August 2004 Ponder This

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Part A

For the case of N=1, the trivial solution is the length F(1) = 1.

For the case of N=2, the intersection of the two paths lies on the circle shown, then the intersection will be a right angle and the euclidean distance criteria will not be violated while maximizing the first path. By maximizing the function $F(2) = cos\theta + sin\theta$, the paths form a right isosceles triangle and $F(2) = \sqrt{2}$.

For higher N, recursion can be used. At the end of the first path, the distance traveled is $cos\theta_N$, while the remaining euclidean distance is $sin\theta_N$. As in figure 1 the criteria forces that the remaining distance remain optimum with the solution to $F(N-1)$, the total distance is

$$F(N) = cos\theta_N + F(N-1)sin\theta_N. \tag{1}$$

Maximizing $F(N)$ with respect to θ_N

$$\frac{dF(N)}{d\theta_N} = -sin\theta_N + F(N-1)cos\theta_N = 0 \tag{2}$$

$$F(N-1) = tan\theta_N \tag{3}$$

Using trig identities, we can find values for sin and cos,

$$cos\theta_N = \frac{1}{\sqrt{F^2(N-1)+1}} \tag{4}$$

$$sin\theta_N = \frac{F(N-1)}{\sqrt{F^2(N-1)+1}} \tag{5}$$

Plugging back into (1) leads to

$$F(N) = \frac{1}{\sqrt{F^2(N-1)+1}} + F(N-1)\frac{F(N-1)}{\sqrt{F^2(N-1)+1}} \tag{6}$$

$$F(N) = \sqrt{F^2(N-1)+1} \tag{7}$$

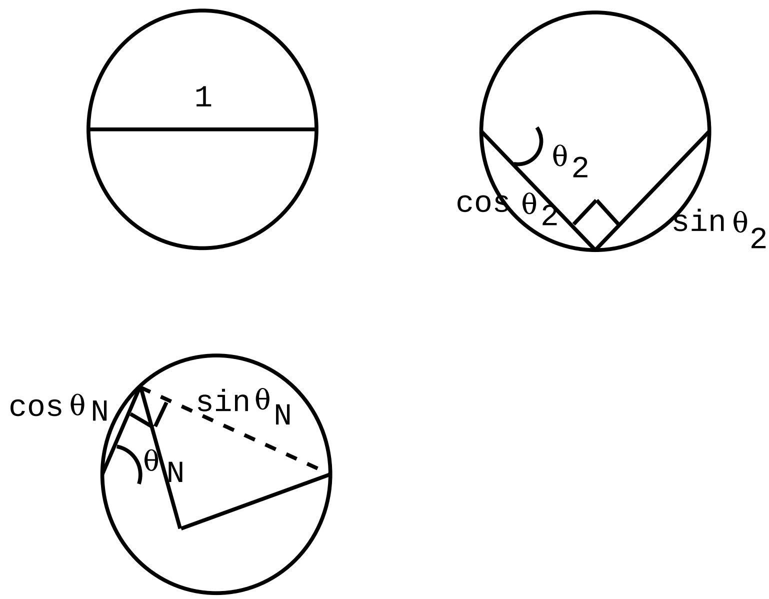

Figure 1: Recursion for part A

$$F^2(N) - F^2(N-1) = 1 \tag{8}$$

With the initial condition of $F(1) = 1$, this quadratic differences equation has the solution $F(N) = \sqrt{N}$.

Part B

To approach point B from the southwest, and having not made any right turns, the only path that allows meets this is spiral the goes east then north then west of the house as shown in Fig 2.

```
N=1:14 % need only 12
theta=atan(sqrt(N-1))
theta2=pi/2-theta
plot([0 sqrt(N)/sqrt(12).*exp(j*(3*pi/2 - cumsum(pi/2-atan(sqrt(N-1)))))])
axis image
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This spiral can be constructed by a minimum of 12 linear segments. The sum of the angles in Fig 2 need to be at least 270°. Using the angles θ_N calculated in (3), cumulating angle angle from point B by incrementing N leads to 0°, 45°, 80°, 110°, 137°, 160°, 183°, 204°, 223°, 241°, 259° and 276°. It takes at least 12 line segments to have a rotation greater than 270°, so there for 12 is the minimum N for part B.

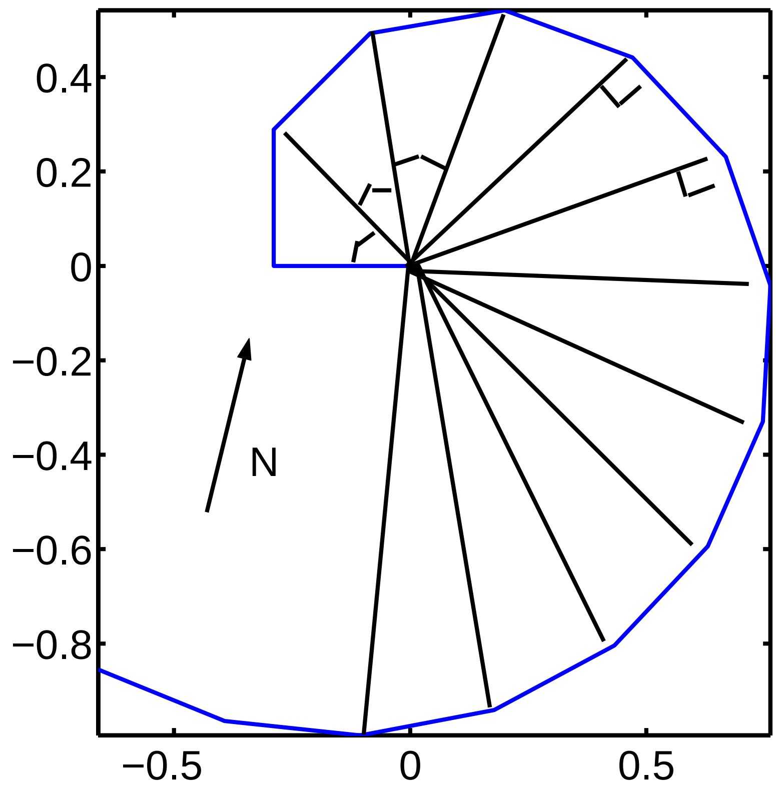

Figure 2: Spiral for Part B

Part C

The first intersection will lie on a circle whose diameter is a line between points A and B. The first leg will have a length of $cos\theta_N$ and will have a euclidean distance to point B of $sin\theta_N$. Since the third leg is due south, a similar triangle is formed and θ_N is reused, as shown in Fig 3.

The total distance traveled is

$$F = cos\theta + sin^2\theta + sin\theta cos\theta \tag{9}$$

Maximizing the distance with respect to θ

$$\frac{dF}{d\theta} = -sin\theta + 2sin\theta cos\theta + cos\theta cos\theta - sin\theta sin\theta = 0 \tag{10}$$

Solving (10) for θ requires the use of double angle formulas and solving for roots of a cubic equation. Numerically, $\theta = .8878$ radians and the total distance traveled is $F = 1.72235799584917$, which is slightly less than $F(3) = \sqrt{3}$.

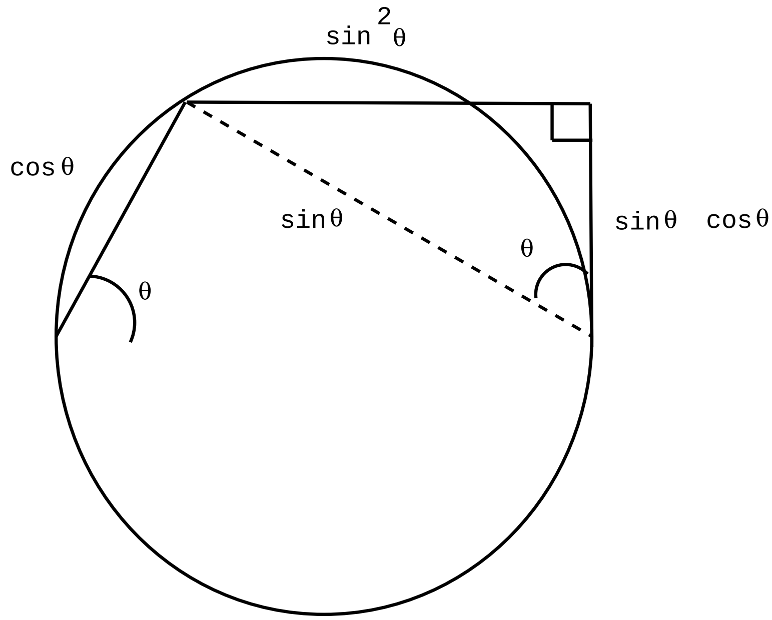

Figure 3: Part C

$$\arctan(\frac{17}{6} - 4 \left(1/12\,\sqrt[3]{-298+6\,\sqrt{-1}\sqrt{237}} + \frac{23}{6}\,\frac{1}{\sqrt[3]{-298+6\,\sqrt{-1}\sqrt{237}}} + 1/6\right)^2 - 1/12\,\sqrt[3]{-298+6\,\sqrt{-1}\sqrt{237}} \tag{11}$$

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