1 ECONOMY WITH DEFAULT

1.1 SETTING THE MODEL:

The households maximize:

$$E_0 \left\{ \sum_t^{\infty} \beta^t U(c_t) \right\} \tag{1}$$

The resource constrain is:

$$c = \begin{cases} y + B - q(B', y)B' & \text{if repay} \\ y^{def} & \text{if they default} \end{cases}$$

Multiple risk neutral lenders in a competitive market, then by zero profits condition:

$$q(B', y) = \frac{1 - \delta(B', y)}{1 - r} \tag{2}$$

The value function for a country with option to default:

$$v^o(B, y) = \max_{\{c,d\}} \left\{ v^c(B, y), v^d(y) \right\} \tag{3}$$

where v^c is the value of paying and v^d is the defaulting value. Both are defining next:

$$v^d(y) = u(y^{def}) + \beta \int_{y'} \left[\theta v^o(0, y') + (1 - \theta) v^d(y') \right] f(y'|y) dy' \tag{4}$$

$$v^c(B, y) = max_{(B')} \left\{ u(y - q(B', y)B' + B) + \beta \int_{y'} v^o(B', y') f(y'|y) dy' \right\} \tag{5}$$

In addition, countries face a borrowing constraint $B' \geq -Z$ to prevent Ponzi schemes. Let $A(B)$ and $D(B)$ be the repayment and default sets, such that:

$$A(B) = \{ y \in Y : v^c(B, y) \geq v^d(y) \} \tag{6}$$

$$D(B) = \{ y \in Y : v^c(B, y) < v^d(y) \} \tag{7}$$

Finally, the default probability is:

$$\delta(B', y) = \int_{D(B')} f(y'|y) dy' \tag{8}$$

1.2 RECURSIVE EQUILIBRIUM:

In this economy the equilibrium is a list of policy functions for consumption, debt, default a repayments sets, and bond prices such that:

- Taking as given everything else consumption maximize utility and satisfy resource constraint
- Taking as given the bond price; the policy function for debt B' and sets $A(B)$, $D(B)$ are in line with the government maximization problem
- Bond prices are consistent with the default set and the zero profit condition.

A clue to solve this model is to know that when $B = 0$ the country is indifferent between defaulting or not, then $v^o = v^c = v^d$. The file $code1.jl$ solves the baseline model.

1.3 RESULTS:

The solution has the form ($y_{low} \approx 0.95E[y]; y_{high} \approx 1.05E[y]$):

1.4 Simulation

1.5 NON LINEAR APPROXIMATION WITH NEURAL NETWORK

Following ..., I approximate the simulated value function and bond price with a neural network with one hidden layer with 16 nodes.

$$\hat{x}_t(s_t;\Theta) = \theta_0 + \sum_{q=1}^{Q} \theta_{q,1}\phi\left(\theta_{q,2} + \sum_{i=1}^{j} \theta_{q,2+i}\, s^{(i)}\right) \tag{9}$$

The activation function is $\phi(k) = \log\left(1+e^k\right)$.
First of all, I normalize all the states that are our inputs (let's $\tilde{s}$ the original variable):

$$s_i := \frac{\tilde{s}_i - \frac{1}{2}(\sup(s)+\inf(s))}{\frac{1}{2}(\sup(s)-\inf(s))}$$

The optimal Θ is chosen to minimize the sum of square residuals (I am assuming a univariate process):

$$\Theta = arg\,min_\theta \xi(\theta;s,\hat{x}) = arg\,min_\theta \frac{1}{2}\sum_{i=1}^{T}(x_i-\hat{x}_i)^2 \tag{10}$$

The first partial derivatives are (lets $e_i = x_i - \hat{x}_i$):

$$\frac{\partial \mathscr{L}}{\partial \theta_0} = -\sum_i e_i \tag{11}$$

$$\frac{\partial \mathscr{L}}{\partial \theta_{j,1}} = -\sum_i e_i\phi(k_{j,i}) \tag{12}$$

$$\frac{\partial \mathscr{L}}{\partial \theta_{j,2}} = -\sum_i \theta_{j,1}e_i\phi'(k_{j,i}) \tag{13}$$

$$\frac{\partial \mathscr{L}}{\partial \theta_{j,2+l}} = -\sum_i \theta_{j,1}e_i\phi'(k_{j,i})s_{i,l} \tag{14}$$

Then, the gradient is:

$$\theta_{m+1} = \theta_m - \varepsilon_m\nabla_\theta\xi(\theta;s,\hat{x}) \tag{15}$$

The minimization algorithm use a backtracking line search with a parameter $\gamma = 0.707$ (put a priori), hence

while

$$\xi(x-\varepsilon_{m,i}\nabla_\theta\xi(x)) > \xi(x) - \frac{\varepsilon_{m,i}}{2}(||\nabla_\theta\xi(x)||_2)^2$$

then

$$\varepsilon_{m,i+1} = \gamma\varepsilon_{m,i}$$

leading:

$$\varepsilon_m^*$$

therefore $\theta_{m+1} = \theta_m - \varepsilon_m^*\nabla_\theta\xi(\theta;s,\hat{x})$ where $||\cdot||_2$ is the common Euclidean norm.