第一题

$a = 2i + 3j - k, b = i - 2j + k$

$\therefore a \cdot b = 2 \cdot 1 + 3 \cdot (-2) + (-1) \cdot 1 = -5$

$a \times b = (2i + 3j - k) \times (i - 2j + k)$

$$= \begin{pmatrix} i & j & k \\ 2 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix}$$

$= i - 3j - 7k$

$b \times a = -(a \times b) = -i + 3j + 7k$

$\therefore ab = -a \cdot b + a \times b = 5 + i - 3j - 7k$

第二题

令 $p = a + bi + cj + dk, q = e + fi + gj + hk, \nu = bi + cj + dk, \mu = fi + gj + hk$

则有 $pq = (a + \nu)(e + \mu) = ae + a\mu + e\nu + \nu\mu = (ae - \nu \cdot \mu) + (a\mu + e\nu + \nu \times \mu)$，实数部分为 $ae - \nu \cdot \mu$

同样 $qp = (e + \mu)(a + \nu) = ea + e\nu + \mu a + \mu\nu = (ea - \mu \cdot \nu) + (e\nu + \mu a + \mu \times \nu)$，实数部分为 $ea - \mu \cdot \nu$

因为内积满足交换律，故实数部分相等，证明完毕

第三题

沿用上一题的结果有 $pq = (ae - \nu \cdot \mu) + (a\mu + e\nu + \nu \times \mu)$

$\therefore (pq)^* = (ae - \nu \cdot \mu) - (a\mu + e\nu + \nu \times \mu)$

$\because q^* = e - (fi + gj + hk) = e - \mu\,, p^* = a - (bi + cj + dk) = a - \nu$

$\therefore q^*p^* = (e - \mu)(a - \nu) = ea - e\nu - a\mu + \mu\nu = (ea - \mu \cdot \nu) - (e\nu + \mu a) + \mu \times \nu = (ae - \nu \cdot \mu) - (a\mu + e\nu + \nu \times \mu)$

$\therefore (pq)^* = q^*p^*$，证明完毕

第四题

由罗德里格斯公式 $\mathbf{v}' = \mathbf{v}\cos\theta + (\mathbf{u} \times \mathbf{v})\sin\theta + \mathbf{u}(\mathbf{u} \cdot \mathbf{v})(1 - \cos\theta)$

其中 $\theta = -\frac{2}{3}\pi,\ u = \frac{\sqrt{3}}{3}i + \frac{\sqrt{3}}{3}j + \frac{\sqrt{3}}{3}k$

$\therefore \mathbf{v}' = \mathbf{v}\cos(-\frac{2}{3}\pi) + (\mathbf{u} \times \mathbf{v})\sin(-\frac{2}{3}\pi) + \mathbf{u}(\mathbf{u} \cdot \mathbf{v})(1 - \cos(-\frac{2}{3}\pi))$

$$= (-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{K} + \frac{3}{2}\mathbf{u}\mathbf{u}^T)\mathbf{v}\text{，其中 } \mathbf{K} = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix}$$

又 $\because \mathbf{u}\mathbf{u}^T = \mathbf{K}^2 + \mathbf{I}$

$\therefore \mathbf{R} = \mathbf{I} - \frac{\sqrt{3}}{2}\mathbf{K} + \frac{3}{2}\mathbf{K}^2$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$\because$ 平移向量 $\mathbf{t} = \begin{pmatrix} 2 & 3 & 4 \end{pmatrix}^T$

$\therefore$ 该变换对应的映射矩阵为 $\begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

第五题

旋转轴的单位向量 $\mathbf{u}' = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{pmatrix}^T$，旋转角度 $\theta = \frac{\pi}{2}$

因此构造四元数 $\mathbf{q} = \cos\frac{\pi}{4} + \sin\frac{\pi}{4}(\frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}j) = \frac{\sqrt{2}}{2} + \frac{1}{2}i + \frac{1}{2}j$

则 $\mathbf{v}' = \mathbf{q}\mathbf{v}\mathbf{q}^{-1}$