Mathematical Model of the Eye Glyphs

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Introduction

If you're reading this, the *eyes* need no introduction. I won't waste your time explaining what they are, but if you did somehow end up downloading this document and have no idea what it's for, you should probably read about this somewhere else.

The best source of eyes info I can point you towards would be Gonzo's The Emerald Tablet, containing as much info as possible about the topic, and hopefully containing this document also.

As I don't need to elaborate on the context of the puzzle itself, I'll use this section to talk about me, and why I chose to write a document about this.
I'm Chris, a third year in Maths University (Aix-Marseille Université, France, I know I'm doxxing myself but trust me, my personal info's basically everywhere anyway), and although that doesn't give me any authority on mathematics as a whole, I haven't seen a lot of variety in the math-based methods to model this puzzle, so *that's what this document will mainly try to do.*

For clarification, I'll model the problem in different ways that all focus on different fields of mathematics, so that *anyone* who has some high-level education in maths can try and have a crack at making up a new method to try and solve the puzzle (not that anyone really needs help with that, the community's been killing it with new ideas).

I'll obviously try to explain how and why some of the concepts I'll mention are useful in the context of solving the puzzle, but no promises on trying to actually use them. And, I'm sorry, but I won't be tackling cryptography as a whole as there's already a **lot** of documents covering that already (cf. The Emerald Tablet).

For convenience, I'll denote the center, upwards, right, downwards, and left eyes as C, U, R, D and L respectively.
The set containing all eye positions will be :

$$\mathcal{E} := \{C, U, R, D, L\}$$

1 The algebraic $\mathbb{Z}_5$ approach

The most obvious way of viewing the problem, that has been used before, is representing the eye positions as numbers mod 5.

Let's add some context to that.

1.1 (not) Group Theory

In group theory, a *group* can be mostly thought of as a combination of a set, and an operation, generally addition, denoted by $+$.
This can be for example $(\mathbb{Z}, +)$, the integers.

A *ring* is the combination of a group and another operation, generally multiplication denoted by $\times$ or $\cdot$ (rings aren't like fields though, so you don't have to worry about inverses being defined).

Consider the ring $(\mathbb{Z}_5, +, \cdot)$.
An element of the ring is, by definition, an element of $\mathbb{Z}_5$, on which we can perform addition $(+)$ and multiplication $(\cdot)$.

Forgive the seemingly unnecessary terms, but we need them to avoid confusion later.

The community who worked on this approach and developed the main ideas surrounding it decided the following attribution [1] :

$$C = 0, \quad U = 1, \quad R = 2, \quad D = 3, \quad L = 4$$

This, alone, does not satisfy a group structure!

The main grudge I have with this vision is that it leads members of the community to come up with ways to "add" the numbers (and maybe take the remainder mod 5 since all the numbers are already mod 5).
But **two rights don't make a left**, and therefore it is incorrect to state that $2 + 2 = 4$ in this system.

This model does have its benefits however, as we'll see shortly after this next section.

1.2 Possible Extensions

This is the part where I wanted to mention vector spaces, but can't due to a lack of group structure in our current system.

This is in my opinion, the part of the model that is most lacking, so I'll try and propose some ideas but honestly, anything goes so long as it follows a certain set of rules.

We want to choose an operation that I'll write as $*$ for now, such that :

- There is a neutral element :

$$\exists p_0 \in \mathcal{E}, \quad \forall p \in \mathcal{E}, \quad p * p_0 = p$$

  Such an element will be noted 0.

- $*$ is commutative :

$$\forall (p_1, p_2) \in \mathcal{E}^2, \quad p_1 * p_2 = p_2 * p_1$$

- $*$ is associative :

$$\forall (p_1, p_2, p_3) \in \mathcal{E}^3, \quad p_1 * (p_2 * p_3) = (p_1 * p_2) * p_3$$

- Every element has an inverse by $*$:

$$\forall p_1 \in \mathcal{E}, \exists p_2 \in \mathcal{E} \quad p_1 * p_2 = 0$$

This assumes $*$ is commutative, but if you want to try using an non-commutative operation, that's fine too, although the rest of this document might not fit just right with such a choice.

For this to work, we necessarily need to pick a 0 element, which the community has more or less agreed should be C.

From there, we need an operation $*$ that will allow us to build an isomorphism from our current system to $(\mathbb{Z}_5, +)$.
Such an operation is the cyclic relationship defined by :

$$U * U = R, \quad U * R = D, \quad U * D = L, \quad U * L = C$$

which allows us to write once and for all $(\mathcal{E}, *) \simeq (\mathbb{Z}_5, +)$, with :

$$C = 0, \quad U = 1, \quad R = 2, \quad D = 3, \quad L = 4$$

This might feel unnecessary to you, but other isomorphisms are possible with different attributions, and you could draw parallels with other groups entirely, so don't hesitate to think of other sets with 5 elements (五行, for example).

1.3 Possible Vector Space, Operations

A lot of evidence points towards regrouping an eye message as trigrams.

Such trigrams contain 3 eyes, each staring in a given direction.

Let $\mathcal{E}^3 = \mathcal{E} \times \mathcal{E} \times \mathcal{E}$ be the Kronecker product on which we can define a more generalised version of the $*$ operation :

$$\forall (t_1, t_2) = \left(\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, \begin{pmatrix} p'_1 \\ p'_2 \\ p'_3 \end{pmatrix} \right) \in \mathcal{E}^3 \times \mathcal{E}^3, \quad t_1 * t_2 = \begin{pmatrix} p_1 * p'_1 \\ p_2 * p'_2 \\ p_3 * p'_3 \end{pmatrix}$$

Obviously, in a similar fashion we have $(\mathcal{E}^3, *) \simeq (\mathbb{Z}_5^3, +)$, so we'll be referring to $\mathbb{Z}_5^3$ now.

We can now extend our system to vector spaces, given *yet another* operation, say $\otimes$, from elements of a field $(S, \circ, \otimes)$, such that :

- $\otimes$ is distributive in regards to $+$:

$$\forall (t_1, t_2) \in \mathbb{Z}_5^3 \times \mathbb{Z}_5^3, \forall \sigma \in S, \quad \sigma \otimes (t_1 + t_2) = \sigma \otimes t_1 + \sigma \otimes t_2$$

- $\otimes$ is distributive in regards to $\circ$:

$$\forall t \in \mathbb{Z}_5^3, \forall (\sigma_1, \sigma_2) \in S^2, \qquad (\sigma_1 \circ \sigma_2) \otimes t = \sigma_1 \otimes t + \sigma_2 \otimes t$$

This is quite harder to find as multiplication by real number is usually the selected operation for $\otimes$, and yet there is no obvious way to multiply trigrams by a real factor.

However, constructing a vector space-like object is possible with some other operations that work on bigger sets of data, namely *composition* of elements from the *permutation group* $(\mathfrak{S}_3)$ [2].

1.3.1 Symmetric Group vector space-ish extension

I won't be showing proof that the operation is valid, as the proof is easy but requires a good understanding of permutations in $\mathfrak{S}_n$ and explaining the details of that is completely pointless in regards to the vector space itself.

In this case, we'll be looking at $(\mathfrak{S}_3, \circ)$, where $\circ$ will act as $\circ$ and $\otimes$ from the previous definition. Composition between permutations is common, and applying a permutation from $\mathfrak{S}_n$ to any n-sized array (in our case, a vector), is rather natural.

Here is an example of how you could apply operations between trigrams in this vector space :

Let $t = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, t' = \begin{pmatrix} p'_1 \\ p'_2 \\ p'_3 \end{pmatrix} \in \mathbb{Z}_5^3$. Consider $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \in \mathfrak{S}_3$.

$$t + t' = \begin{pmatrix} p_1 + p'_1 \\ p_2 + p'_2 \\ p_3 + p'_3 \end{pmatrix}$$

$$\sigma(t) = \begin{pmatrix} p_3 \\ p_1 \\ p_2 \end{pmatrix}$$

Just as a proof of concept, it's distributive :

$$\sigma(t) + \sigma(t') = \begin{pmatrix} p_3 \\ p_1 \\ p_2 \end{pmatrix} + \begin{pmatrix} p'_3 \\ p'_1 \\ p'_2 \end{pmatrix} = \begin{pmatrix} p_3 + p'_3 \\ p_1 + p'_1 \\ p_2 + p'_2 \end{pmatrix} = \sigma(t + t')$$

And each element in $\mathfrak{S}_3$ has an inverse in regards to $\circ$. We'll write τ_{ij} the transpositions between the i-th and j-th element to save space :

σ	σ^{-1}
id	id
τ_{12}	τ_{12}
τ_{13}	τ_{13}
τ_{23}	τ_{23}
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

Don't be mistaken though, this isn't a proper vector space since I don't know how one could use $\mathfrak{S}_3$ to form a ring structure, and therefore cheated this definition to fit the bill a bit better.

1.3.2 Operations on trigrams

The idea of looking at trigrams as vector-like objects of $\mathbb{Z}_5^3$ is still interesting.

To be honest, I don't know what to write in this section, but still need to include it to emphasize how important variety is in this kind of work.

Many have proposed that the cipher changes with position, or some exterior factor, so we can consider a message as a sequence of $\mathbb{Z}_5^3$ elements, say $(t_n)_{n\in\mathbb{N}}$, and then have something of the form :

$$\omega(1) = \sum_{n=0}^{+\infty} a_n t_n$$

be a converging series, where a_n is some sequence (*citation needed*).

If that seems like a cool path you want to try, have fun with it, change what you need in the terminology, you could have $\omega(1)$ only be finite for a finite message, tamper with $\omega(X) = \sum_{n=0}^{+\infty} a_n t_n X^n$, choose a_n to be a periodic function to look for patterns... anything goes, and if you're hesitating, don't hesitate to contact me.

2 Analysis

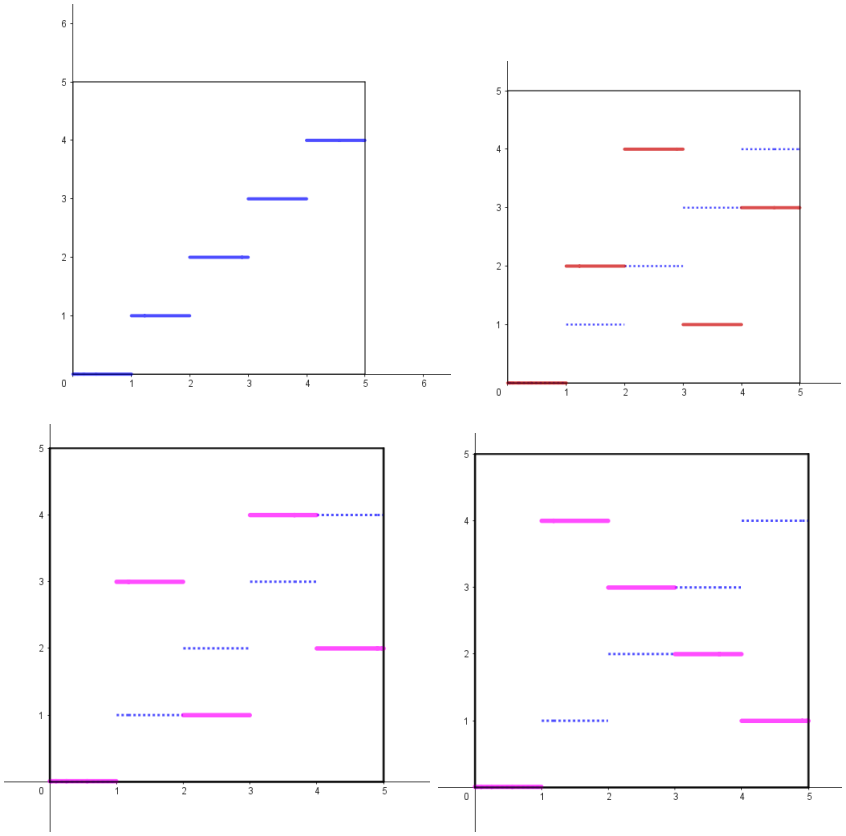
There are multiple ways of analysing the eyes by trying to identify their distribution with a function.
At first I was planning on separating this section so it could fit both real analysis and complex analysis, but real analysis on single variable functions boiled down to number theoretical functions, which, frankly, is boring.

Trivial discrete version
Here is the typical $[\![0,5]\!] \to [\![0,5]\!]$ function that would describe the eye distribution, using $\mathcal{E} \simeq [\![1,5]\!]$ similarly to our algebra section :

$$\forall n \in [\![0,5]\!], \quad f(n) = n$$

There is no really interesting thing to do at this point, so I won't dwell on it, but if you're thinking maybe you can multiply it, $Im\,(\lambda \times f([\![0,5]\!])) = [\![0,5]\!]$ for all λ such that $gcd(\lambda, 5) = 1$, also known as any number that isn't a multiple of 5.

See here for the patterns :



The blue lines are the identity function, and red or magenta just corresponds to different factors.

Top left is identity, top right is $2f$, bottom left is $3f$ and bottom right is $4f$.

2.1 The Complex Bijection

Real analysis doesn't work out too well if we stay in a single dimension, and moving up a dimension is tedious in contrast to just switching to the complex plane.
But is that possible ?

Well, actually, it's more than possible : it's *practical*.
Remember how the eyes attribution was taking $C = 0$ and then moving clockwise ?
It turns out, the complex plane's main character is perfect for rotations.

What I mean by this is that multiplying any complex number by i will effectively rotate it by $\frac{\pi}{2}$ radians on the plane relative to 0.
If you want to see the vector field associated with multiplication by i, head here, and type ix into the function definition (I have a new version where it displays it as z but I'm not going to update it just for that, sorry).
If you want proper norm colouring, you'll need to click on the **update** button, but it's not essential to understand my point.

Note : there's an error in my terminology in the "Instructions" section, due to a copy-paste mistake from my Pólya field applet, this is not a Pólya vector field.

Main takeaway :

> We can identify $\mathcal{E}$ with $\{0, 1, -i, -1, i\}$, by identifying $\mathbb{C} \simeq \mathbb{R}^2$ and noticing that these are precisely the directions the eyes are staring in.
>
> Multiplying by $-i$ rotates clockwise and therefore always yields the "next" element in the list (excluding the center position), but the best part is that now, *up+down=center* and *left+right=center* is justified by a way more natural set of equations :
>
> $$1 - 1 = 0 \quad i - i = 0$$

We could definitely extend $[\![0, 5]\!]$ to $[0, 5]$, but that would seem odd as the numbers between 0 and 5 serve more as indexes, and having numbers between that might require some very personal interpretation, as multiplication between the numbers themselves was very up in the air.

Let's change that.

2.2 A Complex Extension

Let $f : [0, 5] \to S(0, 1)$ where $S(0, 1)$ is the unit sphere in $\mathbb{C}$, defined by :

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ e^{\frac{i(1-x)\pi}{2}} & \text{if } 1 \leq x \leq 5 \end{cases}$$

The only problem with this model is that it isn't continuous in $x = 1$.
I'm not sure this can be, or even *should* be fixed, as it could currently be described as a piecewise smooth function on $[0, 5]$, where each smooth component is either constant or bijective with another space.

For those wondering what this function looks like, here it is.

To be honest, I have work to do so I don't have too much time to experiment, but the point of extending the model to a function like this one proposed here, is to allow for a wider range of operations.
The point of the whole document is to have a couple models that work, and extend them to a more general model each time, to make the math easier to work on, so it won't always be one-to-one.

Final Notes

Will be updated, I have quite a lot of work so I'm afraid I can't be 27/4 on this (yes I saw the typo but it's funny so I'll leave it).

Don't hesitate to contact me if you have something you think might be useful for the document, or something you think is inaccurate.
I'm also available if you need help on something, so yeah, you can DM me on Discord !

Sincerely,
Chris Mzz.#0523

Exterior Sources, Further Reading

Couldn't find any easy to read paper on :

- Non-abelian vector spaces,
- Recursively defined sequences used in whole series.

References

[1] codewarrior0. Noita eye glyph messages.

[2] LUCAS REIS. On the dimension of permutation vector spaces. *Bulletin of the Australian Mathematical Society*, 100(2):256–267, 2019.