Notes for differential equation predictor neural network

I. DIFFERENTIAL EQUATIONS

These notes contain the nonlinear differential equations that the neural network in Solving_ODEs_with_NNs.ipynb is trained on. The equations are

$$\frac{d\rho_{ge}}{dt} = (\mathrm{i}\Delta - 1)\,\rho_{ge} - \mathrm{i}(2\rho_{ee} - 1)\big[\Omega + (V + \mathrm{i}\Gamma)\rho_{ge}\big], \tag{1a}$$

$$\frac{d\rho_{ee}}{dt} = -\,2\rho_{ee} + 2\mathrm{Im}[\Omega^* \rho_{ge}] - 4\Gamma|\rho_{ge}|^2. \tag{1b}$$

where ρ_{ge} is a complex parameter that can be split into a real and imaginary part. The parameters Ω and Δ are variables that are changed when training the neural network. The parameters V and Γ have fixed values $V = 20.4$ and $\Gamma = 22.9$. Eqs. (1) can be used to describe certain physical systems, such as a cloud of atoms with two-energy levels (e.g., the atom can be excited or unexcited), which are then driven by a laser with strength Ω and frequency Δ. The coefficients V and Γ are determined by the sum of interactions between the atoms.

At large enough times, Eqs. (1) eventually reach the steady-state, i.e., $d\rho_{ge}/dt = 0$ and $d\rho_{ee}/dt = 0$. For certain values of Ω and Δ, there can be two steady-states, which is known as bistability. Because Eqs. (1) are fairly simple, we can actually determine the steady-state solutions semi-analytically, and find that

$$\rho_{ge} = \frac{\mathrm{i}\Omega(2\rho_{ee} - 1)}{\mathrm{i}\,[\Delta - (2\rho_{ee} - 1)V] - [1 - (2\rho_{ee} - 1)\Gamma]}, \tag{2}$$

while $(2\rho_{ee} - 1)$ obeys the cubic equation

$$\begin{aligned} &\big[\Gamma^2 + V^2\big](2\rho_{ee} - 1)^3 + \big[\Gamma^2 + V^2 - 2\Delta V - 2\Gamma\big](2\rho_{ee} - 1)^2 + \\ &\big[\Delta^2 + 1 + 2|\Omega|^2 - 2\Delta V - 2\Gamma\big](2\rho_{ee} - 1) + \big(\Delta^2 + 1\big) = 0. \end{aligned} \tag{3}$$

Eq. (3) has two real positive solutions when the differential equations display bistability.