Literature Review: Chaos in Interacting Communities

Original paper by: M. Ostilli and W. Figueiredo in 2014
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Abstract

We provide a review of a paper that applies network theory to study the dynamics of two interacting communities. Specifically, Ostilli and Figueiredo develop an Ising model so that the states of agents can be defined as Ising variables whose transitions are defined stochastically on discrete-time intervals. Ostilli and Figueiredo focus on the mean-field limit case to reduce the equations and study the macro-dynamics. We summarize their findings as the inter- and intra-couplings are varied.

1 Introduction

In 2014, Ostilli and Figueiredo published a formulation of their model under which the socio-economic interactions between two communities could be studied (Ostilli and Figueiredo [2014]). The authors note that in general, the study of chaos is done in the context of mechanical or physical systems. Nonetheless, their aim is to show, with a minimal model, that the interactions between two communities can lead to chaos given the right assumptions. One of the more important assumptions is the discrete-time nature of the dynamics. This is not an unreasonable assumption, as people generally make decisions at discrete and random times based on their interactions with each other and their environment. Although they initially develop a microscopic model, Ostilli and Figueiredo focus on the mean-field limit case in order to reduce the equations and observe the macroscopic dynamics, from which they draw their conclusions about the types of dynamics that can arise from the non-linearities supplied by the presence of two interacting communities.

2 Model Formulation

Ostilli and Figueiredo set the context for their model formulation by making the point that regarding complex networks theory, it is the presence of a community structure, and the interactions between the agents of those communities, that defines the network. Thus to fully formulate the model, they must account for the states of any agent as a function of the neighboring agents, as well as those agents in the other community.

Ostilli and Figueiredo begin by defining two communities of agents $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$ whose sizes are $N^{(1)}$ and $N^{(2)}$ respectively. As is typical when formulating an Ising model, the authors consider the simple case in which each agent can only be in two possible states. In the context of this paper, those states are “friendly” and “unfriendly”. If σ_i is the state of agent i, then $\sigma_i = +1$ and $\sigma_i = -1$ represent each of these states respectively, with $i \in \mathcal{N}^{(1)} \bigcup \mathcal{N}^{(2)}$. According to these signs, the term $\sigma_i \sum' \sigma_j$ is minimized or maximized, where $\sum'$ is defined to run over the set of neighbors of agent i.

To quantify the interactions within and between the two communities, Ostilli and Figueiredo define the 2 x 2 matrix J, where $J^{(1,1)}$ and $J^{(2,2)}$ are the *intra*-couplings, and $J^{(1,2)} = J^{(2,1)}$ are the *inter*-couplings.

For the purposes of generalization, the authors define $\Gamma^{(l,m)}$ as the set of coupled spins (i, j) within or between the communities. Finally β is a global factor that scales all of these couplings.

It is worth noting that an Ising model is conventionally used in the context of a mechanical system. As such, Ostilli and Figueiredo have technically defined a general Ising model on communities with the above definitions M. and J.F. [2009]. As was alluded to previously, though, Ostilli and Figueiredo require that the time dynamics are discrete in nature, and thus will have little to share with the classical results of statistical mechanics. This is due in part to the fact that in the classical realm of physics, time is continuous, whereas humans act on a discrete time scale. In fact, many of the results that are shared in this paper hinge on the realization of three conditions, which are listed here as they define them:

1. the presence of unfriendly couplings
2. the discrete-time nature of the dynamics
3. the existence of at least two interacting communities that rearrange their configurations at alternate, and of course discrete, times

Interestingly, condition (3) produces non-linearities, which leads to chaos once the first two conditions are realized. Here, Ostilli and Figueiredo make the point that for a 1-community model, explicit non-linear reactions need to be imposed in order for chaos to arise. F. and R. [2013] shows this to be the case, where each agent can be in a state of friendliness and unfriendliness simultaneously, depending on its neighbors. The authors aim to show that, for a 2-community system, chaos arises much more naturally.

Let $t \in \mathbb{N}$ be the discrete time at which agents within the communities interact with one another or change states. In order to define a periodic partition on $\mathbb{N}$, Ostilli and Figueiredo let $\mathbb{E} \equiv \{\text{set of even numbers}\}$ and $\mathbb{O} \equiv \{\text{set of odd numbers}\}$, so that $\mathbb{E} \cup \mathbb{O} = \mathbb{N}$, $\mathbb{E} \cap \mathbb{O} = \varnothing$, and $\mathbb{E}$ and $\mathbb{O}$ both have infinite cardinality. They then define the *local transition rate probabilities*, i.e. the probability for any agent with state σ_i to transition to state σ_i', as

$$\omega(\sigma_i \to \sigma_i'; t) = \begin{cases} \delta_{\sigma_i,\sigma_i'},\ i \in \mathcal{N}^{(1)} & t \in \mathbb{O} \\ \frac{1+\sigma_i' \tanh(\beta J^{(1,1)} \sum_{j:(i,j)\in\Gamma^{(1,1)}} \sigma_j + \beta J^{(1,2)} \sum_{j:(i,j)\in\Gamma^{(1,2)}} \sigma_j)}{2}, & \text{for } i \in \mathcal{N}^{(1)}, t \in \mathbb{E} \\ \delta_{\sigma_i,\sigma_i'},\ i \in \mathcal{N}^{(2)} & t \in \mathbb{E} \\ \frac{1+\sigma_i' \tanh(\beta J^{(2,2)} \sum_{j:(i,j)\in\Gamma^{(2,2)}} \sigma_j + \beta J^{(2,1)} \sum_{j:(i,j)\in\Gamma^{(2,1)}} \sigma_j)}{2}, & \text{for } i \in \mathcal{N}^{(2)}, t \in \mathbb{O} \end{cases} \tag{1}$$

We assume that $\delta_{\sigma_i,\sigma_i'}$ is some intrinsic probability for an agent to change states that does not depend on its neighbors or agents from the other community. Notice that the local transition rate probabilities are defined probabilistically and stochastically, as the value depends on which partition of time the agent finds itself in. Further, this choice encourages agents to be in the same state as the majority of its intra- and inter-neighbors so as to minimize the "energy" of the system, and the presence of the $\tanh(\cdot)$ function allows for the rates to be non-negative and normalized at any time $\sum_{\sigma_i'} \omega(\sigma \to \sigma'; t) = \frac{1}{\text{Time Unit}}$. Lastly, when couplings are positive, (1) ensures that the system "satisfies the principle of detailed balance and the principle of maximal entropy for any quadratic interactions", and that the Boltzmann equilibrium can be found for the case of positive couplings. For these reasons, Ostilli and Figueiredo argue that equations (1) are justified.

Ostilli and Figueiredo now formalize the discrete-time probabilistic dynamics by defining $N = N^{(1)} + N^{(2)}$ to be the total number of agents in both communities. Then, $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_N)$ such that $\boldsymbol{\sigma}$ is the list of states for each agent, and the associated probability vector $p(\boldsymbol{\sigma}; t) \equiv$ the probability that every agent is in a particular state as defined by $\boldsymbol{\sigma}$ at time $t \in \mathbb{N}$. They now introduce the *master equation*:

$$\frac{p(\boldsymbol{\sigma};t+1)-p(\boldsymbol{\sigma},t)}{\alpha}=-\sum_{\boldsymbol{\sigma}'}p(\boldsymbol{\sigma};t)W(\boldsymbol{\sigma}\to\boldsymbol{\sigma}')+\sum_{\boldsymbol{\sigma}'}p(\boldsymbol{\sigma}';t)W(\boldsymbol{\sigma}'\to\boldsymbol{\sigma}) \tag{2}$$

where the *global* transition rates are:

$$W(\boldsymbol{\sigma}\to\boldsymbol{\sigma}')=\prod_{i\in\mathcal{N}^{(1)}\bigcup\mathcal{N}^{(2)}}w(\sigma_i\to\sigma_i') \tag{3}$$

and $0<\frac{\alpha}{2}\leq 2$ is the rate at which a free spin, such that $J^{(l,m)}=0$, makes a transition to either state. It can be shown that $\alpha\leq 1$ is a necessary restriction to make so that $p(\boldsymbol{\sigma};\boldsymbol{t})\in[0,1]$.

3 The Mean-Field Limit

At this point the authors have defined the macroscopic dynamics with equations (1) - (3), and they use these to derive what they call the *order parameters* defined by equations (4) and (5), which they use to study the macroscopic dynamics:

$$m^{(1)}(t)=\sum_{\boldsymbol{\sigma}}p(\boldsymbol{\sigma},t)\frac{1}{N}\sum_{i\in\mathcal{N}^{(1)}}\sigma_i \tag{4}$$

$$m^{(2)}(t)=\sum_{\boldsymbol{\sigma}}p(\boldsymbol{\sigma},t)\frac{1}{N}\sum_{i\in\mathcal{N}^{(2)}}\sigma_i \tag{5}$$

As often is the case regarding systems described by an Ising model, it would be difficult to study the details in finite-dimensions. As such, Ostilli and Figueiredo focus on the *mean-field limit* defined by the settings $|\Gamma^{(1,1)}|=\left[N^{(1)},\ 2\right],\ |\Gamma^{(2,2)}|=\left[N^{(2)}\ 2\right],$ and $|\Gamma^{(1,2)}|=N^{(1)}N^{(2)}$ They also rescale the couplings according to the sizes of the communities: $J^{(1,1)}\to J^{(1,1)}/N^{(1)}$, $J^{(2,2)}\to J^{(2,2)}/N^{(2)}$, and $J^{(1,2)}\to J^{(1,2)}(N^{(1)}+N^{(2)})/(2N^{(1)}N^{(2)})$. They then parametrize the sizes of the communities as such:

$$N^{(1)}=N\rho^{(1)},\ N^{(2)}=N\rho^{(2)},\ \rho^{(1)}+\rho^{(2)}=1 \tag{6}$$

Now, letting $N\to\infty$, equations (3) can be rewritten

$$\omega(\sigma_i\to\sigma_i';t)=\begin{cases}\delta_{\sigma_i,\sigma_i'}, & i\in\mathcal{N}^{(1)},t\in\mathbb{O}\\ \frac{1+\sigma_i'\tanh(\beta\tilde{J}^{(1,1)}m^{(1)}(t)+\beta\tilde{J}^{(1,2)}m^{(2)}(t)}{2}, & i\in\mathcal{N}^{(1)},t\in\mathbb{E}\\ \delta_{\sigma_i,\sigma_i'}, & i\in\mathcal{N}^{(2)},t\in\mathbb{E}\\ \frac{1+\sigma_i'\tanh(\beta\tilde{J}^{(2,2)}m^{(2)}(t)+\beta\tilde{J}^{(2,1)}m^{(1)}(t)}{2}, & i\in\mathcal{N}^{(2)},t\in\mathbb{O}\end{cases} \tag{7}$$

Where $\tilde{\boldsymbol{J}}$ is defined

$$\tilde{\boldsymbol{J}}=\begin{bmatrix}J^{(1,1)} & \frac{J^{(1,2)}}{2\rho^{(1)}}\\ \frac{J^{(2,1)}}{2\rho^{(2)}} & J^{(2,2)}\end{bmatrix}$$

By plugging equations (8) into equations (2), the master equation, they introduce the **deterministic evolution equations** for the order parameters:

$$\frac{m^{(1)}(t+1) - m^{(1)}(t)}{\alpha} = \begin{cases} 0, & t \in \mathbb{O} \\ \tanh\left(\beta \tilde{J}^{(1,1)} m^{(1)}(t) + \beta \tilde{J}^{(1,2)} m^{(2)}(t)\right) - m^{(1)}(t), & t \in \mathbb{E} \end{cases} \tag{8}$$

$$\frac{m^{(2)}(t+1) - m^{(2)}(t)}{\alpha} = \begin{cases} 0, & t \in \mathbb{E} \\ \tanh\left(\beta \tilde{J}^{(2,2)} m^{(2)}(t) + \beta \tilde{J}^{(2,1)} m^{(1)}(t)\right) - m^{(2)}(t), & t \in \mathbb{O} \end{cases} \tag{9}$$

And these are the equations that Ostilli and Figueiredo use to approximate the dynamics between two communities by approaching infinite dimensionality via the mean-field limit.

4 Friendly Couplings

Ostilli and Figueiredo proceed to introduce the steady state solutions to equations (9) and (10); i.e, when time t tends to infinity in both of these equations. These steady state solutions are described by *friendly coupling*, which implies that the coupling values, $\tilde{J}^{(l,m)}$, are positive, thus forcing little difference between the present discrete-time dynamics and the continuous Glauber dynamics.

$$\begin{cases} m^{(1)} = \tanh\left(\beta \tilde{J}^{(1,1)} m^{(1)} + \beta \tilde{J}^{(1,2)} m^{(2)}\right) \\ m^{(2)} = \tanh\left(\beta \tilde{J}^{(2,1)} m^{(1)} + \beta \tilde{J}^{(2,2)} m^{(2)}\right) \end{cases} \tag{10}$$

Note that the steady state solutions (above) are no longer dependent on time, and represent the two interacting communities at an equilibrium, $(m^{(1)}, m^{(2)}) = (0, 0)$. This result is included to supplement the following section regarding bifurcations and chaotic trajectories that can occur near this fixed point depenind on initial conditions and paramter values.

5 Competitive Interactions; Emergence of Chaos

Depending on the coupling values of the $\tilde{J}$ matrix in equation (7), the system can reach many different regimes with fixed points, limit cycles, or strange attractors. Ostilli and Figueiredo plot $m^{(1)}(t_l)$ of (8) and (9) , known as the *evolution* equations, as a function of β, where they let the system evolve towards high enough values of time to remove temporary transients.

In Figure 1 below, one can see two examples of bifurcation diagrams and the associated trajectory diagrams for different interaction cases between the communities, and whose bifurcation points coincide with the β values such that $\det\left(\mathbf{1} - \beta \tilde{\boldsymbol{J}}\right) = 0$. Numerical analysis done by Ostilli and Figueiredo showcases that the system can give rise to two types of bifurcations, period-doubling and symmetry breaking. The first scenario takes place when each community is anti-cooperative but the two are mutually cooperative (first row of diagrams). The second scenario takes place when there is anti-cooperation in one community, and small or zero cooperation in the other (bottom row of diagrams). This notion of *cooperation* is simply describing the signs and values of the matrix entries, $\tilde{J}^{(l,m)}$. Specifically, for the first row, they use $\tilde{J}^{(1,1)} = \tilde{J^{(2,2)}} = -1$, and $\tilde{J}^{(1,2)} = \tilde{J}^{(2,1)} = 0.8$. For the second, $\tilde{J}^{(1,1)} = -1$, $\tilde{J^{(2,2)}} = 0.01$, and $\tilde{J}^{(1,2)} = \tilde{J}^{(2,1)} = 1$.

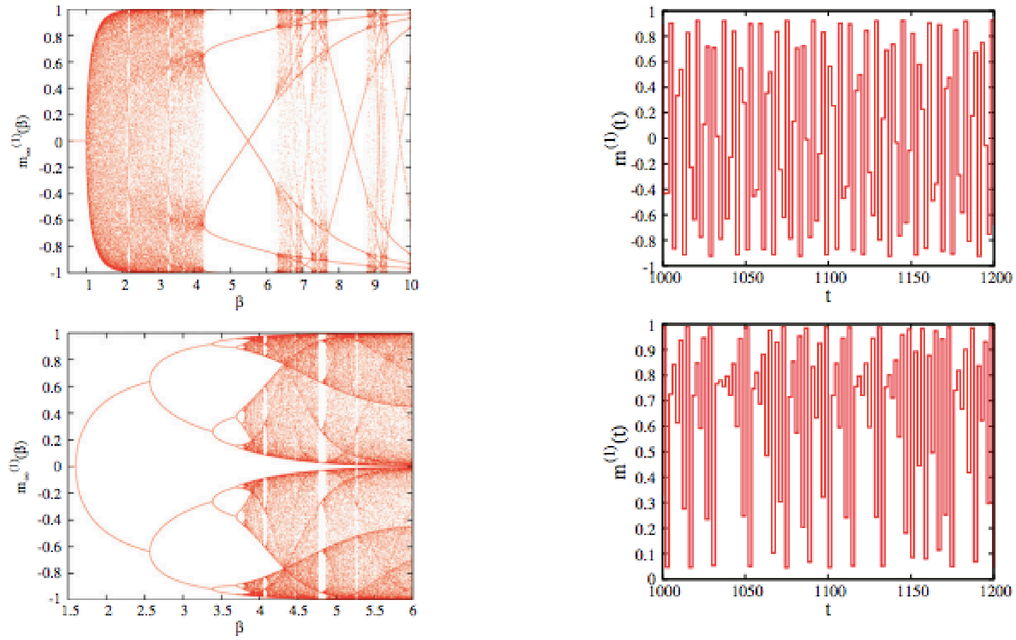

Figure 1: Bifurcation diagrams (left) and a corresponding trajectory (right)

The first scenario results in Lyapunov exponents less than or equal to zero, which implies no chaos in the system. Nonetheless, the trajectory of the diagram, taken at $\beta = 1.5$, are aperiodic and therefore highly sensitive to small changes in the parameter beta, which is chosen to be defined as *marginally chaotic*.

The second scenario had a bifurcation diagram characterized by few windows of stability along with bifurcation cascades, whose structure is similar to that of the familiar logistic map. In this case, the maximal lyapunov exponent is positive which implies chaotic trajectories, which can be seen by the drastically irregular aperiodic oscillations in the trajectory diagram, which is taken at $\beta = 4.5$.

6 Conclusions

Ostilli and Figueiredo published a concise and informative article describing the dynamics of interacting communities based on different cooperation scenarios described mathematically. As they've shown using a simple model based on network theory, when two interacting communities (and the agents within them) have stochastically defined transition rates on discrete time intervals, aperiodic trajectories as well as chaos can arise. The content of their paper has applications to ecological biology, economics, and social dynamics; for this reason, the studies of Ostilli and Figueiredo can be appreciated and educational for a wide variety of individuals in academia.

References

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