AdvancedProblemAnswers

November 12, 2018

1 Advanced Problem Answers

1.1 Metaprogramming Problem

```
In [1]: macro myevalpoly(z,a...)
            isempty(a) && error("You forgot to pass coefficients!")
            ex = :($(a[length(a)]))
            for i in 1:length(a)-1
               ex = :($ex * $(z) + $(a[length(a)-i]) )
            end
            println(ex)
            ex
        end

Out[1]: @myevalpoly (macro with 1 method)

In [2]: @myevalpoly 7 2 3 4 5

((5 * 7 + 4) * 7 + 3) * 7 + 2


Out[2]: 1934

In [3]: @evalpoly 7 2 3 4 5

Out[3]: 1934
```

1.2 Plot the roots of Wilkinson's polynomial with perturbation

First, we need to construct coefficients a_k. For the polynomial $\prod_{i=1}^{4}(x - z_i)$, we have the coefficients

$$\begin{pmatrix} z_1 z_2 z_3 z_4 \\ -z_1 z_2 z_3 - z_1 z_4 z_3 - z_2 z_4 z_3 - z_1 z_2 z_4 \\ z_1 z_2 + z_3 z_2 + z_4 z_2 + z_1 z_3 + z_1 z_4 + z_3 z_4 \\ -z_1 - z_2 - z_3 - z_4 \\ 1 \end{pmatrix},$$

thus we can exploit the structure and write a double `for` loop to calculate the coefficients. A more general formula is

$$\begin{cases} 1 = a_n \\ x_1 + x_2 + \cdots + x_{n-1} + x_n = -a_{n-1} \\ (x_1x_2 + x_1x_3 + \cdots + x_1x_n) + (x_2x_3 + x_2x_4 + \cdots + x_2x_n) + \cdots + x_{n-1}x_n = a_{n-2} \\ \quad\vdots \\ x_1x_2 \ldots x_n = (-1)^n a_0. \end{cases}$$

Checkout Vieta's formulas for more information.

In [4]:
```julia
function root2coeff(z::AbstractVector{T}) where T
    N = length(z)
    co = zeros(T, N+1)
    # The last coefficient is always one
    co[end] = 1
    # The outer loop adds one root at a time
    for j in 1:N, i in j:-1:1
        co[end-i] -= z[j]*co[end-i+1]
    end
    co
end
@show typemax(Int), typemax(Int128)
root2coeff(1:20)
```

```
(typemax(Int), typemax(Int128)) = (9223372036854775807, 170141183460469231731687303715884105727
```

Out[4]:
```
21-element Array{Int64,1}:
  2432902008176640000
 -8752948036761600000
 -4642984320068847616
  5575812828558562816
  8037811822645051776
 -3599979517947607200
  1206647803780373360
  -311333643161390640
    63030812099294896
   -10142299865511450
     1307535010540395
     -135585182899530
       11310276995381
        -756111184500
          40171771630
          -1672280820
             53327946
             -1256850
                20615
```

```
     -210
        1
```

Those numbers are close to `typemax(Int)`, so integer overflows may occur, lets use `Int128` instead.

```
In [5]: root2coeff(Int128(1):20)
```

```
Out[5]: 21-element Array{Int128,1}:
           2432902008176640000
          -8752948036761600000
          13803759753640704000
         -12870931245150988800
           8037811822645051776
          -3599979517947607200
           1206647803780373360
          -311333643161390640
            63030812099294896
          -10142299865511450
            1307535010540395
           -135585182899530
             11310276995381
             -756111184500
               40171771630
               -1672280820
                  53327946
                  -1256850
                     20615
                      -210
                         1
```

Next, we need to construct a companion matrix and solve for roots. A companion matrix is in the form of

$$\begin{bmatrix} 0 & 0 & \dots & 0 & -z_1 \\ 1 & 0 & \dots & 0 & -z_2 \\ 0 & 1 & \dots & 0 & -z_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -z_{n-1} \end{bmatrix}.$$

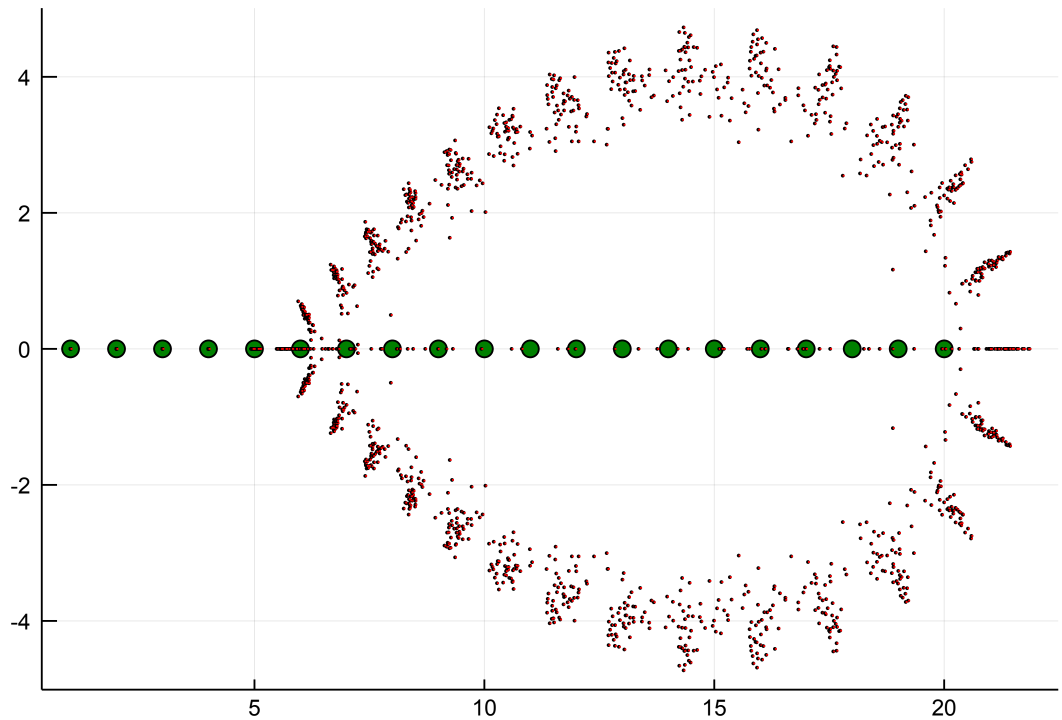
```
In [15]: using LinearAlgebra
         function poly_roots(z)
             len = length(z)
             # construct the ones part
             mat = diagm(-1 => ones(len-2))
             # insert coefficients
             mat[:, end] = -z[1:end-1]
             eigvals(mat)
         end
```

Out[15]: poly_roots (generic function with 1 method)

We have everything ready now. We just need to calculate all the roots and plot it.

In [16]:
```
using Random
Random.seed!(1)
function wilkinson_poly_roots(n=100)
    # original coefficients
    coeff = root2coeff(Int128(1):20)
    rts = Vector{Complex{Float64}}[]
    # add perturbation
    for i in 1:n
        pert_coeff = coeff.*(1 .+ rand(21)*1e-10)
        push!(rts, poly_roots(pert_coeff))
    end
    rts
end
using Plots; gr()
function plt_wilkinson_roots(rts)
    # plot roots without perturbation
    plt = scatter(1:20, zeros(20), color = :green, markersize = 5, legend=false)
    for i in eachindex(rts)
        # plot roots with perturbation
        scatter!(plt, real.(rts[i]), imag.(rts[i]), color = :red, markersize = .5)
    end
    plt
end
wilkinson_poly_roots() |> plt_wilkinson_roots
```

Out[16]:


4
2
0
-2
-4
5
10
15
20