Identification and Evaluation of Magnetic-Field Sources of Magnetic Airborne Detector Equipped Aircraft*

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Summary—**A general method for identifying and evaluating magnetic sources associated with the magnetic airborne detector-equipped aircraft is described. It is derived for the compensation of magnetic noise related to the maneuvers of the aircraft. Mathematical formulas are included with a uniform engineering method, that is independent of the type of magnetic source encountered, for analyzing magnetic airborne detector records. A method for calibrating magnetic sources while in flight is also offered.**

I. Introduction

FOR this investigation, the AN/ASQ-8 magnetic airborne detector (MAD) installation was used. During flight operations, its detector magnetometer is aligned with the earth's magnetic field at the particular geographical location and measures only increments of the magnitude of the magnetic field along the earth's magnetic-field lines. Random fluctuations in output of the detector magnetometer were a maximum of 0.05 gamma (approximately one millionth of the earth's magnetic field, 1 gamma $=10^{-5}$ oersted).

The sources of magnetic noise external to the MAD can be classified into two distinct groups: noise unrelated to the maneuvers of the aircraft, and noise related to the maneuvers of the aircraft. This paper is concerned only with the noise which is related to the maneuvers of the aircraft.

There are three different sources of magnetic interference associated with the airframe. The first source is the permanent magnetism of various ferromagnetic structural parts of the aircraft, which may result from some manufacturing process or from contact with a magnet or magnetized parts or tools. This type of field turns with the aircraft, thereby changing its relation to the earth's field vector and causing a change in the magnetic field surrounding the magnetometer.

A second characteristic source of magnetic interference, produced by airframe parts, is the induced magnetic fields created in the aircraft's ferromagnetic structures by the earth's magnetic field. These fields are generally associated with soft iron parts, although they also occur, to a limited extent, in hard steel parts. Unlike the permanent field, the induced magnetic field does not turn with the aircraft, but its polarity and magnitude are determined by the direction and magnitude of the earth's field.

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The third source of interference is that produced by eddy-current magnetic fields. These occur in all of an aircraft's skins, ribs, frames and other structural units without respect to the magnetic materials from which they are constructed. The eddy currents require only the existence of electrical conductivity and airplane maneuvers, and are generated in the same fashion as the currents produced in a coil or loop of wire rotating in a magnetic field. An electrical conducting path is formed around the outside edge of the aircraft's conducting sheets or closed-loop structures. As the aircraft maneuvers in the magnetic field of the earth, electrical currents are produced in different metallic parts proportional to time changes of the fluxes of the earth's magnetic field through these parts. The flow of current creates a magnetic field in a direction perpendicular to the plane of the sheet or conducting loop.

In the group of noise sources related to the maneuvers of the carrier are displacements through magnetic gradients (almost vertical for pitch maneuvers).

II. Basic Geometry

The reference system used in this paper is defined by the natural axes of motion of the aircraft with the detector located at the origin (O), as shown in Fig. 1. The X axis and its unit vector $\bar{\imath}$ are parallel to the transverse axis (T) of the aircraft and are positive for T left; the Y axis and its unit vector $\bar{\jmath}$ are parallel to the aircraft's longitudinal axis (L) and are positive for L forward; the Z axis and its unit vector $\bar{k}$ are parallel to the vertical axis (V) of the aircraft and are positive for V down.

The maneuvers of the MAD carrier in pitch, roll and yaw are defined as follows:

Roll maneuver: rotation of the aircraft about the longitudinal (L) axis.

Roll angle (ψ): the angle between the transverse (T) axis and the horizontal—positive for left wing down.

Pitch maneuver: rotation of the aircraft about the transverse (T) axis.

Pitch angle (λ): the angle between the longitudinal (L) axis and the horizontal—positive for noise down.

Yaw maneuver: rotation of the aircraft about the vertical (V) axis.

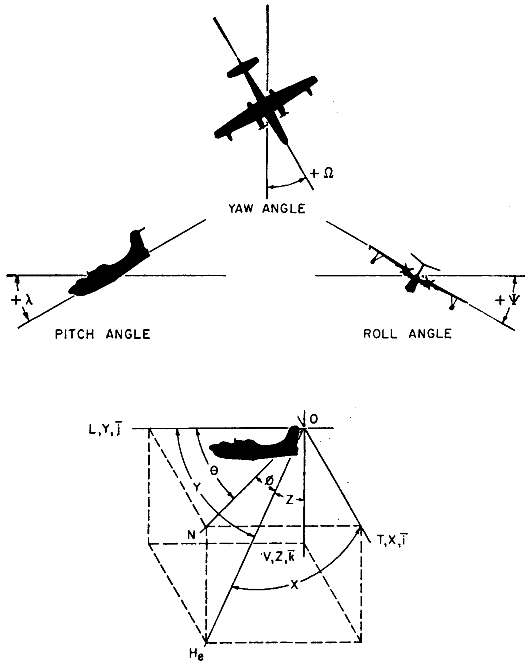

Fig. 1—Basic geometry and definition of maneuver angles.

Yaw angle (Ω): the angle between the longitudinal (L) axis and the flight heading—positive for nose left.

The magnetic heading (θ) of the aircraft is measured clockwise from north. The dip angle (ϕ) is the angle between the earth's magnetic vector $\overline{H}_e$ and the horizontal, positive for down.

The direction of the earth's magnetic field vector $\overline{H}_e$ is determined by the direction angles X, Y and Z, formed by the $\overline{H}_e$ with axes X, Y and Z. The directional cosines ($\cos X$, $\cos Y$, $\cos Z$) can be expressed in terms of the maneuver angles of the aircraft, the magnetic heading and the dip angle of the earth's magnetic field vector $\overline{H}_e$ as follows:

For rolls, when the pitch and yaw angles are zero:

$$\begin{aligned} \cos X &= \cos\phi \sin\theta \cos\psi + \sin\phi \sin\psi \\ \cos Y &= \cos\phi \cos\theta \\ \cos Z &= \sin\phi \cos\psi - \cos\phi \sin\theta \sin\psi. \end{aligned} \tag{1}$$

For pitches, when the roll and yaw angles are zero:

$$\begin{aligned} \cos X &= \cos\phi \sin\theta \\ \cos Y &= \cos\phi \cos\theta \cos\lambda + \sin\phi \sin\lambda \\ \cos Z &= \sin\phi \cos\lambda - \cos\phi \cos\theta \sin\lambda. \end{aligned} \tag{2}$$

For yaws, when the roll and pitch angles are zero:

$$\begin{aligned} \cos X &= \cos\phi \sin\theta \cos\Omega - \cos\phi \cos\theta \sin\Omega \\ \cos Y &= \cos\phi \cos\theta \cos\Omega + \cos\phi \sin\theta \sin\Omega \\ \cos Z &= \sin\phi. \end{aligned} \tag{3}$$

For pitch, roll and yaw angles all zero:

$$\begin{aligned} \cos X &= \cos\phi \sin\theta \\ \cos Y &= \cos\phi \cos\theta \\ \cos Z &= \sin\phi. \end{aligned} \tag{4}$$

III. Permanent Magnetic-Field Equations

The sources of the permanent magnetic field in the aircraft are firmly attached to the aircraft and do not vary with aircraft maneuvers. The total field at the origin O (Fig. 1) due to the permanent magnets is constant in magnitude and can be expressed in terms of the three components T, L and V parallel to the axes of the aircraft. Hence, the permanent magnetic field at the detector input $\overline{H}_p$ may be expressed as

$$\overline{H}_p = T\bar{i} + L\bar{j} + V\bar{k}, \tag{5}$$

where the components T, L and V are subject to the rotation of the aircraft axes and present varying amounts of field to the detector, which is always aligned parallel to the earth's magnetic-field vector $\overline{H}_e$. The amount of field at the detector output H_{pd} can be expressed by

$$H_{pd} = \frac{\overline{H}_p \cdot \overline{H}_e}{H_e} \tag{5a}$$

or

$$H_{pd} = T\cos X + L\cos Y + V\cos Z. \tag{5b}$$

IV. Induced Magnetic-Field Equations

The induced field at the detector input H_i (origin O) can be written as the vector sum of the induced fields due to each component of the earth's field. Assuming linear media, the fields produced by H_x, H_y, H_z at the detector input are proportional to $H_e \cos X$, $H_e \cos Y$, $H_e \cos Z$ and, in general, are not aligned with principal axes of the aircraft. They may be written as in (6), where the first letter of each proportionality coefficient indicates the respective earth's field component and the second one indicates the induced component at the detector input.

$$\begin{aligned} \overline{H}_i = {} & H_e \cos X(TT\bar{i} + TL\bar{j} + TV\bar{k}) \\ & + H_e \cos Y(LT\bar{i} + LL\bar{j} + LV\bar{k}) \\ & + H_e \cos Z(VT\bar{i} + VL\bar{j} + VV\bar{k}). \end{aligned} \tag{6}$$

Thus, for example, TL denotes a longitudinal component produced at the detector by the transverse component of $\overline{H}_e$. Summing expressions for the induced components due to H_x, H_y and H_z, the total induced field at the detector $\overline{H}_i$ is expressed in (6a). However, the magnetic detector sees only the projection of the field of (6a) along itself, that is, in the direction of the earth's magnetic vector. This component H_{id} may be written as in (6b).

$$\begin{aligned} \overline{H}_i = H_e[& (TT\cos X + LT\cos Y + VT\cos Z)\bar{i} \\ & + (TL\cos X + LL\cos Y + VL\cos Z)\bar{j} \\ & + (TV\cos X + LV\cos Y + VV\cos Z)\bar{k}] \end{aligned} \tag{6a}$$

$$H_{id} = \frac{\overline{H}_i \cdot \overline{H}_e}{H_e};$$

expanding and replacing $\cos^2 Y = 1 - \cos^2 X - \cos^2 Z$

$$H_{id} = H_e[(TT - LL)\cos^2 X + (LT + TL)\cos X \cos Y + (VT + TV)\cos X \cos Z + (VL + LV)\cos Y \cos Z + (VV - LL)\cos^2 Z + LL)]. \quad (6b)$$

Since the amplifier associated with the detector passes alternatng current only, the constant term LL need not be considered. The quantities $(TT-LL)$, $(TL+LT)$, $(TV+VT)$, $(LV+VL)$ and $(VV-LL)$ are dependent on the dimensions, location and susceptibility of the aircraft materials exhibiting induced magnetic effects and on the location of the detector in the fields emanating from these materials, but they are independent of the aircraft maneuvers.

Capital letters are used throughout this paper to designate the induced field sources.

V. The Eddy-Current Field Equations

The eddy-current field at the detector input $\overline{H}_E$ (origin O) can be written as the vector sum of the eddy-current magnetic fields due to each component of the earth's field. The eddy-current fields produced by H_x, H_y, H_z at the detector input are proportional to $\dot{H}_x$, $\dot{H}_y$, $\dot{H}_z$, respectively, and, in general, are not aligned with a principal axis of the aircraft.

They may be written as in (7) where the proportionality coefficients may be interpreted in the same manner as those of the induced field expressions. Rearranging expressions for the eddy-current components due to H_x, H_y, H_z, the total eddy-current field at the detector input $\overline{H}_E$ is expressed in (7a). Again, as in the case of the induced and permanent fields, the total intensity seen by the magnetometer is only the projection of this field in the direction of the earth's magnetic vector. This component H_{ED} may be written as in (7b).

$$\overline{H}_E = \dot{H}_x(tt\bar{\imath} + tl\bar{\jmath} + tv\bar{k}) + \dot{H}_y(lt\bar{\imath} + ll\bar{\jmath} + lv\bar{k}) + \dot{H}_z(vt\bar{\imath} + vl\bar{\jmath} + vv\bar{k}) \quad (7)$$

$$\overline{H}_E = (\dot{H}_x tt + \dot{H}_y lt + \dot{H}_z vt)\bar{\imath} + (\dot{H}_x tl + \dot{H}_y ll + \dot{H}_z vl)\bar{\jmath} + (\dot{H}_x tv + \dot{H}_y lv + \dot{H}_z vv)\bar{k} \quad (7a)$$

$$H_{ED} = \frac{\overline{H}_e \cdot \overline{H}_E}{H_e}.$$

Expanding and replacing,

$$\sin Y\dot{Y} = -\sin X\dot{X} - \sin Z\dot{Z}$$

$$H_{ED} = H_e[(tt - ll)\cos X(\cos X)' + lt\cos X(\cos Y)' + vt\cos X(\cos Z)' + tl\cos Y(\cos X)' + vl\cos Y(\cos Z)' + tv\cos Z(\cos X)' + lv\cos Z(\cos Y)' + (vv - ll)\cos Z(\cos Z)']. \quad (7b)$$

The quantities $tt-ll$, lt, vt, tl, vl, tv, lv and $vv-ll$ do not depend on the maneuvers of the aircraft, but are dependent on the electrical conductivity and size of materials used in the aircraft skins, ribs, frames and other structures and on the location of the detector in the aircraft. Lower-case letters are used throughout this paper to designate the eddy-current sources. Magnetic aircraft sources are referred to by the 16 quantities: T, L, V, $(TT-LL)$, $(VV-LL)$, $(TL+LT)$, $(TV+VT)$, $(LV+VL)$, $(tt-ll)$, $(vv-ll)$, lt, vt, tl, vl, tv and lv.

VI. Statement of the Problem and General Solution

In order to attack the problem in general, it is necessary to solve a system of equations relating to the 16 sources of magnetic interference due to the permanent, induced- and eddy-current fields of the aircraft.

All 16 magnetic sources can be related to the aircraft maneuvers and the equations can be developed from data received during sinusoidal manuevers of a MAD-equipped aircraft. For sinusoidal maneuvers, the maneuver angles can be expressed as in (8).

$$\psi = \psi_m \sin \omega t, \quad \lambda = \lambda_m \sin \omega t, \quad \Omega = \Omega_m \sin \omega t \cdots, \quad (8)$$

where ψ_m, λ_m, Ω_m are angular amplitudes of 5° to 10° and ω is an angular frequency of the order of magnitude of 1.

The equations so developed are expressed as follows:

$$\begin{aligned}
&K_1{}^aT + K_2{}^aL + K_3{}^aV + K_4{}^a(TT - LL) + \cdots + K_{16}{}^a(vl) = M_1 \\
&K_1{}^bT + K_2{}^bL + K_3{}^bV + K_4{}^b(TT - LL) + \cdots + K_{16}{}^b(vl) = M_2 \\
&\cdots + \cdots + \cdots + \cdots + \cdots + \cdots + \cdots + \cdots + \cdots \\
&K_1{}^pT + K_2{}^pL + K_3{}^pV + K_4{}^p(TT - LL) + \cdots + K_{16}{}^p(vl) = M_{16} \quad (9)
\end{aligned}$$

The coefficients $K_1{}^aK_2{}^a \cdots K_{16}{}^p$ can be calculated and, in general, depend on the aircraft attitude, heading, type of manuever, period and angular amplitude of the maneuver, magnetic dip angle and the strength of the earth's magnetic field. The expressions for these coefficients are obtained by the substitution of $\cos X$, $\cos Y$, $\cos Z$ from (1)–(4) into (5b), (6b) and (7b). These expressions are presented in Tables I–III.

The terms M_1, M_2, etc., are magnitudes of the MAD signals as indicated on the MAD-equipment pen recorder for particular aircraft attitudes, and they depend on the MAD equipment's sensitivity, phase and frequency response. In pitch maneuvers, however, the signals produced include those caused by the vertical gradient effect (signal due to changes in altitude); these components must be excluded before the true magnitude of pitch signals can be determined.

Initial attempts to solve the general equation (9) by use of IBM digital computers led to the discovery that the system has practically an infinite number of solutions (small changes of M_i or neglect of various harmonics cause decisive changes in the solution of the

TABLE I*

SIGNAL-TO-SOURCE RATIOS FOR ROLL MANEUVERS AS FUNCTIONS OF HEADING θ

Source	$\sin\theta$	$\cos\theta$	$\sin 2\theta$	$\cos 2\theta$	1
T	$\cos\phi\cos\psi$				$\sin\phi\sin\psi$
L		$\cos\phi$			
V	$-\cos\phi\sin\psi$				$\sin\phi\cos\psi$
$TT-LL$	$\frac{1}{2}\sin 2\phi\sin 2\psi$			$-\frac{1}{2}\cos^2\phi\cos^2\psi$	$\sin^2\phi\sin^2\psi+\frac{1}{2}\cos^2\phi\cos^2\psi$
$VV-LL$	$-\frac{1}{2}\sin 2\phi\sin 2\psi$			$-\frac{1}{2}\cos^2\phi\sin^2\psi$	$\sin^2\phi\cos^2\psi+\frac{1}{2}\cos^2\phi\sin^2\psi$
$TL+LT$		$\frac{1}{2}\sin 2\phi\sin\psi$	$\frac{1}{2}\cos^2\phi\cos\psi$		
$TV+VT$	$\frac{1}{2}\sin 2\phi\cos^2\psi-\frac{1}{2}\sin 2\phi\sin^2\psi$			$\frac{1}{4}\cos^2\phi\sin 2\psi$	$\frac{1}{2}\sin^2\phi\sin 2\psi-\frac{1}{4}\cos^2\phi\sin 2\psi$
$LV+VL$		$\frac{1}{2}\sin 2\phi\cos\psi$	$-\frac{1}{2}\cos^2\phi\sin\psi$		
$tt-ll$	$\frac{1}{2}\sin 2\phi\cos^2\psi\dot{\psi}-\frac{1}{2}\sin 2\phi\sin^2\psi\dot{\psi}$			$\frac{1}{4}\cos^2\phi\sin 2\psi\dot{\psi}$	$-\frac{1}{4}\cos^2\phi\sin 2\psi\dot{\psi}+\frac{1}{2}\sin^2\phi\sin 2\psi\dot{\psi}$
tl		$\frac{1}{2}\sin 2\phi\cos\psi\dot{\psi}$	$-\frac{1}{2}\cos^2\phi\sin\psi\dot{\psi}$		
tv	$-\frac{1}{2}\sin 2\phi\sin 2\psi\dot{\psi}$			$-\frac{1}{2}\cos^2\phi\sin^2\psi\dot{\psi}$	$\frac{1}{2}\cos^2\phi\sin^2\psi\dot{\psi}+\sin^2\phi\cos^2\psi\dot{\psi}$
vt	$-\frac{1}{2}\sin 2\phi\sin 2\psi\dot{\psi}$			$\frac{1}{2}\cos^2\phi\cos^2\psi\dot{\psi}$	$-\sin^2\phi\sin^2\psi\dot{\psi}-\frac{1}{2}\cos^2\phi\cos^2\psi\dot{\psi}$
vl		$-\frac{1}{2}\sin 2\phi\sin\psi\dot{\psi}$	$-\frac{1}{2}\cos^2\phi\cos\psi\dot{\psi}$		
$vv-ll$	$-\frac{1}{2}\sin 2\phi\cos^2\psi\dot{\psi}+\frac{1}{2}\sin 2\phi\sin^2\psi\dot{\psi}$			$-\frac{1}{4}\cos^2\phi\sin 2\psi\dot{\psi}$	$-\frac{1}{2}\sin^2\phi\sin 2\psi\dot{\psi}+\frac{1}{4}\cos^2\phi\sin 2\psi\dot{\psi}$

* The proper use of this table can be illustrated by the following example. In (7) the signal created by the source $(TT-LL)$ on roll maneuvers according to this table will be represented by the expression:

$$(TT-LL)H_e[(\tfrac{1}{2}\sin 2\phi\sin 2\psi)\sin\theta-(\tfrac{1}{2}\cos^2\phi\cos^2\psi)\cos 2\theta+\sin^2\phi\sin^2\psi+\tfrac{1}{2}\cos^2\phi\cos^2\psi].$$

TABLE II*

SIGNAL-TO-SOURCE RATIOS FOR PITCH MANEUVERS AS FUNCTIONS OF HEADING θ

Source	$\sin\theta$	$\cos\theta$	$\sin 2\theta$	$\cos 2\theta$	1
T	$\cos\phi$				
L		$\cos\phi\cos\lambda$			$\sin\phi\sin\lambda$
V		$-\cos\phi\sin\lambda$			$\sin\phi\cos\lambda$
$TT-LL$				$-\frac{1}{2}\cos^2\phi$	$\frac{1}{2}\cos^2\phi$
$VV-LL$		$-\frac{1}{2}\sin 2\phi\sin 2\lambda$		$\frac{1}{2}\cos^2\phi\sin^2\lambda$	$\sin^2\phi\cos^2\lambda+\frac{1}{2}\cos^2\phi\sin^2\lambda$
$TL+LT$	$\frac{1}{2}\sin 2\phi\sin\lambda$		$\frac{1}{2}\cos^2\phi\cos\lambda$		

TABLE II (*Cont'd*)

$TV+VT$	$\frac{1}{2}\sin 2\phi\cos\lambda$		$-\frac{1}{2}\cos^2\phi\sin\lambda$		
$LV+VL$		$\frac{1}{2}\sin 2\phi\cos^2\lambda-\frac{1}{2}\sin 2\phi\sin^2\lambda$		$-\frac{1}{4}\cos^2\phi\sin 2\lambda$	$\frac{1}{2}\sin^2\phi\sin 2\lambda-\frac{1}{4}\cos^2\phi\sin 2\lambda$
lt	$\frac{1}{2}\sin 2\phi\cos\lambda\dot{\lambda}$		$\frac{1}{2}\cos^2\phi\sin\lambda\dot{\lambda}$		
lv		$-\frac{1}{2}\sin 2\phi\sin 2\lambda\dot{\lambda}$		$\frac{1}{2}\cos^2\phi\sin^2\lambda\dot{\lambda}$	$\sin^2\phi\cos^2\lambda\dot{\lambda}+\frac{1}{2}\cos^2\phi\sin^2\lambda\dot{\lambda}$
vt	$-\frac{1}{2}\sin 2\phi\sin\lambda\dot{\lambda}$		$-\frac{1}{2}\cos^2\phi\cos\lambda\dot{\lambda}$		
vl		$-\frac{1}{2}\sin 2\phi\sin 2\lambda\dot{\lambda}$		$-\frac{1}{2}\cos^2\phi\cos^2\lambda\dot{\lambda}$	$-\sin^2\phi\sin^2\lambda\dot{\lambda}-\frac{1}{2}\cos^2\phi\cos^2\lambda\dot{\lambda}$
$vv-ll$		$-\frac{1}{2}\sin 2\phi\cos^2\lambda\dot{\lambda}+\frac{1}{2}\sin 2\phi\sin^2\lambda\dot{\lambda}$		$\frac{1}{4}\cos^2\phi\sin 2\lambda\dot{\lambda}$	$-\frac{1}{2}\sin^2\phi\sin 2\lambda\dot{\lambda}+\frac{1}{4}\cos^2\phi\sin 2\lambda\dot{\lambda}$

* The proper use of this table can be illustrated by the following example. In (9) the signal created by the source $(vv-ll)$ on pitch maneuvers according to this table will be represented by the expression:

$$(vv-ll)H_e[(-\tfrac{1}{2}\sin 2\phi\cos^2\lambda\dot{\lambda}+\tfrac{1}{2}\sin 2\phi\sin^2\lambda\dot{\lambda})\cos\theta+(\tfrac{1}{4}\cos^2\phi\sin 2\lambda\dot{\lambda})\cos 2\theta-\tfrac{1}{2}\sin^2\phi\sin 2\lambda\dot{\lambda}+\tfrac{1}{4}\cos^2\phi\sin\lambda\dot{\lambda}]$$

TABLE III*

SIGNAL-TO-SOURCE RATIOS FOR YAW MANEUVERS AS FUNCTIONS OF HEADING θ

Source	$\sin\theta$	$\cos\theta$	$\sin 2\theta$	$\cos 2\theta$	1
T	$\cos\phi\cos\Omega$	$-\cos\phi\sin\Omega$			
L	$\cos\phi\sin\Omega$	$\cos\phi\cos\Omega$			
V					$\sin\phi$
$TT-LL$			$-\frac{1}{2}\cos^2\phi\sin 2\Omega$	$-\frac{1}{2}\cos^2\phi\cos 2\Omega$	$\frac{1}{2}\cos^2\phi$
$VV-LL$					$2\sin^2\phi$
$TL+LT$			$\frac{1}{2}\cos^2\phi\cos 2\Omega$	$-\frac{1}{2}\cos^2\phi\sin 2\Omega$	
$TV+VT$	$\frac{1}{2}\sin 2\phi\cos\Omega$	$-\frac{1}{2}\sin 2\phi\sin\Omega$			
$LV+VL$	$\frac{1}{2}\sin 2\phi\sin\Omega$	$\frac{1}{2}\sin 2\phi\cos\Omega$			
$tt-ll$			$-\frac{1}{2}\cos^2\phi\cos 2\Omega\dot{\Omega}$	$\frac{1}{2}\cos^2\phi\sin 2\Omega\dot{\Omega}$	
lv	$\frac{1}{2}\sin 2\phi\cos\Omega\dot{\Omega}$	$-\frac{1}{2}\sin 2\phi\sin\Omega\dot{\Omega}$			
lt			$-\frac{1}{2}\cos^2\phi\sin 2\Omega\dot{\Omega}$	$-\frac{1}{2}\cos^2\phi\cos 2\Omega\dot{\Omega}$	$\frac{1}{2}\cos^2\phi\dot{\Omega}$
tl			$-\frac{1}{2}\cos^2\phi\sin 2\Omega\dot{\Omega}$	$-\frac{1}{2}\cos^2\phi\cos 2\Omega\dot{\Omega}$	$-\frac{1}{2}\cos^2\phi\dot{\Omega}$
tv	$-\frac{1}{2}\sin 2\phi\sin\Omega\dot{\Omega}$	$-\frac{1}{2}\sin 2\phi\cos\Omega\dot{\Omega}$			

* The proper use of this table can be illustrated by the following example. In (5) the signal created by the source T on yaw maneuvers according to this table will be represented by the expression:

$$T(\cos\phi\cos\Omega\sin\theta-\cos\phi\sin\Omega\cos\theta).$$

system). The errors in the determination of roots of (9) (magnetic sources of the aircraft) depend on the accuracy of signal measurements and the ratio of signal to source for each term. As this ratio becomes smaller, a greater error is introduced into the final root solution for the source. Thus, in order to reduce or eliminate such errors, it is essential to exclude from (9) harmonics of signals in which the ratio of signal to source is especially small.

It can be shown that the fundamental component of every permanent or induced magnetic source signal at the output of the detector magnetometer, due to any pitch, roll, or yaw sinusoidal manuever accomplished at small angular amplitudes, can be expressed by

kS_{pi} sin ωt (*i.e.*, always in phase with the maneuver angle), where k is a function of the heading and angular amplitude of the maneuver as well as the dip angle in the geographical area of the maneuver.
$\omega=(2\pi/T)$ is the angular frequency of the maneuver.
S_{pi} is the value of magnetic source for the permanent magnetic field (sources T, V, L).
or S_{pi} is (source$\times H_\phi$) for the induced fields sources $(TT-LL)$, $(VV-LL)$, $(TL+LT)$, $(TV+VT)$ and $(LV+VL)$, where H_ϕ is the strength of the earth's magnetic field in the maneuver location at dip angle ϕ.

The second harmonic component of the signal at the detector magnetometer output for every permanent or induced source can be expressed as $mS_{pi}\cos 2\omega t$, where m is also a function of the heading and angular amplitude of the maneuver as well as of the dip angle.

The fundamental and second harmonic components of the eddy-current magnetic-source signals due to small angular amplitude sinusoidal maneuvers are expressed as $nS_{ed}\cos\omega t$ and $pS_{ed}\sin 2\omega t$, respectively, where S_{ed} is equal to the product of eddy-current source$\times H_\phi$; n and p are coefficients dependent on the heading, frequency and angle amplitude of the maneuver as well as of the dip angle.

In Tables IV and V the coefficients $2k$, $2m$, $2n$ and $2p$ are given as functions of the magnetic dip angle ϕ for $\omega=1$ ($T=6.28$ seconds) for sinusoidal pitch and yaw manueuvers of 5° amplitude and for roll maneuvers of 10° amplitude. All maneuvers are performed on North-South, East-West cardinal headings. In the remainder of this article, these maneuvers are called reference maneuvers.

The expressions $kS_{pi}\sin\omega t$, $mS_{pi}\cos 2\omega t$, $nS_{ed}\cos\omega t$, $pS_{ed}\sin 2\omega t$, and the coefficients of Tables IV and V can be derived by expanding the expressions $\sin\psi$, $\sin 2\psi$, $\cos\psi$, $\cos 2\psi$, $\sin\lambda$, $\sin 2\lambda$, $\cos\lambda$, $\cos 2\lambda$, $\sin\Omega$, $\sin 2\Omega$, $\cos\Omega$ and $\cos 2\Omega$ found in Tables I–III into Taylor's

TABLE IV
VALUES OF THE FUNDAMENTAL PERM-INDUCED AND EDDY-CURRENT SIGNAL COEFFICIENTS† FOR CARDINAL HEADINGS
($\omega=1$, $\psi_m=10°$, $\lambda_m=5°$, $\Omega_m=5°$)

Source	North-South			East-West			
	Roll	Pitch	Yaw	Roll	Pitch	Yaw	
T	$0.349\sin\phi$		$\mp 0.175\cos\phi$	$0.349\sin\phi$			Values of $2k$
L		$0.175\sin\phi$			$0.175\sin\phi$	$\pm 0.175\cos\phi$	
V		$\mp 0.175\cos\phi$		$\mp 0.349\cos\phi$			
$TT-LL$				$\pm 0.349\sin 2\phi$			
$VV-LL$		$\mp 0.175\sin 2\phi$		$\mp 0.349\sin 2\phi$			
$TL+LT$	$\pm 0.175\sin 2\phi$		$-0.175\cos^2\phi$		$\pm 0.087\sin 2\phi$	$0.175\cos^2\phi$	
$TV+VT$	$0.349\sin^2\phi$		$\mp 0.087\sin 2\phi$	$-0.349\cos 2\phi$			
$LV+VT$		$-0.175\cos 2\phi$			$0.175\sin^2\phi$	$\pm 0.087\sin 2\phi$	
$tt-ll$				$\pm 0.172\sin 2\phi$			Values of $2n$
$vv-ll$		$\mp 0.087\sin 2\phi$		$\mp 0.172\sin 2\phi$			
vl		$-0.175\cos^2\phi$					

TABLE IV (*Cont'd*)

tl	$\pm 0.174 \sin 2\phi$		$-0.175 \cos^2\phi$				Values of $2n$
lt					$\pm 0.087 \sin 2\phi$	$0.175 \cos^2\phi$	
tv	$0.346 \sin^2\phi$		$\mp 0.087 \sin 2\phi$	$0.34(9) \sin^2\phi$*			
vt				$-0.34(9) \cos^2\phi$*			
lv		$0.175 \sin^2\phi$			$0.175 \sin^2\phi$	$\pm 0.087 \sin 2\phi$	

* The third place of the coefficient is correct for $\phi = 45°$ only.
† In cases where the coefficients are preceded by two polarity signs, the upper sign refers to North or East headings and the lower sign refers to South or West headings.

TABLE V
VALUES OF THE SECOND-HARMONIC PERM-INDUCED AND EDDY-CURRENT SIGNAL COEFFICIENTS† FOR CARDINAL HEADINGS
($\omega = 1$, $\psi_m = 10°$, $\lambda_m = 5°$, $\Omega_m = 5°$)

Source	North-South			East-West			
	Roll	Pitch	Yaw	Roll	Pitch	Yaw	
T				$\pm 0.015 \cos\phi$		$\pm 0.004 \cos\phi$	Values of $2m$
L		$\pm 0.004 \cos\phi$	$\pm 0.004 \cos\phi$				
V	$0.015 \sin\phi$	$0.004 \sin\phi$		$0.015 \sin\phi$	$0.004 \sin\phi$		
$TT - LL$	$-0.030 \sin^2\phi$		$-0.008 \cos^2\phi$	$0.030 \cos 2\phi$		$0.008 \cos^2\phi$	
$VV - LL$	$0.030 \sin^2\phi$	$-0.008 \cos 2\phi$		$-0.030 \cos 2\phi$	$0.008 \sin^2\phi$		
$TL + LT$							
$TV + VT$				$\mp 0.030 \sin 2\phi$	$\pm 0.002 \sin 2\phi$	$\pm 0.002 \sin 2\phi$	
$LV + VL$	$\pm 0.008 \sin 2\phi$	$\pm 0.008 \sin 2\phi$	$\pm 0.002 \sin 2\phi$				
$tt - ll$	$0.030 \sin^2\phi$		$0.008 \cos^2\phi$	$-0.030 \cos 2\phi$		$-0.008 \cos^2\phi$	Values of $2p$
$vv - ll$	$-0.030 \sin^2\phi$	$0.008 \cos 2\phi$		$0.030 \cos 2\phi$	$-0.008 \sin^2\phi$		
vl	$\mp 0.015 \sin 2\phi$	$\mp 0.008 \sin 2\phi$					
tl							
lt							
tv				$\mp 0.030 \sin 2\phi$		$\mp 0.004 \sin 2\phi$	
vt				$\mp 0.030 \sin 2\phi$	$\mp 0.004 \sin 2\phi$		
lv		$\mp 0.008 \sin 2\phi$	$\mp 0.004 \sin 2\phi$				

† In cases where the coefficients are preceded by two polarity signs, the upper sign refers to North or East headings and the lower sign refers to South or West headings.

series, assuming $\psi=\psi_m \sin \omega t$, $\lambda=\lambda_m \sin \omega t$, and $\Omega=\Omega_m \sin \omega t$ and breaking these series after the first term for sin's and after the second term for cos's.

By neglecting signal harmonics greater than the second (this is feasible for the small angular amplitude maneuvers) and by assuming that, during sinusoidal pitch maneuvers, the detector magnetometer accomplishes vertical sinusoidal oscillations at the pitch frequency, the signal at the MAD magnetometer output at any moment during the reference maneuver on a particular heading is expressed by (10),

$$hA_m \sin(\omega t+\varphi) + \sum F_{pi} \sin \omega t + \sum F_{ed} \cos \omega t + \sum f_{pi} \cos 2\omega t + \sum f_{ed} \sin 2\omega t, \tag{10}$$

where A_m and φ are, respectively, the amplitude (in feet) and the phase shift of vertical sinusoidal oscillations of the detector magnetometer relative to the pitch attitude sinusoidal; h is the vertical gradient of the earth's magnetic field in gammas per foot,

$$\omega = \frac{2\pi}{T} = 1,$$

where T is the period of the maneuvers and

$$\sum F_{pi} = \sum k_{pi} S_{pi},$$
$$\sum f_{pi} = \sum m S_{pi},$$
$$\sum F_{ed} = \sum n S_{ed}, \qquad \sum f_{ed} = \sum p S_{ef}.$$

The summations $\sum$ include all perm-induced and eddy-current terms, respectively, for all sources which create signals on one particular heading. Thus, for instance on East pitch maneuvers $\sum F_{pi}$, $\sum F_{ed}$, $\sum f_{pi}$ and $\sum f_{ed}$ are expressed by (11), where coefficients k, n, m, p are those in Tables IV and V.

$$\begin{aligned} \sum F_{pi} &= k_1 L + k_2(TL+LT)H_\phi \\ \sum F_{ed} &= n_1 ltH_\phi + n_2 lvH_\phi \\ \sum f_{pi} &= m_1 V + m_2(VV-LL)H_\phi + m_3(TV+VT)H_\phi \\ \sum f_{ed} &= p_1(vv-ll)H_\phi + p_2 vtH_\phi. \end{aligned} \tag{11}$$

When applying (10) to roll and yaw maneuvers, the term $hA_m \times \sin(\omega t+\varphi)$ is absent.

The expressions for the signals at the detector-magnetometer output at any moment t during any maneuver performed at arbitrary angular frequency and angular amplitudes (approximately those of the reference maneuvers) are modified.

For pitches:

$$hA_m(\omega,\lambda_m)\sin[\omega t+\varphi(\omega,\lambda_m)] + \frac{\lambda_m}{5}\sum F_{pi}\sin\omega t + \frac{\lambda_m}{5}\omega\sum F_{ed}\cos\omega t + \left(\frac{\lambda_m}{5}\right)^2 \sum f_{pi}\cos 2\omega t + \left(\frac{\lambda_m}{5}\right)^2 \omega \sum f_{ed}\sin 2\omega t, \tag{12}$$

for rolls:

$$\frac{\psi_m}{10}\sum F_{pi}\sin\omega t + \frac{\psi_m}{10}\omega\sum F_{ed}\cos\omega t + \left(\frac{\psi_m}{10}\right)^2 \sum f_{pi}\cos 2\omega t + \left(\frac{\psi_m}{10}\right)^2 \omega\sum f_{ed}\sin 2\omega t, \tag{13}$$

for yaws:

$$\frac{\Omega_m}{5}\sum F_{pi}\sin\omega t + \frac{\Omega_m}{5}\omega\sum F_{ed}\cos\omega t + \left(\frac{\Omega_m}{5}\right)^2 \sum f_{pi}\cos 2\omega t + \left(\frac{\Omega_m}{5}\right)^2 \omega\sum F_{ed}\sin 2\omega t, \tag{14}$$

where the expressions $\sum F_{pi}$, $\sum F_{ed}$, $\sum f_{pi}$, and $\sum f_{ed}$ are made up *for the coefficients of Tables IV and V divided by two.* The expressions $A_m(\omega, \lambda_m)$ and $\varphi(\omega_1\lambda_m)$ indicate only that the vertical displacement of the detector magnetometer and the phase shift of the sinusoid of this displacement relative to the pitch attitude sinusoid are functions of the angular frequency and amplitude of the pitch maneuvers.

The phase shift between the detector magnetometer output and the AN/ASQ-8 pen recorder may be designated as α for the signal fundamentals and β for the second harmonics. Attenuation of the system may be denoted as ϵ for the fundamentals and δ for the second harmonic components while the MAD sensitivity is determined in record divisions per gamma (s div/gamma). Thus, the magnitude of the maneuver signals indicated on the MAD recorder for any moment t, any angular frequency ω, and angular amplitudes ψ_m, λ_m, and Ω_m for the roll, pitch, and yaw maneuvers, respectively, is expressed in record divisions as follows:

For pitches:

$$\left\{ hA_m(\omega,\lambda_m)\sin[\omega t+\varphi(\omega,\lambda_m)+\alpha] + \epsilon\frac{\lambda_m}{5}\sum F_{pi}\sin(\omega t+\alpha) + \epsilon\frac{\lambda_m}{5}\omega\sum f_{ed}\cos(\omega t+\alpha) + \delta\left(\frac{\lambda_m}{5}\right)^2\sum f_{pi}\cos(2\omega t+\beta) + \delta\left(\frac{\lambda_m}{5}\right)^2\omega\sum f_{ed}\sin(2\omega t+\beta)\right\} \times s, \tag{15}$$

for rolls:

$$\left[\epsilon \frac{\psi_m}{10} \sum F_{pi} \sin(\omega t + \alpha) + \epsilon \frac{\psi_m}{10} \omega \sum F_{ed} \cos(\omega t + \alpha) + \delta\left(\frac{\psi_m}{10}\right)^2 \sum f_{pi} \cos(2\omega t + \beta) + \delta\left(\frac{\psi_m}{10}\right)^2 \omega \sum f_{ed} \sin(2\omega t + \beta)\right] \times s, \tag{16}$$

for yaws:

$$\left[\epsilon \frac{\Omega_m}{5} \sum F_{pi} \sin(\omega t + \alpha) + \epsilon \frac{\Omega_m}{5} \omega \sum F_{ed} \cos(\omega t + \alpha) + \delta\left(\frac{\Omega_m}{5}\right)^2 \omega \sum f_{pi} \cos(2\omega t + \beta) + \delta\left(\frac{\Omega_m}{5}\right)^2 \omega \sum f_{ed} \sin(2\omega t + \beta)\right] \times s. \tag{17}$$

Expression (15) may be rewritten for $t=0$, $T/4$, $T/2$ and $3T/4$ where 0, $T/4$, $T/2$ and $3T/4$ correspond to various attitudes of the aircraft during the maneuver.

For $t=0$:

$$\epsilon h A_m(\omega, \lambda_m) \sin[\varphi(\omega, \lambda_m) + \alpha] + \epsilon \frac{\lambda_m}{5} \sum F_{pi} \sin\alpha + \epsilon \frac{\lambda_m}{5} \omega \sum F_{ed} \cos\alpha + \delta\left(\frac{\lambda_m}{5}\right)^2 \sum f_{pi} \cos\beta + \delta\left(\frac{\lambda_m}{5}\right)^2 \omega \sum f_{ed} \sin\beta = \frac{M_0}{s}, \tag{18}$$

for $t=T/4$:

$$\epsilon h A_m(\omega, \lambda_m) \cos[\varphi(\omega, \lambda_m) + \alpha] + \epsilon \frac{\lambda_m}{5} \sum F_{pi} \cos\alpha - \epsilon \frac{\lambda_m}{5} \omega \sum F_{ed} \sin\alpha - \delta\left(\frac{\lambda_m}{5}\right)^2 \sum f_{pi} \cos\beta - \delta\left(\frac{\lambda_m}{5}\right)^2 \omega \sum f_{ed} \sin\beta = \frac{M_{T/4}}{s}, \tag{19}$$

for $t=T/2$:

$$-\epsilon h A_m(\omega, \lambda_m) \sin[\varphi(\omega, \lambda_m) + \alpha] - \epsilon \frac{\lambda_m}{5} \sum F_{pi} \sin\alpha - \epsilon \frac{\lambda_m}{5} \omega \sum F_{ed} \cos\alpha + \delta\left(\frac{\lambda_m}{5}\right)^2 \sum f_{pi} \cos\beta + \delta\left(\frac{\lambda_m}{5}\right)^2 \omega \sum f_{ed} \sin\beta = \frac{M_{T/2}}{s}, \tag{20}$$

for $t=3T/4$:

$$-\epsilon h A_m(\omega, \lambda_m) \cos[\varphi(\omega, \lambda_m) + \alpha] - \epsilon \frac{\lambda_m}{5} \sum F_{pi} \cos\alpha + \epsilon \frac{\lambda_m}{5} \omega \sum F_{ed} \sin\alpha - \delta\left(\frac{\lambda_m}{5}\right)^2 \sum F_{pi} \cos\beta - \delta\left(\frac{\lambda_m}{5}\right)^2 \omega \sum f_{ed} \sin\beta = \frac{M_{3T/4}}{s}, \tag{21}$$

where the terms M_0, $M_{T/4}$, $M_{T/2}$ and $M_{3T/4}$ are readings in record divisions relative to the zero reference line on the MAD pitch signal record for $t=0$, $T/4$, $T/2$ and $3T/4$.

Subtracting (20) from (18) and (21) from (19), the following expressions are obtained:

Subtracting (20) from (18):

$$\frac{5}{\lambda_m} h A_m(\omega, \lambda_m) \sin[\varphi(\omega, \lambda_m) + \alpha] + \sum F_{pi} \sin\alpha + \omega \sum F_{ed} \cos\alpha = \frac{M_0 - M_{T/2}}{2s\epsilon \frac{\lambda_m}{5}}. \tag{22}$$

Subtracting (21) from (19):

$$\frac{5}{\lambda_m} h A_m(\omega, \lambda_m) \cos[\varphi(\omega, \lambda_m) + \alpha] + \sum F_{pi} \cos\alpha - \omega \sum F_{ed} \sin\alpha = \frac{M_{T/4} - M_{3T/4}}{2s\epsilon \frac{\lambda_m}{5}}. \tag{23}$$

From (22) and (23), it can be seen that the harmonic terms have been eliminated, and that the right-hand sides of the formulas indicate the differences $(M_0 - M_{T/2})$ and $(M_{T/4} - M_{3T/4})$ which, for their evaluation, do not require the determination of the signal record zero reference line. Solving (22) and (23) with respect to $\sum F_{pi}$ and $\sum F_{ed}$ results in the following expressions:

$$\frac{5}{\lambda_m} h A_m(\omega, \lambda_m) \cos(\omega, \lambda_m) + \sum F_{pi} = \frac{(M_0 - M_{T/2}) \sin\alpha + (M_{T/4} - M_{3T/4}) \cos}{2s\epsilon \frac{\lambda_m}{5}} \tag{24}$$

$$\frac{5}{\omega\lambda_m} h A_m(\omega, \lambda_m) \sin(\omega, \lambda_m) + \sum F_{ed} = \frac{(M_0 - M_{T/2}) \cos - (M_{T/4} - M_{3T/4}) \sin}{2s\epsilon \frac{\lambda_m}{5} \omega}. \tag{25}$$

The same procedures used with (15) can be repeated with (16) and (17) to obtain similar formulas for roll and yaw maneuvers with the exception of the vertical-gradient term, which is absent during these maneuvers.

For roll maneuvers:

$$\sum F_{pi} = \frac{(M_0 - M_{T/2}) \sin \alpha + (M_{T/4} - M_{3T/4}) \cos \alpha}{2s\epsilon \dfrac{\psi_m}{10}} \tag{26}$$

$$\sum F_{ed} = \frac{(M_0 - M_{T/2}) \cos \alpha - (M_{T/4} - M_{3T/4}) \sin \alpha}{2s\epsilon \dfrac{\psi_m}{10} \omega} \cdot \tag{27}$$

For yaw maneuvers:

$$\sum F_{pi} = \frac{(M_0 - M_{T/2}) \sin \alpha + (M_{T/4} - M_{3T/4}) \cos \alpha}{2s\epsilon \dfrac{\Omega_m}{5}} \tag{28}$$

$$\sum F_{ed} = \frac{(M_0 - M_{T/2}) \cos \alpha - (M_{T/4} - M_{3T/4}) \sin \alpha}{2s\epsilon \dfrac{\Omega_m}{5} \omega} \cdot \tag{29}$$

By varying cardinal magnetic headings θ or even dip angles ϕ it is possible to obtain simple mathematical expressions for all the magnetic sources.

VII. Calculation of the Aircraft Magnetic Sources

In order to apply the formulas just derived to the analysis of MAD records and to the calculation of magnetic sources, it is necessary that the following information be obtained during aircraft maneuvers:

(1) Simultaneously with the MAD signal, the instantaneous aircraft attitudes must be recorded to enable the frequency and angle amplitudes of the maneuvers to be determined (Fig. 2).

(2) The MAD-system phase response must be known to determine the phase shift α of the signal fundamentals relative to the attitude sinusoid (see Fig. 3).

(3) The attenuation of the MAD equipment must be known to determine the coefficient ϵ (see Fig. 4).

(4) A table of computed ratios—fundamental signal-amplitude to source for 10° rolls, 5° pitches and 5° yaws at $\omega = 1$ for cardinal compass headings as a function of the magnetic dip angle ϕ should be compiled (see Tables IV and V).

(5) Values for the terms $A_m(\omega, \lambda_m)$ and $\varphi(\omega, \lambda_m)$ the amplitude and phase shift of the detector magnetometer oscillations during pitch maneuvers for various ω and λ_m should be compiled into tabular form. It is necessary to know these values for every new construction of the aircraft. If such a table is not available, it is possible to exclude the vertical-gradient terms appearing in the pitch equations in many cases by performing pitch maneuvers of identical frequency and angular amplitude on opposite cardinal headings.

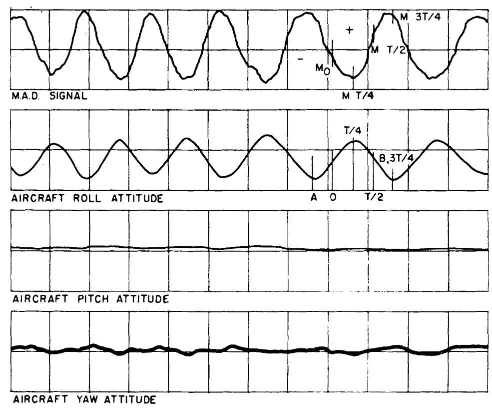

Fig. 2—MAD flight record—north roll maneuver.

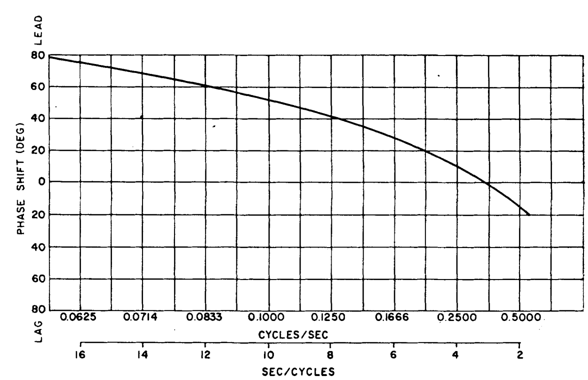

Fig. 3—AN/ASQ-8 system phase shift (HTA band).

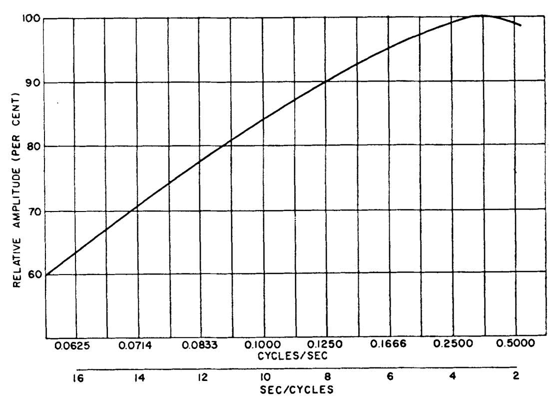

Fig. 4—AN/ASQ-8 system frequency response (HTA band).

VIII. Calibration of Compensating Sources

To compensate the original magnetic fields associated with the aircraft, it is necessary to apply compensating sources which are equal to the aircraft sources as determined from (24)–(29) but opposite in sign. Hence, it is necessary to calibrate these sources prior to the compensation procedure.

The source-calibration procedure requires that two similar maneuvers (pitch, roll or yaw) be peformed on any heading with and without a compensating source S_x, where S_x is to be calibrated. S_x is created by coils, rings or permalloy strips, which serve as compensating devices and are attached or mounted on the aircraft.

MAD signals received from the compensation maneuvers are analyzed in accordance with (24)–(29) and Table IV. From these equations and Table IV, the source-calibration formulas are readily derived.

IX. Practical Notes

The main source of error with which this method is concerned is nonsinusoidal maneuvers of the aircraft. Usually, the maneuvers indicate amplitude and frequency modulation, and often they are nonperiodical. Also, it is not easy practically to obtain pure single-axis maneuvers. However, practical applications proved that a sufficiently trained pilot can satisfy the basic requirements of this method of compensation. Of course, development of devices for indicating and controlling the frequency and angular amplitude of the aircraft maneuvers could improve considerably the accuracy of this method.

The method developed is applicable to any type airplane or airship capable of performing single-axis sinusoidal maneuvers.

References

[1] W. E. Tolley and J. D. Lawson, "Magnetic Compensation of MAD Equipped Aircraft," Airborne Instruments Lab. Inc., Mineola, N. Y., Rept. 201-1; June, 1950.

[2] P. Leliak, "Identification and Evaluation of Magnetic Field Sources Associated with MAD Equipped Aircraft," Electronics Div., The Martin Company, Baltimore, Md., Rept. No. ER 7362; September, 1955.

Correction

W. K. Saunders, author of "Post-War Developments in Continuous-Wave and Frequency-Modulated Radar," which appeared on pages 7–19 of the March, 1961, issue of these Transactions, has called the following to the attention of the *Editor*.

On page 7 replace the last 8 lines of the 3rd paragraph with:

"through signal three-thousandths of an electrical degree, given certain phase relationship may produce as large a low-frequency signal in the mixer as a return wave fully modulated by Doppler, but 142 db below the transmitter in power. At X-band, three-thousandths of an electrical degree is represented by 0.00001 inch on a direct path, or one-half that on a path involving a reflection from a missile fin. Even if the fundamental vibrations of the mechanical system have frequencies which are well below the Doppler band of interest, it can be shown that some geometries produce harmonics of the fundamental vibration frequency in the path length. Owing to the very small path variations that are significant, these harmonics cannot always be ignored."

On p. 11, insert + in second displayed formula.

On p. 12, second line, second paragraph: "AN-APN-100 [16]."

On p. 14, section D, second paragraph, third line: "Fig. 14 [25], [36], [50]."

On p. 14, section D, second paragraph, 28th line: "[25], [36], [50]."

On p. 14, section D, second paragraph, last line: "by Witmer [50] . . . ."

On p. 15, next to last paragraph, sixth line: replace $V(t)$ with $U(t)$.

On p. 16, Fig. 15 caption: replace $V(t)$ with $U(t)$.