A Theory of Information Structure
I. General Principles*

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Current cognitive theories are founded on the *processing computer analogy*, i.e., that a cognitive system is analogous to a computer. In contrast to this view, the following paper proposes a *content computer analogy* which states that cognition is the attempt to model phenomena as computers.

In fact, the *content computer analogy* is extended to a *content machine analogy*, because the term *machine* is understood as having two kinds of structure: (1) computational, or algebraic structure, and (2) stability. The paper develops a theory of the cognitive role of these two factors.

Algebraic structure determines the generative or connective aspects of stimulus sets. However, stability is shown here to crucially constrain the allowable decomposition and ordering of that generative structure. It is argued that an understanding of this constraint is important to the study of cognition. The constraint results in (1) a cognized structure of nested environmental *control*, and (2) the imposition of a *reference frame*. It is thereby argued that the concept of reference frame is strongly related to that of control. A general theory of reference frames is offered. The constraint just described forces reference frames to be *asymmetric-sequential*; and it is argued that this explains a number of phenomena in perceptual organization, categorization, linguistic grammar, and planning structure. © 1986 Academic Press, Inc.

1. INTRODUCTION

1.1. *The Computational Component*

This series of papers elaborates a view of cognition which differs from the information-processing approach in the following fundamental way: While both theories are built on a proposed relationship between computers and cognition, the relationships specified by the two theories are crucially different.

The relationship that is currently of great concern (e.g., Newell & Simon, 1972) will be called the *processing computer analogy*. The relationship to be proposed here will be called the *content computer analogy*. The difference between the two can be expressed simply like this: The processing computer analogy states that the cognitive *system* is structured like a computer, whereas the content computer analogy states that cognitive *representations* are structured as computers.

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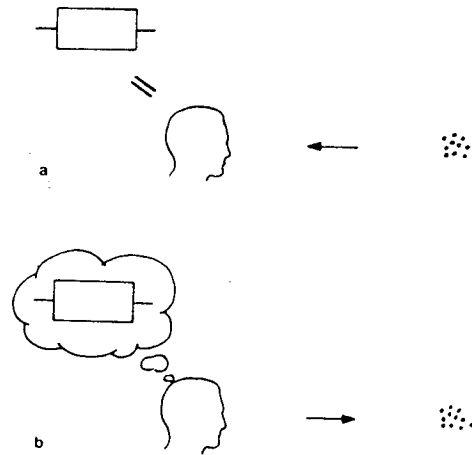
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FIG. 1. (a) The Process Machine Analogy: The head is equivalent to a machine. This is illustrated by the " = " sign between the head and the machine in the upper left. The stimuli, on the right, enter the head, or equivalently the machine. (b) The Content Machine Analogy: The head looks at a set of stimuli (shown on the right), and sees a machine (shown in the cloud above the head).

To understand this further consider the following three objects: (1) a brain, (2) a computer, and (3) a stimulus set. The two analogies define two different relationships between these objects, as illustrated in Fig. 1. In the upper diagram, the processing computer analogy has been represented. Here, the brain is regarded as equivalent to a computer. A stimulus set, the set of dots on the right, is inputted into the head just as if it were inputted into a computer. Figure 1b, however, represents the content computer analogy. Here, the brain looks at a stimulus set, and what it sees is a computer. Furthermore, the claim to be made is that the brain cannot help but see stimulus sets, or phenomena, as computers. For I will argue that this is what it means to extract information from the stimulus set.

Again, one can express the difference between the two analogies using this somewhat over-simplified characterization. In the information-processing paradigm, there are essentially two components: (a) the *information-processing system* and (b) the *commodity*, information, which is being manipulated in that system. The computer is the generally agreed-upon model for the *former*. However, the view I will offer here is that the computer is the required model for the *latter*, the manipulated commodity.

There are a number of ways that the Content Computer Analogy can be intuitively understood, depending upon ones interests. For example, as we shall see, the Content Computer Analogy can be regarded as the statement that cognition and planning are equivalent. That is, planning is not just a high-level cognitive activity, but is the very essence of any cognitive representation, i.e., on any level of the cognitive system. Again, according to Leyton (1974, 1982), the Content Computer Analogy can be regarded as stating that cognition and science are equivalent.

This is because, under the analogy, cognition is regarded as the structuring of empirical data in terms of logical languages.

An explicit formalization of the relation between scientific theories and computation has been worked out by Solomonoff (1964) who regards a scientific theory as a means by which a data set S can be generated by a program on some suitable Turing machine. That is, the program is an explanation for S. The *amount of information* in S is then regarded as the length of the shortest such program (see also Chaitin, 1966, Kolmogorov, 1968).

It is important to understand that, unlike statistical information theory, which equates the amount of information extracted from S with the number of possible alternatives to S, the information measure just described equates the information extracted from S with a specific kind of *representation* of S, i.e., a representation in terms of a program and a computer. Thus the Content Computer Analogy is a statement in which the constructs *information extraction* and *representation* will be regarded as equivalent.

1.2. *The Stability Component*

I have said that the content computer analogy can be regarded as saying that the brain represents anything as a computer. In other words, the brain structures representations so that they look like computers.

In fact, the content computer analogy is only half of the basic postulate from which the following cognitive theory will be elaborated. Computers are the embodiment of the notion of *computability*. However, it will be claimed that cognition attempts to define also another type of structure: *stability*[1]. The stability of a phenomenon will be taken to be its *observer-defined persistence*. The theoretical consequences of this added factor will be several. For example, it will be possible to offer a general theory of reference frames which will be applied to a wide range of phenomena from Gestalt perception to linguistics.

1.3. *The Content Machine Analogy*

These two forms of structure, computability and stability, are the central concerns of two major contemporary theories of machines: *computer science*, which studies computation, and *dynamical systems theory*, which studies stability. Since machines, as they are currently investigated, embody these two kinds of structure, and the basic proposal to be made is that cognition attempts to define these two kinds of structure, it would be more accurate to call the basic proposal, the Content *Machine* Analogy:

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[1] Relationships between a computational view of information and Shannon information theory are presented in Fine (1973). A relationship between a stability view of information and Shannon information theory is presented in Footnote 13.

CONTENT MACHINE ANALOGY: *The cognition of, or extraction of information from, a phenomenon is the description of the phenomenon as a machine*[2].

The two disciplines, computer science and dynamical systems theory, rarely communicate with each other, as they have very different concerns. However, it will be claimed here that, in cognitive representations, their two forms of structure, computation and stability, mutually constrain each other.

The type of structure which emerges from this mutual constraint will be called a *reference frame*. This series of papers attempts to show that the several uses of the term "reference frame" in cognitive psychology are examples of this structure. For example, it will be argued that Marr-Nishihara shape perception and Roschian categorization are examples of such structures.

Simultaneously, the mutual constaint of computation and stability will be seen to define any environment as a nested structure of *control*. A conclusion of the theory will be that reference frames are nested structures of control.

Finally, in conjunction with the statement of the Content Machine Analogy, the following evaluation principle is offered. Observe first that some of the alternative descriptions of a particular phenomenon can yield more information than others. Furthermore, one can assume that the cognitive system attempts to choose those descriptions which have more information than less. Thus one has, quite simply:

COGNITIVE MAXIMIZATION PRINCIPLE: *The cognitive system attempts to produce descriptions which maximize machine structure.*

1.4. Information, Reference Frames, and Planning

A central outcome of this series of papers will be the development of a precise equivalence between the notions of information, reference frame, and planning structure. Let us now briefly sketch how this will be done.

With respect to the notion of reference, an important result which will be incorporated into the following work will be Rosch's discovery that reference in categories is *asymmetric*. Rosch (1975) found that natural categories—such as colors, line orientations, and numbers—have reference point stimuli—such as focal colors, vertical and horizontal lines, and number multiples of 10—with respect to which the other category members are judged. For example, off-red is referred to red, a leaning object to the vertical, and 99 to 100. The crucial result is that the reverse references do not usually happen; e.g., 100 is not referred to 99, etc.

However, now consider the following phenomenon which was demonstrated in a set of experiments I conducted in Leyton (1984a). I found that, when subjects are

[2] It should be understood that these papers are about the structuring of phenomena in *human cognition*, and not, for example, in mathematical disciplines. The simplest illustration of this distinction is the theory of categorization or set membership. Mathematics is based on set theory, where set membership corresponds to the fulfillment of a set of properties. However, Rosch has shown that, in *human cognition*, set-membership has a profoundly different structure: it corresponds to graded similarity to a prototype. In these papers, we are interested only in results of the latter type.

presented with a rotated parallelogram, they reference it to a non-rotated one, which they then reference to a rectangle, which they then reference to a square. Figure 15 shows this successive reference sequence. In other words, there is not just one reference point in a category, but a whole succession of them. How can one explain this phenomenon?

Recall that this paper will attempt to show that, in a cognitive structure, computation and stability mutually constrain each other. What will be argued is that this mutual constraint causes a reference frame to have a nested structure where, within each level of the structure, there is a reference point. The sequence of nesting will be denoted by $G_1 \cdot G_2 \cdots G_n$, where each G_i is a level of the structure and e_i is the reference point in that level. Thus it will be argued that in the succession phenomenon, described above, the sequence of shapes results because the cognitive system creates a path through the nested layers $G_n, G_{n-1}, \ldots, G_1$, picking out the reference points $e_n, e_{n-1}, \ldots, e_1$, from each successive layer. The series of shapes are the successive reference points, e_i, encountered through the levels.

Remarkaby, one can argue that this phenomenon also occurs in linguistics. It is well known that transformations can be removed from a sentence only in a particular order [see Akmajian and Heny (1975), for an extensive review]. This, in fact, creates the same type of layered structure, $G_1 \cdot G_2 \cdots G_n$, where each level G_i is associated with a particular transformation. The sequence of sentences (or sentence representations) that results from the removal of the transformations is a path of reference points through the levels, to the kernel. This phenomenon occurs whether one believes in transformational grammar (e.g., Chomsky, 1957) or in a rival theory such as relational grammar (e.g., Perlmutter & Postal, 1983).

I will attempt to show that this type of layered asymmetric sequence, $G_1 \cdot G_2 \cdots G_n$, is general, i.e., that the many phenomena, about which cognitive psychologists have used the term *reference frame*, all have this same underlying structure. It is in fact, far from obvious that these psychological phenomena have this nested asymmetric reference structure. For example, consider the square/diamond problem. Mach was the first to demonstrate that if one places a cartesian reference frame over a square in two different particular orientations, the square looks like a square with one orientation and like a diamond with the other (Fig. 10). This problem has remained unsolved for almost a century and it clearly rests on the phenomenon of a cartesian reference frame. In the second paper, I will attempt to show that a cartesian reference frame involves, cognitively, a quite intricate successive asymmetric-reference structure, $G_1 \cdot G_2 \cdots G_n$, and that it is this structure that causes the square/diamond effect. The theory offered correctly predicts more striking examples than the square/diamond phenomenon and explains them.

Again, in an entirely different area, a major unsolved problem in Gestalt psychology is to explain what a *part* or *grouping* is. In the second paper, I will attempt to show that a part of grouping is, cognitively, a level in successive asymmetric reference sequence, i.e., that it is a layer in a reference frame.

Let us now turn away from perception, and look at planning. It is generally acknowledged that planning is hierarchical; i.e., that major steps precede finer ones.

For example, if one wishes to play the piano, one moves one's entire body across the room to the piano; then one moves one's arm so that the hands are above the keyboard; and finally, one moves one's fingers on the keys.

What is remarkable is that the standard representation of this motor space is a sequence $G_1 \cdot G_2 \cdots G_n$, where each component G_i is (1) the motor space of one of the components, and (2) the *reference frame* for the next component of the sequence (e.g., Paul, 1981). That is, the movements in any component are referred to the movements in the previous component of the sequence. Thus the motor space is structured as a *nested asymmetric reference sequence* of the exact type I proposed earlier for phenomena in other cognitive areas such as perception, categorization, and linguistics.

A crucial conceptual result of this is the following: Because it will be argued that a planning hierarchy has the layered reference structure of the type I describe, one will be able to invert this conclusion and regard *all reference frames as planning hierarchies*. For example, because I will argue that, in perception, a Gestalt organization is a nested asymmetric reference sequence, one can conclude that *a Gestalt organization is a planning hierarchy*. Even something as simple as a square will be shown to be structured as a planning hierarchy, and it is this hierarchy that will be shown to be responsible for the square/diamond effect.

Let us now see how two of the main proposals of the following paper will be put together. (1) The first is the Content Machine Analogy which states that the *extraction of information*, from a stimulus set, is the description of that set as a machine. (2) The second is the proposal that a machine, used as a cognitive description, involves the mutual constraint of computability and stability, and that this constraint results in a successive asymmetric structure which will be taken to be a reference frame. Putting these two proposals together, one concludes the following: The Content Machine Analogy forces an equivalence between the notion of *information* and *reference frame*; i.e., under this view, the extraction of information is exactly the imposition of a reference frame. For example, one extracts the information "a square" when one embeds it in the successive shape sequence shown in Fig. 15. This embedding will distinguish it from the information "a diamond," which involves a different embedding.

We have just put together the Content Machine Analogy with the proposal that reference frames are asymmetric-sequential, and we have thereby derived a proposal concerning the nature of information-extraction, or stimulus definition. If one now puts these notions together with the claim that *planning hierarchies* are asymmetric-sequential, one obtains this result: The Content Machine Analogy implies that the extraction of information from a stimulus set is exactly the description of that set as a planning hierarchy.

This view will lead us to the final main conclusion of these papers: that *all cognition is planning*. Most importantly, we should observe that in current cognitive psychology there is a division between *perception*, i.e., the extraction of a roughly neutral data base, and *planning*, a highly goal-directed activity. Under the system to be elaborated in these papers, there is no such division. At any level of the nervous

system, all that ever happens is planning. This result is conceptually forced upon us as soon as we realize that a planning hierarchy has the same structure as the reference frames we find in perception.

2. OVERVIEW

This series of papers is divided into six main parts. Part I (the present paper) develops the general principles of the structural theory. The next four parts are applications of the general theory to four ascending levels of cognition: (1) *perceptual organization*, (2) *categorization*, (3) *linguistic grammar and function*, and (4) *planning*. The final part turns from the structural theory to a theory of what criteria the cognitve system uses to evaluate and choose between alternative representations which all conform to the structural theory. Thus, whereas the first five papers examine the realization of the Content Machine Analogy, the final paper examines the realization of the Cognitive Maximization Principle (end of Sect. 1.3). A brief overview now follows.

General Theory

The purpose of this first paper is to develop a detailed statement of the Content Machine Analogy. From this analogy the paper elaborates a system of general proposals that will be tested in the subsequent papers under the specific cognitive areas examined in those papers.[3]

We begin by investigating the way in which the first of the two forms of structure, *computability*, can be the structure of a cognitive representation. The space of programs of a computer forms a highly connected system of the type called an *algebraic* structure. From this point onwards, computational aspects will be referred to as *algebraic*, to emphasize our concern with this system of connectivity. Part of the basic postulate will state that the cognition of a stimulus set is the description of that set as the algebraic structure of a machine. One should note here that no mathematical knowledge will be assumed in this paper.

The discussion of algebraic aspects shows that the Content Machine Analogy can be realized in two very different types of representation. These two types allow us to characterize, in a very precise way, some important dichotomies in cognitive psychology: (a) One of the dichotomies is that of whether objects are referred to each other *symmetrically* (as, for example, in the usual view of similarity) or *asymmetrically* (as in the examples given at the beginning of Sect. 1.4). (b) A second dichotomy is whether an object is characterized by a set of properties that are *invariant* to all its instances (as in the classical theory of categorization) or by the properties of what is called a *prototype* instance in Rosch's categorization theory.

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[3] Principles and Proposals are, in most cases in the text, hypotheses that are open to empirical testing, and will be tested within the specific cognitive areas of the subsequent papers. The word Principle is chosen when a statement is more weighty than a Proposal.

(c) A third dichotomy is this: A possible implication of Rosch's results (recall the beginning of Sect. 1.4) is that a stimulus is characterized as a *deviation* from a referent. For example, one's cognitive representation of off-red seems to be that off-red is a deviation from red. Thus, off-red seems to be *represented as its referential relation to red.* We shall say, generally, the reference is *explicit* in the representation of the stimulus. It will be seen that prominent models of information structure do not permit this psychological phenomenon to occur.

In relation to these issues, we find that the two alternative types of representation, which realize the Content Machine Analogy, have the following properties: (1) In the first type, (a) reference is *symmetric*, (b) objects are defined as *invariants*, and (c) reference is *non-explicit*. (2) In the second type, (a) reference is *asymmetric*, (b) objects are defined in a way that accords with the *prototype* phenomenon, and (c) reference is *explicit*. It is the algebraic analysis of these two forms of representation that will allow us to give a clarification of the above dichotomies. We then choose (2) as the basis of the following theory because, in all the areas to be examined (areas within perceptual organization, categorization, grammar, and planning), the properties of (1) are violated and the properties of (2) are corroborated.

Having understood the Content Machine Analogy as implying asymmetric and explicit reference, on algebraic (computational) grounds, we then introduce the phenomenon of *sequential* asymmetric reference. The claim to be made is that this phenomenon is explained by stability factors.

Therefore, we then move from *algebra* and investigate the way in which the second of the two forms of structure, *stability*, characterizes aspects of cognitive descriptions. As I said, the stability of a property is taken here to be its defined persistence. A proposal is offered which is crucial to the theory developed in the series of papers. It will be called the Cognitive Stability Principle, and it states that the cognitive system attempts to *remove instabilities* in the order of their decreasing instability. Let us consider this statement more closely.

I will argue that the cognitive system attempts to define the persistence of each of the behaviors of a machine which cognition is using as a model for a stimulus set. This causes the machine's behaviors to be partitioned into classes with respect to persistence; that is, each behavior in a class has the same stability value.

One can in fact put together the behaviors of a class and regard them as the behavioral structure G_i of a single component machine. Thus I will argue that the attempt to define the stability value of each of the behaviors of the machine, G, causes the machine to be decomposed into a product of component machines $G = G_1 \cdot G_2 \cdots G_n$, where any G_i, in the sequence, has greater stability than any G_j, to the right of it in the sequence.

The Cognitive Stability Principle then states that, given a machine structure, G, which cognition has taken as the structure of a representation, the cognitive system attempts to remove, from the representation, the component structures G_i, one by one, in order of decreasing instability (i.e., from right to left). Several examples of this will be seen. For instance, in shape perception, we shall see that the cognitive

system attempts to remove properties of a figure in a particular sequence. This sequence successively stabilizes the figure. Hence, for example, one obtains the successive shape reference described earlier. Again, in linguistics, syntax transformations can be removed from a sentence only in a particular order. A variety of functional data will be used to argue that this successive transformational removal corresponds to the successive reduction of a type of *discourse instability*.

Note that it is the Cognitive Stability Principle that will be regarded as producing the successive asymmetric sequence of a reference frame; i.e., the latter reference sequence will be seen as successive removal of instabilities.

Following from the Cognitive Stability Principle, a crucial proposal is then offered which will be called the Principle of Nested Control. This states that, given a machine factorization $G = G_1 \cdot G_2 \cdots G_n$, into stability levels, the cognitive system regards any component machine G_i as *controlling* the machines with less stability than itself—that is, all the machines to the right of it in the sequence. The result is that G is given a structure of nested control. But, of course, because G is the machine structure imposed by the cognitive system on the environment, the fundamental consequence is that the cognitive system sees the *environment* as a nested structure of control. It will be argued that the many examples of *reference frames*, in cognitive psychology, are such structurings of the environment.

Observe that central to all these concepts is a single principle which is, in fact, a particular detailed statement of the Content Machine Analogy. This statement will be called the Description Postulate. Roughly speaking, the Description Postulate states that cognition, or the extraction of information, is the attempt to model a phenomenon as the algebraic structure of a machine, with the powerful condition that the model conform to the Cognitive Stability Principle; that is, the model is decomposed into a sequence of levels, $G = G_1 \cdot G_2 \cdots G_n$ with respect to stability. From this, one obtains all the results concerning the perception of nested control, reference frames, etc.

A Unified Theory of Perceptual Organization

With respect to perception, an investigation into these decomposition sequences reveals that any such sequence is split into two subsequences, one structuring the way in which the stimulus set can change under *external* action (e.g., deformation, rotation), and the other describing the way in which the set is perceived as *internally* generated from a subset. An examination of the nested control structure of the latter yields a theory of *grouping*. The external and internal sequences are shown to be strongly related to each other in that the symmetry axes of the internal (generating) sequence are lines of flexibility of the most stable components of the external (e.g., deforming) sequence. It is claimed that cartesian reference frames are subsequences of the full sequences. Using these principles, a unified theory is offered of several apparently quite separate perceptual areas, e.g., Marr-Nishihara shape perception, Gestalt grouping, the orientation-and-form problem, and motion perception. A technical elaboration of this theory has appeared in Leyton (1984b); and a neuronal model in Leyton (1985).

The Structure of Categories

In this part, the cognitive principles developed in Part I are applied in order to develop a unification of Rosch's prototype theory with the criterial definition theory of categorization. Rosch's category properties are redescribed in terms of notions in our cognitive theory, and so too is the criterial theory. Essentialy, with respect to categories, it is proposed that two descriptions in terms of machines are cognitively formed: (1) an internal reference structure, which yields the prototype phenomena, and (2) the action of this structure as a cue space to a criterial structure. A 10-part definition of category is offered.

Grammatical Structure and Function

In this part it is argued that linguistic grammar is an example of an asymmetric-sequential structure, $G = G_1 \cdot G_2 \cdots G_n$. Furthermore, it is argued that, as with perceptual organization, the structure $G = G_1 \cdot G_2 \cdots G_n$ is split into two subsequences, one defining the internal generative structure of a sentence and the other defining its external transformational structure, i.e., how it can be altered. These proposals will be corroborated independently with respect to a number of linguistic theories; in particular, with respect to Chomskian theory and the radically different approach of functional linguistics. The data will include not only English but African Bantu languages and Philippine languages.

Planning

It will be argued that the central criterion which partitions a plan into a hierarchy of levels is *stabilty*; i.e., the major steps are more stable than the fine-tuning steps. Second, because each level is essentially a generative structure (i.e., a space of algorithms to reach a goal), each level corresponds to a computational (algebraic) space. Thus the conclusion will be that a planning hierarchy has the structure $G_1 \cdot G_2 \cdots G_n$ introduced above. This claim will be remarkably easy to corroborate in terms of modern views of trajectory planning (e.g., Paul, 1981; Brooks, 1983).

The general result of this argument will be that, because all the other cognitive levels we have examined also have that structure, *the cognitive system is a planning-definition system at all levels.*

Cognitive Maximization

While the preceding parts tell us how a cognitive representation is structured, the final part examines how the cognitive system adjudicates between competing representations which all conform to the same structural principles (i.e., which all have an asymmetric-explicit-sequential machine structure $G_1 \cdot G_2 \cdots G_n$). The Cognitive Maximization Principle tells us that the cognitive system chooses the representation which *maximizes* machine structure (i.e., which maximizes the above kind of structure). We will explore what this means, and offer a theory that

attempts to explain the evaluation criteria which cognition appears to apply in the four domains we studied in the preceding parts: perceptual organization, categorization, linguistic grammar, and planning.

PART I: GENERAL THEORY

3. MACHINES: PRELIMINARY COMMENTS

3.1. *Basic Structure*

Recall that it is an assumption of this series of papers that the concept of a machine, as it emerges in cognitive representations, embodies two forms of structure: (1) *algebraic structure*, which describes the computational connectivity of the machine; and (2) *stability*, which concerns the hypothesized persistence of the machine. The next few sections examine the first of these.

Because this series of papers argues that the cognition of a phenomenon is a description of the latter as a machine, we need to understand how machines are structured in order to propose here a theory of the cognitive structure of phenomena. The *algebraic* structure of a machine is its *space of algorithms*. It will be argued that this space defines some fundamental properties of cognitive representations; for example: (1) the internal *reference* structure of a representation, and (2) what is cognitively understood to be an *object*. Thus an understanding of algebraic aspects is crucial to the theory to be developed. No prior mathematical knowledge is required and we begin at the most basic level.

Every machine has a simple format which is diagrammed in Fig. 2. The box in the center of the figure is an object. Inputs (i.e., operations) are applied to the object and outputs (i.e., results) are obtained. Two factors are regarded as being responsible for the outputs: (a) The inputs P, and (b) certain properties, Q, of the object, itself, at the time of input. These latter properties are called the object's *state*. When an input is applied, the machine changes its state. Thus, for example, a computer is a machine with the program as an input, the memory as a state, and the print on the printout paper as the output.[4]
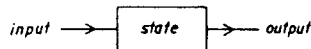

input $\longrightarrow$ state $\longrightarrow$ output

FIG. 2.   The basic structure of a machine.

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[4] Theoretically, a machine is a quintuple $M = (Q, P, Y, \delta, \lambda)$, where Q, P, and Y are *sets* called states, inputs, and outputs, respectively; and the functions δ: $Q \times P \to Q$ and λ: $Q \times P \to Y$ are interpreted as determining the next state and output, respectively.

However, a theorem (e.g., Bobrow & Arbib, 1974) states that any machine can be redescribed such that its output is considered part of the state (e.g., the print produced by a computer can be considered to be part of its memory). The resulting redescribed machine is called a *state-output machine*, and it has the convenience of allowing one to describe everything in terms of inputs and states, ignoring the concept of output.

This means that a machine can be regarded as comprising essentially three components:

(1) an **object**;

(2) an **input space**: a set P of inputs, or actions, which can be applied to the object;

(3) a **state space**: a set Q of states into which the object can be put by applying the inputs.

3.2. Cognition: Two Preliminary Comments

(1) Recall that the Content Machine Analogy asserts that information is extracted from an object exactly when the latter is viewed as a machine. For example, one can let cups or footballs be viewed as machine objects, i.e., examples of the box in the center of Fig. 2. Kicking is a usual input operation applied to a football. The states of the ball are its positions. Thus, kicking the ball causes it to change its state.

Similarly, one can view a cup as a machine. One can apply certain actions to it (e.g., drinking), and it will change its state (i.e., become emptier). Alternatively, one can apply the same input operations to a cup as one did to a football, i.e., kicking, and it will change its state from not broken to broken. Note that, when kicking is applied to a ball, it retains its non-broken state. Thus even though one can have the same inputs {kicking} and the same *space* of results {broken, non-broken} for both a cup and a football, the two objects have a different *inter-relationship* or correspondence *between* inputs and results.

Thus, describing an object as a machine is a means of defining it as a specific inter-relation between inputs and results.

(2) We have just been considering the inter-relation represented by Fig. 2. The Content Machine Analogy claims that the cognition of an object or phenomenon must be a description of it such that it takes on that form. However, Fig. 2 represents only one aspect of what a machine is.

How much further we still have to go can be seen if we recall that the computational structure of any machine can be represented by a computer. The consequence is that the Content Machine Analogy implies that one cognizes an object, such as a football, only when one views it as a computer.

This assertion seems to be patently ridiculous. And yet this series of papers will attempt to demonstrate that this view is not only correct, but that it is an important theoretical idea needed for the understanding of cognition.

The essential difference between merely asserting that all objects look like Fig. 2,

and claiming that they actually look like computers, resides in the concept of *algorithm*; that is, *program* or *effective procedure*. An algorithm is a string of inputs, i.e., $x_1, x_2,..., x_n$. Each input x_i can be regarded as an operation or action on the machine, or as an *instruction* or *rule* prescribing how the present state of the computer is to change (i.e., its state-transition). The structure of algorithms will form much of the concern of this paper, because it will be argued that there is substantial psychological data to indicate that this structure is the structure of *reference* in a cognitive representation. For example, an analysis of the space of algorithms will lead to an understanding of the *prototype* structure if a category.

3.3. *Clarifying Some Terminology*

Before we continue with this concept, some readers might require the following terminological clarification.

An input prescribes a change of state. For example, an instruction in a program is an input. It prescribes a step from one stage in the computer's calculation to the next. Thus the reader should take note that the following terms will be used interchangeably:

An *input* = a *transformation* of one state to another

= an *operation* on the machine

= an *action* on the machine.

Furthermore, we have

The *input-space* = the set of all inputs.

Thus the following familiar terms can be described in this way:

A *program*, *algorithm* or *effective procedure*

= a selection of members from the input space,

applied sequentially to the machine.
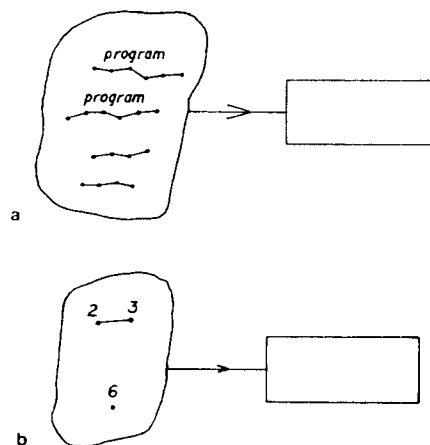
Figure 3a diagrams this notion.

FIG. 3.   (a) The input space as a collection of programs. (b) The algebraic connectivity between programs.

3.4. *What Is a Parameterization?*

It will be argued that the cognition of a stimulus set is the modeling of the latter as a machine. Some of the concepts to be developed in this view can be illustrated by looking at the phenomenon of *parameterization*. I will offer here a particular characterization of this phenomenon: In fact, I will claim that parameterization is the modeling of stimulus sets as machines, i.e., that parameterization conforms to the Content Machine Analogy.

A typical example of a parameterization is the placement of a cartesian coordinate system over a plane. In the general case, the system can be lain over a curved space (e.g., the surface of a cylinder), and can be many-dimensional. Consider a two-dimensional parameterization of a set. Select a point, for example (5, 3). The label (5, 3), in fact, contains a considerable amount of information. It means:

> "This point is obtained by moving one unit up from (5, 2)"
>
> "This point is obtained by moving one unit right from (4, 3)"
>
> "This point is obtained by moving one unit right and one unit up from (4, 2)"
>
> etc.

Thus the label (5, 3) is a name for the collection of all possible *algorithms* (strings of instructions) for getting from any other point in the underlying set to (5, 3). The label (5, 3) therefore describes a structure of programs; in fact, a machine.

The other machine constructs are assigned as follows. Because the instructions (i.e., inputs) are rules for changing position in the underlying set, the positions must be the *states* of the machine. Finally, the *machine object* (the box in Fig. 2) must be what is pushed into the various positions; and this can be regarded as what is *invariant*, from position to position. Clearly the common object which can be found at all positions is a *point*. So the machine is a point.

Thus a *parameterization* of a set of points appears to be a means of viewing the set as a single point moved into several states. Furthermore, and crucially, the changes of state in a parameterization are determined by a space of algorithms (programs). This turns the point into a machine. These concepts will be elaborated and illustrated in what follows.

4. ALGEBRAIC CONCEPTS

The phenomena, introduced briefly in the previous section, in fact arise from the *algebraic* structure of a machine. As we shall see, the algebraic structure is the *space of algorithms* which can be applied to the machine. That structure will be argued to be central to cognitive representations. Consider, for example, the following phenomenon. Recall, that in the previous section on parameterization, it was seen that algorithms are an important means by which points can be *referred* to each other. Therefore, the space of algorithms can be regarded as a *reference frame* that is imposed upon a stimulus set. Thus, this series of papers will claim that, by study-

ing the algebraic structure of a machine (the space of algorithms), one can explain a number of aspects generally of the internal reference structure of a cognitive representation.

However, we will find that the algebraic structure can be used to define reference in two entirely different ways, corresponding to two very different realizations of the Content Machine Analogy. In fact, these two different uses of the machine's algebraic structure will enable us to obtain a very clear characterization of some important debates in current cognitive psychology.

One such debate is between whether reference structures are *symmetric* or *asymmetric*. The two opposing positions can be summarized briefly as follows. The view that reference structures are symmetric is often based on the presupposition that *similarity* is symmetric, i.e., that the extent to which stimulus A is similar to stimulus B is the same as the extent to which stimulus B is similar to stimulus A (e.g., Shepard, 1962a, 1962b). However, in contrast to this, Rosch (1975, 1978) and Tversky (1977; Tversky & Gati, 1982) have put forward important evidence that similarly is asymmetric. Some of this evidence was mentioned at the beginning of Section 1.4; for example, category members are judged as similar to their reference point stimuli (e.g., 99 is similar to 100) but the converse does not usually happen (e.g., 100 is not judged as similar to 99).

One thing that will be argued is that very simple algebraic considerations can characterize, in quite a profound way, the difference between these positions.

Again, we shall find that algebraic considerations can characterize two opposing views concerning the cognitive nature of an object. One view states that an object is a set of properties common to all its instances (e.g., Piaget, 1969; Gibson, 1979). The opposing position states that the instances of an object vary in the number of characterizing properties they posses. For example, prototype instances have *more* of the characterizing properties than the non-prototypical instances (e.g., Rosch, 1978).

Another issue which has not concerned cognitive psychology, but will be considered here to be important, is this: As we said earlier, some of Rosch's results seem to indicate that a stimulus is characterized as a deviation from the prototype. For example, off-red is characterized as a deviation from red. Therefore, off-red seems to be cognitively represented as its referential relation to red. I described this phenomenon earlier by saying that reference is *explicit* in the representation of the stimulus. However, most theories of representation (e.g., multi-dimensional scaling; constancy theories, multi-resolution theories, etc.) do not permit the representation of a stimulus to explicitly encode its referential relationships.

A claim to be made later is that algebra clarifies the dichotomy between non-explicit and explicit reference.

In fact, it will be regarded that algebra clarifies the close *inter-dependence* between the above three dichotomies: symmetric vs. non-symmetric reference; objects as invariants vs. objects as prototypes; non-explicit vs. explicit reference.

Let us therefore take a preparatory look at algebra and begin to consider how algebraic criteria could determine the structures created by cognition.

4.1. *Input Spaces as Algebraic Structures*

As pointed out above, the algebraic structure of a machine is the space of its algorithms. Furthermore, as illustrated in the discussion on parameterization, algorithms can be used to refer points to each other. In particular, it is important to recall from Section 1.4 that a central feature of the theory to be developed will be the claim that cognitive reference structures involve not just one reference point but a whole sequence of them. How then will this sequential reference be described?

Recall that the Content Machine Analogy states that the cognition of a phenomenon is the description of the latter as a machine. The behavior over time of a machine—the sequential change from state to state—is prescribed by a string of inputs, $x_1 \cdot x_2 \cdot x_3 \cdots x_n$. It is a string of this type that is called a program, or algorithm, or effective procedure. With a number of structural concepts to be introduced later, a particular class of such sequences will eventually be used to embody sequential reference. In preparation, it is necessary therefore to consider the sequentiality of algorithms in greater detail.

A somewhat hidden but important aspect defining this sequentiality is the means by which algorithms are built, as sequences, out of elements or shorter strings of inputs.

Algebra is the study of systems of combination. The means of combining a number of elements to produce another is given by a *combination rule*. For example, the rule "multiplication" combines the two numbers, 2 and 3, to produce 6. Figure 3b illustrates a machine where the input space is a set of numbers. For example, the machine might be the Gross National Product and the inputs might be numbers applied to it. Suppose now that these numbers are combined using the multiplication rule. This rule would combine 2 and 3 into a sequence of inputs 2×3. This is the program shown in Fig. 3b. However, the program has the same effect as another input, 6, elsewhere in the space.

Inputs—or more generally, strings of inputs—can be combined as follows: If two strings are represented by the symbols, $w_1 = x_1 \cdot x_2 \cdots x_n$ and $w_2 = y_1 \cdot y_2 \cdots y_m$, then the combined string can be simply the first string followed by the second; i.e., $w_1 \cdot w_2 = x_1 \cdots x_n \cdot y_1 \cdots y_m$.

Observe now that, using this concatenation process, three successive strings can be bracketed in any way, as can be illustrated using numbers:

$$(2 \times 3) \times 5 = 30 = 2 \times (3 \times 5).$$

The brackets indicate which numbers are combined first. Generally, the property just described can be stated as

$$A1: [w_1 \cdot w_2] \cdot w_3 = w_1 \cdot [w_2 \cdot w_3].$$

However, one can assume a further property to hold for the input set of a machine: there is a member of that set which does *nothing* when applied to the

machine. That is, an input, called the *identity* element, which satisfies a property that one can again illustrate using numbers:

$$5 \times 1 = 5 = 1 \times 5.$$

The number 1 is an identity element and, in the general case, the symbol e will be used to represent the identity. Thus, the property we have just defined is

A2: $w \cdot e = w = e \cdot w$ for all w in the input space.

Now assume that, for every sequence w of inputs, there is an *inverse* sequence $1/w$ which undoes the action of w. Again, one can illustrate this property using numbers:

$$5 \times 1/5 = 1 = 1/5 \times 5.$$

The general statement is

A3: For any input w, there is an input $1/w$ such that

$$w \times 1/w = e = 1/w \times w.$$

It is clear that properties of this type, i.e., *algebraic* properties, express the way inputs can combine, and that the properties therefore express essential aspects of the programs (strings) which can be built out of the inputs.

The set of inputs of any machine satisfies A1. The compliance with A2 can also be assumed[5], quite trivially. Many machines also satisfy A3. To simplify the discussion, this final condition will be assumed to hold for the machines used in cognitive descriptions. Any set which satisfies A1, A2, and A3 is mathematically called a *group*. Thus the discussion will refer to the machine input space as an *input group*. Recall that the input group is the space of algorithms that are applied to the machine. The word *group* refers to the fact that the algorithms will satisfy the simple algebraic properties A1, A2, and A3.

Now recall that algorithms create a referential structure; e.g., in a parameterization they refer points to each other. Thus later in this paper, the input group (space of algorithms) will become the reference frame that structures a stimulus set. However, we will find that the group can be used as a reference frame in two very different ways, which will correspond to two very different realizations of the Content Machine Analogy. One of the alternatives will turn out to have symmetric reference, and the other alternative will turn out to have asymmetric reference. As a fore-taste of the symmetric case consider the following.

Symmetric reference in a machine can be actualized by what is called the *state-transition diagram of* the machine. To illustrate this, let us look carefully at a simple example. Figure 4 shows the state-transition diagram of a light switch, which can be

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[5] See Bobrow & Arbib, (1974) for an extensive discussion of the algebraic structure of machines.
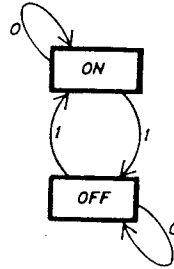
FIG. 4.   Thr state-transition diagram of a light switch.

explained as follows. In any state-transition diagram, one simply draws a box for each state, and draws an arrow between each box to represent the state-transition caused by each input. Thus, an input labels each arrow. A *switch* is a machine with a state space consisting of two states {ON, OFF}. Therefore, there are two boxes in the state-transition diagram shown in Fig. 4. Furthermore, at any time, one of two inputs can be applied—DON'T SWITCH and SWITCH—which can be represented by the numbers 0 and 1, respectively. Therefore, for each state, there are two arrows leading from the state, one for each input. Each arrow describes the effect of one of the inputs.

Let us therefore look closely at Fig. 4. Suppose that the switch were in the state marked ON. If one applied the input 0, the switch would follow the arrow marked 0, emerging from ON and returning back to ON. That is, the input 0 has no effect on the state. However, if one applied the input 1, the switch would follow the arrow marked 1 to the state marked OFF. Exactly the same would happen if one were at OFF. That is, the input (arrow) named 0 would not alter the state, and the input (arrow) named 1 would send the switch to the only other state.

Thus, the state-transition diagram, described above, is *symmetrical*, because the actions at one state equal the actions at the other state. This means that the state OFF is *related* to the state ON, in exactly the same way as the state ON is related to the state OFF. Therefore, if cognition imposed, on a pair of stimuli, a reference frame which has the same structure as that shown in Fig. 4, the stimuli would be *symmetrically referred* to each other.

Observe that the state space, in this diagram, is {ON, OFF} and the input space is {0, 1}. In this case the input space is a group called $Z_2 = \{0, 1\}$, the cyclic group of size 2. It is called *cyclic* because the repeated application of the only non-trivial element 1 causes the system to cycle symmetrically backwards and forwards between ON and OFF. The close relation between the *symmetry* of the diagram and the *use of a group* is no accident. In particular, any symmetric reference frame in cognition must presuppose a group of transformations, as we shall soon see.

4.2. Conventional Group Concepts

The Content Machine Analogy states that the extraction of information from a stimulus set is the imposition of a machine structure on that set. This structure is a

schema which looks like Fig. 2. What we have been dealing with, in the previous section, is the structure of the left hand side of Fig. 2, the input space. The assumption will be that the input space is structured as a group—i.e., sequences of inputs (algorithms) obey laws A1–3.

Let us now look more carefully at two particular aspects of the way groups are used in mathematics: (a) They are used to describe *symmetries*. [This makes them basic, for example, to quantum physics and general relativity.] (b) They identify an *object* as a set of invariant properties through changes of state. These two concepts are discussed, in turn, in the next two sections. But first we need to understand why they will be discussed.

Recall that the group of a machine will allow us to characterize at least two important aspects of a cognitive representation: (1) the internal structure of reference, and (2) what is cognitively understood to be an object. When groups are used to define *symmetries* and *objects as invariants*, then they can be used to characterize some important contemporary theories of representation. For example, one class of such theories are the current transformational theories of perception (e.g., Piaget, 1969; Gibson, 1979; Hoffman, 1966, 1968). These theories either explicitly or implicitly base themselves on the above two group-theoretic concepts, i.e., *symmetry* and *objects as invariants*. I will attempt to show that these two concepts provide a *invalid* basis for cognitive theory: For example, transformational grammar seems to structure sentence-spaces *asymmetrically* with respect to the kernel (Chomsky, 1957); and, likewise, categories seem to structure stimuli asymmetrically with respect to the prototype (Rosch, 1975). Again, as we shall see, the *objects-are-invariants* viewpoint accords with the classical theory of category-membership, not with Rosch's prototype view. In the subsequent sections, I will develop a very different use of group theory which strongly prescribes asymmetry and prototype structure; and predicts a number of phenomena which will be corroborated. However, first let us look closely at a group-theoretic characterization of the cognitive theories based upon symmetric reference and objects as invariants. In the remainder of Section 4, I shall also describe most of the group theory that will be required in the subsequent papers.

4.2.1. Groups and Symmetry

We will require a very clear notion of what symmetric reference means. In order to obtain this, we first require a clear view of how symmetry is characterized in mathematics.

Consider the hexagon shown in Fig. 5a. The symmetry of the figure is, in fact, expressed in the allowable rigid mappings of the figure to itself; as we shall see: The most obvious such mapping is t, the reflection (flip) of the figure about its vertical axis; as shown in Fig. 5b. It is the fact that there is a reflection *transformation* t, sending the figure onto itself that allows one to say that the figure is reflectionally symmetric.

Another such mapping is the rotation, r_{60}, of the figure by 60° about its center. This transformation gives Fig. 5c, and expresses what one can call a *rotational sym-*
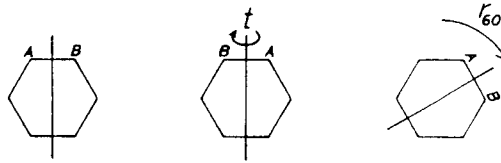
FIG. 5. (a) A hexagon. (b) One of its reflections. (c) One of its rotations.

metry of the figure. In fact the entire set of rigid mappings, and therefore symmetries, can be elaborated as follows: First, not only is there rotation by 60°, but there is rotation by 120°, rotation by 180°, rotation by 240°, and rotation by 300°; that is, in increments of 60° from 0° up to 300°. Second, there is not only reflection *t* about the vertical axis, but also reflection about any of the other axes. One can obtain any such reflection by simply regarding it as a reflection about the *rotated* vertical axis. This means that such a reflection is the product of *t* and some rotation; for example, it can be tr_{120}.

The complete list of all these transformations is, in fact, quite simple to understand:

$$D_6 = \{e, r_{60}, r_{120}, r_{180}, r_{240}, r_{300}$$

$$t, tr_{60}, tr_{120}, tr_{180}, tr_{240}, tr_{300}\}.$$

It consists of 12 transformations: The first six, the upper row, are the six possible rotations; and the second six, the lower row, are the six possible reflections. In fact, if one compares the upper row carefully with the lower row, one discovers that *the members of the lower row are the members of the upper row, each multiplied by t.*

It turns out that this set of 12 elements forms a group. That is, the elements obey the three axioms A1–3. The group is called D_6. A corresponding group, D_4, exists for a square. In fact, any regular polygon, of *n* sides, has an associated group, D_n, describing its symmetries. The first *n* members of D_n are the *n* possible rotations and the second *n* members are the *n* possible reflections.

The above illustrates the fact that, if a stimulus set is symmetrically inter-related, then there is an associated group of transformations. The group is the structure of inter-relations. In fact, it will be argued later that groups are the reference frames that cognition uses to structure stimulus sets. An important point should now be recalled from Section 1.4: A central claim to be made is that *reference frames are decomposed into levels.* The consequence is that the groups used are decomposed. It will be useful therefore, to conclude this section by illustrating certain aspects of the decomposition of a group.

We return to D_6. The first six elements (the upper row of elements shown above) themselves form a group in their own right. This is a cyclic group Z_6. Cyclicity expresses the fact that applying r_{60} six times not only successively obtains each of the other rotations in turn—for applying r_{60} twice gives r_{120}, and applying r_{60} three times gives r_{180}, etc.—but that applying r_{60} six times returns one to the starting position.

The second set of elements (the lower row) in D_6 do not together form a group. Nevertheless, as noted above, they are the members of Z_6 multiplied by t. One can write this simply as tZ_6. Therefore we see that D_6 divides into two halves:

$$Z_6 = \{e, r_{60}, r_{120}, r_{180}, r_{240}, r_{300}\}$$

$$tZ_6 = \{t, tr_{60}, tr_{120}, tr_{180}, tr_{240}, tr_{300}\}.$$

Because Z_6 is a group, in its own right, one says that it is a *subgroup* of D_6. Furthermore, a set such as tZ_6—that is, a subgroup multiplied by one element—is called a *coset*. The multiplying element (in this case t) is called a *coset leader*. Observe that the subgroup Z_6 is also a coset; quite trivially it is eZ_6.

For any polygon of n sides, the associated group D_n consists of a cyclic subgroup Z_n and a coset tZ_n, in exactly the same way. The Z_n subgroup is the collection of rotations and the tZ_n coset is the collection of reflections. For example, according to mathematicians, the symmetry group, D_4, of the square is the set

$$D_4 = \{e, r_{90}, r_{180}, r_{270},$$

$$t, tr_{90}, tr_{180}, tr_{270}\}.$$

The upper row is the subgroup of rotations which is the cyclic group Z_4. The lower set is the coset tZ_4. The coset leader is again t.

Given a group G and a subgroup H, there may be several cosets g_1H, g_2H, $g_3H,...$ of H. These would constitute several "rows" of elements in the same way as D_6 above consists of two rows. The rows, or cosets, *partition* the entire group.

The notion of a coset will be useful to us later. I will argue, for example, that Gestalt groupings, in perception, are cosets. Furthermore, it can be seen quite easily that morphological classes in linguistics are also cosets.

4.2.2. An Object as Invariant

In the previous section, we saw how groups can be used to characterize symmetry. This will enable us quite soon to use groups to characterize symmetric *reference*. However, an associated concept will also be needed.

Recall the Content Machine Analogy which states that the cognition of a phenomenon is a description of the phenomenon as a machine. We have said that a machine consists of

(1) an **object**;

(2) an **input space**: a set P of inputs, or actions, which can be applied to the object;

(3) a **state space**: a set Q of states into which the object can be put by applying the inputs.

In Section 4, so far, we have been considering the nature of the second item, the input space. This will provide the referential structure of a representation. Let us now begin to consider what is cognitively understood as an *object*, the first item.

It is clear that if something can be followed as it undergoes changes of state, then it is recognized as an *entity* which, in some sense, remains present during the changes. In fact, there can be two very different answers to the question of what an entity, or object, is. The modern role of a group provides the first of these, and this answer has been adopted by all group-theoretic psychologists such as Piaget (1969) and Hoffman (1966, 1968). In this series of papers, I will oppose this view. However, it is necessary to recapitulate it here.

Consider the properties of the triangle in Fig. 6. For example, the figure is (a) three-sided, (b) closed, (c) equilateral, (d) pointing up, (e) positioned in the upper right. Note, however, that the first three properties are crucially different from the last two. They express the figure's identity as a triangle whereas the others appear incidental. For example, one would say that the figure would be the same if it were pointing to the left or positioned in the lower right area.

The problem then is to establish a means by which, given a property, one can distinguish whether it belongs to the first or second class. The solution was given by the great mathematician Klein (1893) who thereby laid the foundations of the modern view of geometry. He claimed that the distinction is made by an assumed group of transformations. Those properties which change under the transformations are regarded as incidental; those which remain invariant are regarded as essential. For example, in the above case, an allowable group had been assumed in making the distinction. It was that which consists of all possible, translations, rotations, and dilations acting on the figure. These operations can be seen as destroying properties (d) and (e) but leaving (a), (b), and (c) invariant.

An invariant property under a group of transformations is called, in mathematics, a *geometric object*. It is clear that, in an important sense, a machine object is that which remains invariant under changes of state, i.e., under the applied inputs. Thus because the input spaces are assumed here to be groups, the machine object, in this sense, corresponds to Klein's notion of the geometric object.

This view is useful when one considers *processing* aspects of cognition. With respect to bottom-up processing criteria, detecting the invariants across two stimuli is a first step needed before conjecturing some transformational relationship between the two stimuli. It is this idea that makes the concept of invariant identical to that of object in the transformational theories of Piaget (1969), Gibson (1966, 1979), and Hoffman (1966, 1968). The invariant is what is tracked; and the fact that it can be tracked makes it an object.
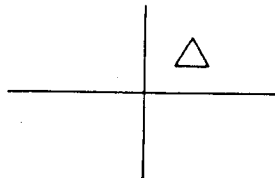

FIG. 6. A triangle with a number of properties, some of which are essential to its geometry (e.g., three-sideness) and some of which are not (e.g., being in the upper-right quadrant).

However, observe that this type of theory leads automatically to the classical view of category membership; i.e., that there is an invariant set of properties held by each member of a category. (Note that a category such as *bird* can be regarded as an object.) Rosch (1978) shows, however, that a category (i.e., object) is cognitively defined not by a set of properties invariant to all of its exemplars (instances), but by the properties of a *prototypical* exemplar (instance); i.e., this latter member has *more* of the characterizing properties than a non-prototypical member.

Thus although, in processing terms, invariants might be useful in keeping track of an object, in cognitive terms, they do not correspond to an object. This does not mean the group theory cannot define what is cognitively understood as an object. It only means that the wrong group-theoretic construct was chosen. Therefore, to account for Rosch's and other data, I will later give a very different group-theoretic definition of a machine object. Thus the term *object* will not be used in the sense defined by Klein, and the term *invariant* will be used instead for this sense.

DEFINITION 1. A **machine invariant** is any property which remains unchanged under the total collection of machine inputs.

These issues will be examined a number of times in this paper.

4.2.3. *Examples of Groups*

Recall that groups will enable us to characterize at least two fundamental aspects of a cognitive representation; the *reference* structure, and what is cognitively understood as an *object*. The present section is the final preparatory one, and will describe several of the groups to be used throughout the series of papers. In the following section, we can at last develop a characterization of symmetric reference.

(1) *The cyclic groups.* We have seen that any symmetry group D_n, of a regular n-sided polygon, contains a cyclic group Z_n, which rotates the polygon to its n equivalent positions. Cyclic groups need not arise only in such a context. For example, we met Z_2, the cyclic group of size 2, as the group of movements of a light switch.

A central property of any cyclic group Z_n is that it can always be represented by a clock with n positions around the circumference. For example, Z_{12} can be represented by the usual *day clock*. After reaching 11, one returns to 12 which is also the 0 position. Thus any cyclic group Z_n can be put in a canonical form:

$$Z_n = \{0, 1, 2, 3,..., n-1\}$$

where it is understood that one loops back to 0 after reaching $n-1$.

The largest cyclic group is

$$Z = \{..., -3, -2, -1, 0, 1, 2, 3,...\}$$

which is the clock of infinite circumference, or simply the infinite set of whole numbers.

(2) *Some continuous groups.* The groups we have been considering so far are *discrete* groups; e.g., they can be represented by *separated* notches on a clock. Many groups are *continuous.* That is, their elements form a continuum. Simple examples are as follows. More complex ones are used later.

(2i) *The group R.* The group R consists of all the numbers along a continuous infinite line. This contains all the whole numbers in Z above, and all the decimal numbers in between them. Furthermore, the entire set forms a group under addition. For example

$$7.351 + 1.002 = 8.353.$$

Note that Z is a subgroup of R.

(2ii) *The group* R^2. Not only can the line be turned into a group but so can the plane, which one can simply denote by R^2. Any point in the plane has coordinates (x_1, y_1) and can be added to any other point (x_2, y_2) by simply adding the x-coordinates together and adding the y coordinates together. Under this rule, the plane forms a group.

Recall that the group Z is a subgroup of R. Similarly, the group Z^2 (not the light-switch group Z_2) is a subgroup of R^2. This subgroup is the set of points which constitute the grid contained in the continuous plane. It is the set of coordinates which have only whole numbers as coordinate entries.

(2iii) *R and* R^2 *as translation groups.* If one understands a number x as an *instruction to move a distance x units,* then the groups R and R^2 can be understood as *translation groups.* For example, the element 5 (taken from R) can be understood as *move 5 units along a line,* and this can be added to the element 6 (meaning: move 6 units along a line) to yield 11 (meaning: move 11 units along a line).

Similarly, the element (x, y) taken from R^2, can be understood as *move x units horizontally and y units vertically in the plane.* These two-dimensional translations can be added to each other to form a group. What Section 3.4, in fact, tried to show is that a parameterization can be regarded as a machine, where the inputs form the group R^2, and the object (being moved about) is the single point.

(2iv) *The continuous cyclic group* SO_2. We examined earlier the discrete cyclic group Z_n; that is, where there are n equally spaced notches around the clock. If one goes to the continuous case, one has a continuous circle where one has all the points in between the notches. This means that all rotations r_θ are possible, where θ is any angle. Again, any two members of this set of rotations can be added as shown in Fig. 7. The set forms a group denoted by SO_2.
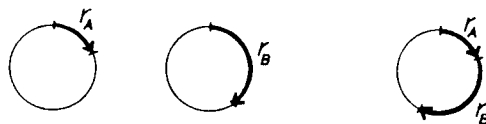

Fig. 7. Addition of members of the rotation group.

5. STATE-SPACE DESCRIPTIONS

Let us now return to the Content Machine Analogy, which is going to be the basis of the cognitive theory to be developed. The analogy is the claim that cognition is the attempt to describe phenomena as machines. Therefore, what is required is a proposal of the precise form such a description could take.

To obtain an intuitive view of what we are going to do, consider again Fig. 1b (which represents the Content Machine Analogy). In this figure, the head looks at a set of stimuli, S, and sees the stimuli as a machine, M. The way this is going to be represented is by using the following notation:

$$M \longrightarrow S$$

which simply reads, from left to right:

The machine M is "imaged" onto the stimulus set S.

(What is meant by the term "imaged" will be defined soon.) Since the above will be claimed to be general structure of a cognitive description or representation, the symbol D (*description*) will be used for the above sequence of symbols. That is, we have

$$D: M \longrightarrow S.$$

This notation simply reads:

The description D takes the machine M and "images" it onto the stimulus set S.

Note that each of the symbols in this map correspond to a crucial component of Fig. 1b.

We are now going to consider these statements more thoroughly.

Let us begin with an example. Suppose one is presented with a collection of stimuli such as those comprising a hexagon (Fig. 5a). We have seen in Section 4.2.1 how the entire hexagon can be regarded as an object to which a certain group of inputs, D_6, can be applied, i.e., rotations and reflections. This helps us towards understanding how the hexagon, as an object, can be regarded as a machine. However, there is another way in which the hexagon can be turned into a machine; and it is upon this latter way that we now concentrate. (In fact, both ways are cognitively relevant, as will be argued in the perceptual organizational theory). The latter way of describing the hexagon as a machine is a follows: Instead of letting the *object* of the machine be the entire hexagon, let the object be only a *side*. That is, the side is the object to which inputs (actions) are applied. Thus, rather than regarding the hexagon as comprising six sides, one views it as a single side being pushed into six states. The hexagon then becomes the set of alternative states, the state space, of the side. Finally, the changes between the states (i.e., changes between positions) of this object, are prescribed by the inputs. These inputs are in fact the 12 members of D_6 listed in Section 4.2.1.

Underlying the simple construction just described is, in fact, a viewpoint on objects, reference frames, etc., strongly akin to cognitive theories which have symmetric reference and the classical notion of a category. Before we can see this, it is necessary to clarify the above construction in detail.

Recall, from Section 3.1, that a machine consists of

(1) an **object**;

(2) an **input space**: a set P of inputs, or actions, which can be applied to the object;

(3) a **state space**: a set Q of states into which the object can be put by applying the inputs.

Here, for the hexagon, the following three entities have been assigned to the above three constructs:

(1) object   = a side;

(2) input space   = the 12 transformations in D_6;

(3) state space   = the 6 positions of the sides.

One can visualize more clearly what is happening here if one constructs the state-transition diagram for this machine, as follows. Recall the state-transition diagram for a light switch illustrated in Fig. 4. One can see from Fig. 4 that if one is given the state ON, then by applying the entire input group $Z_2 = \{0, 1\}$, one obtains the states reachable from ON. The same is true of the state, OFF. Similarly, if one is given a position for the side in the hexagon, then, by applying all 12 inputs (the members of D_6) to this state, one obtains all the states reachable from that single position. Figure 8 depicts the state-transition diagram for the hexagon seen as a machine. This diagram can be more clearly understood by comparing it with the state-transition diagram of a light switch (in Fig. 4). Recall that in Fig. 4 each box represents a different state and each input is an arrow leading from one state to another (i.e., causing a change of state). Figure 8 expresses the hexagon as a state-transition diagram in exactly the same way: The sides become the boxes representing states; the arrows between the sides are the inputs causing transitions between the states. In fact there should be 12 arrows starting at each side, because there are 12 inputs, i.e., members of D_6, that can be applied to any side. For legibility, however, Fig. 8 shows only three arrows emerging from each side. Thus, as an example, consider the *upper right side*. Three arrows have their stems originating at this side; (1) the arrow marked e, which both starts and terminates at that side (signifying no change); (2) the arrow drawn just below that, marked r_{60}, which starts at the side and terminates at the lower right side, and (3) the arrow marked r_{300} (drawn inside the shape and near the top), which again starts at that side but terminates at the top side. Nine more arrows should be drawn in, starting from this side. The same considerations apply to each of the sides. Thus the hexagon is represented as the state-transition diagram of a machine.
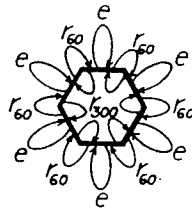
FIG. 8. A state space description of a hexagon.

From Fig. 8, it is important to understand that a set of stimuli, the sides of a hexagon, are the members of the state space of a machine. One can represent this by the symbols

$$D: Q \longrightarrow S$$

where Q is the state space of a machine, and S is the stimulus set. The string of symbols just given should be read from left to right and translated simply as:

The description, D, takes a state space, Q, and images it (via the arrow) onto the stimulus set, S.

It is now obvious that, in using the term *image*, no reference is meant to the phenomenon of *imagery*, i.e., internal visualization. The term will be used in the sense of *mapping onto*, as takes place, for example, when a set of coordinates are lain over, or imaged onto, a globe of the earth. (Of course, D is a mapping in the mathematical sense.)

Now note that all of the states of Q might not be instantiated in S. For example, the stimulus set might be perceived as a hexagon with one of its sides missing. So the symbol $\{0\}$ representing OFF, or EMPTY, has to be added to the right hand side of the map, in order to absorb those members of Q (the machine's state space) which are not taken up in S, when Q is imaged onto S. It is rather like a movie screen which has a hole in it even though the film which is projected onto it is complete. The symbol $\{0\}$ will be added to represent any "holes" in the stimulus set. Thus one has

DEFINITION 2. A **state-space description** of a stimulus set S is a map from the state space Q of a machine onto S. That is, a map

$$D: Q \longrightarrow S \cup \{0\}.$$

Now let us look at three psychological implications of state-space descriptions.

Symmetric reference. Observe that in the state-space description of the hexagon (Fig. 8), for every set of arrows relating one side to another, there is an exactly symmetrical set going in the opposite direction. For example, for each arrow r_{60} which sends one side to the next, there is an arrow r_{300} which sends the latter back to the former. That is, each side is symmetrically referred to any other side.

Non-explicit reference. The referential relations, i.e., the arrows, exist *in between* the stimuli, i.e., the sides. Thus the stimuli and the referential relationships are encoded as separate entities. Therefore, the stimuli are not explicitly encoded as the referential relations.

Objects as invariants. Observe that, in the state space description of the hexagon, every side is descriptively the same as any other side. There is nothing in the labeling of one side that is different from the labeling of any other. Each has exactly the same set of arrows. Therefore, the notion of *invariant* has some very real meaning here. Thus one can consider the *object*, which is pushed into the six positions, to be the invariant, that is, the side.

The consequences just discussed, of state-space descriptions, form the basis of several important psychological theories. For example, multi-dimensional scaling is a much used technique for creating a multi-dimensional space in which a set of psychological data points can be embedded. This resulting space is often supposed to represent the psychological space in which the points exist. A central assumption is that the distance between a pair of points A and B, in the space, represents the psychological similarity of the two data points. Consequently, the similarity of A to B is assumed to be that of B to A; because the distance and its reverse are the same (Shepard, 1962a, 1962b, 1974, 1980; Coombs, 1964; Kruscal, 1964; Torgerson, 1965; Guttman, 1971; Carrol & Wish, 1974; Carrol & Arabie, 1980).

Observe also that, in the space produced by multi-dimensional scaling, reference is non-explicit. That is, any stimulus in such a representation is not explicitly defined as its relationship to another stimulus (e.g., like off-red is defined as its relation to red.)

In the present series of papers, such theories will be characterized as being based on *state-space descriptions.* That is, a multi-dimensional space, of the type described above, will be understood as a machine in which the input group is a group of transformations (e.g., tranlations) that preserve distance.

However, it will be claimed that state-space descriptions are an invalid basis for cognitive theory. One reason is that in each of the areas to be examined, we shall see that symmetric reference is violated. For example, in Gestalt perception, an irregular shape is seen as a deformation of a regular one (e.g., Goldmeier, 1982); but not the reverse. Again, in categorization, Rosch (1975) has shown that non-prototypical members (e.g., the number 99) are judged as similar to the prototype (e.g., the number 100), rather than the reverse. Again, one can see that, in linguistics, sentences are considered to be related to a neutral sentence called a kernel, but not the reverse. Thus the assumption of symmetrical reference seems to be significantly violated where sufficient psychological investigation has taken place. (See also Tversky (1977) and Tversky and Gati (1982) for a further discussion and support of asymmetry in similarity judgement.)

Again, there appears to be evidence, from categorization, that a stimulus is cognitively characterized as a deviation from a prototype. For example, off-red is cognitively defined as a deviation of red (Rosch, 1975); i.e., off-red is understood as

its referential relationship to red. That is, the referent is *explicit* in the representation of the stimulus. This violates the second property, *non-explicitness*, given above for state-space descriptions.

Concerning *objecthood*, as was noted earlier, the theories of Piaget (1969), Gibson (1966, 1979), and Hoffman (1966, 1968) have taken as crucial the notion that an object is an invariant. However, in the detailed discussion to be carried out later, we shall see that in each of the areas to be examined, invariance is violated. For example, in perception, an object can be recognized even after deformation which has destroyed invariants. Again, in categorization, membership is not given by the fulfillment of a set of properties common to all members. Again, in linguistics, there is no invariant sentence common to a kernel and its transforms.

6. Input-Space Descriptions

We thus require a view of representation that will accord with the condition of asymmetric explicit reference, and will imply a view of objecthood that strongly predicts constructs such as prototype and kernel.

Recall that it was proposed, in Section 3.4, that the parameterization of a set is a description of the set as a machine. The argument, which was used to substantiate this claim, is parallel to the argument which led to a state-space description. As with the view given above of a hexagon, a parameterization causes a set of objects to be reduced to only one object moved into several states. The space of inputs also forms a group. In the case of a two-dimensional space, the inputs form the group, R^2, of translations on a plane. At any position, e.g. (5, 3), the entire space of inputs describes the effective procedures which connect this position to any other. This procedural space is implied by the label (5, 3).

However, what we failed to take into account was that the label (5, 3) implies something more strongly even than the above considerations. It implies that, given the set of effective procedures, there is one procedure which is particularly distinguished—namely, that procedure using the numbers 5 and 3; i.e., move right 5 units and up 3 units. The starting point for this procedure is the position (0, 0). From this concept, the implication is that any other point (x, y) will use the *same* position as a starting point for the corresponding procedure "move right x units, and up y units." Thus the implication, in the labeling provided by a parameterization, is that there is distinguished position at which the machine starts.

The choice of a particular starting state in a machine description has a crucial consequence: *All the other states can be labeled by the inputs which obtained it from the starting state.* The label (5, 3) is the input which obtains that position from the (0, 0) point.

As a second example, consider again the state-space description of the hexagon in Fig. 8. None of the six sides was chosen as a starting state. However, suppose that the top side were chosen as such. Then every other side could be labeled by the inputs which produced that side from the top one. Figure 9 shows this labeling. The

12 inputs are distributed around the hexagon (2 per side), showing which inputs were used to reach that side from the top side. For example, one of the labels of the upper right side is r_{60}, because it is obtained from the top side by rotating the latter 60° to the right. The other label of the upper right side is tr_{300}, because the side is also obtained from the top side by a 300° rotation together with a reflection about the central axis. Again, the top side itself is the image both of e and of t, for the following reasons: As was said, it is the image of e because it is the starting side. However, it is the image also of t because, when one reflects the top side about the vertical axis, one obtains the top side, only with its left and right halves reflected. Note finally, that the same input cannot be used for two sides, because an input cannot move a state (e.g., the top side) into two states; and the hexagon is labeled here only with respect to movements from the starting top side. (Note that this means there are only 12 labels instead of the 72 in Fig. 8).

What I have done above is to take the input group

$$D_6 = \{e, r_{60}, r_{120}, r_{180}, r_{240}, r_{300}$$

$$t, tr_{60}, tr_{120}, tr_{180}, tr_{240}, tr_{300}\}$$

and image it onto a stimulus set. That is, the 12 elements of D_6 were imaged onto the six sides of the hexagon.

Thus we obtain here a new view of what is meant by a description of something in terms of a machine. Instead of imaging just the *states* onto a set of stimuli, as we did in Definition 2, we image the *inputs* onto the stimuli:

DEFINITION 3.   An **input-space description** of a stimulus set, S, is a map from the input group G of a machine onto S; in fact, the map

$$D: G \longrightarrow S \cup \{0\}.$$

To simplify the notation, the set $\{0\}$ will often not be mentioned. One reads the string of symbols, $D: G \longrightarrow S$, from left to right simply as:

The description, D, takes the members of an input group G, and images them (via the arrow) onto the stimulus set S.

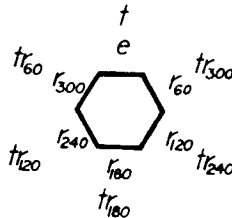In other words, Definition 2, that of state-space description has been replaced

FIG. 9.   An input space description of a hexagon.

here by Definition 3, that of an input-space description. It will be argued that the latter corresponds more accurately and usefully to the cognition of a phenomenon.

Before we continue, however, note that the map can be many–one. Figure 9 gives a map from the group D_6, of 12 elements, onto a stimulus set containing only 6 elements. This many–one property, and the direction of the map, seem to be in direct opposition to the contemporary coding-theoretic approaches, for example, that of Leeuwenberg (1971), which is aimed at converting a figure into a string of letters. I will attempt to show that the theory, which is to be developed from Definition 3, explains perceptual (and, generally, cognitive) phenomena in a way that does not seem possible using a coding approach.

Now let us see what an input-space description implies in terms of reference and objecthood.

Asymmetric reference. Consider the input-space description in Fig. 9. In this view, every side is asymmetrically, i.e., unidirectionally, related to the top side. For example, the label r_{60}, on the upper right side, means that the side is regarded as the result of rotation by $60°$. But this can only mean that the side is understood as starting in the top position. However, this relationship is asymmetric, because the label, $e =$ non-rotation, given to the top side, does not refer to the r_{60} side. Similarly, any other side is unidirectionally referred to the top side. Thus the entire description is an asymmetric one.

Explicit reference. The lable given to any stimulus (any side) is the referential relationship itself (e.g., r_{60}, for the upper right side). Thus the referential relation is explicit in the representation of the stimulus.

The object as group identity. Observe that, in the input-space description, no side receives the same labeling as any other side. For example, there is only one side that is the image of r_{60} and tr_{300}. Thus, every stimulus is descriptively different. This means that the notion of *invariant* becomes unimportant.

We therefore require a viable definition of *object*, that accords with the notion of input-space description. Recall that I took the machine object to be that which is acted upon by the inputs. Observe that in the input-space description of the hexagon (Fig. 9), the sides are defined as transformations of, that is *actions on*, the top side; the side labeled e. This must logically be the case for any input-space description: The stimuli are characterized as the result of actions on the stimulus labeled by the identity element. Therefore, generally, I propose the appropriate construct for *object* to be the *machine in the non-input or initial state*. For example, in the case of the hexagon, the machine object should be regarded as the top side. This therefore motivates the following:

DEFINITION 4. Given an input-space description $D: G \longrightarrow S$, the **machine object** is that member of S which receives the identity element of G [that is, $D(e)$].

Another example of a machine object is the point $(0, 0)$ in a parameterization of a plane, because $(0, 0)$ is the identity element of the two-dimensional group of the

plane. Again, in Roschian categorization, I will propose that the prototype is the machine object, because, as will be argued later, this receives the identity element in the input-space description that structures the category. When the notion of stability is discussed later, the identification of the prototype with the machine object will allow us to understand why the prototype has such a characterizing role with respect to the category. Again, in transformational grammar, the machine object will be the kernel sentence; because this receives the identity element in the input-space description that provides the transformational structure. Again, when stability considerations are brought in later, the characterizing aspects of the kernel will be explained by understanding the kernel as the machine object.

Much more will be said on this latter. Extensive support will also be offered of this proposal of the cognitive nature of an object.

Points of clarification. (1) For illustration in this section a very simple shape, a hexagon, was chosen. The second paper (i.e., on perceptual organization) develops the framework considerably further to handle highly complex shapes, e.g., the shapes of animals, birds, plants. The only reason these examples have not been examined here is that one requires several theoretical developments from the second paper to describe the corresponding analysis of such shapes.

(2) Some computational issues: Observe first that input-space descriptions are specified by considerably less bits of information than state-space descriptions, for this reason: In a state-space description, each stimulus in the representation must carry with it one copy of the entire input space. In an input-space description, the entire stimulus set shares only one copy of the input space. Other computational issues are handled in the footnote.[6]

(3) It should be understood that at no time will the term *input* refer to the stimuli which enter the organism. To understand this more clearly, consider what has been done in the previous two sections. We began with Fig. 1b which shows a head looking at a stimulus set and seeing a machine. The machine is in the "cloud" on the left. Now, any machine has two substructures: a state space and an input space. This means that what is on the left of Fig. 1b, "in the cloud," has two substructures:

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[6] Two computational questions suggest themselves: (1) How many groups does the cognitive system need to form the many descriptions it does? The answer I have proposed in Leyton (1986) is beyond the mathematical level of the present papers, but concludes that in fact one needs very few. Speaking technically: In the discrete case, the cognitive system only needs free groups; because any discrete G is the homomorphic image of a free group F. All that is required to obtain any G is to specify those elements in F which become equated with the identity element. Second, one requires only a small number of free groups, perhaps only three of four: those that correspond to a generator set of size 1, 2, 3 or 4. This is because, from such a small set, sufficiently complex groups can be constructed. An equivalent set of considerations exist for the continuous case, by substituting the notion of Lie Algebra, L, for free group; i.e., any required group G is then specified by factoring a discrete central normal subgroup from the universal covering group associated with L. (2) Given a set of alternative descriptions that the cognitive system can apply to a stimulus set, by what criteria does the system make its choice of description? This is a question of the *evaluation* of alternative structural descriptions, and is a different issue from the nature of a structural description—the subject of the present paper. The separate subject of evaluation is to be handled in the final paper of this series.

a state space and an input space. What we have asked is how these are sent (imaged) onto the stimulus set on the right of Fig. 1b. Two possibilities were considered: (1) The machine's *state space* is sent directly onto the stimulus set. (2) The machine's *input space* is sent directly onto the stimulus set. It is the latter alternative that will be taken as the basis of the cognitive theory of these papers.

7. The Description Postulate: Algebraic Statement

Recall that the Content Machine Analogy is the claim that cognition is the description of phenomena as machines. Recall also that I have taken the machine here to embody two types of structure: algebraic structure and stability. Thus the Content Machine Analogy has essentially two components corresponding to these two forms of structure. I now claim that Definition 3 provides the first component. That is, one is able to state with greater detail:

> Description Postulate Algebraic component: *The cognition of, or extraction of information from, a phenomenon is a map from the input group of a machine onto the phenomenon; i.e., an input-space description of the phenomenon.*[7]

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[7] The Content Machine Analogy claims that cognition is the description of a phenomenon as a machine; but the Description Postulate claims that cognition is the description of a phenomenon *only* in terms of an input group. Thus the latter might seem to ignore most of what comprises a machine and therefore not seem to be a full realization of the Content Machine Analogy. In fact, this is not the case. Recall that any machine can be redescribed as a *state output machine* (i.e., the set of outputs can be understood as a component of the state space). Thus, in Section 3.1, we defined a machine as comprising three components: (1) an object, (2) an input space, and (3) a state space. In Sections 5 and 6, we have shown that a state-space description and an input-space description each define two different ways in which *all three components* are mapped onto the stimulus set. In particular, an input-space description maps all three components onto the set as follows. First, the input-space description explicitly maps the input group to the stimulus set. Second, recall that the machine object was defined a priori as the entity on which the inputs act. Thus we argued that, due to the inherent asymmetry of an input-space description, this entity is the stimulus receiving the identity element. Third, because the input-space description is essentially a description of the stimulus set as an initialized *state space*, the input-space description implicitly maps the state space onto the stimulus set. Thus *all three components*, comprising a machine, are mapped onto the stimulus set via an input-space description; and the Description Postulate can truly be regarded as a full realization of the Content Machine Analogy.

It might be thought, however, that we could replace machine-theoretic concepts completely with algebraic ones. For example, it might be thought that the term "algebraic identity element" could be used instead of "machine object." However, such a replacement would severely hinder one of the main purposes of our discussion. First, recall that we are trying to characterize what is cognitively understood as an object. Thus we cannot abandon the term *object*. Second, we are trying to see how different theoretical viewpoints characterize the cognitive nature of an object. In viewpoints of the type we have understood as based on state-space descriptions, the object is realized as an invariant whereas in input-space descriptions, the object is realized as an algebraic identity. Thus if we spoke not of objects but of algebraic identities, we would not be able to make the required comparison between theoretical systems.

Observe that the postulate states that the cognition of something is a *map*, $D: G \longrightarrow S$ (recall Definition 3, and its relation to the postulate). The map consists essentially of *three* components: a machine group G, a stimulus set S, and a correspondence D which sends (i.e., images) the members of G onto the members of S. G will turn out to be the reference frame that structures the stimulus set. Observe that a cognition of S will change if a different G (reference frame) is used. But observe that an alteration in the cognition of S can occur without altering the G. For, even though G and S might be fixed, the mapping correspondence D, between G and S, might alter—thus altering the description; and as was said above, it is the description which defines the cognition of S. Examples where G and S are fixed, and D is altered, will be given throughout this series of papers. For instance, in the square/diamond effect, one has the same underlying stimulus set S and the same reference frame G; however, the frame G is sent onto S in one of two different ways, thus creating a square in one case and a diamond in the other.

8. REFERENCE FRAMES: ALGEBRAIC THEORY

8.1. *Introduction*

Let us now see how the machine-theoretic view being developed may provide a theory of cognitive reference frames. Assuming that a reference frame can be regarded at least as a set of objects, the term "reference" is usually used in one of two senses: (1) as the act of embedding an object into the set, i.e., referencing an object into the frame, or (2) as the act of judging or defining the objects within the frame with respect to each other, i.e., referencing the objects to each other. In what follows, the former will be called *external reference*, and the latter will be called *internal reference*. It will be a central purpose of these papers to provide a theory of reference frames such that, using either sense of reference, the term *reference frame* denotes the same cognitive structure. That is, one references (embeds) objects into structures where the objects are referenced (judged with respect to) each other. However, to make the discussion simpler, in what follows, although I will be speaking about the construct of *reference frame* in both senses, when the term *reference* is mentioned separately, it will be used in the sense of internal reference.

The principal usage of the term reference frame, in psychology, seems to be to explain the different descriptions, given by subjects, of the same set of stimuli in different circumstances: the researchers argue that the stimuli are described differently because they are assigned to different "frames." In some sense, therefore, frames are considered, in these situations, to be crucial in determining the descriptions. Let us review some examples.

(1) *Orientation and form.* A number of researchers have shown that the perception of the shape of a figure can be different if the figure is assigned different underlying Cartesian frames [see Rock (1973) for an extensive discussion]. The classic example, due to Mach (1897), is that of the phenomenological difference
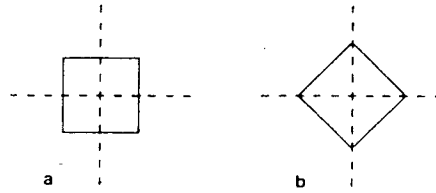
FIG. 10. (a) A square. (b) A diamond.

between a square in its horizontal orientation Fig. 10a, and in a 45° orientation where it ceases to be square-like, and becomes diamond-like (Fig. 10b); that is, the angles need no longer be 90°. The effect of orientation on form was subtly investigated in a remarkable series of studies by Goldmeier (1937/1972)—where it was demonstrated to be a theoretically substantial phenomenon. For example, in Fig. 11, Goldmeier showed that the central figure is perceived by subjects to be dissimilar to the outer two figures (Goldmeier, 1972, p. 88), even though the three are geometrically identical. Rock (1973) has extensively analyzed the relationship between the various frames which are involved in the perception of shape, i.e., the Cartesian system defined by the retinal frame, the gravitational or ecological frame, the bodily frame, etc. One should note that the importance of the orientation and form studies is that they show that perception is a process of *describing* stimuli, rather than representing the outside world—that is, there is no sense in which the perceptual structure corresponds to a structure in the real world; the only structure that occurs is that yielded by the cognitive system—an issue I will attempt to make rigorous in this series of papers.

(2) *Binford and Marr–Nishihara shape description.* Marr and Nishihara (1978) have claimed that the perceptual description of shape (e.g., the shapes of animals) has several levels of detail of a specific sort: Any level is the construction of the figure as a concatenation of approximately cylindrical modules, in the manner described by Binford (1971), with the specific relative widths and lengths specified (Fig. 12). Such a description is achieved by assigning a collection of object-centered local reference frames to regions of the stimulus configuration. One axis of each local frame becomes the rotation axis of the associated cylinder.

(3) *Induced motion.* Duncker (1929) has shown that the perceived speed of a moving object depends on the assigned reference frame. For example, Fig. 13 shows a rectangular frame moved relative to an observer, while a point inside the frame is
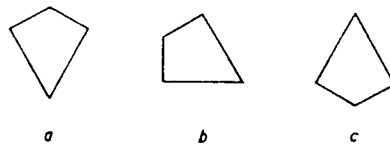

FIG. 11. An example of the orientation-and-form phenomenon, due to Goldmeier (1937, 1982): The outer two figures are judged as having greater similarity to each other than to the inner figure.
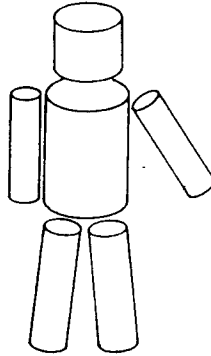
FIG. 12. An example of Binford/Marr–Nishihara shape analysis.

kept still. Duncker nevertheless found that the point is perceived to be moving and the frame is perceived to be at rest. Thus the rectangle is said to provide a reference frame for the moving point.

(4) *Johansson motion perception.* Consider Fig. 14a, two stimuli moving perpendicularly and in phase. Johansson (1950) showed that, if only one is presented, it is seen as a stimulus moving along its path of motion. However, he showed that if both are presented, they are seen as the ends of a diagonal "invisible rod" being stretched and contracted along its length, while moving in the opposite diagonal direction (Fig. 14b). This means that the configuration of stimuli form a reference frame which defines the stimuli in terms of each other.

(5) *Musical systems.* As has been known for centuries, musical relationships produce a reference structure (e.g., a structure of resolutions) which, at varying points in the musical work, are assigned by the perceiver, in varying ways, to the same auditory unit to alter its perceptual effect (e.g., the same diminished 7th chord can imply a number of different resolutions depending on context).

(6) *Pitch structure.* In several of the world's musical systems (both melodic and harmonic) a distinguished tone, a "tonic," is chosen as a reference point which defines the perceptual values to be taken by the other pitches used. The pitch scale thus becomes a reference frame.

(7) *Categorization.* Rosch (1975, 1978) has proposed that natural categories have reference point stimuli (prototypes) with respect to which the other category members are judged. For example, in the category of colors, focal colors act as reference points (e.g., off-red is referenced to red); in the category of line-orientations, the

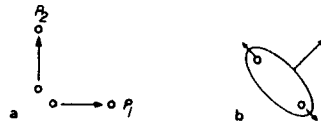FIG. 13. A movement of the outer rectangle induces perceived motion of a stationary point (Duncker, 1929).

Fig. 14. An example of the Johansson (1950) motion phenomenon: (a) Two points P_1 and P_2, moving perpendicularly and in phase, are actually seen as (b) two points moving to and from each other along a diagonal line that is itself seen as moving.

reference points are vertical and horizontal orientations; in the category of numbers, the reference points are multiples of 10 (e.g., 99 is referenced to 100). Rosch further showed that the reverse references do not occur (e.g., 100 is not referred to 99). Thus there is an asymmetry in the reference process.

From the above examples it appears that the term reference frame possesses a number of quite different specific meanings: In most shape and orientation studies, it appears to mean a cartesian coordinate system; with respect to the Duncker phenomenon, it is a rectangular surround; in the Johansson motion studies, it becomes a configuration of stimuli (Fig. 14a); in the auditory cases considered, it has been used to denote a musical system of development, and a pitch structure. Again because Rosch claims that members of categories are referenced to prototypes, one can regard categories as reference frames.

However, during the course of this series of papers, I will try to show that all the above phenomena are examples of exactly the same precise structure: namely, an input-space description. To do this, the argument will require (1) further theoretical analysis of the structure of input-space descriptions, and (2) a number of theoretical developments in the areas concerned. Much of the remainder of Part I will be devoted to the first of these topics. This discussion will eventually allow the development of a more detailed statement of the Content Machine Analogy, which, as we shall see, makes the concepts of cognition and reference frame strongly interdependent.

8.2. Reference as an Algebraic Phenomenon

8.2.1. Simple Reference

One important advantage of input-space descriptions to our theory is that, unlike state-space descriptions, they have an asymmetric *reference* structure. The rest of this paper is a deeper analysis of this asymmetry.

Recall that under an input-space description each stimulus is asymmetrically defined with respect to the stimulus labeled by the identity element. That is, any stimulus in the set is *referred* to the element labeled by e. For example, in the input-space description of a hexagon (Fig. 9), the r_{60}-labeled side is defined with respect to the top side, thus complying with the meaning of *internal reference* (beginning of Sect. 8.1). Hence, under the input-space description, the hexagon is structured by a reference frame. In this frame, the top side is the reference point.

The content of the next two paragraphs are crucial to our discussion:

Observe that one references the upper right side to the top one by *removing* or *factoring* the transformation r_{60}, which obtained the upper right side. This *removal* or *factorization* returns one to the top side. An alternative way of understanding this is that one is *retracing the generation algorithm* that lead from the top side to the upper right side. That is, the side was *generated* from the top side by r_{60} and one is simply retracing that generation path, or algorithm. Thus, under this view: *The structure of reference is the reversal of the structure of generation.*

As a second example, recall that, under a parameterization, a point labeled, for example, (5, 3) distinguishes a particular algorithm, "move across 5 units and up 3 units," with a particular starting point (0, 0) for that algorithm. Furthermore, the assumption is that any other point (x, y) also *refers back* to that labeled by (0, 0). Thus, a parameterization is a reference frame. Observe that once again the reference point (0, 0) is labeled by the identity element—this time, of the two-dimensional translation group, R^2. Furthermore, referencing (5, 3) to (0, 0) is the removal or *factorization* of the transformation sequence, "across 5 and up 3 units," from the point (5, 3). Equivalently, it is the reversal or retracing of that generation algorithm.

Thus, according to a theory built on input-space descriptions, one would have:

PROPOSAL 5. *In human cognition, internal reference, or judgement-with-respect-to, is the factorization of an input group to the identity, i.e., the retracing of a generation algorithm.*

DEFINITION 6. The factorization of a group to the identity is called **simple reference**.

However, there is a fundamental cognitive phenomenon that I believe exists in the reference structures which have just been described, and defines what I will propose to be equivalent to the referencing process. In order to see what this phenomenon is, we will consider a more complicated reference frame than the two which have so far been described.

8.2.2. *Proposing the Existence of Successive Reference*

In Leyton (1982), I proposed that an input group G is cognitively stratified into levels

$$G = G_1 \cdot G_2 \cdots G_n \qquad (8.2.1)$$

where each level G_i is assumed to be an input group. I argued that reference acts by removing each level G_i in turn, from the last level G_n down to the first G_1. Thus, whereas in the previous section, we simply had reference as the removal of a single group, now we have reference as the removal of a sequence of groups. Observe that because each G_i has its own reference point e_i, what one obtains is a sequence of reference points $e_n, e_{n-1}, ..., e_1$, as one goes down levels.

In order to illustrate these concepts, I will describe here a very simple experiment and analyze the results in terms of the above concepts. The reader should note that the experiment is used here only to illustrate these concepts. The actual corroboration of the above proposals can be given only in the subsequent papers where I concentrate in detail on four cognitive areas in turn.

In Leyton (1984a, Experiment 1), I presented subjects with a parallellogram, as shown in Fig. 15b. They asked which *other* abstract geometrical figure came to mind. The usual answer given was a rectangle. They were then presented with a rectangle and asked which other abstract geometrical figure came to mind. The usual answer was a square. The statistical significance with which the entire sequence, parallelogram → rectangle → square, occurred was considerable. [$n = 12$; 8 subjects created this sequence; each remaining subject chose a different sequence. Taking the severely stringent assumption that only the five *chosen* sequences are possible, expected mean = 2.4 per sequence. The significance of the score of 8 is thereby $t(11) = 3.289$, $p < .005$, one-tailed.]

To restate the result, the subjects constructed the sequence of three shapes represented by Figs. 15b, c, and d, where the next figure in the sequence was the one which was *called to mind* on the presentation of the previous figure.

The question which concerns us here is what the subjects were actually doing in calling to mind the next shape. My claim is that they were in fact *judging* the presented figure with respect to the one which they had reported had come to mind. In fact, I claim that the reference sequence can be extended further back to an earlier starting point, Fig. 15a, where the parallelogram has been rotated off the horizontal line. Starting at this earlier stage was not possible in the first experiment because the calling-to-mind task referred to a *different shape*, and Figs. 15a and b are not distinguished by shape.

Thus the next experiment tested four successive percepts. My hypothesis was that *subjects judge the parallelogram in Fig. 15a with respect to that in Fig. 15b; and the latter is, in turn, judged with respect to a rectangle; and this latter figure is, in turn, judged with respect to a square.* Thus one obtains the full sequence shown in Fig. 15. Experiment 3 in Leyton (1984a) was designed to test this. Subjects were presented with the successive pairs along the sequence: i.e., ⟨rotated parallelogram, parallelogram⟩; ⟨parallelogram, rectangle⟩; ⟨rectangle, square⟩. They were asked which member of each pair was the one which they judged the other with respect
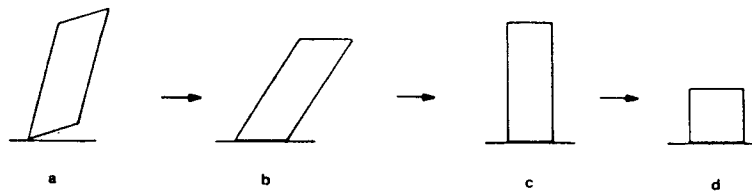

FIG. 15. A successive reference phenomenon: The rotated parallelogram is referred to a non-rotated one which is referred to a rectangle which is referred to a square.

to. The claim was that they would always choose the second member in each of the pairs (as listed above), thus establishing the hypothesized sequence. (Order effects were controlled for by appropriate randomization.) This claim was corroborated with considerable statistical significance [$n = 12$; there are eight possible pairwise orderings; expected mean $= 1.5$ per ordering; hypothesized ordering occurred nine times; $t(11) = 4.771$, $p < .005$, one-tailed].

The reader should note that although we began (in the first experiment) with a sequence of three figures, we have now extended the sequence to that of four. It is this latter full sequence that will be discussed for the remainder of this section.

In what way can these experiments be used to illustrate the concepts developed in the present paper?

Recall the Description Postulate, which states the cognition, or the extraction of information, is an imaging of an input group onto a stimulus set (Sect. 7). In the experiment considered, I will argue that the relevant group is the one called $SL_2 R$, which is defined as follows: $SL_2 R$ is a group of transformations of a particular type: First, the transformations are linear, which means that straight subspaces (e.g., lines) are sent to straight subspaces. Second, the transformations preserve area.

What is important now to observe it that it can be mathematically demonstrated that the group can be decomposed into three subgroups. These three subgroups are illustrated in Figs. 16a, b, and c as

(1) SO_2, the group of rotations (Fig. 16a);

(2) N, the group of shears; i.e., where rectangles are mapped to parallelograms (Fig. 16b);
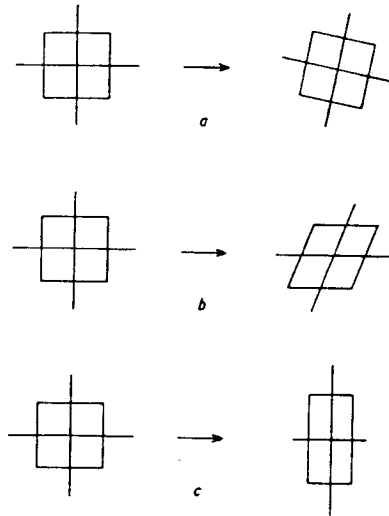

FIG. 16. Three subgroups of $SL_2 R$: (a) The rotation group; (b) the shear group; and (c) the pure deformation group.
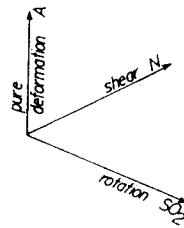
FIG. 17. A representation of $SL_2 R$ (near the identity element).

(3) A, the group of pure deformations—transformations which stretch and contract along perpendicular axes, i.e., map squares into rectangles[8] (Fig. 16c).

These three subgroups, into which $SL_2 R$ factors, are each given by one parameter, respectively, (1) the angle of rotation, (2) the angle of shear, and (3) the ratio of stretch along the axes.[9] The three parameters can thus be represented by the three axes in the three-dimensional space shown in Fig. 17. Therefore, this space will be used to represent $SL_2 R$ itself. That is, any point in the space is a transformation that is defined by a shear, stretch, and rotation value.

What I will now try to show is that, given the rotated parallelogram in Fig. 15a, $SL_2 R$ provides the reference frame for this stimulus, and that it is this frame which produces the results. Figure 18 shows the group $SL_2 R$ as previously depicted in Fig. 17, but with some extra letters and lines which will now be explained. We begin by representing the rotated parallelogram shown in Fig. 15a by the point, x_1, in the group shown in Fig. 18. This point represents a quantity of rotation, shear, and pure deformation, because the point has a value on each of the three dimensions. Now the first reference is that from the rotated parallelogram to its non-rotated version (Fig. 15b). We can identify the non-rotated version with the point x_2 in the group as depicted in Fig. 18. Thus the reference from the first figure to the second is modeled by the movement in the group from point x_1 to x_2 along the bold line shown, because the bold line is parallel to the axis representing the subgroup SO_2 of rotations in Fig. 18. Thus we have started at the point x_1 and moved from it along the rotations dimension, in the direction of *decreasing rotation* until we reached the zero point on the rotations dimension, that is x_2. At this point all rotation has been completely removed (factorized) from the percept, and one obtains the non-rotated parallelogram (Fig. 15b).

The same applies to the remaining stages of the referencing process. At x_2, one has the parallelogram in its second orientation, and this is referenced to the straightened version, the rectangle. One can model this reference by a trajectory in the group in Fig. 18. This second reference stage starts with the figure at point x_2. By moving along the bold arrow from x_2 to x_3, one is going in the direction of *decreasing shear*—i.e., parallel to the axis marked N—until one reaches point x_3,

[8] The term *pure deformation* is used in a standard text such as Guggenheimer's (1977, p. 95).

[9] In fact, I will use the symbol $SL_2 R$ to stand for *any* three-parameter group, where the parameters are those just defined. Area preservation need not be complied with. Pilot studies of mine have indicated that the *intention* of some subjects is to preserve area, whether or not this is actually accomplished.
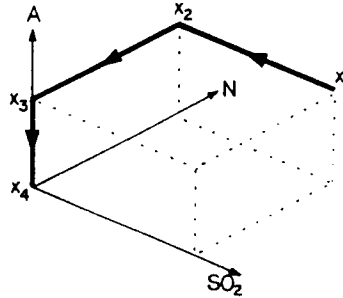
Fig. 18.   The successive shape reference of Fig. 15 represented as a trajectory in $SL_2 R$.

where shear has a zero value on that dimension. The point x_3 is thus the stimulus, the rectangle.

Again, the same applies to the final reference stage where the rectangle is referenced to its nonelongated version, a square. In this case the relevant group is A, the pure deformation group which is the vertical dimension in $SL_2 R$ as depicted in Fig. 18. Thus the referential action is that of moving down this subgroup, i.e., along the bold arrow from x_3 to x_4. The point x_3 in Fig. 18 represents the rectangle in Fig. 15c and the point x_4 represents the square in Fig. 15d.

Piecing together these stages, one observes that the complete successive reference example in Fig. 15 is given by the entire bold trajectory in Fig. 18. The next section offers a general reason why the ordering occurs, and the next paper offers a specific translation of that reason into perceptual organizational terms.

However, here, what we need to observe is that movement along each segment of the trajectory is equivalent to the removal of one of the dimensions, i.e., one of the groups. Thus, it appears that reference acts by the *successive factorization* of the groups, SO_2 (rotation), N (shear), and A (pure deformation), from the full group $SL_2 R$; that is, the input space is decomposed as

$$SL_2 R = A \cdot N \cdot SO_2 = G_1 \cdot G_2 \cdot G_3$$

which is an example of the general structure defined in Equation (8.2.1). Furthermore, the removal of any of these group factors is equivalent to referencing back to the point labeled by the identity or zero element in that group; i.e., within any group factor one has simple reference. Observe now that, because the rotated parallelogram has values in each of the three dimensions, it has coordinates $g_1 \cdot g_2 \cdot g_3$. However, as the referencing occurs, each of these elements is replaced successively by an identity element in one of the subgroups. That is, one has

$$\text{rotated parallelogram} = g_1 \cdot g_2 \cdot g_3$$

$$\text{parallelogram} = g_1 \cdot g_2 \cdot (e_3)$$

$$\text{rectangle} = g_1 \cdot (e_2 \cdot e_3)$$

$$\text{square} = (e_1 \cdot e_2 \cdot e_3).$$

Therefore, from the above discussion, one appears to have here an illustration of the *successive reference-points phenomenon*, which was proposed at the beginning of this section. That is, the reference process goes successively to the identity elements, e_3, e_2, and e_1, of the groups, SO_2, A, and N.

9. STABILITY

9.1. *The Cognitive Stability Principle*

So far in this paper we have done the following: Two alternative realizations of the Content Machine Analogy were proposed: state-space descriptions and input-space descriptions. Because input-space descriptions have asymmetric explicit reference, it was this alternative that was chosen. Next, the notion of successive reference was incorporated into input-space descriptions by inducing a *particular decomposition* and a *particular ordering* on the domain G, of such descriptions. Thus, we obtained a structure $G_1 \cdot G_2 \cdots G_n$ which is *asymmetric, explicit,* and *sequential.*

An important theoretical problem is evident concerning the structure, $G = G_1 \cdot G_2 \cdots G_n$: What determines the *particular decomposition* that is chosen by a cognitive system, and what determines the *particular ordering*? For example, there is no a priori reason, in the $SL_2 R$ example, why the decomposition had to be into the above subgroups, and why these had to have the ordering that was found. In order to settle this question, let us turn to the other type of structure which has been assumed here to be embodied in the notion of machine: stability.

In these papers, the *stability* of a property will be taken to be its *observer-defined persistence.* Conversely, *instability* will be taken to be *observer-defined transience.*

It is a fact, little understood outside mathematics, that the most important criterion determining all mathematical modeling is stability. The mathematical arguments invoking stability have a particular form presented in, for example, Hirsch and Smale (1974); Marsden and McCracken (1976); and Thom (1975). As the mathematicians Marsden and McCracken (1976, p. 3) declare: "It is only *stable* mathematical models, or features of models that can be relevant in 'describing' nature." Essentially, as Thom (1975, p. 320) argues: nothing can be *known* unless it is stable. The claim made here is that this proposal can be regarded as stating that information is extractable only from a stability. That is:

PRINCIPLE 7.   *Instabilities reduce information.*

Since one can assume that the cognitive system attempts always to maximize the extraction of information, then based on Principle 7, I propose:

PRINCIPLE 8.   *Given a description, D, of a phenomenon, the cognitive system attempts to remove instabilities from D, leaving stabilities.*

Although an extensive attempt will be made to corroborate this principle in each of the cognitive areas to be examined in these papers, one may observe here that the principle can be regarded as the basis of all constancy phenomena. For example, the size changes which an object undergoes, due to its different distances from the observer, are unstable whereas the "actual" size of the object is not. Thus the size changes are *removed*, leaving the impression of actual size. Again, in the field of color constancy, an experiment by Hering (1905) may be considered to illustrate the principle particularly clearly. Hering demonstrated that a fuzzy grey shadow-spot cast on a piece of white paper still allows that part of the paper underlying the spot to be perceived as white. However, if a line is drawn around the spot such that the contour is coincident with the shadow's penumbra, the underlying section is perceived as grey. Using Principle 8, one can give an interpretation to these results in the following way: In the former case the shadow is perceived as an instability, due to the supposed incidental positioning of the paper with respect to the light sources. Therefore it is factored from the color of the paper. (This is the equivalent argument to the above size constancy argument.) However, in the second case, the boundary is regarded as a *stable* cue signifying the partition of the paper into different colors. Thus the grey is not factored as it is in the former case.

Two crucial points are evident from the last example. First, it is the observer that defines the stability values of properties of the representation. The reader should note that the specific *criteria*, used by an observer in making this decision, will be examined in the subsequent papers under the specific area of cognition where each criterion is cognitively used. However, as far as we are concerned in the present paper, the criterion can be completely arbitrary; what will be important for us first to recognize is that the cognitive system has a *choice* with respect to the value to be assigned. Thus, as we saw in the case of the grey shadow-spot, the observer *chooses* either one of, at least, two different stability values for the stimulus.

The second point to be made is so important that it occupies us for the remainder of this paper: The claim to be put forward here is that *the choice of stability value has structural consequences upon the cognitive representation*. That is, while, as far as this first paper is concerned, the criteria for deciding upon a stability value can be arbitrary, what concerns us is that, once a set of stability (persistence) values have been decided upon, there are resulting structural effects upon the representation.

An illustration of this was given in the shadow-spot experiment. The sheet is structured differently depending on the stability value assigned to the shadow spot. That is, the structural consequence of regarding the shadow-spot as unstable is that it is not seen as part of the paper's surface; i.e., it is removed or partitioned from the paper's surface. The organizational consequence of regarding the shadow-spot as stable is that it is seen as being a part of the paper's surface.

Before the structuring effect is illustrated further, let us gain a better understanding of the nature of the structurations.

What is now taken into account is that one property of a phenomenon can have *relatively* greater or lesser stability in comparison with another property of the

phenomenon. Putting the concept of relative stability together with Principle 8, I propose that properties with greater instability are removed by the cognitive system before properties with lesser instability. This conclusion will be important to the theory being developed and is stated as follows:

COGNITIVE STABILITY PRINCIPLE: *Given any phenomenon, S, the cognitive system attempts to remove its properties in order of decreasing instability.*

The ordering provided in this principle will be called the *stability ordering.*

Recall now that this section began by asking what determines the particular decomposition $G_1 \cdot G_2 \cdots G_n$ of the input group, G, used in an input-space description; and the particular ordering of the levels, G_i. The claim to be made now is that the principle just given determines these structural aspects. That is, the decomposition $G_1 \cdot G_2 \cdots G_n$ is a decomposition into stability levels. Furthermore, the group G_n is cognized as the least stable and so is removed first; level G_{n-1} is cognized as the next most unstable and therefore is removed next; and so forth down to G_1, which is removed last because it is recognized as the most stable.

The shadow-spot example is too simple to illustrate this. However, as an immediate illustration let us return to the succession of four figures found in the experiments in Section 8 (Fig. 15). The reader will recall that when a subject is presented with a parallelogram, he or she calls to mind a rectangle, and when presented with a rectangle, he or she calls to mind a square. As a subsequent experiment [Experiment 4 in Leyton (1984a)] a separate group of subjects were asked a different type of question concerning these shapes. They were presented with the parallelogram and asked to state which property of the figure's shape was "most unstable, transient, or likely to change." With statistical significance, the chosen reply for this figure was its *slant* [$n = 12$, $p < .003^{10}$]. A further group of subjects were asked the same question concerning the rectangle. With statistical significance, the chosen reply for this figure was its *height* [$n = 12$, $p < .019$]. The next paper will develop a specific perceptual-organizational reason why this ordering occurred. However, here it is sufficient to observe that the "persistence" judgement task had a significant effect.

Two points of comparison, between the first experiment and the one just described, are worth noting.

(1) Observer that the two experiments asked fundamentally different questions. In the former case the subjects were asked a question concerning a figure *other* than the one presented, i.e., which other figure was called to mind. In the second case, no second or hypothetical figure was alluded to. The subject was ased a question concerning properties of the *presented* figure; i.e., which was the most unstable property. Note furthermore that the issue of a property of the presented figure did not arise in the first experiment.

(2) Comparing the two sets of results, one observes the following interesting

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[10] A binomial table was used because subjects gave only two categories of replies.

phenomenon: In the first experiment, when presented with a figure, the subjects called to mind another figure which was the same as the presented figure except for one property (e.g., when presented with a parallelogram, they called to mind a rectangle, which is the same as the parallelogram except for the slant). It was just this property that the second group of subjects had independently chosen as the most unstable of the presented figure. Thus, in calling to mind a hypothetical figure, the subjects in the first experiment had removed the instability identified in the second experiment.

A crucial point to observe now is that there was a *successive* removal of the instabilities. In the first experiment, the initial figure in the series was a parallelogram. A parallelogram has both *slant* and *height*. However, the subjects factorized these *in succession* rather than together. Furthermore, the same order of factorization was consistently chosen.

One therefore sees here what appears to be an example of the Cognitive Stability Principle: that cognition attempts to remove properties in order of decreasing instability. In fact, because of the transformational relatedness of the figures, one can consider the subjects as moving progressively to *models* of greater and greater stability; that is, *describing* the percept as *something else* with an instability (e.g., a parallelogram is a rectangle with slant).

This type of phenomenon will appear repeatedly throughout this series of papers. For example, in the second paper, we shall see several more such examples in shape perception. Again in transformational grammar, it will be claimed that the transformational system is partitioned with respect to stability and the components are factored in order of decreasing instability. Again it will be claimed that the levels of a planning hierarchy are induced by a stability ordering and one obtains the levels of ever more major ground work by successively factorizing the levels in order of decreasing instability (recall the overviews given in Sect. 1.4 and Sect. 2). It is the general nature of the occurrence of this factorization phenomenon that encourages us to give it central importance in the remainder of this paper.

Although the SL_2R example will be discussed in much greater detail in the perceptual-organizational paper, one further test should, however, be mentioned here. Even though, it was concluded above that the subjects were successively factorizing instabilities, i.e., that the results corroborate the Cognitive Stability Principle, we still have to check that subjects were going to what they regarded as progressively more stable constructs. For example, there is still the possibility that although the subject factorizes one instability, its actual removal might result in a construct which appears more unstable.

In the Experiment 5 of Leyton (1984a), I extended the sequence, once again, back to an earlier starting point, the rotated parallelogram. Subjects were presented with the successive pairs of figures in Fig. 15—that is, ⟨rotated parallelogram, parallelogram⟩, ⟨parallelogram, rectangle⟩, ⟨rectangle, square⟩—and asked which member of each pair seemed to be "more unstable, transient, or likely to change." The conjecture was that the subjects would always choose the second member in each pair. This was corroborated with considerable statistical

significance [$n = 12$; all 12 gave the hypothesized order; therefore, significance was greater than the pairwise ordering on reference judgements; i.e., greater than $t(11) = 4.771$, $p < .0005$, one-tailed; order effects were controlled for]. Thus the hypothesis that the subjects tried to create successively more stable constructs, in this shape case, was supported.

9.2. The Nature of Internal Reference

The experiment, just described, also allows one to return to the concept of reference or judgement-with-respect-to. Recall that the Cognitive Stability Principle was used to solve the problem of the order with which input groups are factorized in a reference frame. I claimed that the order is that of decreasing instability.

Let us once again turn to the $SL_2 R$ illustration, because the other examples of successive factorization, put forward in these papers, can be described only with the theoretical structure developed later for their respective cognitive domains. In Section 8.2.2, it was argued that

$$SL_2 R = A \cdot N \cdot SO_2$$

constitutes a reference frame into which a parallelogram is placed. The reference ordering is given by factorization of the rotation group, SO_2, the shear group, N, and pure deformation group, A, in succession, each factorization being a movement of the figure to the identity of the corresponding group. One question which then had to be settled was *why* this particular factorization sequence was chosen and not another. This question can now be answered. In Experiment 3 (Sect. 8.2.2) and the experiment just described, the same pairs of figures were shown to the subjects (the successive pairs in Fig. 15). In the former experiment the subject were asked which member of each pair was the *referent*; in the latter experiment the subjects were asked which member was more *stable*. The answers were found to be the same. Thus the particular route which a subject takes through $SL_2 R$, when referencing, is also the route determined by the Cognitive Stability Principle, i.e., the stability order.

It seems therefore that, in these experiments, judgement-with-respect-to and factorization-of-instability correspond. In fact, we will find, throughout this series of papers, that this correspondence exists quite extensively. For example, again in shape perception, we will find that the points on a side are both judged with respect to, and considered less stable than, the central point. Again, we shall find that the sides of a polygon are judged with respect to, and considered to be less stable than, a horizontal side; etc. Again in linguistics we shall find that the transformational system is decomposed with respect to stability, and movement to succesively more stable levels corresponds to the direction of reference. Again, in a planning hierarchy, we shall see that the hierarchy is induced by stability, and the factorization of instability corresponds to the reference ordering between levels. In anticipation of these results, I therefore propose:

PROPOSAL 9.   *The phenomenon of* **judgement-with-respect-to** *and* **factorization-of-instability** *are equivalent.*

Note that Proposal 9 is an assertion which concerns not only the ordering of the subgroups G_1, G_2,..., G_n, but also concerns the ordering within any one factor G_i. For example, in the SL_2R illustration, recall that within a group factor, one has simple reference (Definition 6); i.e., movement to the group identity (for example, along any one of the three segments in Fig. 18). Proposal 9 asserts that the within-group transformational reversal, which constitutes simple reference, is also the result of the removal of instability.

The type of structure, $G = G_1 \cdot G_2 \cdots G_n$, that I have been defining will be called an algebraic stability ordering:

DEFINITION 10.   A stability ordering on an input group G (in fact, a dynamical system[11] on G) is called an **algebraic stability ordering** if it decomposes G into a

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[11] An algebraic stability ordering is a dynamical system in the strict sense of the term: Let us consider the continuous case only. Let G be a Lie group, and TG its tangent bundle with associated bundle projection $\pi: TG \to G$. Then $\sigma: G \to TG$ is a vector field if it is a *section* of the bundle; that is, σ is a continuous function such that $\pi * \sigma = I_g$, at each point $g \in G$. Thus we can define an algebraic stability ordering rigorously as a dynamical system as follows:

DEFINITION 10.   Let G be a Lie group, TG its tangent bundle and $|\cdot|$ a norm on G induced by a Riemannian metric on G. Then an **algebraic stability ordering** on G is a section $\sigma: G \to TG$ of the tangent bundle such that G can be written as a product $G = G_1 \cdot G_2 \cdots G_n$, and σ can be written as a sum, $\sum_{i=1}^{i=n} \{\sigma_i: G \to TG\}$, of n vector fields, where, given any left coset $g_1 \cdot g_2 \cdots g_{i-1} \cdot G_i$ $(g_j \in G_j)$, the following conditions hold: (i) in all but a small neighborhood of the coset leader $g_1 \cdot g_2 \cdots g_{i-1}$, one has $|\sigma_i(g)| \geqslant |\sigma_{i-1}(g)|$, for all g in the coset; (ii) $g_1 \cdot g_2 \cdots g_{i-1}$ is a sink for σ_i, within the coset; and the coset is the basin of attraction for $g_1 \cdot g_2 \cdots g_{i-1}$; and (iii) $|\sigma_i(\cdot)|$ decreases on any trajectory defined on $g_1 \cdot g_2 \cdots g_{i-1} \cdot G_i$.

Observe that if we let Γ be the Lie algebra of G, then, using the natural Γ-valued 1-form, we can consider σ to map into the Lie algebra.

Thus the vector field σ models the reference structure we have been developing above. In particular, it is important to observe that each vector $\sigma(g) \in TG_g$ models the strength and direction of cognitive reference at g. The reader should note that *time*, in a real sense, is not necessarily involved: The field σ defines the reference structure and indeed there is a parameter that defines the induced flow structure. But this parameter simply shows how the points interconnect referentially in the reference structure; something that the field does not directly show.

It is now possible to give a rigorous definition of the *instability* of a transformation g. It seems reasonable to call the norm $|\sigma(g)|$ of the reference vector at g, the instability of g, because in the present case, the norm can be regarded as a Liapunov function of the vector field, and a Liapunov function can be regarded as a measure of distance from stabilization. In particular, each coset leader $g_1 \cdot g_2 \cdots g_{i-1}$ is an asymptotically stable equilibrium with respect to the dynamical system σ_i restricted to the coset; and the Liapunov function is minimized at $g_1 \cdot g_2 \cdots g_{i-1}$.

The reader should note that, due to the low mathematical level assumed in these papers, and the fact that still more mathematical material should not be explained in the paper, the above more rigorous definitions will not be incorporated into the main body of the paper. The reader should also note that by an abuse of notation $G_1 \cdot G_2 \cdots G_n$ will usually be called the algebraic stability ordering. [The above is presented in greater detail in Leyton (1986)].

sequence $G = G_1 \cdot G_2 \cdots G_n$, of group factors, G_i; and there is increasing stability in the following two directions:

(1) componentwise, G_n to G_1, and

(2) within each component, G_i, elementwise to the identity, e_i, of that component.

Thus, for example, to go back to our illustrating experiment, $SL_2 R = A \cdot N \cdot SO_2$ is an algebraic stability ordering because stability increases in the direction from SO_2 to N to A; and also within each group as the shapes move to the identity elements of each respective group. For example, in the shear group, N as the parallelogram straightens to a rectangle, it is cognized as becoming more stable.

Definition 10 therefore embodies one of the central constructs of our theory: a reference structure that is *asymmetric-sequential*.

10. THE PRINCIPLE OF NESTED CONTROL

In this section, I will argue that the imposition of an algebraic stability ordering, on a stimulus set, results in the cognition of that set as a *nested structure of control*. This proposal will be called the Principle of Nested Control. Since this principle will be crucially important to our cognitive theory, it is worth spending some time clarifying it.

Suppose that a man is at some point, A, on the surface of the earth. He holds out his arm, and lets an object drop to the ground. The object undergoes a group of downward movements G_2 as it falls.

Then supposing that the man moves to another point, B, and does the same thing. The object falls to the ground at this new point. So the object undergoes the action of the same group, G_2, at point B: as shown in Fig. 19.

Observe now that in moving from point A to point B, another group G_1 has been introduced—that of movements across the earth. In fact, G_1 (the movement of the man) moves the group of object movements, G_2 (the gravitational trajectory), from the first point A, to another point B (see Fig. 19). The group G_1 will be said to be a *control* variable with respect to G_2, because it controls the position of the G_2 machine. In control theory, one way of understanding a control variable is that it *sets* a value (e.g., position) of the action of a system (i.e., it is a boundary value). One resulting effect is illustrated in the present example: the control variable G_1 sends a machine of the form G_2 at A, onto a machine of the form G_2 at B. Equivalently, it sends gravitational trajectories onto each other. That is, because the control variable, here, sets the position of machine G_2, it can also be regarded as moving it about.[12]

There is, in fact, an important way of representing each of the downward trajec-

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[12] Of course, in control theory, the general case has a control parameter continuously changing so that the above effect is hidden.
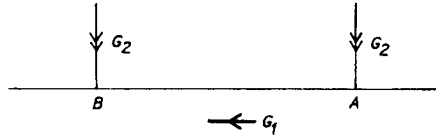
FIG. 19.   An example of nested control: The group G_1 moves trajectories of the group G_2 around.

tories in this system. Let us denote the translation across the earth, that moved the man to the first point A, by t_A; and the translation across the earth, which moved the man to point B, by t_B. Then, one can simply label the gravitational trajectory at A by

$$t_A \cdot G_2,$$

that is, G_2 translated, or acted on, by t_A. And one can represent the translation of G_2 to B by

$$t_B \cdot G_2,$$

that is, G_2 translated, or acted on, by t_B. This means that the two trajectories, $t_A \cdot G_2$ and $t_B \cdot G_2$, are *cosets* of G_2. Recall, from Section 4.2.1, that a coset, of some H, is a set of the form gH.

Now observe that the two values t_A and t_B come from the group G_1, the movements across the earth. So one can represent the fact that G_1 moves the gravitational G_2 around, simply by the product $G = G_1 \cdot G_2$—as long as we understand that G_1 acts to the right of it in the product, that is, on G_2, moving the latter around. If the reader has any problems with the product, $G_1 \cdot G_2$, he or she simply has to remember that it means exactly what Fig. 19 means. The diagram simply represents the product.

The whole process can, in fact, be built up further. For example, the earth is slowly moving through space. So it is carrying the above entire system $G_1 \cdot G_2$ from one part of the solar system to another. Now, because the earth is moving under a group G_0, one can say that the latter acts on the $G_1 \cdot G_2$ system. Again, as before, one can represent this simply by adding G_0 to the left of the sequence, thus;

$$G_0 \cdot G_1 \cdot G_2.$$

Again, one must remember that G_0 acts on the sequence to the right of it; that is, it acts on $G_1 \cdot G_2$, the terrestrial system shown in Fig. 19, and moves it around the solar system. Again, because G_0 moves the terrestrial system about, I will say that it acts as a *control* variable, with respect to that system. Each occurrence of the terrestrial system (i.e., at different places in the earth's orbit) will again be a coset of $G_1 \cdot G_2$. Furthermore, a coset will be represented by $t_0 \cdot G_1 \cdot G_2$, where t_0 is the translation through space which brought $G_1 \cdot G_2$ to that part of the solar system.

Now for the next concept. An important thing to observe is that we have here a *nested* system of control. G_1, the man, controls the position of the gravitational trajectory, G_2, moving it about on the surface of the earth; and G_0 controls the combined terrestrial system $G_1 \cdot G_2$, moving it about in the solar system. This nested effect is represented by the fact that every group in the sequence, $G_0 \cdot G_1 \cdot G_2$, acts on the sequence to the right of it. Such a right hand sequence will be called an *i-subsequence*. (The letter i, will stand for "initial" in the reference ordering, or "internal" in the nested structure.)

Now let us involve the notion of *stability*.

First, note that each position in the solar system orbit G_0 is cognized as changing more slowly, i.e., is more stable, than each position of the walking man, the system G_1 to the right of G_0 in the sequence. Again, each position in the man's path, G_1, changes more slowly, and is thus more stable, than each position in the gravitational trajectory, G_2, the system to the right of G_1 in the sequence. Thus, any group in the sequence is slower to change than the system to the right of it, i.e., the system which it controls.

I will now simply reverse the above argument, in the case of algebraic stability orderings. That is, I will argue that an *algebraic stability ordering creates the impression of nested control.* The next paragraph attempts to derive this proposal, and then it will be stated properly.

Suppose that a machine input-group decomposition were $G = G_1 \cdot G_2$, where this product is an algebraic stability ordering. Then the machine would always travel towards the identity of G_2 before any significant movement in G_1 occurs. For instance, if the above terrestrial system were an algebraic stability ordering, the object would complete its gravitational trajectory to the earth, before any significant movement of the man, across the earth, takes place. This would mean that while movement in G_2 takes place, the value in G_1 would remain fixed; e.g., the position on the surface of the earth would be the same while the object descends. Phenomenologically, one has a machine of the form G_2 with a fixed value in G_1 (e.g., a trajectory with a fixed position). Essentially, then, the algebraic stability ordering on $G_1 \cdot G_2$ causes the product to be viewed as shown in Fig. 19; that is, as a set of distinct machines, all of the form G_2, but each with a different value in G_1; e.g., the set of gravitational trajectories, each with a different position on the earth's surface. The group G_1 can then be viewed as *controlling* these G_2-machines; that is, moving them about. Thus I claim that an algebraic stability ordering on the decomposition $G = G_1 \cdot G_2$ has the phenomenological result of causing G_1 to be viewed as a control parameter with respect to G_2.

To generalize this conclusion, one requires the definition mentioned earlier in passing:

DEFINITION 11. Given a group decomposition $G = G_1 \cdot G_2 \cdots G_n$, with an algebraic stability ordering, the subsequence $G_{j+1} \cdot G_{j+2} \cdots G_n$ is called the *i*-**subsequence** of the factor G_j.

That is, given any factor G_j, the subsequence to the right of it, in the decomposition, is called the i-subsequence of G_j. Thus, as a general statement of the conclusion given above I propose:

PRINCIPLE OF NESTED CONTROL: *Given an input group decomposition* $G = G_1 \cdot G_2 \cdots G_n$, *with an algebraic stability ordering, any factor* G_j *is cognized as a control group of its i-subsequence* $G_{j+1} \cdot G_{j+2} \cdots G_n$.

That is, any factor G_j is cognized as a control group of the subsequence $G_{j+1} \cdot G_{j+2} \cdots G_n$, to the right of it in the overall sequence. In other words, G_j is cognized as moving machines of the form $G_{j+1} \cdot G_{j+2} \cdots G_n$, onto other machines of that form.

As an example, consider again Fig. 18 which represents the $SL_2 R$ sequence. In particular, consider the SO_2 axis in that diagram. Parallel to and above that axis, there is a dotted line emerging from x_3. That is, one has two parallel lines, one starting from x_4 and the other starting from x_3, and both in the SO_2 direction. Observe now that the point x_4 is a square, and the point x_3 is a rectangle. Thus the two lines are, respectively, the square with its rotation space and the rectangle with its rotation space. Therefore, the A axis *sets* the stretch value of any figure that is to have the dynamical space of the rotation group. (Recall that the term control will be used in these papers in the sense of *setting* a value of a dynamic; i.e., a sense used in control theory). Note further that the A axis sends the SO_2 lines onto each other; e.g., it sends the square with its rotation space onto the rectangle with its rotation space. In the next paper, the perceptual consequences of the control phenomenon will be shown to be very significant. Indeed, it will become the entire backbone of the theory of Gestalt grouping.

11. THE DESCRIPTION POSTULATE

It is now possible to state the Description Postulate, which will be the basis of the cognitive theory of these papers. It is a more detailed statement of the Content Machine Analogy, which states that cognition is the description of phenomena as machines. Recall that the structure defined by a machine is understood here as having two forms, algebraic structure and stability. In Section 7, the algebraic component of the Description Postulate claimed that the cognition of a stimulus set is the structuring of the set as the input group of a machine. It is now possible to add the stability component of the Description Postulate. This addition is the statement that the input group has an ordering of the type given in Definition 10.

DESCRIPTION POSTULATE: *The cognition of, or extraction of information from, a phenomenon, S, is a map* $D: G \longrightarrow S \cup \{0\}$ *from the input group* G *of a machine onto* S, *such that* G *is given a decomposition,*

$G = G_1 \cdot G_2 \cdots G_n$, *defined by an algebraic stability ordering on the machine*[13]. (*Such a map will be called a* **machine description.**)

It is now possible to offer a definition of reference frame based upon the theory being developed here. In the present series of papers, I will try to show that this definition is behind all the disparate examples of reference frames given in Section 8.1.

DEFINITION 12. A **reference frame** is an input group, G, with an algebraic stability ordering, $G = G_1 \cdot G_2 \cdots G_n$.

The basic concept to be gained from the above definition is stated in the following proposal:

PROPOSAL 13. *Cognition* (*or the extraction of information*) *is a map from a reference frame*, $G = G_1 \cdot G_2 \cdots G_n$, *onto a stimulus set.*

Thus, in this view, one must conclude that the concept of *cognition*, or *information*, cannot be separated from the concept of *reference frame*.

12. THE PRINCIPLE OF STIMULUS IDENTITY

The notion of stability is central to the *dynamical* structure of a machine. Dynamical systems concepts enter crucially in the theory which I have been developing in this paper. According to the mathemathical theory of dynamical systems (e.g., Hirsch & Smale, 1974, p. 190), any such system has a particular property which I take as fundamental to the theory being elaborated here:

The entire state space of a dynamical system can essentially be identified with just a few states within that space: the stable states of the system.

The reason is that the system will typically be at only those states.

In fact, the identification of a system with its stable states can be viewed as one of

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[13] It is now possible to see a relationship between Shannon information theory and the construct *stability* in the present paper. Following the generalizations developed by Resnikoff (1985) of the Shannon measure and the Boltzman equation, we can define the change in Shannon information in going from g to h along an algebraic stability ordering to be $\log(|\sigma(h)|/|\sigma(g)|)$; i.e., the ratio of the norms of the associated vectors in the tangent bundle. Thus, because we earlier equated stability with information, our measure of information is the negative of the Shannon measure. That is, to obtain the Shannon information function, one simply inverts the graph of the Liapunov function we defined on the algebraic stability ordering (Footnote 11). Thus, structurally, the functions are the same.

What are our main reasons for considering the information function as the inversion of the Shannon measure? As stated earlier, a primary reason follows from Thom's thesis that nothing can be known or recognized unless it is stable. However, a second reason is actually related to this: We shall find that the more cognitively stable an object is, the greater is the number of trajectories that are associated with it. (For example, a kernel sentence allows more transformations than a derived sentence.) Thus upon the presentation of a more stable object, there is a greater reduction in uncertainty concerning the behaviors of the object under a set of actions prior to the object's presentation.

the implications of the Cognitive Stability Principle, which proposes that cognition attempts to remove instabilities leaving stabilities; for this removal means that what is being represented becomes identified with those states that are defined as stable.

Seeing the above dynamical systems principle as an application of the Cognitive Stability Principle allows one to understand why I did not define a machine object as an invariant under the input group. The invariants are *part* of the presented stimulus; whereas, according to the Cognitive Stability Principle, the machine object is actually a *different* stimulus. Let us consider this in more detail.

The $SL_2 R$ example, examined in Section 8 and 9, illustrates this well; for recall the conclusion that, in the factorization process, the subject was "*describing*" the percept as *something else* with an instability (e.g., "a parallelogram is a rectangle with slant") (Sect. 9). Thus consider the transformation of a rectangle into a parallelogram. Suppose that one assumed that the cognized object which has undergone the transformation is the set of properties which are *invariant* in the transformation from the rectangle to the parallelogram. Since both figures have four straight sides, one such invariant is the property: *four straight sides*. Another invariant is: *opposite sides are equal and parallel.* However, observe that the *sizes of the angles* are not invariant in going from the rectangle to the parallelogram; i.e., the angles change. Thus, from the invariance viewpoint of an object, what one has here is a four-sided object with opposite sides equal and parallel—but no specification being made with respect to angle size. However, suppose instead that one adopted the above dynamical systems' principle, which states that a system is identified with just its stable states. Then one would conclude that the cognized object is the set of properties of the more stable stimulus, the rectangle. This includes "having right angles"—a property *not* possessed by the parallelogram. Thus, if one adopted the above dynamical systems' principle, one would have to conclude that when looking at the *parallelogram*, one cognizes it as an object with right angles, which has undergone an unstable change: *even though there are no right angles evident in the presented stimulus*!

As was pointed out earlier, contemporary psychologists such as Gibson (1966, 1979), Piaget (1969), and Hoffman (1966, 1968) have proposed that a stimulus is identified as an invariant under a set of transformations. This is the Klein view of geometry which has dominated all branches of mathematics in this century. However, the theory presented in this series of papers develops an opposing position. Fundamentally, this latter position is based on the Cognitive Stability Principle, one implication of which is that a stimulus is identified not with its invariants, but with its stable states. This identification is embodied in the concept of an algebraic stability ordering (Definition 10). One property of such an ordering is that a stable state has to be a *group identity element*. The reader can now see a still more important reason why I define the machine object (Sect. 6) as being that stimulus labeled by the identity element. It is because the algebraic stability ordering ensures that this stimulus is the stable point in the system, and the Cognitive Stability Principle implies that the system is identified with the stable point. To summarize:

PRINCIPLE OF STIMULUS IDENTITY: *A stimulus is identified as that (other) stimulus which receives the algebraic identity element under a common input-space description, i.e., as the machine object.*

This principle will be corroborated several times during the course of the series of papers. For example, it will be claimed that the principle explains why categories are identified with their prototypes and why, in Transformational Grammar, kernel sentences are understood as capturing the essence of a derived sentence—i.e., the latter's propositional content.

13. SUMMARY

This paper began by proposing the *Content Machine Analogy*, which states that cognition is the description of phenomena as machines. Two alternative realizations of this analogy were proposed and explored: state-space descriptions, which have *symmetric non-explicit reference* and define *objects as group invariants*; and input space descriptions, which have *asymmetric explicit reference* and define *objects as group identity elements*. The latter was chosen as the basis for the cognitive theory because these papers attempt to account for asymmetric explict reference and phenomena such as prototypes and kernel. Thus the Content Machine Analogy received a particular form in the *algebraic* component of the Description Postulate. This part of the postulate states that the cognition of a stimulus set S is a map from a machine's input group onto S. Because the input group manifests asymmetric explicit internal reference, we began our exploration of it as the reference frame of a representation.

It was then proposed that the input group decomposes cognitively into subgroups which are successively factorized from the total input group. The successive factorization is in fact the successive initialization of machine components. The theory predicts that the sequence of initializations results in stimuli having successive reference points in the total frame. The important question remained, however, of the nature of the criterion which determined both the frame decomposition and the ordering of the factorization sequence. An answer was provided by investigating *stability* issues as follows.

It was argued that instabilities reduce information. This proposal lead to the Cognitive Stability Principle which is of considerable importance to the theory developed in these papers. The principle states that, given a description (i.e., "representation"), the cognitive system attempts to remove instabilities from it, in order of decreasing instability. This was claimed to have a powerful structuring effect on the machine groups used in cognizing a phenomenon. The effect is the group decomposition introduced earlier, and the factorization ordering is that of decreasing instability. This stability ordering together with a within-subgroup stability ordering towards each subgroup identity, was called an *algebraic stability ordering*. This type of structure embodies one of the central constructs of the proposed cognitive theory; a reference structure that is *asymmetric-sequential*.

I then derived and proposed a Principle of Nested Control which claimed that an algebraic stability ordering on an input group is cognized as a nested structure of control. The principle will be crucially useful in these papers.

At this point it was possible to give a particular detailed statement of the Content Machine Analogy, called the Description Postulate: The cognition of a stimulus set is a map from the input group of a machine onto the set such that the machine structure has a decomposition determined by an algebraic stability ordering. Because it was claimed that a stimulus is described as a machine, moving under an algebraic stability ordering, the cognitive effect of the description is that of embedding the stimulus in a reference frame. Thus a reference frame was defined as the domain of the description map. Observe that the Principle of Nested Control specifies a powerful relationship between the cognition of control and the phenomenon of reference frame.

Finally, the Cognitive Stability Principle was seen as implying the dynamical systems principle that a system can be identified with its stable state. This lead to the Principle of Stimulus Identity, which proposes that a stimulus is identified as that (other) stimulus which receives the identity element in a common input-space description.

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