon}} - 1 \\ &= \ln(1+x) + 1 - e^{-\frac{x}{\epsilon}} \end{aligned}$$

Part (b)

Below are the graphs of all three solutions to (19), for $\epsilon = 0.1$ and $\epsilon = 0.05$.



Problem 4: Uniform, Exact, Inner and Outer Solns for ϵ = 0.1
Uniform
Outer
Inner
y
x
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Problem 4: Uniform, Exact, Inner and Outer Solns for ϵ = 0.05
Uniform
Outer
Inner
y
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.2
0.4
0.6
0.8
1.2
1.4
1.6
1.8

Problem 5

Use singular perturbation methods to obtain a uniform approximation to the solution of the BVP:

$$\epsilon u'' - (2x+1)u' + 2u = 0, \;\; u(0) = 1, \;\; u(1) = 0, \;\; 0 < \epsilon << 1 \tag{22}$$

State clearly both the inner and outer solutions that you derived.

Part (a)

From the boundary condition $u(1) = 0$, we get that $u_o(x) = 0$. To save the reader some time, we note that this would lead to a contradiction when studying the limit as $\epsilon \to 0$ of the inner and outer solutions in terms of the intermediate variable. In essence, our outer solution becomes our inner solution, and our inner becomes our outer solution. Thus we employ methods previously used to find the outer solution, to find the inner, which is setting $\epsilon = 0$ and applying the boundary condition $u(0) = 1$. In this way, (22) becomes

$$\begin{aligned} -(2x+1)u_i' + 2u_i &= 0 \\ \implies u_i(x) &= C(2x+1) \end{aligned}$$

$$u_i(0) = C = 1$$

So that our outer solution becomes

$$u_i(x) = 2x + 1 \tag{23}$$

For the inner solution, we have to consider changes in the boundary layer by making a change of variables

$$\xi = \frac{1-x}{\delta(\epsilon)} \quad \text{and} \quad U(\xi) = u(\delta(\epsilon)\xi) = u(1-x)$$

$$\implies x = 1 - \delta(\epsilon)\xi$$

As these subsitutions differ than previous problems, the first derivative also change:

$$\frac{du}{dx} = -\frac{1}{\delta(\epsilon)}\frac{dU}{d\xi} \quad \text{and} \quad \frac{d^2u}{dx^2} = \frac{1}{\delta(\epsilon)^2}\frac{d^2U}{d\xi^2}$$

So that (19) becomes

$$\begin{aligned} \frac{\epsilon}{\delta(\epsilon)^2}U''(\xi) + (2(1-\delta(\epsilon)\xi)+1)\frac{U'(\xi)}{\delta(\epsilon)} + 2U(\xi) &= 0 \\ \frac{\epsilon}{\delta(\epsilon)^2}U''(\xi) + (3-2\delta(\epsilon)\xi)\frac{U'(\xi)}{\delta(\epsilon)} + 2U(\xi) &= 0 \\ \frac{\epsilon}{\delta(\epsilon)^2}U''(\xi) + 3\frac{U'(\xi)}{\delta(\epsilon)} - 2\delta(\epsilon)\xi\frac{U'(\xi)}{\delta(\epsilon)} + 2U(\xi) &= 0 \\ \frac{\epsilon}{\delta(\epsilon)^2}U''(\xi) + 3\frac{U'(\xi)}{\delta(\epsilon)} - 2\xi U'(\xi) + 2U(\xi) &= 0 \end{aligned}$$

We want to ensure that no (non-zeroth order) terms reach asymptotic behavior in relation to the others. So we compare the coefficients $\frac{\epsilon}{\delta(\epsilon)^2}$, $\frac{1}{\delta(\epsilon)}$, 1 and ξ. This leaves us with the cases

1. $\dfrac{\epsilon}{\delta(\epsilon)^2}$ and $\dfrac{1}{\delta(\epsilon)}$ have the same order. Then $\delta(\epsilon) = \mathcal{O}(\epsilon)$ so that ξ and 1 are comparatively small.
2. $\dfrac{\epsilon}{\delta(\epsilon)^2}$ and ξ have the same order. Then $\delta(\epsilon) = \mathcal{O}(1)$ which provides the outer solution and thus is not of interest.
3. $\dfrac{\epsilon}{\delta(\epsilon)^2}$ and 1 have the same order. Then $\delta(\epsilon) = \mathcal{O}(\epsilon^{1/2})$, so that they each have order 1 and thus $\frac{1}{\epsilon}$ is not comparatively small.

Thus (1) is the only possible case, so we let $\delta(\epsilon) = \mathcal{O}(\epsilon)$. Then our scaled ODE becomes

$$U'' - 2\epsilon\xi U' + 3U' + 2\epsilon U' = 0$$

which is amenable to regular perturbation. So we observe the leading-order approximation, where $\epsilon = 0$, which yields the new ODE

$$U'' + 3U' = 0, \quad U(1) = 0$$

which has the solution

$$U(\xi) = C_1(-\frac{1}{e^3} + e^{-3\xi})$$

which provides our inner approximation

$$u_o(x) = C_1(-\frac{1}{e^3} + e^{-3\frac{1-x}{\epsilon}}) \tag{24}$$

Collecting out outer and inner solutions,

$$\begin{cases} u_i(x) = 2x + 1 & x = \mathcal{O}(1) \\ u_o(x) = C_1(-\frac{1}{e^3} + e^{-3\frac{1-x}{\epsilon}}) & x = \mathcal{O}(\epsilon) \end{cases}$$

for $x \in [0, 1]$. We now study the following limits to find C_1, by employing the matching condition:

$$\lim_{x \to x_0 = 1} u_i(x) = \lim_{\epsilon \to 0+} u_o(x)$$

Computing $\lim_{x \to x_i = 1} u_i(x)$:

$$\lim_{x \to 1} u_i(x) = \lim_{x \to 1} 2x + 1 = 3$$

since these must match,

$$\lim_{\epsilon \to 0+} u_o(x) = \lim_{\epsilon \to 0+} C_1(-\frac{1}{e^3} + e^{-3\frac{1-x}{\epsilon}}) = C_1(\frac{-1}{e^3}) = 3 \implies C_1 = -3e^3$$

And so our updated list of solutions becomes

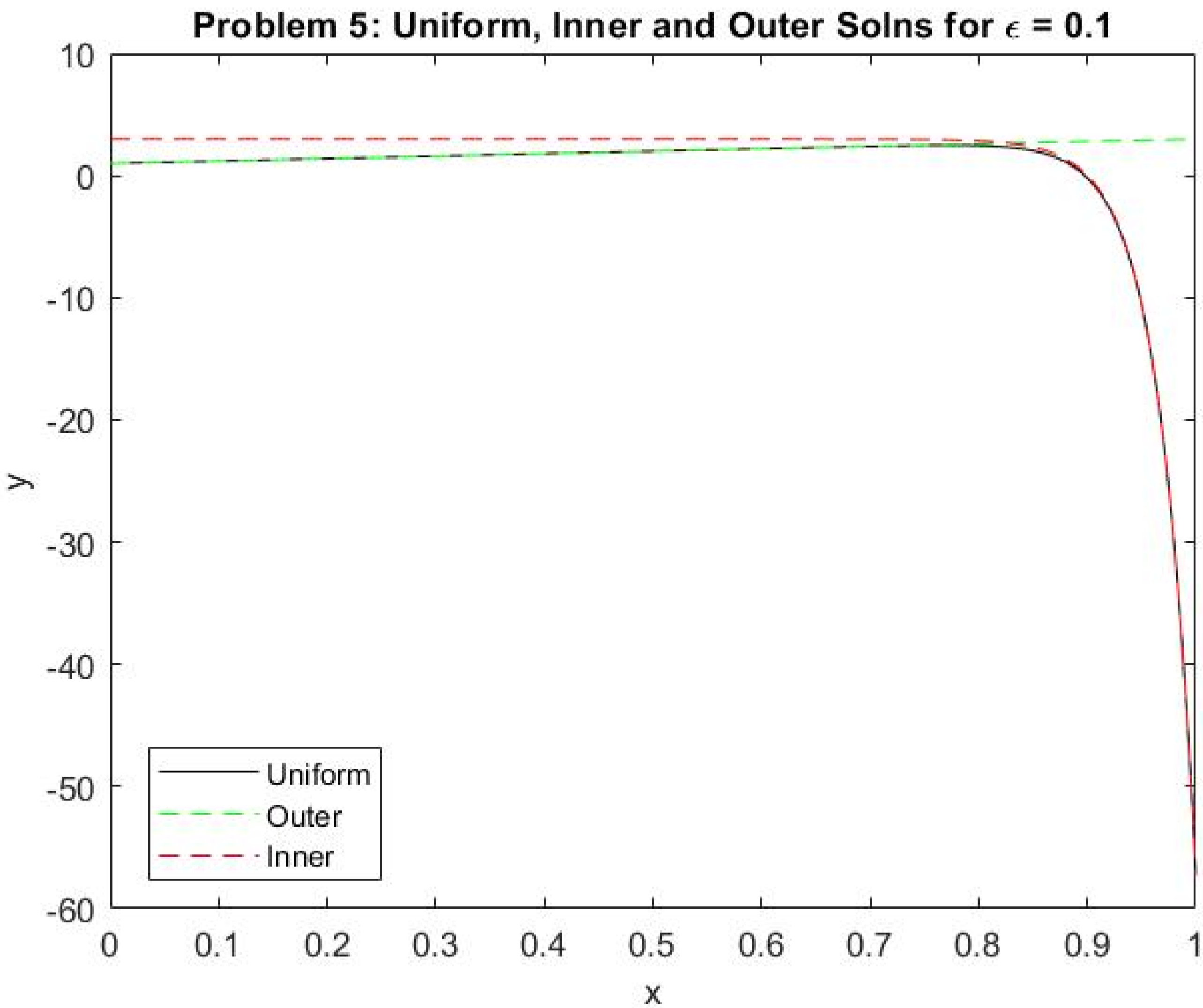
$$\begin{cases} u_i(x) = 2x + 1 & x = \mathcal{O}(1) \\ u_o(x) = 3 - 3e^{\frac{-3}{\epsilon}(1-x)+3} & x = \mathcal{O}(\epsilon) \end{cases}$$

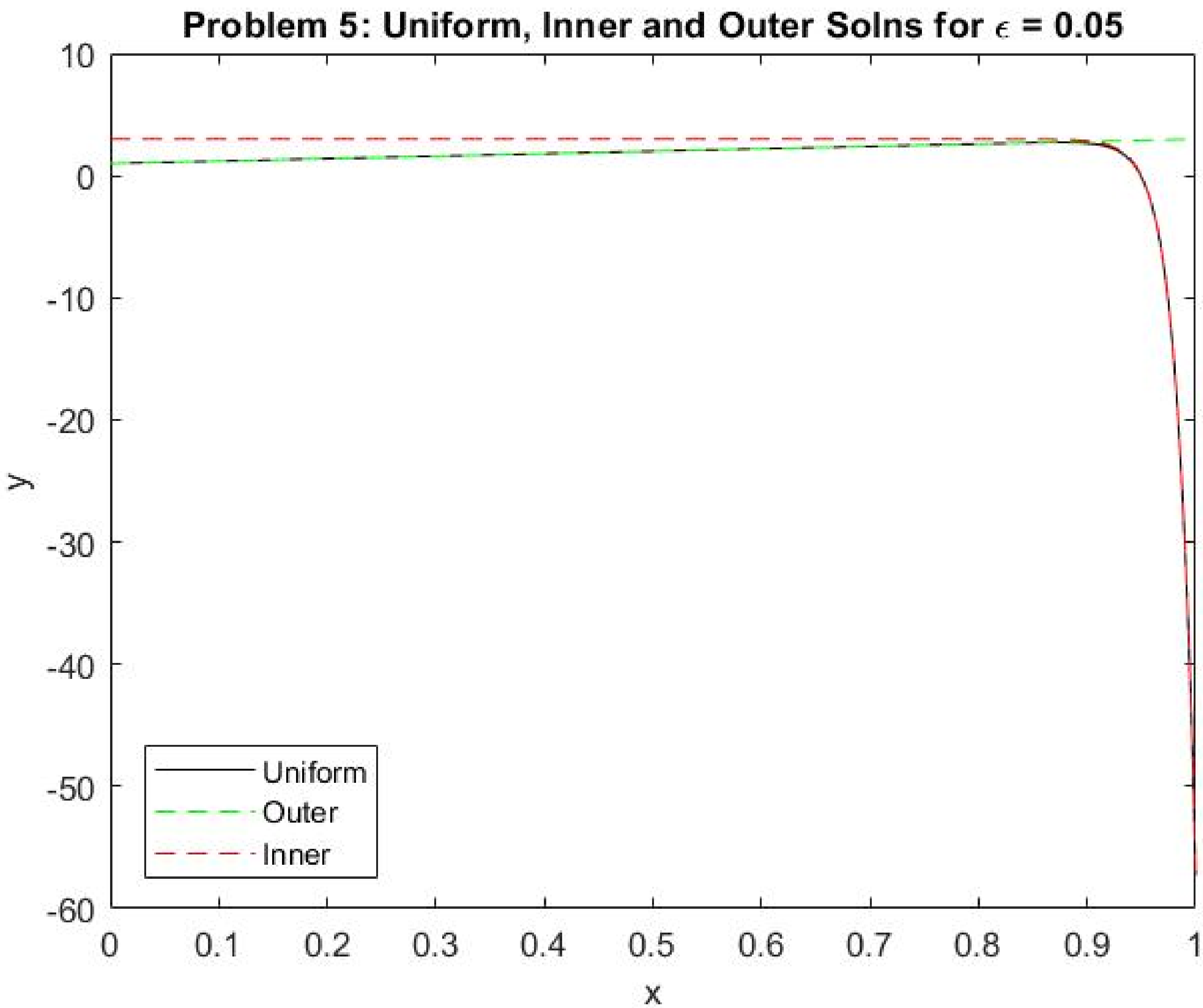
We now take the sum of the inner and outer approximations, and subtract the matching condition, to find our uniform approximation:

$$\begin{aligned} u_u(x) &= u_o(x) + u_i(x) - 3 \\ &= 2x + 1 - 3e^{\frac{-3}{\epsilon}(1-x)+3} \end{aligned}$$

Part (b)

Note that, for the graphs below, the inner and outer approximations should be swapped.


Problem 5: Uniform, Inner and Outer Solns for ϵ = 0.1
10
0
-10
-20
-30
-40
-50
-60
y
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x
Uniform
Outer
Inner


Problem 5: Uniform, Inner and Outer Solns for ϵ = 0.05
10
0
-10
-20
-30
-40
-50
-60
y
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x
Uniform
Outer
Inner

Problem 6

Use singular perturbation methods to obtain a uniform approximation to the solution of the BVP:

$$\epsilon y'' + (t+1)^2 y' = 1, \;\; y(0) = 1, \;\; \epsilon y(0) = 0, \;\; 0 < \epsilon << 1 \tag{25}$$

State clearly both the inner and outer solutions that you derived.

Part (a)

We can find our outer approximation by setting $\epsilon = 0$ so that (25) becomes

$$\begin{aligned}
&(t+1)^2 y' = 1 \\
\implies &y_o(t) = -\frac{1}{t+1} + C
\end{aligned}$$

Since we don't have a nice condition for our outer solution, we leave it in its general form for now.

For the inner solution, we have to consider changes in the boundary layer by making a change of variables

$$\xi = \frac{t}{\delta(\epsilon)} \quad \text{and} \quad Y(\xi) = y(\delta(\epsilon)\xi)$$

So that (25) becomes

$$\begin{aligned}
\frac{\epsilon}{\delta(\epsilon)^2} Y''(\xi) + (\xi\delta(\epsilon) + 1)^2 \frac{Y'(\xi)}{\delta(\epsilon)} &= 1 \\
\frac{\epsilon}{\delta(\epsilon)^2} Y''(\xi) + \xi^2 \delta(\epsilon)^2 \frac{Y'(\xi)}{\delta(\epsilon)} + 2\xi\delta(\epsilon)\frac{Y'(\xi)}{\delta(\epsilon)} + \frac{Y'(\xi)}{\delta(\epsilon)} &= 1 \\
\frac{\epsilon}{\delta(\epsilon)^2} Y''(\xi) + \xi^2 \delta(\epsilon) Y'(\xi) + 2\xi Y'(\xi) + \frac{Y'(\xi)}{\delta(\epsilon)} &= 1
\end{aligned}$$

Choosing $\delta(\epsilon) = \mathcal{O}(\epsilon)$ yields the scaled ODE

$$Y''(\xi) + Y'(\xi) = 0$$

which has the solution

$$Y(\xi) = c_1 + c_2 e^{-\xi)}$$

which corresponds to the inner solution

$$y_i(t) = c_1 + c_2 e^{-t/\epsilon}$$

and applying our initial condition $y(0) = 1$,

$$\begin{aligned}
&y_i(0) = c_1 + c_2 = 1 \\
\implies &c_2 = 1 - c_1
\end{aligned}$$

yielding an updated inner solution

$$y_i(t) = c_1 + (1 - c_1) e^{-t/\epsilon}$$

and by $y'(0) = \frac{1}{\epsilon}$,

$$y_i'(0) = \frac{c_1 - 1}{\epsilon} = \frac{1}{\epsilon}$$
$$\implies c_1 = 2$$

yielding the final inner solution

$$y_i(t) = 2 - e^{-t/\epsilon}$$

Collecting the inner and outer solutions:

$$\begin{cases} y_o(t) = -\frac{1}{t+1} + C & t = \mathcal{O}(1) \\ y_i(t) = 2 - e^{-t/\epsilon} & t = \mathcal{O}(\epsilon) \end{cases}$$

for $x \in [0, 1]$. To study the overlap region, we create the intermediate variable

$$\eta = \frac{t}{\sqrt{\epsilon}}$$

We should have that, as a matching condition:

$$\lim_{\epsilon \to 0+} y_o(\sqrt{\epsilon}\eta) = \lim_{\epsilon \to 0+} y_i(\sqrt{\epsilon}\eta)$$

Computing $\lim_{\epsilon \to 0+} y_o(\sqrt{\epsilon}\eta)$:

$$\lim_{\epsilon \to 0+} y_o(\sqrt{\epsilon}\eta) = -1 + C$$

Then by our matching condition,

$$\begin{aligned} \lim_{\epsilon \to 0+} y_i(\sqrt{\epsilon}\eta) = \lim_{\epsilon \to 0+} 2 - e^{-\sqrt{\epsilon}\eta/\epsilon} &= 2 \\ &= -1 + C \implies C = 3 \end{aligned}$$

Thus our final list of outer and inner solutions becomes

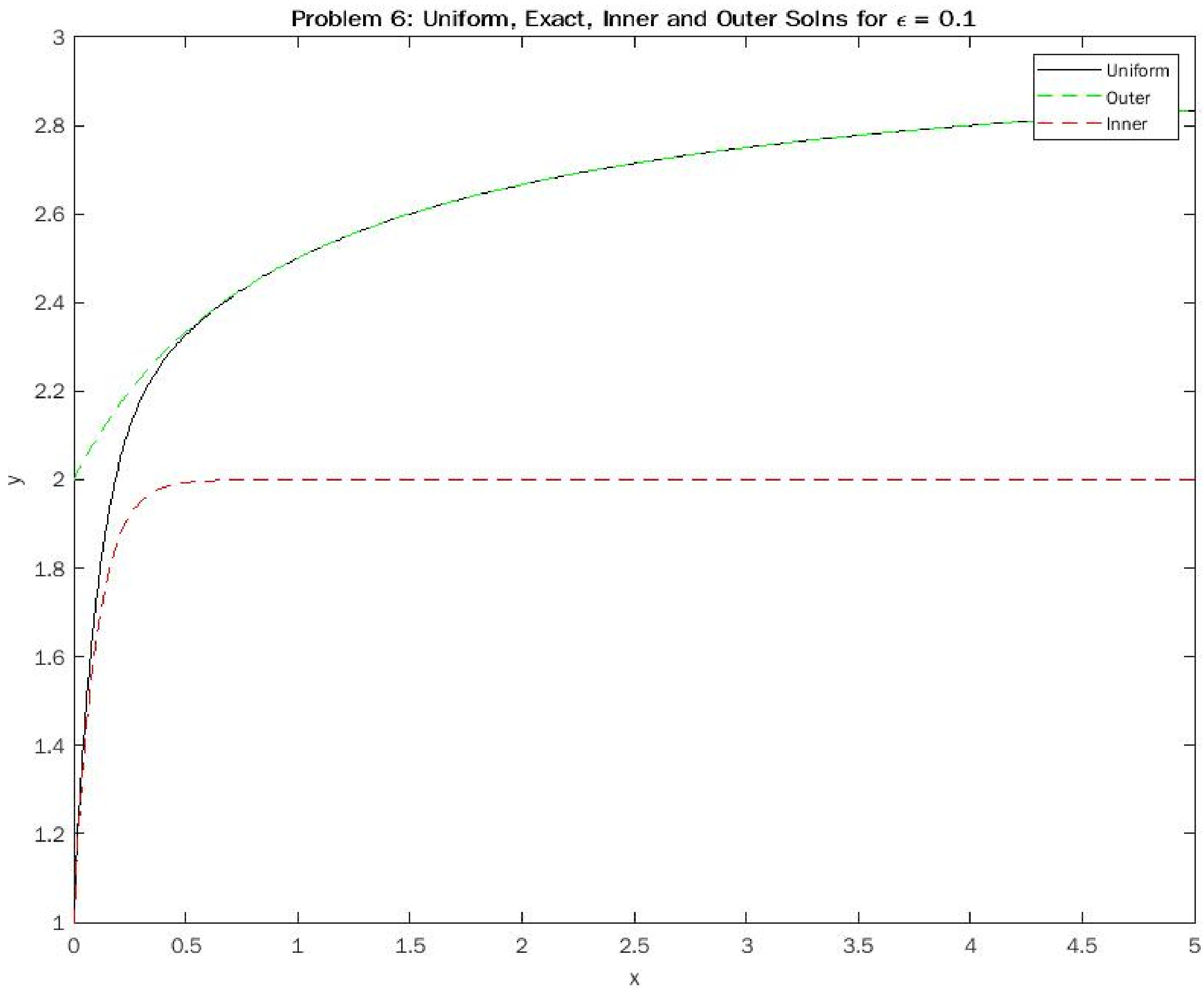
$$\begin{cases} y_o(t) = -\frac{1}{t+1} + 3 & t = \mathcal{O}(1) \\ y_i(t) = 2 - e^{-t/\epsilon} & t = \mathcal{O}(\epsilon) \end{cases}$$

We now take the sum of the inner and outer approximations, and subtract the matching condition, to find our uniform approximation:

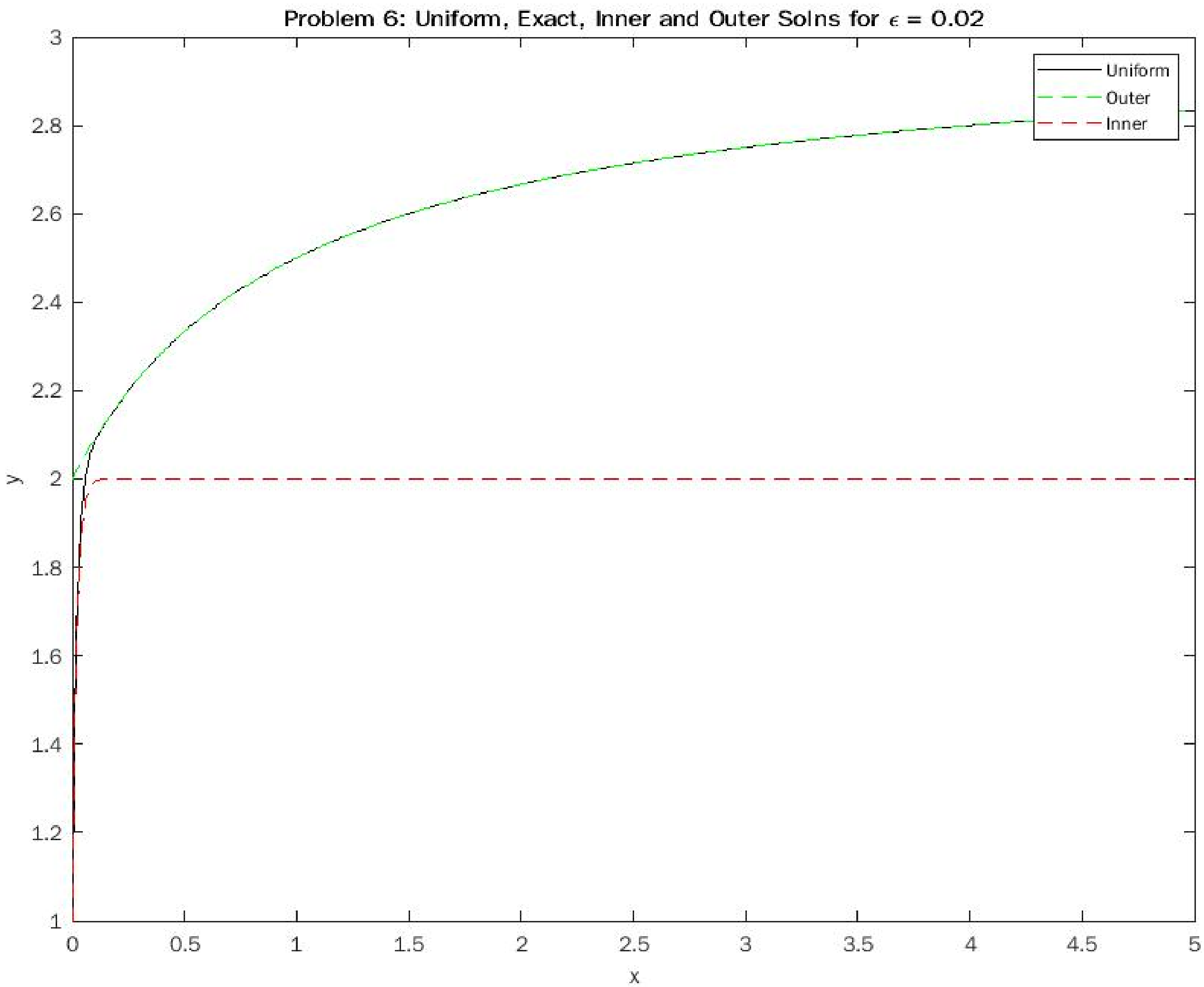
$$\begin{aligned} y_u(x) &= y_o(x) + y_i(x) - 2 \\ &= -\frac{1}{t+1} + 3 + 2 - e^{-t/\epsilon} - 2 \\ &= -\frac{1}{t+1} + 3 - e^{-t/\epsilon} \end{aligned}$$

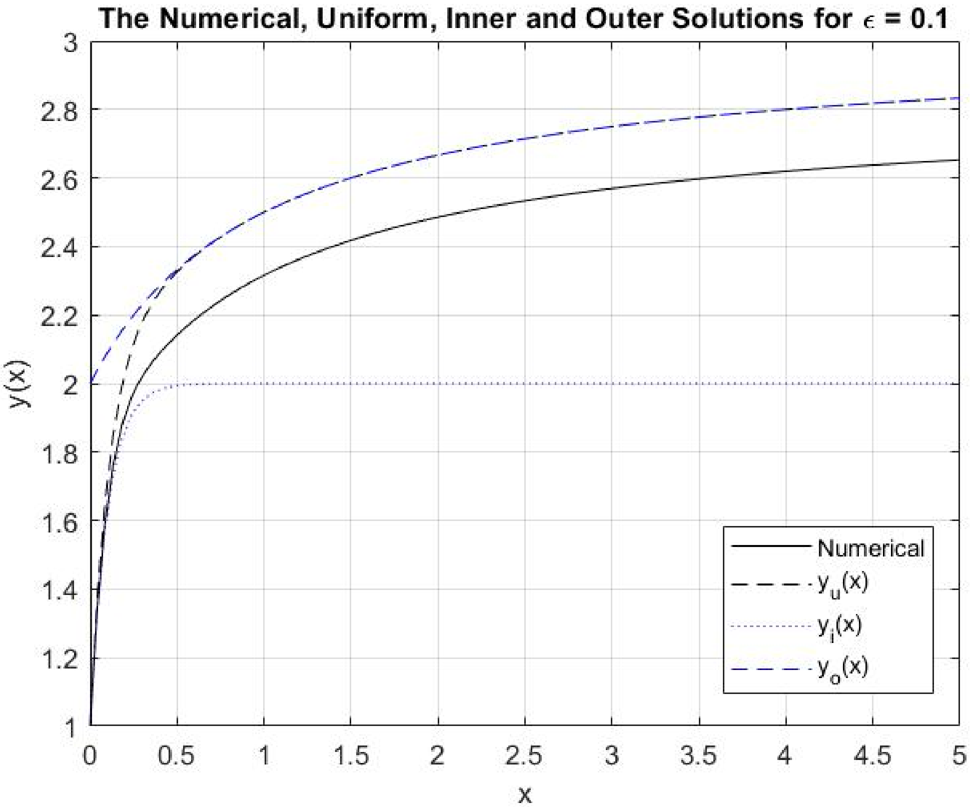
Part (b)

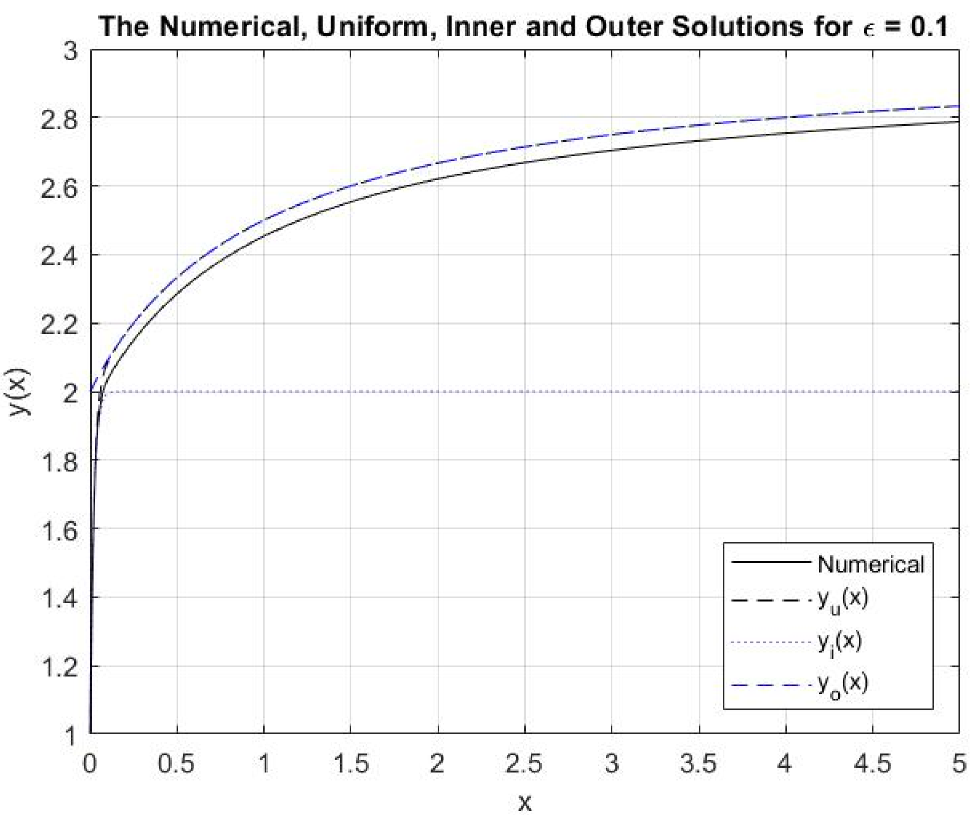
It is interesting that in the graphs below without the numerical solutions, the overlap domain is quite small. In the case that $\epsilon = 0.2$, it essentially disappears! It appears that the uniform approximation trails above the numerical solution, especially in the boundary layer. Of course, this gap closes slightly with lower epsilon.



Problem 6: Uniform, Exact, Inner and Outer Solns for ϵ = 0.1
Uniform
Outer
Inner
y
x
3
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5


Problem 6: Uniform, Exact, Inner and Outer Solns for ϵ = 0.02
Uniform
Outer
Inner
y
x
3
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5


The Numerical, Uniform, Inner and Outer Solutions for ϵ = 0.1
y(x)
x
Numerical
$y_u(x)$
$y_i(x)$
$y_o(x)$


The Numerical, Uniform, Inner and Outer Solutions for ϵ = 0.1
y(x)
x
Numerical
$y_u(x)$
$y_i(x)$
$y_o(x)$

Problem 7

Consider the initial value problem given by the system:

$$x' = ky - x + \epsilon xy, \qquad x(0) = 1 \tag{26}$$

$$\epsilon y' = x - \epsilon xy - y, \qquad y(0) = 0, \quad 0 < \epsilon << 1 \tag{27}$$

Part (a)

Use singular perturbation methods to approximate the solution. State clearly both the inner and outer solutions that you derived for both $x(t)$ and $y(t)$. Discuss why the convergence to the actual solution differs between the cases $k < 1$ and $k > 1$. Does either case provide a uniformly valid approximation for $t > 0$? Explain.

Let $x = x_0 + \epsilon x_1 + \mathcal{O}(\epsilon^2)$, and $y = y_0 + \epsilon y_1 + \mathcal{O}(\epsilon)$. Setting $\epsilon = 0$ yields the zeroth order approximation:

$$\begin{cases} \frac{dx_0'}{dt} &= ky_0 - x_0 \\ 0 &= x_0 - y_0 \end{cases}$$

$$\begin{cases} \frac{dx_0'}{dt} &= ky_0 - x_0 \\ y_0 &= x_0 \end{cases}$$

By substitution, we have that

$$\frac{dx_0'}{dt} = x_0(k-1)$$

which is a separable ODE of the form

$$\begin{aligned} \frac{dx_0'}{x_0} &= (k-1)dt \\ \ln(x_0) &= (k-1)t + c_1 \\ x_0 &= e^{(k-1)t}e^{c_1} \end{aligned}$$

letting $e^{c_1} = C_1$,

$$x_0 = C_1 e^{(k-1)t}$$

By the condition that $x(0) = 1$, we know that $x_0(0) = 1$ thus

$$\begin{aligned} x_0(0) = C_1 e^{(k-1)\cdot 0} &= 1 \\ C_1 &= 1 \end{aligned}$$

thus our zeroth order outer approximations become

$$\begin{aligned} x_0(t) &= e^{(k-1)t} \\ &= y_0(t) \end{aligned}$$

Now we find the inner approximation, first by creating the scaled time variable:

$$\bar{t} = \frac{t}{\epsilon}$$

And defining the substitutions:

$$X(\bar{t}) = x(\epsilon\bar{t})$$
$$Y(\bar{t}) = y(\epsilon\bar{t})$$

To arrive at our scaled system

$$\begin{cases} \frac{dX}{dt} & = \epsilon(kY - X + \epsilon XY) \\ \frac{dY}{dt} & = X - \epsilon XY - Y \end{cases}$$

Finding the zeroth order approximation of this new scaled system, let $X = X_0 + \epsilon X_1 + \mathcal{O}(\epsilon^2)$ and $Y = Y_0 + \epsilon Y_1 + \mathcal{O}(\epsilon^2)$ so that the system becomes (at $\epsilon = 0$)

$$\begin{cases} \frac{dX_0}{dt} & = 0 \\ \frac{dY_0}{dt} & = X_0 - Y_0 \end{cases}$$

By our initial conditions, $X_0(0) = x_0(0) = 1$ and $Y_0(0) = y(0) = 0$ so that the first equation gives

$$X_0(\bar{t}) = c_2 = 1$$

and the second becomes

$$Y_0'(\bar{t}) = 1 - Y_0$$

which is a linear ODE of the form

$$Y_0' + Y_0 = 1$$

which, with the initial condition, has the solution

$$Y_0(\bar{t}) = -e^{-\bar{t}} + 1$$

Collecting our inner approximations:

$$\begin{cases} x_i(t) & = 1 \\ y_i(t) & = -e^{-t/\epsilon} + 1 \end{cases}$$

and the outer approximations:

$$\begin{cases} x_o(t) & = e^{(k-1)t} \\ & = y_o(t) \end{cases}$$

We now verify that the matching conditions for each set of inner and outer solutions hold, the matching condition being for x being:

$$\lim_{t\to 0} x_0(t) = \lim_{\epsilon \to 0} x_i(t)$$

which is clearly satisfied since

$$\lim_{t\to 0} x_0(t) = \lim_{t\to 0} e^{(k-1)t} = 1 \equiv x_i(t)$$

Similarly, the matching condition for y is

$$\lim_{t\to 0} y_0(t) = \lim_{\epsilon \to 0} y_i(t)$$

which is also satisfied since

$$\lim_{t\to 0} y_0(t) = \lim_{t\to 0} e^{(k-1)t} = 1 = \lim_{\epsilon\to 0} y_i(t) = 0 + 1$$

Now, we can find our uniform approximations by summing the inner and outer approximation and subtracting the common limit for each variable:

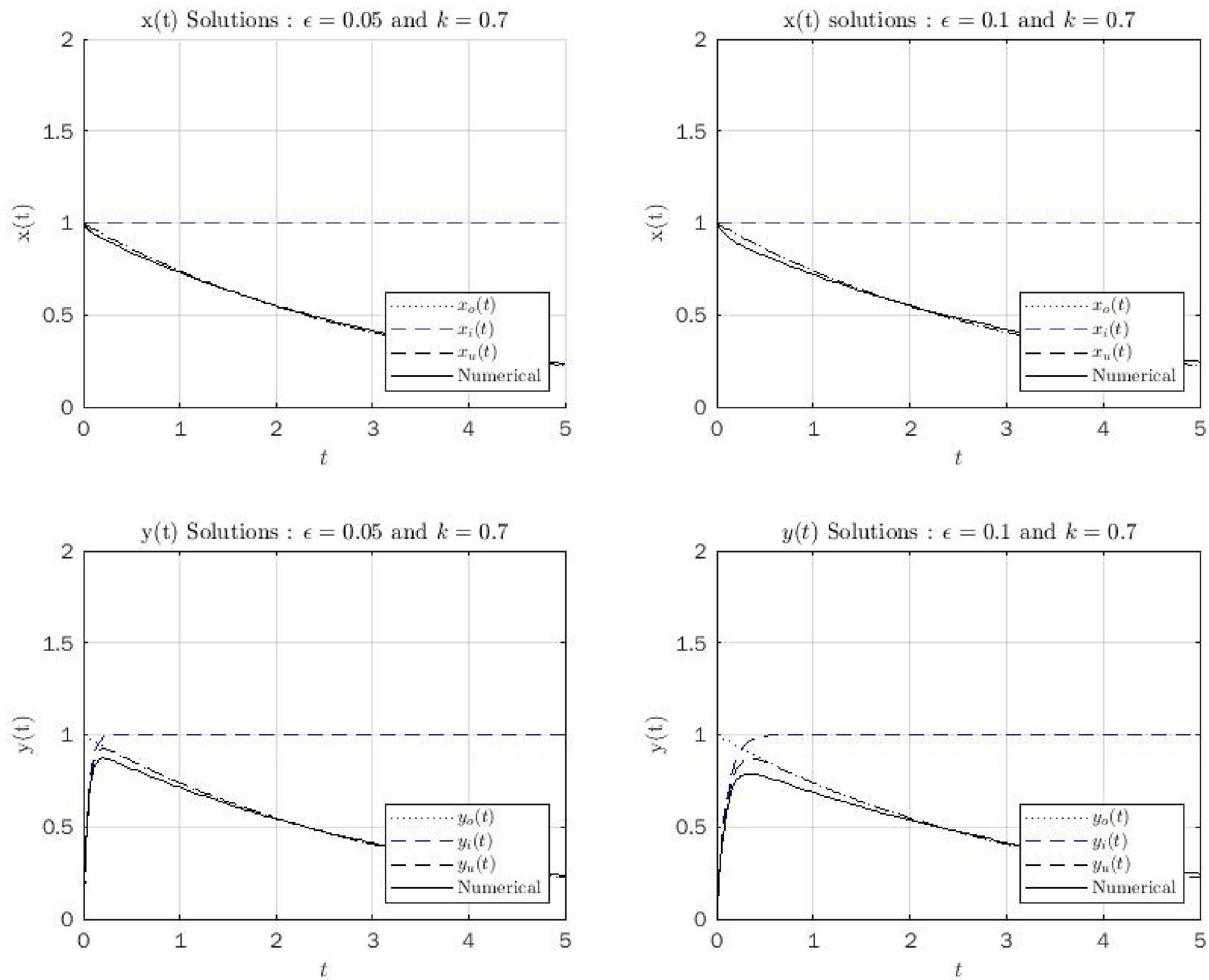
$$\begin{cases} x_u(t) & = e^{(k-1)t} \\ y_u(t) & = -e^{t/\epsilon} + e^{(k-1)t} \end{cases}$$

We now analyze, qualitatively, the following cases of k values:

1. $k > 1 \implies$ the uniform approximations will diverge exponentially from the exact solutions, since the exponent of the exponential term will be positive

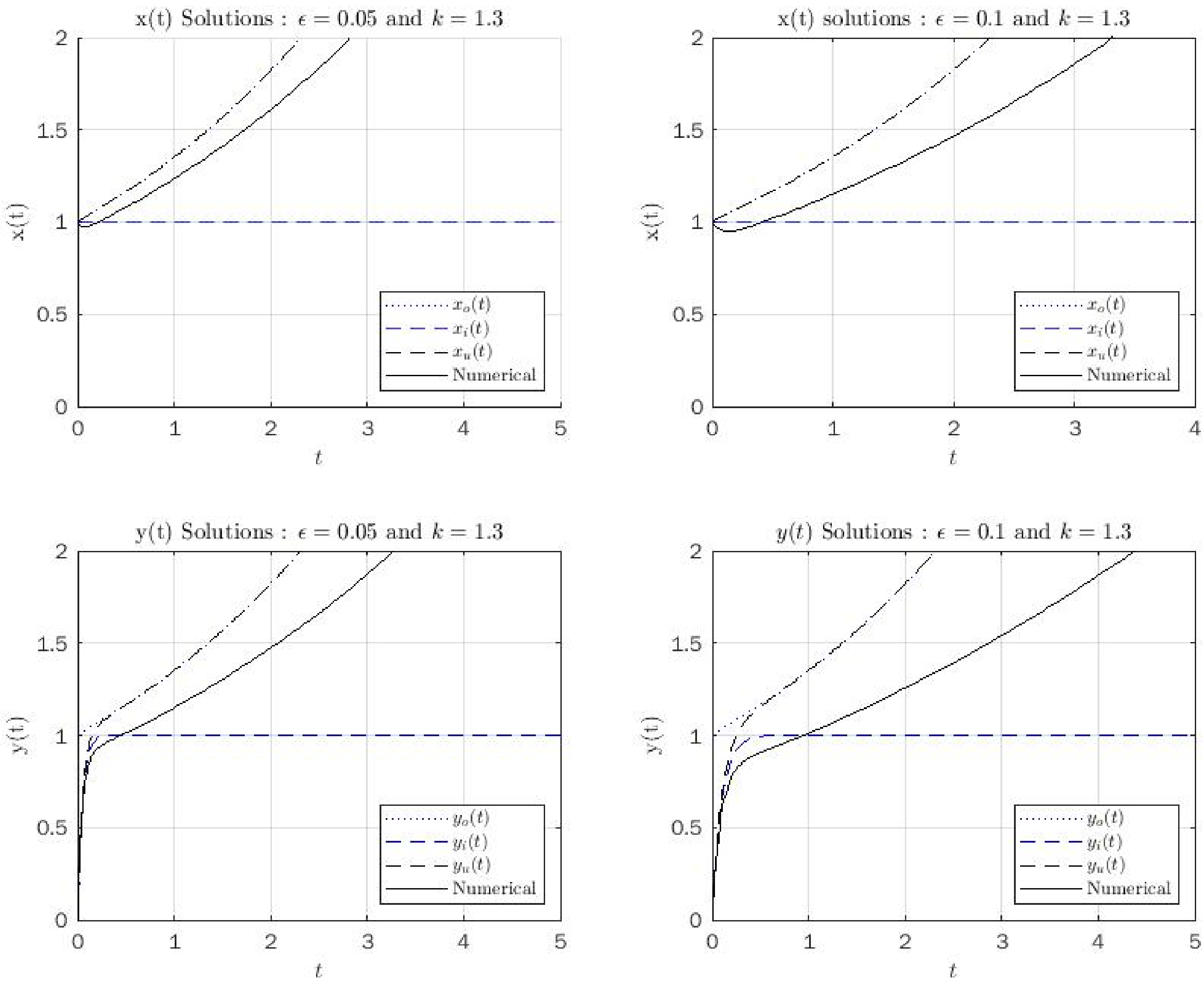
2. $k < 1 \implies$ the uniform approximations will converge; the error decreases exponentially as the exponential terms will have negative exponents. As such, this case should provide a uniformly valid approximation

Part (b)



These results confirm our suspicions from part (a), that the uniform approximations would be close to the numerical solutions for k < 1, which has the expected decay behavior. Of course, smaller epsilon makes for a more accurate approximation.

Part (c)



Clearly, k > 1 does not allow for quite as accurate approximations when compared to the results from part **(b)**, for k < 1. In this case, it seems that the uniform solutions continue to diverge from the numerical solutions as t increases.