From GBM SDE to the Conditional Density and Likelihood

1. The SDE for GBM

Consider the geometric Brownian motion (GBM) process $\{S_t : t \geq 0\}$ defined by the stochastic differential equation

$$dS_t \;=\; \mu\, S_t\, dt \;+\; \sigma\, S_t\, dW_t,$$

2. Conditional Distribution is Lognormal

Given $S_t = s_t$, we have

$$\ln\!\Big(\tfrac{S_{t+\Delta t}}{s_t}\Big) \;=\; \big(\mu - \tfrac{1}{2}\,\sigma^2\big)\,\Delta t \;+\; \sigma\,\sqrt{\Delta t}\,Z, \quad Z \sim \mathcal{N}(0,\,1).$$

Hence,

$$\ln\!\big(S_{t+\Delta t}\big) \;\mid\; S_t = s_t \;\;\sim\;\; \ln(s_t) \;+\; \mathcal{N}\Big(\big(\mu - \tfrac{1}{2}\,\sigma^2\big)\,\Delta t,\; \sigma^2\,\Delta t\Big).$$

Therefore, $S_{t+\Delta t} \mid S_t = s_t$ follows a lognormal distribution with:

$$\mathbb{E}\big[\ln(S_{t+\Delta t}) \mid S_t = s_t\big] \;=\; \ln(s_t) \;+\; \big(\mu - \tfrac{1}{2}\,\sigma^2\big)\,\Delta t, \quad \mathrm{Var}\big[\ln(S_{t+\Delta t}) \mid S_t = s_t\big] \;=\; \sigma^2\,\Delta t.$$

3. Lognormal PDF of $S_{t+\Delta t}\,|\,S_t$S(t+t)—S(t)

The conditional probability density function is therefore:

$$f\big(s_{t+\Delta t}\,\big|\,s_t\big) \;=\; \frac{1}{s_{t+\Delta t}\,\sigma\,\sqrt{2\pi\,\Delta t}}\;\exp\!\left(-\,\frac{\big[\ln(s_{t+\Delta t}) - \ln(s_t) - (\mu - \tfrac{1}{2}\,\sigma^2)\,\Delta t\big]^2}{2\,\sigma^2\,\Delta t}\right).$$

Equivalently, letting $x = \ln\!\big(s_{t+\Delta t}\big) - \ln\!\big(s_t\big)$, we have $x \sim \mathcal{N}\big((\mu - \tfrac{1}{2}\,\sigma^2)\,\Delta t,\; \sigma^2\,\Delta t\big)$.

4. Constructing the Likelihood for Observed Data

Suppose we have discrete observations $\{s_0, s_1, \ldots, s_n\}$ of the process at uniform time intervals Δt (so $t_i = i\,\Delta t$). Under the GBM model, the observations satisfy

$$S_{t_{i+1}} \;\Big|\; S_{t_i} = s_i \;\;\sim\;\; \mathrm{Lognormal}\Big(\ln(s_i) + (\mu - \tfrac{1}{2}\sigma^2)\,\Delta t,\; \sigma^2\,\Delta t\Big).$$

The *likelihood function* for parameters (μ, σ) is given by the product of conditional densities:

$$L(\mu,\sigma) \;=\; \prod_{i=0}^{n-1} f\big(s_{i+1}\,\big|\,s_i;\,\mu,\sigma\big).$$

Often it is more convenient to work with the *log-likelihood*:

$$\ell(\mu,\sigma) \;=\; \ln L(\mu,\sigma) \;=\; \sum_{i=0}^{n-1} \ln\!\Big[f\big(s_{i+1} \mid s_i;\,\mu,\sigma\big)\Big].$$

Using the explicit lognormal PDF above, each term in the sum is

$$\ln\!\Big[f\big(s_{i+1} \mid s_i\big)\Big] \;=\; -\,\ln(s_{i+1}) \;-\; \ln\!\big[\sigma\,\sqrt{2\pi\,\Delta t}\big] \;-\; \frac{\Big(\ln(s_{i+1}) - \ln(s_i) - (\mu - \tfrac{1}{2}\,\sigma^2)\,\Delta t\Big)^2}{2\,\sigma^2\,\Delta t}.$$

Hence the total log-likelihood is

$$\ell(\mu,\sigma) \;=\; \sum_{i=0}^{n-1}\left[-\,\ln(s_{i+1}) \;-\; \ln\!\big(\sigma\,\sqrt{2\pi\,\Delta t}\big) \;-\; \frac{\big(\ln(s_{i+1}) - \ln(s_i) - (\mu - \tfrac{1}{2}\,\sigma^2)\,\Delta t\big)^2}{2\,\sigma^2\,\Delta t}\right].$$