Definition of floor: $\lfloor x \rfloor \leq x \leq \lfloor x \rfloor + 1$. If this was $\in \mathbb{R}$then it could be anything wavy, so we make $\lfloor x \rfloor \in \mathbb{Z}$.

So what is $a_1 = \lfloor \frac{\sqrt{n}+a}{d} \rfloor$?

$\lfloor \frac{\sqrt{n}+a}{d} \rfloor \leq \frac{\sqrt{n}+a}{d} < \lfloor \frac{\sqrt{n}+a}{d} \rfloor + 1$

We're trying to find out why $d \mid n - (a - a_1 d)^2$. Because our recurance relation

$\frac{\sqrt{n}+a_i}{d_i} = c_i + \frac{1}{x_i}$, where $x_i = c_{i+1} + \frac{1}{x_{i+1}}$.

$\frac{\sqrt{n}+a_i}{d_i} = \frac{\sqrt{n}+a_i}{d_i} - c_i + c_i$

$= \frac{\sqrt{n}+a_i-c_i d_i}{d_i} + c_i$

$= \frac{1}{\frac{d_i}{\sqrt{n}+a_i-c_i d_i}} + c_i$

$= \frac{1}{\frac{d_i}{\sqrt{n}+a_i-c_i d_i} * \frac{\sqrt{n}-a_i+c_i d_i}{\sqrt{n}-a_i+c_i d_i}} + c_i$

$= \frac{1}{\frac{d_i(\sqrt{n}-a_i+c_i d_i)}{n-(a_i-c_i d_i)^2}} + c_i$

$= \frac{1}{\frac{\sqrt{n}-a_i+c_i d_i}{\frac{n-(a_i-c_i d_i)^2}{d_i}}} + c_i$.

$\frac{\sqrt{n}+a_i}{d_i} = c_i + \frac{1}{\frac{\sqrt{n}-a_i+c_i d_i}{\frac{n-(a_i-c_i d_i)^2}{d_i}}}$. We recurse with$\lfloor \frac{\lfloor \sqrt{n} \rfloor + a_i}{d_i} \rfloor = c_i$, $-a_i + c_i d_i \rightarrow a_{i+1}$, and $\frac{n-(a_i-c_i d_i)^2}{d_i} \rightarrow d_{i+1}$, which is the same as$\frac{n-a_{i+1}^2}{d_i} \rightarrow d_{i+1}$.

OK, how about this. Take$-a_i + c_i d_i = a_{i+1} \mod (d_i)$

$-a_i = a_{i+1} \mod (d_i)$.

$a_i^2 = a_{i+1}^2 \mod (d_i)$

Now, we have $\frac{n-a_{i+1}^2}{d_i} \rightarrow d_{i+1}$, so we're looking to see $n - a_{i+1}^2 \mod (d_i)$.

$n - a_{i+1}^2 = n - a_i^2 \mod (d_i)$.

We can use induction to go all the way back to

$n - a_1^2 = n - a_0^2 \mod (d_0)$

$= n - 0^2 \mod (1) = 0$!!!

$d_{i+1} = \frac{n-a_{i+1}^2}{d_i}$

$d_1 = \frac{n-a_1^2}{d_0}$

$d_1 = n - a_1^2$

$d_2 = \frac{n-a_2^2}{d_1}$

$d_2 = \frac{n-a_2^2}{n-a_1^2} = \frac{n-(c_1 d_1 - a_1)^2}{n-a_1^2} = \frac{n-(c_1^2 d_1^2 - 2c_1 d_1 a_1 + a_1^2)}{n-a_1^2} = \frac{n-a_1^2-c_1^2 d_1^2+2c_1 d_1 a_1}{n-a_1^2} = 1 + \frac{-c_1^2 d_1^2 + 2c_1 d_1 a_1}{n-a_1^2} = 1 + \frac{-c_1 d_1(1+2c_1 d_1 a_1}{n-a_1^2}$

$d_2 = \frac{n-a_2^2}{d_1} = \frac{n-(c_1 d_1 - a_1)^2}{d_1} = \frac{n-(c_1^2 d_1^2 - 2c_1 d_1 a_1 + a_1^2)}{n-a_1^2} = \frac{n-a_1^2-c_1^2 d_1^2+2c_1 d_1 a_1}{n-a_1^2} = 1 + \frac{-c_1^2 d_1^2 + 2c_1 d_1 a_1}{d_1} = 1 + -c_1^2 d_1 + 2c_1 a_1$

$d_{i+1} = \frac{n-a_{i+1}^2}{d_i} = \frac{n-(c_i d_i - a_i)^2}{d_i} = \frac{n-a_i^2}{d_i} + -c_i^2 d_i + 2c_i a_i$.

AHA!

Since $d_{i+1} = \frac{n-a_{i+1}^2}{d_i}$. That implies $d_i = \frac{n-a_i^2}{d_{i-1}}$, and rearrange to $d_{i-1} = \frac{n-a_i^2}{d_i}$.

So $d_{i+1} = \frac{n-a_{i+1}^2}{d_i} = \frac{n-(c_i d_i - a_i)^2}{d_i} = \frac{n-a_i^2}{d_i} + -c_i^2 d_i + 2c_i a_i = d_{i-1} + -c_i^2 d_i + 2c_i a_i$.

This will prove that if we start with $d_0$and $d_1 \in \mathbb{Z}$, then all $d_i \in \mathbb{Z}$

What about $n - a_{i+1}^2 = d_i d_{i+1}$. Implying that there's a factoring going on.

We are interested in the series c_i, when $a_0 = 0$ and $d_0 = 1$. That is $\sqrt{n}$.

$c_0 = \lfloor\sqrt{n}\rfloor$.

$a_1 = -a_0 + c_0 d_0 = \lfloor\sqrt{n}\rfloor = c_0$

$d_1 = \frac{n-(a_0 - c_0 d_0)^2}{d_0} = n - \lfloor\sqrt{n}\rfloor^2$.

$c_1 = \lfloor\frac{\lfloor\sqrt{n}\rfloor + a_1}{d_1}\rfloor = \lfloor\frac{\lfloor\sqrt{n}\rfloor + \lfloor\sqrt{n}\rfloor}{n - \lfloor\sqrt{n}\rfloor^2}\rfloor = \lfloor\frac{2\lfloor\sqrt{n}\rfloor}{n - \lfloor\sqrt{n}\rfloor^2}\rfloor$

$a_2 = -a_1 + c_1 d_1 = -\lfloor\sqrt{n}\rfloor + \lfloor\frac{2\lfloor\sqrt{n}\rfloor}{n - \lfloor\sqrt{n}\rfloor^2}\rfloor(n - \lfloor\sqrt{n}\rfloor^2)$

$d_2 = \frac{n-(a_1 - c_1 d_1)^2}{d_1}$

Start with $a_1 d$, subbing in the inequality expression for a_1.

$a_1 d = \lfloor\frac{\sqrt{n}+a}{d}\rfloor d \le \frac{\sqrt{n}+a}{d} d < \lfloor\frac{\sqrt{n}+a}{d}\rfloor d + d$

$\lfloor\frac{\sqrt{n}+a}{d}\rfloor d \le \sqrt{n} + a < \lfloor\frac{\sqrt{n}+a}{d}\rfloor d + d$

Since $(a - a_1 d)^2 = (a_1 d - a)^2$

$(a - a_1 d)^2 = (\lfloor\frac{\sqrt{n}+a}{d}\rfloor d - a)^2 \le (\sqrt{n} + a - a)^2 < (\lfloor\frac{\sqrt{n}+a}{d}\rfloor d + d - a)^2$

$(a - a_1 d)^2 = (\lfloor\frac{\sqrt{n}+a}{d}\rfloor d - a)^2 \le n < (\lfloor\frac{\sqrt{n}+a}{d}\rfloor d + d - a)^2$

$n - (a - a_1 d)^2 = n - (\lfloor\frac{\sqrt{n}+a}{d}\rfloor d - a)^2 \ge n - n > n - (\lfloor\frac{\sqrt{n}+a}{d}\rfloor d + d - a)^2$

$n - (a - a_1 d)^2 = n - (\lfloor\frac{\sqrt{n}+a}{d}\rfloor d - a)^2 \ge 0 > n - (\lfloor\frac{\sqrt{n}+a}{d}\rfloor d + d - a)^2$. Hmmm, should take take $\mod (d)$? We are really trying to show that $n - (a - a_1 d)^2 \equiv 0 \mod (d)$. But I'm not sure it does... $n = 14$, $a = 3$, $d = 5$. Leads to$a_1 = 1$, $14 - (3 - 1 * 5)^2 = 10$. OK, $10 \equiv 0 \mod (d)$. What if we choose a different a_1,say 2? $14 - (3 - 2 * 5)^2 = -35$. I think that's just a coincidence?!!! Because if doesn't work for other n, say $n = 13$,then dshould have been 4. Maybe we're conflating our ds???

Ah!!! $n - (a - a_1 d)^2$ does not necessarily = 0 mod d! If we choose a random d, it won't work. It seems to only work if you start with d=1.

So, assuming we start with $d_1 = 1$ and $a = 0$. Then $d_2 = n - (a - a_1 d_1)^2 = n - a_1^2$. And the first $a = a_1 = \lfloor\sqrt{n}\rfloor$

$\frac{n-(a - a_1 d_2)^2}{d_2} \to d_3$

$n - (a - a_1 d)^2$

$= n - (a^2 - 2a_1 d + a_1^2 d^2)$

If we take this mod d, only $n - a^2$remains. That's the same equation for the first d because $a_1 = 0$.

OK, everything that's multiplied by d goes away

$n - (\lfloor\frac{\sqrt{n}+a}{d}\rfloor d - a)^2 \ge 0 > n - (\lfloor\frac{\sqrt{n}+a}{d}\rfloor d + d - a)^2 \mod (d)$

$n - (-a)^2 \ge 0 > n - (-a)^2 \mod (d)$

$n - a^2 \ge 0 > n - a^2 \mod (d)$

Let $a_1 = \lfloor\frac{\sqrt{n}+a}{d}\rfloor$which is $\in \mathbb{Z}$.

$n - (a_1 d - a)^2 \ge 0 > n - (a_1 d + d - a)^2 \mod (d)$

$n - (a_1 d - a)^2 \ge 0 > n - ((a_1 + 1)d - a)^2 \mod (d)$. We don't even need to do this mod(d).

$n - ((a_1 d)^2 - 2aa_1 d + a^2) \geq 0 > n - ((a_1+1)^2 d^2 - 2a(a_1+1)d + a^2)$
$n - (a_1^2 d^2 - 2aa_1 d + a^2) \geq 0 > n - ((a_1^2 + 2a_1 + 1)d^2 - 2aa_1 d - 2ad + a^2)$
$n - (a_1^2 d^2 - 2aa_1 d + a^2) \geq 0 > n - (a_1^2 d^2 + 2a_1 d^2 + d^2 - 2aa_1 d - 2ad + a^2)$.
$n - (\lfloor \frac{\sqrt{n}+a}{d} \rfloor d - a)^2 > n - (\lfloor \frac{\sqrt{n}+a}{d} \rfloor d + d - a)^2$
$-(\lfloor \frac{\sqrt{n}+a}{d} \rfloor d - a)^2 > -(\lfloor \frac{\sqrt{n}+a}{d} \rfloor d + d - a)^2$
$(\lfloor \frac{\sqrt{n}+a}{d} \rfloor d - a)^2 < (\lfloor \frac{\sqrt{n}+a}{d} \rfloor d + d - a)^2$
$\lfloor \frac{\sqrt{n}+a}{d} \rfloor d - a < \lfloor \frac{\sqrt{n}+a}{d} \rfloor d + d - a$
$\lfloor \frac{\sqrt{n}+a}{d} \rfloor d < \lfloor \frac{\sqrt{n}+a}{d} \rfloor d + d$
Both sides are multiples of d.