University of Cyprus

Department of Physics

Master Thesis

Magnetic Black Holes

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1 | Introduction

The following work is part of the original thesis on ”Magnetic Black holes”. The thesis was submitted as a fulfillment of the requirements for the completion of the Master’s degree in Physics at the University of Cyprus. This preview of the original thesis has been drafted and translated (from Greek) for the application of admission in the doctoral program in Quantum Computing at the Cyprus Institute. A large portion of the thesis was left out, in compliance to the 3000 words limit.

Chapter 2 analyses the problem of a non relativistic electrically charged particle coupled to the field of a monopole magnetic charge. The particle is constrained on a spherical shell of unit radius. The analysis elaborates on the spinor coordinates introduced by Haldane [5] which map the points of the unit spherical shell on the bloch sphere. The algebra of rotations is constructed in agreement with Haldane. The construction and illustration of the eigenstates is also included in the preview of this chapter.

Chapter 3, describes, in the presence of strong magnetic fields, the instabilities of the conventional ground state of the Standart model vacuum, which breaks the Electroweak Symmetry. These instabilities are the result of the coupling of the W bosons’ spin with the magnetic field. They indicate that the conventional ground state is unstable. It then continues on the minimization of the energy corresponding to the fully symmetric lagragian of the electroweak theory, and from there to the construction of the stable ground state. The stable ground state is symmetric with respect to SU(2)×U(1).

2 | Spherical Landau Problem

2.1 Lowest Landau level

The eigenvalues of the mechanical angular momentum make evident the constraint that $\vec{\Lambda}$ can not vanish, in the presence of a magnetic monopole. Specifically, at the lowest Landau level (LLL), $n = 0$, and the the eigenvalues are

$$\begin{aligned} |L|^2 &= \hbar^2 s_0(s_0+1) \\ |\Lambda|^2 &= \hbar^2 s_0 \end{aligned} \tag{2.1}$$

with $2s_o + 1$ degenerate eigenstates. The $2s_o + 1$ eigenstates have energy $\frac{e\hbar|B|}{2m}$ and are described by homogeneous polynomials of rank $2s_o$ [5]. The eigenfunctions are analysed in the next section.

2.2 Haldane polynomials

A spinor vector, with coordinates $(a(\theta,\phi), b(\theta,\phi))$, is used to map every point (θ,ϕ) of the unit sphere, onto the bloch sphere. Choosing the symmetric vector potential

$$\vec{A} = \frac{\hbar s_o}{er}\cot\theta\hat{\phi} \tag{2.2}$$

the spinor vector is

$$|r\rangle \equiv \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2}e^{i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}e^{-i\frac{\phi}{2}} \end{pmatrix}, \qquad \langle r| \equiv (a^*, b^*) \tag{2.3}$$

For other vector potentials used in the literature, the corresponding forms of the spinor vector are

$$\begin{aligned} \vec{A} = \frac{\hbar s_o}{er}\frac{1-\cos\theta}{\sin\theta}\hat{\phi} \quad &\longrightarrow \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{-i\phi} \end{pmatrix} \\ \vec{A} = -\frac{\hbar s_o}{er}\frac{1+\cos\theta}{\sin\theta}\hat{\phi} \quad &\longrightarrow \begin{pmatrix} \cos\frac{\theta}{2}e^{i\phi} \\ \sin\frac{\theta}{2} \end{pmatrix} \end{aligned} \tag{2.4}$$

The position vector of an electron is related to the spinor vector as follows

$$\vec{\Omega}_r(\theta,\phi) \equiv (a,b)\,\vec{\sigma}\begin{pmatrix} a^* \\ b^* \end{pmatrix} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{2.5}$$

The spinor is an eigenfunction of the spin operator with orientation evident from the following equation

$$(\vec{\Omega}_r\cdot\vec{\sigma})\,|r\rangle = -|r\rangle$$

A rotation of angle ω with respect to an axis parallel to the vector $\vec{\Omega}_n$ is represented, in spinor space, as the action of the exponential operator with the appropriate generators on the spinor vector (2.3)

$$|r\rangle \rightarrow e^{i\frac{\sigma\cdot\vec{\Omega}_n}{2}\omega}|r\rangle$$

If the axis is parallel to $\vec{\Omega}_r$, the result of the rotation in spinor space gives

$$|r\rangle \rightarrow e^{-i\omega/2}|r\rangle \tag{2.6}$$

Apart from the difference of a non-measurable phase, the end spinor is equivalent to the original. The corresponding vector in 3D is invariant to the rotation since it is parallel to the rotation axis.

In spinor space, the generators of rotation, L_i, are represented as

$$\begin{aligned} L_x &= \frac{\hbar}{2}\left(v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}\right) \\ L_y &= \frac{i\hbar}{2}\left(v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}\right) \\ L_z &= \frac{\hbar}{2}\left(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right) \end{aligned} \tag{2.7}$$

and in the concise form

$$\vec{L} = \frac{\hbar}{2}(u,v)\vec{\sigma}\begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \end{pmatrix} \tag{2.8}$$

The preferable basis is that containing the ladder operators, with the raising operator defined as $L_+ \equiv L_x + iL_y$ and the lowering operator as $L_- \equiv L_x - iL_y$

$$\begin{aligned} L_+ &= \hbar u\frac{\partial}{\partial v} \\ L_- &= \hbar v\frac{\partial}{\partial u} \\ L_z &= \frac{\hbar}{2}\left(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right) \end{aligned} \tag{2.9}$$

Their commutators satisfy

$$\begin{aligned} [L_z, L_\pm] &= \pm L_\pm \\ [L_+, L_-] &= 2L_z \end{aligned} \tag{2.10}$$

and the square of the angular momentum operator is given as

$$L^2 = \frac{1}{2}(L_+L_- + L_-L_+) + L_z^2 \tag{2.11}$$

The eigenfunctions of L_z and L^2 satisfy the relations

$$\begin{aligned} L^2\,\psi_{\ell,m} &= \hbar^2\ell(\ell+1)\,\psi_{\ell,m} \\ L_z\,\psi_{\ell,m} &= \hbar m\,\psi_{\ell,m} \qquad -\ell \le m \le \ell \end{aligned} \tag{2.12}$$

Employing the method of separation of variables, defining $U(u)V(v) \equiv \psi_{s_0,m}$, the partial differential equation becomes

$$\underbrace{\frac{u}{U}\frac{\partial U}{\partial u}}_{k} - \underbrace{\frac{v}{V}\frac{\partial V}{\partial v}}_{c} = 2m$$

Since $2m$ is integral, the first and second terms have to equal the integers k and c. Solving the two resulting differential equations, one obtains

$$\begin{aligned} U(u) &\sim u^{c+2m} \\ V(v) &\sim v^{c} \end{aligned}$$

Acting with the operator L^2, and demanding that the eqigenvalue equals that of the LLL (2.1), results to $c = s_0 - m$. The resulting, unnormalized eigenfunctions of L^2 and L_z are

$$\psi_{s_0,m} = u^{s_0+m}v^{s_0-m} = \left(\cos\frac{\theta}{2}\right)^{s_0+m}\left(\sin\frac{\theta}{2}\right)^{s_0-m} e^{im\phi} \tag{2.13}$$

These are common eigenfunctions of Λ^2 and the hamiltonian

$$\Lambda^2\,\psi_{s_0,m} = \hbar^2 s_0\psi_{s_0,m} \tag{2.14}$$

The functions (2.13) constitute a complete orthogonal base which spans the space with dimensionality $2s_o + 1$. The base is represented by the polynomials

$$u^{2s_0}, u^{2s_0-1}v, \ldots, u^{s_0+m}v^{s_0-m}, \ldots, v^{2s_0} \qquad -s_0 \le m \le s_0 \tag{2.15}$$

The orthogonality relation is

$$\frac{1}{4\pi}\int_{S^2}\overline{\psi}_{s_0,m}\psi_{s_0,n}\sin\theta d\theta d\phi = \frac{(s_0+m)!(s_0-m)!}{(2s_0+1)!}\delta_{mn} \tag{2.16}$$

and the normalised form of the eigenfunctions is

$$|s_0,m\rangle = \sqrt{\frac{(2s_0+1)!}{4\pi(s_0+m)!(s_0-m)!}}\,u^{s_0+m}v^{s_0-m} \tag{2.17}$$

Searching for eigenfunctions of the eigenvalue equation [5]

$$\{\vec{\Omega}_r \cdot \vec{L}\}\Psi(u,v) = \hbar s_0 \Psi(u,v) \tag{2.18}$$

the following are obtained

$$\Psi_r^{s_0}(u,v) = (a^* u + b^* v)^{2s_0} \equiv \langle r \,|\, s \rangle^{2s_0} \tag{2.19}$$

These functions describe a particle occupying the LLL and they are eigenfunctions of the operator L with orientation parallel to the vector $\vec{\Omega}_r(a,b)$ with eigenvalue $\hbar s_0$. The polynomials are written in terms of the binomial expansion coefficients, yielding the form

$$\Psi_r^{s_0} = \sum_{m=-s_0}^{s_0} \binom{2s_0}{s_0+m} (a^* u)^{s_0+m} (b^* v)^{s_0-m} \tag{2.20}$$

Integrating the probability density on the sphere gives

$$\int_{S^2} d\Omega \, \overline{\Psi}_r^{s_0} \Psi_r^{s_0} = \frac{4\pi}{2s_0+1} \tag{2.21}$$

2.3 Haldane sphere

The eigenfunctions of L_z are represented graphically in figure 2.1, for the case of magnetic charge $s_0 = 4$. Specifically, the figure shows the probability density $|u^{s_0+m} v^{s_0-m}|^2$ and its azimouthial symmetry. From the $2s_o + 1$ eigenfunctions

$$u^8,\, u^7 v,\, u^6 v^2,\, u^5 v^3,\, u^4 v^4,\, u^3 v^5,\, u^2 v^6,\, u v^7,\, v^8 \tag{2.22}$$

only the first 4 are shown graphically. Figure 2.1 shows the xy plane and the unit sphere centred at the origin. Every function is represented by a single surface of specific colour. The distance between a point with coordinates (θ, ϕ) on a coloured surface and the corresponding point on the unit sphere is analogous to the probability density. For example, the eigenfunction with $m = 4$ (red) has a maximum at polar angle $\theta = 0$, where as the maximum for $m = 3$ (orange) is located at $\theta \approx 0.2\pi$. In general the maximum of the probability density is at polar angle calculated by

$$\theta_{max} = 2\tan^{-1}\left(\sqrt{\frac{s_0 - m}{s_0 + m}}\right) \qquad -s_0 < m \leq s_0 \tag{2.23}$$

For $m = -s_0$ the maximum is given by the limit $\lim_{x \to \infty} (2\tan^{-1} x) = \pi$.

For the graphical representation of the Haldane polynomials (2.19), it is necessary to define the spinor coordinates (a, b) of the spinor $|r\rangle$, which is then entered in (2.19). For example, assigning the values $s_0 = 1$ and $|r\rangle = (\cos\frac{\pi}{6} e^{i\frac{\pi}{6}}, \sin\frac{\pi}{6} e^{-i\frac{\pi}{6}})$ in (2.20), the eigenfunction is

$$\Psi_r^1 = \sum_{m=-1}^{1} \binom{2}{1+m} \left(\cos\frac{\pi}{6}\cos\frac{\theta}{2} e^{i(\frac{\phi}{2} - \frac{\pi}{6})}\right)^{1+m} \left(\sin\frac{\pi}{6}\sin\frac{\theta}{2} e^{-i(\frac{\phi}{2} - \frac{\pi}{6})}\right)^{1-m} \tag{2.24}$$

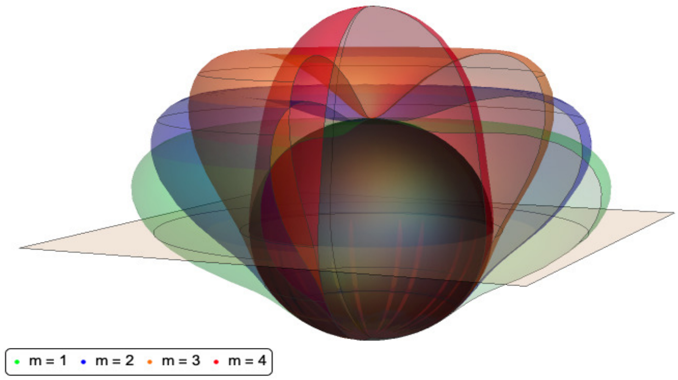


Figure 2.1: *Graphical representation of the eigenfunctions when $s_0 = 4$. Only the first four of the total $2s+1$ eigenfunctions of the operator L_z with eigenvalues $m\hbar$. (Green) u^5v^3 with $m = 1$. (Blue) u^6v^2, $m = 2$. (Orange) u^7v, $m = 3$. (Red) u^8, $m = 4$.*

The graphical representation of the aforementioned assignment is given in 2.2 where the azimuthial symmetry is no longer manifest. In the right half of 2.2, a part of the function is shown to easily distinguish extremal points. These points are the intersections of the unit sphere and the vector $\vec{\Omega}_r$. The diagrams of 2.2 also contain the unit sphere for reference.

2.4 Multiparticle states

Ignoring the electrodynamic interactions between the charged particles, the hamiltonian is simply the sum

$$H \approx \sum_{i=1}^{N} H_i$$

where

$$H_i = \frac{|\Lambda_i|^2}{2m}\frac{eB}{\hbar s_0}$$

is the single particle hamiltonian (index i iterates the particles). The single particle states are occupied according to the Pauli exclusion principle rendering the total wavefunction antisymetric with respect to the interchange between any two particles. When $N = 2s_0 + 1$, the LLL is fully occupied. Individual particles occupy a single orthogonal one particle state of the LLL, see (2.15). The total wavefunction, Ψ_N, is defined as the slater determinant

$$\Psi_N = \begin{vmatrix} u_1^{2s_0} & u_1^{2s_0-1}v_1 & \dots & v_1^{2s_0} \\ u_2^{2s_0} & \dots & \dots & v_2^{2s_0} \\ \vdots & \ddots & & \vdots \\ u_N^{2s_0} & u_N^{2s_0-1}v_N & \dots & v_N^{2s_0} \end{vmatrix} \tag{2.25}$$

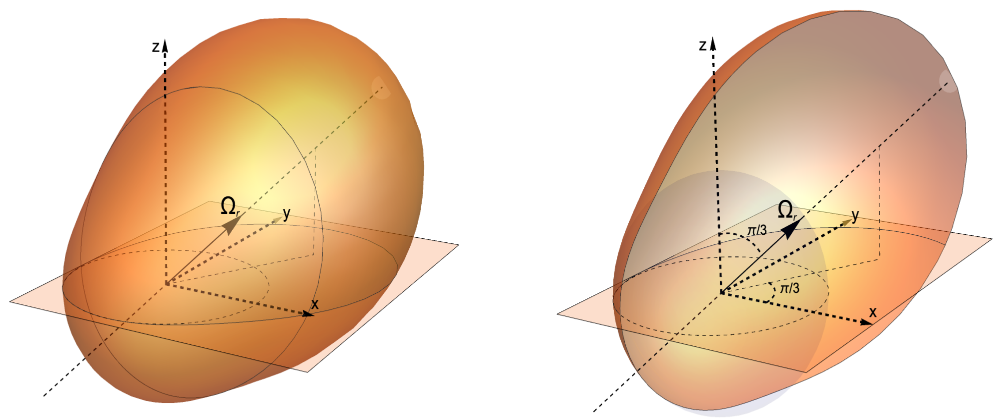


Figure 2.2: *Left, Graphical representation of the probability dens.* $|\Psi^1_{(a,b)}|^2$ *with* $s_0 = 1$ *and* $|r\rangle = (\cos\frac{\pi}{6}e^{i\frac{\pi}{6}}, \sin\frac{\pi}{6}e^{-i\frac{\pi}{6}})$. *Right, the function is cut by the plane* $\phi = \pi/3$. *Maximum at* $\theta = \frac{\pi}{3}, \phi = \frac{\pi}{3}$.

or in the simpler form [5]

$$\Psi_N = \prod_{i<j}^{N}(u_i v_j - u_j v_i) \tag{2.26}$$

where is the degeneracy of the LLL or equivalently the number of particles

$$N = 2s_0 + 1 \tag{2.27}$$

2.5 Dispersion relation for relativistic case

For relativistic fermions with spin 1/2 and charge $-e$, the dispersion relation is found using the Dirac equation. In euclidean spacetime, the equation has the form

$$\left(i\gamma^\mu D_\mu - m\right)\psi = 0 \tag{2.28}$$

where ψ is the Dirac spinor, D_μ is the covariant derivative

$$\begin{aligned} D_\mu &\equiv \partial_\mu - ieA_\mu \\ [D_\mu, D_\nu] &= -ieF_{\mu\nu} \end{aligned} \tag{2.29}$$

and $F_{\mu\nu}$ is the Maxwell tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.30}$$

Acting on each side of the Dirac equation (2.28) with its complex conjugate gives rise to the equation

$$(\gamma^\mu\gamma^\nu D_\mu D_\nu + m^2)\psi = 0 \tag{2.31}$$

Utilizing the commutation and anticommutation relations of the Dirac matrices, equation (2.31) gives

$$\frac{1}{2}(-4iS^{\mu\nu}D_\mu D_\nu + 2D^\mu D_\mu)\psi + m^2\psi = 0 \tag{2.32}$$

and by the substitution of (2.29) the Maxwell tensor enters the equation bringing it to the form

$$\left[D^2 - iS^{\mu\nu}(-ieF_{\mu\nu}) + m^2\right]\psi = 0 \tag{2.33}$$

For non-zero magnetic field

$$\begin{aligned} F_{0i} &= F_{i0} = 0 \\ F_{ij} &= \varepsilon_{ijk}B_k \end{aligned} \tag{2.34}$$

where B_k are the components of the magnetic field, and by using the definition of $S^{\mu\nu}$ [7], the relativistic expression is written as

$$\left[D_0^2 + (-iD_i)^2 - 2e\vec{B}\cdot\vec{S} + m^2\right]\psi = 0 \tag{2.35}$$

Transforming into momentum space, the first term contributes the negative sign in the energy $-E^2$, whereas the second term involves the operator of mechanical momentum which is related to angular momentum via

$$\vec{\Pi}^2 = \frac{1}{r^2}\left[\vec{\Lambda}^2 - i\vec{r}\cdot\vec{\Pi} + (\vec{r}\cdot\vec{\Pi})^2\right] = \frac{\Lambda^2}{r^2} - \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} \tag{2.36}$$

The radial momentum is written as p_3 and the total mechanical momentum is

$$\vec{\Pi}^2 = \frac{\Lambda^2}{r^2} + p_3^2 \tag{2.37}$$

With the eigenvalue of the operator Λ^2 (equation (2.1)) we find that

$$\frac{\Lambda^2}{r^2} = \frac{s_0}{r^2} = eB \equiv |F| \tag{2.38}$$

The third term describes the coupling of the magnetic field with the spin $\vec{S}$

$$2e\vec{B}\cdot\vec{S} = 2eBs \equiv 2|F|s \tag{2.39}$$

For a single fermion the eigenvalues of spin are $\pm\frac{1}{2}$ ($\hbar = 1$) from which, in the case of a strong magnetic field, only the positive one is selected. This results in the cancelling between the two energy contributions, the first being due to the coupling between the angular momentum and the magnetic field and the second due to the coupling between the spin and the magnetic field. All the above give rise to the relativistic dispersion relation in the form

$$E^2(\text{spinor}) - p_3^2 = m^2 + |F|(1-2s) = m^2 + 0, \qquad s = 1/2 \tag{2.40}$$

For scalar fields (spin 0) and charge $-e$, the analogous relation is

$$E^2(\text{scalar}) - p_3^2 = m^2 + |F|, \qquad s = 0 \tag{2.41}$$

For vector fields (spin 1) the relation is [4]

$$E^2(\text{vector}) - p_3^2 = m^2 + |F|(1-2s) = m^2 - |F|, \qquad s = 1 \tag{2.42}$$

Summing up, at the Lowest Landau level $n = 0$, the dispersion relations for the various Lorentz representations are collectively shown below

$$E^2 - p_3^2 = \begin{cases} m^2 + |F|, & s = 0 \\ m^2 + 0, & s = 1/2 \\ m^2 - |F|, & s = 1 \end{cases} \tag{2.43}$$

3 | EW Symmetry Restoration

The degeneracy of the vacuum in the electroweak theory is lifted in the presence of strong magnetic fields. These fields suppress the vev of the higgs field to zero: $\langle\phi\rangle \to 0$. The zero vev is invariant under the gauge transformations of the electroweak symmetry group and thus the corresponding vacuum state is symmetric.

3.1 Electroweak corona

The magnetic field of an extremal, magnetically charged black hole varies according to the inverse square law

$$|F| \equiv eB = \frac{Q}{2r^2} \tag{3.1}$$

where the proper magnitute of the magnetic field has units $(eV)^2$. The maximum is located at the horizon $r_e = \frac{Q\sqrt{\pi G}}{e}$ and is given according to

$$|F| = \frac{e^2}{2\pi G^2 Q} \tag{3.2}$$

where the inverse relation between the magnitute and the charge Q is manifest.

Various phenomena leading to the restoration of the electroweak (EW) symmetry are initiated, when the magnetic charge is smaller than a threshold value Q_{ew}, defined by the energy scale at which the symmetry is expected to break. If $Q < Q_{ew}$, there are two critical radii, r_h and r_w, at which the magnitute of the magnetic field is equal to the square of the higgs and W boson mass, $m_h^2 \equiv 4\lambda v^2$ and $m_w^2 \equiv \frac{g^2v^2}{2}$, respectively

$$r_h = \sqrt{\frac{Q}{2}}\frac{1}{m_h} \quad < \quad r_w = \sqrt{\frac{Q}{2}}\frac{1}{m_w} \tag{3.3}$$

When the magnetic charge is equal to the critical value Q_{ew}, the radius of the horizon is equal to the radius r_w

$$r_e(Q_{ew}) = r_w(Q_{ew}) \qquad Q_{ew} = \frac{e^2}{2\pi G m_w^2} \tag{3.4}$$

At this value the corona is not formed and the phenomena cannot take place.

At radii $r \le r_w$, the conventional vacuum ground state (which breaks the electroweak symmetry) is not stable as the W bosons appear to become tachyonic. The stable ground state is found to be one with a condensate of W bosons, which lowers the energy of the system. As the radius is decreased, the expectation value of the Higgs field is decreased, driven to zero when $r = r_h$. For radii $r_e < r < r_h$ the vev of the Higgs field is zero and thus the EW symmetry is restored. The region $r_h \le r \le r_w$ is called the electroweak corona. For radii $r \le r_w$, the W condensate screens the SU(2) component of the magnetic field. For $|F| > m_h^2$ (equivalently $r < r_h$) only the U(1) component contributes to the magnetic field.

3.2 Transition to the region with Electroweak restoration

The broken Electroweak theory, in the presence of strong magnetic fields, leads to instabilities due to the coupling of the spin of the charged vector bosons (spin 1) with the magnetic field. The square of the effective mass of the Ws is [2, 1]

$$m_{\text{eff}}^2 = -(|F| - m_W^2) \tag{3.5}$$

It is thereby evident that, in relatively strong magnetic fields, $|F| > m_W^2 \sim 10^{20}\,T$, the effective mass squared becomes negative and the Ws are rendered tachyonic. This appearance of tachyonic particles is makes clear that the conventional (symmetry breaking) ground state, with the vevs $\langle\phi\rangle > 0$, $\langle W_\mu\rangle = \langle Z_\mu\rangle = 0$, is unstable. The correct ground state is defined by the minimization of the energy. In the subsequent work, the Higgs field is treated as a dynamic variable and we choose the gauge in which the field is of the form

$$\begin{pmatrix} 0 \\ \phi \end{pmatrix} \tag{3.6}$$

with only the second component non-zero and real. The vev of the field will be determined in terms of the W condensate. The only non-zero components of the tensors

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.7a}$$

$$Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu \tag{3.7b}$$

are the magnetic ones $F_{12} = -F_{21}$ and $Z_{12} = -Z_{21}$, which are to be defined by the equations of the theory. Choosing the Landau gauge, the electromagnetic potential is

$$A_\mu = x_1 \frac{|F|}{e} \delta_{\mu 2} \tag{3.8}$$

The magnetic field is considered to be constant across the transverse plane. The generalisation of the results to a spherical surface is immediate.

The lagragian of the electroweak theory is given by

$$\begin{aligned}
\mathcal{L} = &-\frac{1}{2}|\overline{D}_\mu W_\nu - \overline{D}_\nu W_\mu|^2 - \frac{1}{4}F_{\mu\nu}^2 - \frac{1}{4}Z_{\mu\nu}^2 + (\partial_\mu\phi)^2 \\
&+ \frac{1}{2}g^2\phi^2|W_\mu|^2 + \frac{1}{2}\frac{g^2\phi^2}{2\cos^2\theta_w}Z_\mu^2 \\
&- ig\left(\sin\theta_w F_{\mu\nu} + \cos\theta_w Z_{\mu\nu}\right)W^{\dagger\mu}W^\nu \\
&+ \frac{1}{2}g^2\left((W_\mu^\dagger W_\nu)^2 - (W_\mu^\dagger W^\mu)^2\right) \\
&- \lambda(\phi^2 - v^2)^2
\end{aligned} \tag{3.9}$$

and the corresponding energy density can be written as a sum of perfect squares [2, 1], thus its minimization can be attained by setting these terms equal to zero. Under the assumption of the constrain between the coupling constants λ (Higgs potential) and g ($SU(2)_L$)

$$\lambda = \frac{g^2}{8\cos^2\theta_w} \tag{3.10}$$

the minimization is achieved when the transverse components, to the magnetic field, of the potential W_μ are non-zero and of the form

$$W_0 = W_3 = 0, \quad W_1 = -iW_2 = W(x_1 + ix_2), \qquad W \in \mathbb{C} \tag{3.11}$$

In regard to (3.11), the field equations of the model reduce to [2, 1]

$$(D_1 + iD_2)W = 0, \qquad D_i \equiv \partial_i - ig(\sin\theta_w A_i + \cos\theta_w Z_i) \tag{3.12a}$$

$$F_{12} = \frac{g}{2\sin\theta_w}v^2 + 2g\sin\theta_w|W|^2 \tag{3.12b}$$

$$Z_{12} = \frac{g}{2\cos\theta_w}(\phi^2 - v^2) + 2g\cos\theta_w|W|^2 \tag{3.12c}$$

$$Z_i = -\frac{2\cos\theta_w}{g}\varepsilon_{ij}\partial_j \ln\phi \qquad i = 1, 2 \tag{3.12d}$$

where $v \equiv |\langle\phi_0\rangle|$ is the conventional Higgs vev which breaks the symmetry.

In the small region $eF_{12} \gtrsim m_w^2$, equation (3.12a) leads to the analytical function solution

$$W(x_1, x_2) = \sum_{n=-\infty}^{\infty} C_n \exp\left\{-\frac{\pi}{2}(z+\bar{z})^2 - \pi n^2 + 2\pi n z\right\} \qquad z \equiv \sqrt{\frac{m_W^2}{2\pi}},\ (x_1 + ix_2) \tag{3.13}$$

The amplitude $|W|$ vanishes at points which form a periodic lattice structure on the transverse plane $x_1 - x_2$. The coefficients C_n are defined so that the energy is minimized [3]. By using the constrain $C_{n+1} = C_n$ the solution is a square lattice whereas for $C_{n+2} = C_n$ it is hexagonal. The hexagonal lattice, is the configuration of lowest energy [6]. Figure 3.1a illustrates the amplitude $|\mathrm{W}(x_1, x_2)|$ and the lattice with hexagonal cells. The complex phase is also illustrated in figure 3.1b. The bright lines correspond to the discontinuities of the phase, with one end at the points of zeroes of $|\mathrm{W}(x_1, x_2)|$ and extending to infinity. These lines are sensitive to gauge transformations whereas the roots of the amplitude remain invariant.
Performing the gauge transformation

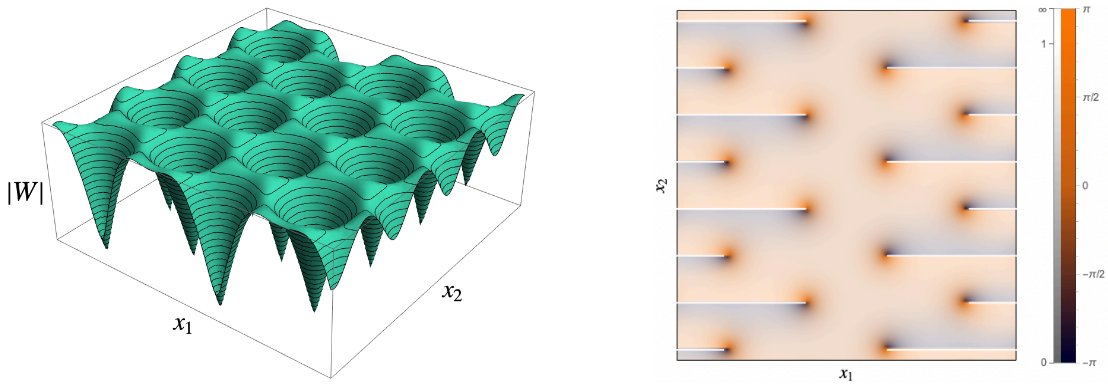
$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \chi \tag{3.14}$$

the phase of the potential $W = |W|e^{i\chi}$ can be eliminated. In this way the electromagnetic potential is given in terms of the real quantities $|W|, \phi$ in the form [1, 2]

$$A_i = \frac{1}{e}\varepsilon_{ij}\partial_j(\ln|W| + 2\cos^2\theta_w \ln\phi) \qquad i = 1, 2 \tag{3.15}$$

and thus does not contain discontinuities, in contrast with the phase $\mathrm{Arg(W)} = \chi$. Considering the line integral over a closed curve, which includes points of discontinuity, only the second term of (3.14) contributes

$$\oint (A_\mu + \partial_\mu \chi)dx^\mu = \oint \partial_\mu \chi dx^\mu \tag{3.16}$$



(a) (b)

Figure 3.1: *Αριστερά: Η πλάτος $|W|$ ως συνάρτηση στο κάθετο στο μαγνητικό πεδίο επίπεδο. Οι ρίζες του πλάτους αντιστοιχούν στα κέντρα των εξαγωνικών δινών. Δεξιά: Το γράφημα της φάσης* Arg(W), *με βάση βαθμονομημένη χρωματική κλίμακα. Διαμέσου των φωτεινών ευθειών εκδηλώνεται ασυνέχεια 2π στη φάση. Οι ευθείες έχουν άκρο σε ρίζα του πλάτους και καταλήγουν στο άπειρο.*

The magnetic flux through the surface Σ enclosed by the curve, is quantised and equal to

$$\int_{\Sigma} F_{12} d^2x \;=\; n\Phi_o \;=\; \oint \partial_\mu \chi \, dx^\mu \tag{3.17}$$

where $n \in \mathbb{Z}$ and $\Phi_o \equiv \frac{2\pi}{e}$ is the fundamental flux. The difference in phase, resulting from the integration on the closed curve is quantised.

The above can be put in contrast to the case of superconducting materials in magnetic fields. The vortices of Cooper pairs in these materials inhibit the penetration of magnetic fields - Meissner effect. In contrast, the vortices of the W condensate enhance the abelian magnetic field F_{12}, evident from the positive sign of the condensate contribution in (3.12b).

Going to even stronger magnetic fields, that is $F_{12} > m_w^2$, the condensate retains its periodic structure, (3.13). This and can be proofed perturbatively. Inside a single cell of the lattice, the following differential equations are satisfied (for the chosen gauge (3.14))

$$\partial^2 \ln|W| = -\frac{g^2\phi^2}{2} - 2g^2|W|^2 \qquad \partial^2 \equiv \partial_1^2 + \partial_2^2 \tag{3.18a}$$

$$\partial^2 \ln\phi^2 = \frac{g^2}{2\cos^2\theta_w}(\phi^2 - v^2) + 2g^2|W|^2 \tag{3.18b}$$

Using these equations, the expectation value $\langle\phi^2\rangle$ is determined, in therms of the cell surface area A

$$\left\langle\phi^2\right\rangle \equiv \frac{1}{A}\int_A \phi^2 d^2x \tag{3.19}$$

Integration of (3.18b), cancels out the contribution of the left part, and the result is

$$\int_A \partial^2 \ln \phi^2 \, d^2x = 0 \tag{3.20}$$

Integrating now equation (3.12b) and using the integral flux equation (3.17) leads to

$$\frac{1}{A}\int_A |W|^2 \, d^2x = \frac{e\overline{F_{12}} - m_W^2}{2e^2} \qquad \overline{F_{12}} \equiv \frac{2\pi n}{eA} \tag{3.21}$$

where $\overline{F_{12}}$ is the average magnetic field inside the cell. Combine the above equations yields the average value

$$\left\langle \phi^2 \right\rangle = v^2 \left(\frac{m_h^2}{m_h^2 \sin^2\theta_w} - \frac{2e\cos^2\theta_w \overline{F_{12}}}{v^2 g^2 \sin^2\theta_w} \right) = v^2 \frac{m_h^2 - e\overline{F_{12}}}{m_h^2 - m_W^2}, \qquad \overline{F_{12}} \leq \frac{m_h^2}{e} \tag{3.22}$$

3.3 Symmetry restoration

According to (3.22), the average value $\langle\phi^2\rangle \to 0$ as $e\overline{F_{12}} \to m_h^2$, and thus the Higgs vev is expected to decrease, $\langle\phi\rangle \to 0$. As the vev decrease but is still non-zero, the symmetry continues to break. When $\langle\phi^2\rangle = 0$, the symmetry breaking ceases since the Higgs vev is now symmetric with respect to $SU(2) \times U(1)$. The vanishing of the value (3.22) signals the transition into the symmetric phase of the electroweak theory, where the appropriate basis of potentials to be used is the one with the three non abelian W_μ^i and the abelian B_μ. The following equations serve as a reminder of their relations and the corresponding strength tensors

$$B_\mu = \cos\theta_w A_\mu - \sin\theta_w Z_\mu \tag{3.23a}$$

$$W_\mu^3 = \sin\theta_w A_\mu + \cos\theta_w Z_\mu \tag{3.23b}$$

$$W_\mu^1 = \frac{1}{\sqrt{2}}(W_\mu + W_\mu^\dagger) \qquad W_\mu^2 = \frac{i}{\sqrt{2}}(W_\mu - W_\mu^\dagger) \tag{3.23c}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{3.23d}$$

$$G_{\mu\nu}^i = \partial_\mu W_\nu^i - \partial_\nu W_\mu^i + g\varepsilon^{ijk} W_\mu^j W_\nu^k \tag{3.23e}$$

The $SU(2) \times U(1)$ symmetric electroweak lagragian is

$$\mathcal{L} = -\frac{1}{4} G_{\mu\nu}^i G^{i\,\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + |D_\mu \phi|^2 - \lambda(\phi^2 - v^2)^2 \tag{3.24}$$

It can be deducted from equations (3.11), (3.12b) and (3.12c) that, in the region $m_W^2 \leq e\overline{F_{12}} \leq m_h^2$, the only non zero magnetic components of the tensors are

$$B_{12} = \frac{g v^2}{2\tan\theta_w} + \frac{g}{2}\tan\theta_w (v^2 - \phi^2) \tag{3.25a}$$

$$G^3_{12} = -G^3_{21} = \frac{1}{2} g^2 \phi^2 \tag{3.25b}$$

This implies that, as $e\overline{F_{12}} \to m_h^2$, only the abelian hypermagnetic field $U(1)_Y$ survives. The kinetic terms for the potentials W^i_μ vanish and thus they can be gauged away from the lagragian. Since the tensors $G^i_{\mu\nu}$ vanish, the $SU(2)$ component of the magnetic field also vanishes. In this way, the $SU(2)$ component is screened. The surviving hypermagnetic field is

$$g_y B_{12} \geq m_h^2 \equiv 4\lambda v^2 \quad (= 2\mu^2) \tag{3.26}$$

The lagragian description of the coupling between the abelian U(1)$_Y$ potential and the scalar Higgs field is

$$\mathcal{L} = -B^2_{\mu\nu} + \underbrace{|(\partial_\mu - i\frac{1}{2} g_y B_\mu)\phi|^2 + \mu^2|\phi|^2}_{\mathcal{L}_o} - \lambda(\phi^*\phi)^2 - \lambda v^2, \qquad (\mu^2 = 2\lambda v^2) \tag{3.27}$$

where the Higgs hypercharge is set to $Y{=}\frac{1}{2}$. As for the coupling constant of U(1)$_Y$, $g_y{=}g\tan\theta_w$. The potential is defined in the Landau gauge

$$B_\mu = x_1 B_{12} \delta_{\mu 2} \qquad \left(\Rightarrow B^2_\mu = -(B_{12}x_1)^2\right) \tag{3.28}$$

where B_{12} the constant magnetic field. The equations of motion of the Higgs field, for $\phi \sim 0$, are

$$\frac{\delta \mathcal{L}_o}{\delta \phi^*} - \partial_\mu \frac{\delta \mathcal{L}_o}{\delta(\partial_\mu \phi^*)} = 0 \quad \Rightarrow \quad \partial^2 \phi = \left(-\frac{g_y^2}{4}(B_{12}x_1)^2 - i g_y B_\mu \partial_\mu + \mu^2\right)\phi \tag{3.29}$$

Separating variables $\phi = e^{iE_n x_o}\phi_n(x_1, x_2, x_3)$, the equations of motion reduce to

$$E_n^2 \phi_n = -\left(\partial_i^2 - g_y^2 \left(\frac{B_{12}x_1}{2}\right)^2 - i g_y (B_{12}x_1)\partial_2 + \mu^2\right)\phi_n \tag{3.30}$$

Imposing a translationally invariant form in the directions x_2, x_3

$$\phi_n \sim e^{ip_2x_2} e^{ip_3x_3} \tilde{\phi}_n(x_1) \tag{3.31}$$

the equation (3.30) takes the form

$$E_n^2 \tilde{\phi}_n = \left(p_3^2 - \mu^2 - \partial_1^2 + \left(\frac{g_y B_{12}}{2} x_1 - p_2\right)^2\right)\tilde{\phi}_n \tag{3.32}$$

The last two terms in the right part of (3.32) are simultaneously diagonalized by the eiqenfunctions of a harmonic oscillator, with frequency $\frac{g_y B_{12}}{2}$, in the x_1 direction. The eiqenvalues read

$$g_y B_{12}\left(n + \frac{1}{2}\right) \tag{3.33}$$

where the integer n is the Landau level. At the lowest Landau level the energy is

$$E_o^2 = p_3^2 - \mu^2 + \frac{g_{\mathrm{y}} B_{12}}{2} \tag{3.34}$$

The hypermagnetic field satisfies the relation (3.26) so the Higgs effective mass, in the dispersion relation (3.34), is positive. This insures that the symmetric state ϕ=0 is stable, in the case of magnetic fields in the range (3.26). The electroweak symmetry is restored.

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