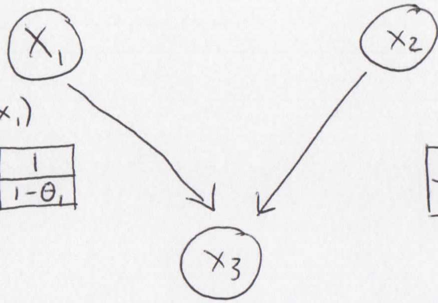
The Bayesian Network, G:



$P(x_1)$

0	1
θ_1	$1-\theta_1$

$P(x_2)$

0	1
θ_2	$1-\theta_2$

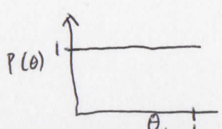
$P(x_3 \mid x_1, x_2)$

x_1	x_2	0	1
0	0	$\theta_{3,0,0}$	$1-\theta_{3,0,0}$
0	1	$\theta_{3,0,1}$	$1-\theta_{3,0,1}$
1	0	$\theta_{3,1,0}$	$1-\theta_{3,1,0}$
1	1	$\theta_{3,1,1}$	$1-\theta_{3,1,1}$

We will assume a flat Beta(1, 1) prior distribution for all parameters.

e.g.

$\theta_1 \sim \text{Beta}(1,1) \Rightarrow P(\theta_1) = 1$



The likelihood $P(D \mid \theta, G)$

$$P(D \mid \theta) = \prod_i P(D_i \mid \theta)$$

$$= \prod_i P(x_{1,i} \mid \theta)\, P(x_{2,i} \mid \theta)\, P(x_{3,i} \mid x_{1,i}, x_{2,i}, \theta)$$

$$= \prod_i \left(\theta_1^{1-x_{1,i}}(1-\theta_1)^{x_{1,i}}\right)\left(\theta_2^{1-x_{2,i}}(1-\theta_2)^{x_{2,i}}\right)\left(\theta_{3,x_{1,i},x_{2,i}}^{1-x_{3,i}}(1-\theta_{3,x_{1,i},x_{2,i}})^{x_{3,i}}\right)$$

$$= \theta_1^{n_{1,0}}(1-\theta_1)^{n_{1,1}}\,\theta_2^{n_{2,0}}(1-\theta_2)^{n_{2,1}} \prod_{a\in\{0,1\}} \prod_{b\in\{0,1\}} \theta_{3,a,b}^{n_{3,a,b,0}}(1-\theta)^{n_{3,a,b,1}}$$

where

$n_{1,a} = |\{i : x_{1,i} = a\}|$ e.g. the number of observations with $x_1 = a$

$n_{2,a} = |\{i : x_{2,i} = a\}|$

$n_{3,a,b,c} = |\{i : x_{1,i} = a, x_{2,i} = b, x_{3,i} = c\}|$

$n_{3,a,b} = |\{i : x_{1,i} = a, x_{2,i} = b\}|$

$n = |\{i\}|$ (the number of observations)

The model evidence $P(D|G)$

$$P(D|G) = \int_\theta P(D|\theta,G)\,P(\theta)\,d\theta$$

$$= \int_{\theta_1}\int_{\theta_2}\int_{\theta_{3,0,0}}\int_{\theta_{3,0,1}}\int_{\theta_{3,1,0}}\int_{\theta_{3,1,1}} \theta_1^{n_{1,0}}(1-\theta_1)^{n_{1,1}}\,\theta_2^{n_{2,0}}(1-\theta_2)^{n_{2,1}} \prod_{a\in\{0,1\}}\prod_{b\in\{0,1\}} \theta_{3,a,b}^{n_{3,a,b,0}}(1-\theta_{3,a,b})^{n_{3,a,b,1}}\, d\theta_1\, d\theta_2\, d\theta_{3,0,0}\cdots d\theta_{3,1,1}$$

$$= \left(\int_{\theta_1}\theta_1^{n_{1,0}}(1-\theta_1)^{n_{1,1}}\,d\theta_1\right)\left(\int_{\theta_2}\theta_2^{n_{2,0}}(1-\theta_2)^{n_{2,1}}\,d\theta_2\right)\prod_a\prod_b\left(\int_{\theta_{3,a,b}}\theta_{3,a,b}^{n_{3,a,b,0}}(1-\theta_{3,a,b})^{n_{3,a,b,1}}\,d\theta_{3,a,b}\right)$$

$$= \left(\frac{\Gamma(n_{1,0}+1)\Gamma(n_{1,1}+1)}{\Gamma(n_{1,0}+n_{1,1}+2)}\right)\left(\frac{\Gamma(n_{2,0}+1)\Gamma(n_{2,1}+1)}{\Gamma(n_{2,0}+n_{2,1}+2)}\right)\prod_{a\in\{0,1\}}\prod_{b\in\{0,1\}}\left(\frac{\Gamma(n_{3,a,b,0}+1)\Gamma(n_{3,a,b,1}+1)}{\Gamma(n_{3,a,b,0}+n_{3,a,b,1}+2)}\right)$$

$$= \left(\frac{n_{1,0}!\,n_{1,1}!}{(n_{1,0}+n_{1,1}+1)!}\right)\left(\frac{n_{2,0}!\,n_{2,1}!}{(n_{2,0}+n_{2,1}+1)!}\right)\prod_{a\in\{0,1\}}\prod_{b\in\{0,1\}}\left(\frac{n_{3,a,b,0}!\,n_{3,a,b,1}!}{(n_{3,a,b,0}+n_{3,a,b,1}+1)!}\right)$$

$$= \left((n+1)\binom{n}{n_{1,0}}\right)^{-1}\left((n+1)\binom{n}{n_{2,0}}\right)^{-1}\prod_{a\in\{0,1\}}\prod_{b\in\{0,1\}}\left((n_{3,a,b}+1)\binom{n_{3,a,b}}{n_{3,a,b,0}}\right)^{-1}$$

$$\log P(D|G) = -\left(\log(n+1) + \log\binom{n}{n_{1,0}} + \log(n+1) + \log\binom{n}{n_{2,0}} + \sum_{a\in\{0,1\}}\sum_{b\in\{0,1\}}\left(\log(n_{3,a,b}+1) + \log\binom{n_{3,a,b}}{n_{3,a,b,0}}\right)\right)$$

binomial coefficients