538 Literature Review: Simple Mathematics Models with Very Complicated Dynamics by Robert M. May

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1 Introduction

May begins his article by arguing that first-order difference equations can be very useful in approximating the dynamics of physical systems. Further, in spite of their simplicity, these models can possess non-trivial and unexpected dynamics.

In order to keep this review at a reasonable length, I will focus my attention on a subset of what May states are the aims of his article. In particular, I go into detail on the nature in which fixed points and cycles arise as the parameters of equation (1) are tuned. I will share examples of fields that benefit from the use of different forms of (1) to model real-life systems, and touch on the apparent chaotic behaviour of these equations.

2 Methodology

2.1 "First-Order Difference Equations : Some Examples and Basic Dynamics"

Imagine an ecological system in which the population of the inhabitants at time $t+1$ is determined strictly by the population of the preceding time interval, t. May makes the claim that an appropriate model for such a system can be expressed in the form

$$X_{t+1} = F(X_t) \tag{1}$$

Which is a first-order, *nonlinear difference equation*. Applications of equation (1) are not limited to population, though. May mentions genetics, economics, and epidemiology as examples where (1) can be used to describe the dynamics of various types of systems, such as the spreading of rumours where X represents the number of people who have who have heard the rumour. In spite of this, I, like May, will generally refer to X as population.

May goes on to describe basic characteristics of (1) such as its monotonic behavior; $F(X)$ will often be strictly increasing for $0 < X < A$ and then decreasing for $X > A$. Hence the nonlinearity of F. I would point out that while May suggests here that $F(X)$ attains an absolute maximum value at $X = A$, it may well be the case that $F(X)$ has several local extrema as population varies. The details of the behavior of $F(X)$, such as the number of extrema and steepness of its tangent lines, are dependent on the parameters present in the definition of $F(X)$. These parameters are defined based on the conditions or properties of the system that $F(X)$ is modeling.

May provides the *logistic difference equation* and introduces two parameters a and b:

$$N_{t+1} = N_t(a - bN_t) \tag{2}$$

May points out that population modeled by (2) would grow exponentially if $b = 0$ and $a > 1$. Otherwise, we have a quadratic equation that produces a negatively oriented parabolic curve. As mentioned, the magnitudes of the slopes of the tangent lines of the curve is directly affected by the parameter a. May then defines $\frac{bN}{A} = X$ to rewrite equation (2) so that we have what May calls the simplest form of (1):

$$X_{t+1} = aX_t(1 - X_t) \tag{3}$$

This form is mainly useful for mathematical analysis as it requires the normalization of X so that $0 < X < 1$ for all t. A simple calculation shows that equation (3) reaches a maximum value of $\frac{a}{4}$ at $X = \frac{1}{2}$. May also points out that solutions trend towards $X = 0$ if $a < 1$ thus for the purposes of our discussions, we restrict a, such that $1 < a < 4$, in order to avoid trivial dynamics. May introduces another example with generally similar behaviour to equation (3), as it exhibits exponential growth for low values of X, and negative growth for high values:

$$X_{t+1} = X_t e^{r(1-X_t)} \tag{4}$$

In this definition, the parameter r is analogous to a of equation (3) as it determines the steepness of the curve. May points out that this model originates in the biological sciences, and may be applied to a large, single species population, being affected by an epidemic disease. Of course, there are many other examples of what May calls "single-humped functions $F(X)$ for equation (1) besides those of (3) and (4).

2.2 "Dynamic Properties of (1)" : Equilibria and Stability

As mentioned earlier, $F(X)$ of (1) is an iterative function. Thus, values of $F(X)$ will remain unchanged upon iteration in the case that

$$X^* = F(X^*) \tag{5}$$

Such values of X of equation (1) are what May calls *fixed points* or equilibrium values. We may find these fixed points graphically by plotting the curve of $F(X)$ on the X_{t+1} vs. X_t plane, and locating points where the curve intersects the 45° line represented by $X_{t+1} = X_t$. These are values such that the population of a species at time $t + 1$ remains unchanged from the population at time t. May points out that, for equations (3) and (4), two such points exist. One, of course, is the trivial solution of zero population. For equation (3), May provides the non-trivial solution to be $X^* = 1 - \frac{1}{a}$.

We then can discuss when populations trend towards or away from their equilibrium values, which is to discuss the stability of X^*. May cites the work of others to take for granted that stability of fixed points is entirely dependent on the slope of the tangent line of $F(X)$ at X^*. May defines this slope in the following manner:

$$\lambda^{(1)}(X^*) = [\frac{dF}{dX}]_{x=x^*} \tag{6}$$

For $-1 < \lambda < 1$, the fixed point $X^* = F(X^*)$ will attract values of X that lie in some ϵ-neighborhood of X^*. That is, iterations of $F(X)$ will converge to $F(X^*)$ if X is near X^*; in this case the fixed point is *locally stable.* For equation (3), we can define $\lambda^{(1)} = 2 - a$ by using the fact that $X^* = 1 - \frac{1}{a}$. Thus, as May explains, a fixed point is stable if and only if $1 < a < 3$, and as it varies beyone this interval, the curve $F(X)$ steepens beyond -45°.

Clearly, a population corresponding to a fixed point would recur every generation so that $F(X_t) = X_{t+1} = X_t$. It is possible, though, for certain populations to recur every second, third, or k^{th} generation. The values of these populations correspond to period-2, period-3, and period-k fixed points respectively. Similar to graphically identifying fixed points, we can find a period-2 fixed point by relating X_{t+2} to X_t, and identifying values of X_t where the curve $F^{(2)}(X)$ intersects the 45° line. Analytically, we want to find X_2^* (representing a period-2 fixed point) such that

$$X_2^* = F^{(2)}(X_2^*) = F[F(X_2^*)] \tag{7}$$

May makes the note that a period-1 fixed point is a "degenerate case of a period-2 solution", since of course X^* satisfies equation (7). Now, applying the chain rule, we have that the slope of $F^{(2)}(X)$ at X^* can be written

$$\lambda^{(2)}(X^*) = [\lambda^{(1)}(X^*)]^2 \tag{8}$$

Now, suppose $-1 < \lambda^{(1)} < 0$ so that X^* is stable and the slope of the tangent line at $F(X^*)$ is between $-45°$ and $0°$. It would follow that $0 < \lambda^{(2)} < 1$ from equation (8). However, consider if $\lambda^{(1)} < -1$ (i.e. the slope of $F(X)$ at X^* steepens beyond the $-45°$ line rendering X^* an unstable fixed point). This would imply $\lambda^{(2)} > 1$. It is at this point, when the fixed point becomes unstable and the slope of $F^{(2)}(X)$ at X^* steepens beyond the $+45°$ line, when the system births two period-2 fixed points as defined by equation (7). Further, May argues that at this moment, the system alternates in what is initially a stable cycle between these two period-2 fixed points.

As a note to the reader, there is a small error at this point in the article: the caption for Figure 2 is meant for Figure 3, and that of Fig 3 is meant for Fig 2. The figures will not be included in this review, but I thought it relevant to mention.

May goes on to explain that we can discuss the stability of period-k orbits in the same way we outlined the stability of fixed points; the stability of a period-k cycle is dependent on the slope of the curve $F^{(k)}$ at the k points. In fact, it turns out that the slope of $F^{(k)}$ is the same at each of these points. May does not show the details of this fact, and instead cites a reference [REF 1.20]. Returning to the example of $k = 2$, hopefully it is clear to the reader that the slope of the curve $F^{(2)}(X)$ at each of the period-2 fixed points is *not* equivalent to values of $\lambda^{(2)}(X^*)$ from equation (8).

So what does it mean for the period-2 cycle to be "initially stable"? Recall that the period-2 cycle is birthed at the moment (i.e. value of the parameter a, according to May) $\lambda^{(2)}$ becomes greater than $+1$, corresponding to when $\lambda^{(1)}$ becomes less than -1. At the same time, $\lambda = +1$, and then as a varies, "decreases through zero towards $\lambda = -1$ as the hump in $F(X)$ continues to steepen" even further beyond the $-45°$ line (where λ represents the slope of the curve of $F^{(2)}(X)$ at the period-2 fixed points). During this interval of values, the period-2 orbits are stable. However, after the threshold of $= -1$ is crossed, the points become unstable, and in the same way as the $k = 1$ case, bifurcate and produce an initially stable period-4 cycle. The pattern predictably continues as cycles of period 8, 16, 32, ... 2^n are created.

This "hierarchy of bifurcating stable cycles" with periods 2^n, as May calls them, are infinite in number as n approaches infinity. What about the stability of these cycles? May explains that for *most* variations of equation (1), the interval of parameter values for which the cycles are stable trends towards zero as n trends to infinity. This results in a "critical parameter value" where, beyond it, an infinite number of cycles with different periods are birthed–corresponding do an infinite number of fixed points with varying periodicities. Some of these cycles have lengthy period of transient behavior during which the alternation of values does not show upon iteration. Along with this are an uncountably infinite number of chaotic trajectories, according to May.

So far, we have limited our discussion to cycles of even period, since these are the first to arise under May's assumptions. It is even further beyond the aforementioned critical parameter value that the first odd period cycle appears, and then further beyond that is when the first period-3 cycle appears, which May claims is at $a = 3.8284$ for equation (3). This is then followed by cycles with every integer period along with asymptotically aperiodic trajectories, which leads may to his discussion on chaos.

2.3 "Fine Structure of the Chaotic Regime"

To begin, we extend our discussion to the dynamics and origins of period-3 cycles. Similar to before, we examine when the following equation is satisfied:

$$X_{t+3} = F^{(3)}(X_t) \tag{9}$$

Which yields period-3 fixed points. Assuming large enough parameter values, we will see that the curve $F^{(3)}(X)$ has pronounced enough hills and valleys so that it intersects the 45° line at 6 non-degenerate points, which make up two distinct period-3 cycles. Each of these orbits are birthed when $\lambda^{(3)} = +1$ (being the slope of $F^{(3)}(X)$ at the period-3 points). One of these cycles is never stable as $\lambda^{(3)}$ increases beyond $+1$ as the curves of $F(X)$ steepen. On the other hand, the other orbit begins unstable and transitions to stability as $\lambda^{(3)}$ at those points decreases towards zero. The orbit remains stable until $\lambda^{(3)}$ decreases further to -1, where the cycle bifurcates to create stable cycles of period 6, 12, 24 ... $3 * 2^n$. This process is similar to that outline for $k = 2$.

May then defines the two basic bifurcation processes discussed above that occur for first-order difference equations. One is called a *tangent bifurcation*, which occurs when $^{(k)}$, the slope of the curve $F^{(k)}(X)$ at the period-k points, becomes $+1$, giving rise to one unstable and one initially stable cycle of period k. As this value decreases towards $\lambda^{(k)} = -1$,, the initially stable cycle becomes unstable, giving rise to a new and initially stable cycle of period $2k$, which is called a *pitchfork bifurcation*.

Once the parameter values of $F(X)$ are large enough, $F^{(k)}$ will intersect the 45° line at 2^k points which are called fixed points of period k. As alluded towards previously, this list of fixed points includes degenerate points that are fixed points of a previous iteration, in particular those iterations which are submultiples of k. I include May's table that lists the number of period k points along with their classifications, see Figure 1.

I thought it worthwhile to include this figure since it nicely illustrates the nature in which fixed points occur. One can imagine that, as the parameters increase in value, larger k-values will be unlocked. May discusses, in particular, the fact that "there are $2^6 = 64$ points with period 6 which include two points of period 1, the period 2 harmonic cycle, and the stable and unstable pair of triplets of points with period 3." This adds up to $1 * 2 + 2 * 1 + 2 * 3 = 10$ degenerate points, leaving 54 points whose basic period is 6. Furthermore, note that the listings of the fourth row, corresponding to number of non-degenerate cycles, divides those of the second row, corresponding to total number of non-degenerate periods. This is true since there are k (non-degenerate) period k points in a (non-degenerate) cycle of period k. The third and fourth rows include cycles that arise either by pitchfork or tangent bifurcation. Recall that the pitchfork bifurcations produce stable cycles, and the tangent bifurcations pairs of stable/unstable cycles. Thus the fifth row subtracts cycles that are unstable from the birth of tangent bifurcations. Lastly, the sixth row subtracts cycles produced by pitchfork bifurcations as harmonics of what May calls a "simpler cycle", which corresponds to the number of stable cycles with basic period k.

May goes on to explain that it is possible to determine the order in which cycles of certain periods appear, through labelling tricks or combinatorial theory. We can also find the intervals of parameter values in which certain cycles are stable, although May claims that this is rather difficult. However, parameter values of significance can be identified easily such as the value in which the first period 3 cycle appears: $a = 1 + \sqrt{8}$. We end the review of his theoretical discussions with when a period-k cycle is maximally stable: when $\lambda^{(k)} = 0$. Note that this condition is satisfied if and only if the curve $F(X)$ has a slope of zero at the fixed points of period k.

2.4 "Practical Problems" and "Applications"

As mentioned earlier, the models analyzed in May's article and this review are a simplification of real-life systems. The dynamics of functions $F(X)$ which can be explained analytically and mathematically sometimes contradict one's intuition about system. May explains that this is seen if we consider values of a from equation (3) that lie outside of the interval $1 < a < 3.5700$, noting that such values produce lengthy periods of transient behavior which produce a density plot that resembles the values of a random process. As the windows in which stable cycles rapidly decrease, and we approach the value in which the first odd-cycle appears, this behavior becomes more evident.

May continues by explaining how the long-term behavior of some models depend heavily on their initial conditions, and that even a slight deviation from a condition can result in wildly different results after several iterations. Weather forecasting and fluid turbulence are some examples that exhibit this behavior, where it is very difficult to make predictions over the long-term. May also warns of the shortcomings of equation (1) when when used to fit data in a laboratory setting due to it being a controlled environment.

3 Conclusion

May ends by emphasizing the importance of teaching these concepts in the classroom, as he believes "nonlinear systems are the rule, not the exception, outside the physical sciences". He argues that students of mathematics are ill-equipped by spending too much time studying linear systems, as is the convention, to analyze the behaviours of nonlinear systems that have been discussed throughout his paper and this review.

Table 2 Catalogue of the number of periodic points, and of the various cycles (with periods $k = 1$ up to 12), arising from equation (1) with a single-humped function $F(X)$

k	1	2	3	4	5	6	7	8	9	10	11	12
Possible total number of points with period k	2	4	8	16	32	64	128	256	512	1,024	2,048	4,096
Possible total number of points with non-degenerate period k	2	2	6	12	30	54	126	240	504	990	2,046	4,020
Total number of cycles of period k, including those which are degenerate and/or harmonics and/or never locally stable	2	3	4	6	8	14	20	36	60	108	188	352
Total number of non-degenerate cycles (including harmonics and unstable cycles)	2	1	2	3	6	9	18	30	56	99	186	335
Total number of non-degenerate, stable cycles (including harmonics)	1	1	1	2	3	5	9	16	28	51	93	170
Total number of non-degenerate, stable cycles whose basic period is k (that is, excluding harmonics)	1	–	1	1	3	4	9	14	28	48	93	165

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