Fourier Transform
formulas for f and $\omega=2\pi f$

Jean Baptiste Fourier

x(t) – signal in time domain
X(f) = F[x(t)] – its Fourier transform

Fourier Transform at frequency f is a correlation of x(t) and $e^{-j2\pi ft} = \cos(2\pi ft) - j\sin(2\pi ft)$

$$\boxed{X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft} dt}$$

$$\tilde{X}(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt = X\left(\frac{\omega}{2\pi}\right)$$

Given X(f) or X(ω), we may obtain x(t) using the inverse Fourier transform:
$x(t)=F^{-1}[X(f)]$

$$\boxed{x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft} df}$$

$$x(t) = \int_{-\infty}^{\infty} X\left(\frac{\omega}{2\pi}\right) e^{j\omega t} d\,\frac{\omega}{2\pi} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{X}(\omega)\, e^{j\omega t} d\omega$$

Amplitude spectrum and phase spectrum

x(t) – signal in time domain
X(f) = F[x(t)] – its Fourier transform (spectrum)

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt = |X(f)| e^{j\varphi(f)}$$

$$X(-f) = \int_{-\infty}^{\infty} x(t) e^{+j2\pi ft} dt = X^{*}(f) = |X(f)| e^{-j\phi(f)}$$

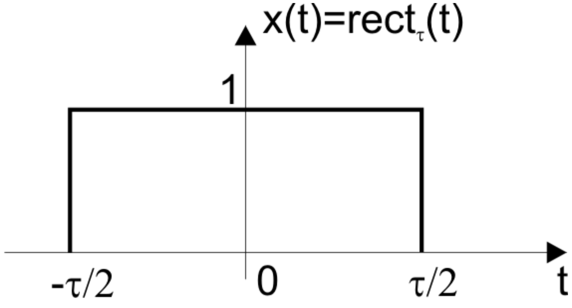
$|X(f)|$ - amplitude spectrum (magnitude spectrum)

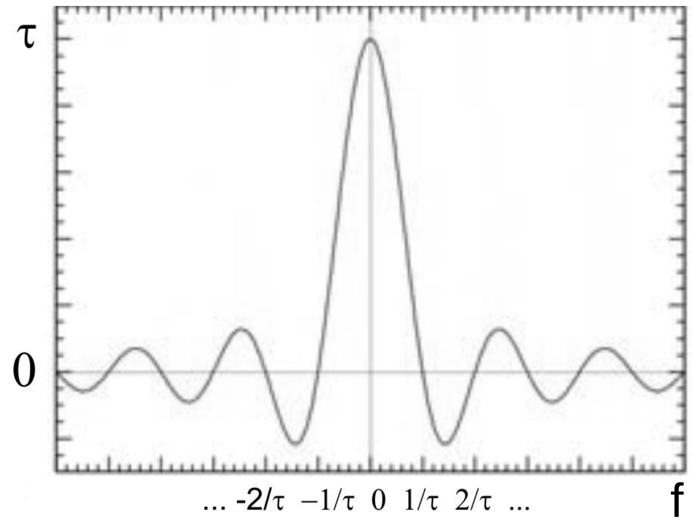
$\varphi(f)$ - phase spectrum

$|X(-f)| = |X(f)|$ even (symmetric) function

$\varphi(-f) = -\varphi(f)$ odd (nonsymmetric) function

An example – rectangular function





$$X(f) = F[x(t)] = \int_{-\tau/2}^{\tau/2} e^{-j2\pi f t} dt = \frac{1}{-j2\pi f} e^{-j2\pi f t}\Big|_{-\tau/2}^{\tau/2} = \frac{1}{-j2\pi f}\left[e^{-j\pi f\tau} - e^{j\pi f \tau}\right] =$$

$$= \frac{1}{\pi f}\left[\frac{e^{+j\pi f\tau} - e^{-j\pi f\tau}}{2j}\right] = \frac{\sin(\pi f \tau)}{\pi f} = \tau \frac{\sin(\pi f\tau)}{\pi f \tau}$$

Properties of Fourier Transform

X(f)=F[x(t)], x(t)=F^{-1}[X(f)]

Shift in time domain:

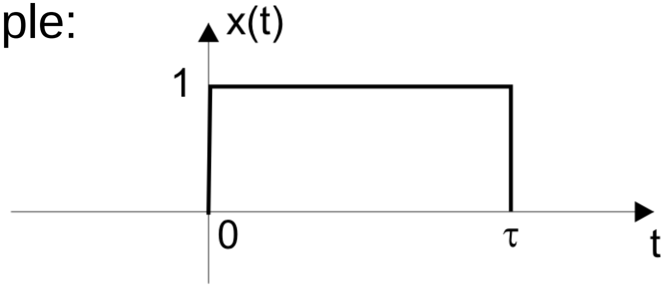
Given X(f)=F[x(t)], what is the spectrum of x(t-t_0) ?

$$e^{a}e^{-a}=1$$

$$F[x(t-t_0)]=\int_{-\infty}^{\infty}x(t-t_0)e^{-j2\pi ft}dt=e^{-j2\pi ft_0}\int_{-\infty}^{\infty}x(t-t_0)e^{-j2\pi f(t-t_0)}d(t-t_0)=$$

$$=e^{-j2\pi ft_0}F[x(t)]=e^{-j2\pi ft_0}X(f)$$

Example:



$$x(t)=rect_\tau(t-\tfrac{\tau}{2})$$

$$F[rect_\tau(t)]=\tau\frac{\sin(\pi f\tau)}{\pi f\tau}$$

$$F[rect_\tau(t-\tfrac{\tau}{2})]=\tau\frac{\sin(\pi f\tau)}{\pi f\tau}e^{-j\pi f\tau}$$

Properties of Fourier Transform

$X(f)=F[x(t)]$, $x(t)=F^{-1}[X(f)]$

Shift in frequency domain:

Given $x(t)=F^{-1}[X(f)]$, what is the inverse Fourier transform of $X(f-f_0)$?

$$F^{-1}[X(f-f_0)] = \int_{-\infty}^{\infty} X(f-f_0)e^{j2\pi ft}df =$$

$$= e^{j2\pi f_0 t} \int_{-\infty}^{\infty} X(f-f_0)e^{j2\pi(f-f_0)t}d(f-f_0) =$$

$$= e^{j2\pi f_0 t} F^{-1}[X(f)] = e^{j2\pi f_0 t}x(t)$$

Properties of Fourier Transform

X(f)=F[x(t)], x(t)=F^{-1}[X(f)]

Differentiation:

Given X(f)=F[x(t)], what is the Fourier transform of $\frac{d}{dt}x(t)$?

$$x(t) = F^{-1}[X(f)] = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

$$\frac{d}{dt}x(t) = \int_{-\infty}^{\infty} X(f)[\frac{d}{dt}e^{j2\pi ft}]df = \int_{-\infty}^{\infty} X(f)[j2\pi f\, e^{j2\pi ft}]df =$$

$$= \int_{-\infty}^{\infty}[j2\pi f\, X(f)]e^{j2\pi ft}df = F^{-1}[j2\pi f\, X(f)]$$

$$F[\frac{d}{dt}x(t)] = F\{F^{-1}[j2\pi f\, X(f)]\} = j2\pi f\, X(f)$$

Differentiation in time domain → multiplication by $j2\pi f = j\omega$ in frequency domain

Convolution

Convolution:

$$x(t)*y(t)=\int_{-\infty}^{\infty}x(\tau)y(t-\tau)\,d\tau=\int_{-\infty}^{\infty}y(\tau)x(t-\tau)\,d\tau$$

Its Fourier Transform:

$$\boxed{F[x(t)*y(t)]=F[x(t)]F[y(t)]=X(f)Y(f)}$$

Proof:

$$F[x(t)*y(t)]=F\left[\int_{-\infty}^{\infty}x(\tau)y(t-\tau)\,d\tau\right]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x(\tau)y(t-\tau)\,d\tau\,e^{-j2\pi ft}dt=$$

$$=\int_{-\infty}^{\infty}x(\tau)e^{-j2\pi f\tau}d\tau\int_{-\infty}^{\infty}y(t-\tau)e^{-j2\pi f(t-\tau)}d(t-\tau)=X(f)Y(f)$$

Convolution in frequency domain:

$$F^{-1}[X(f)*Y(f)]=F^{-1}[X(f)]F^{-1}[Y(f)]=x(t)\,y(t)$$

$$\boxed{F\big[x(t)\,y(t)\big]=X(f)*Y(f)}$$

Autocorrelation

Autocorrelation:

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x(t-\tau)\,dt$$

Its Fourier Transform:

$$F[R_x(\tau)] = X(f)\,X^*(f) = |X(f)|^2$$

Proof:

$$F[R_x(\tau)] = \int_{-\infty}^{\infty} R_x(\tau)e^{-j2\pi f\tau}\,d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)\,x(t-\tau)\,dt\,e^{-j2\pi f\tau}d\tau =$$

$$= \int_{-\infty}^{\infty} x(t)\int_{-\infty}^{\infty} x(t-\tau)e^{-j2\pi f\tau}d\tau\,dt = \int_{-\infty}^{\infty} x(t)\,e^{-j2\pi ft}\int_{-\infty}^{\infty} x(t-\tau)e^{j2\pi f(t-\tau)}d\tau\,dt =$$

$$= \int_{-\infty}^{\infty} x(t)\,e^{-j2\pi ft}dt\int_{-\infty}^{\infty} x(t-\tau)e^{j2\pi f(t-\tau)}d(t-\tau) = X(f)\,X^*(f)$$

This is the Wiener Kchinchin theorem

Spectral density of energy

Function $$F[R_x(\tau)] = X(f)\,X^*(f) = |X(f)|^2$$

is the energy spectral density of signal x(t).

Signal energy may be calculated in time domain and in frequency domain:

$$\boxed{E = \int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-\infty}^{\infty} |X(f)|^2\,df}$$

Parseval theorem

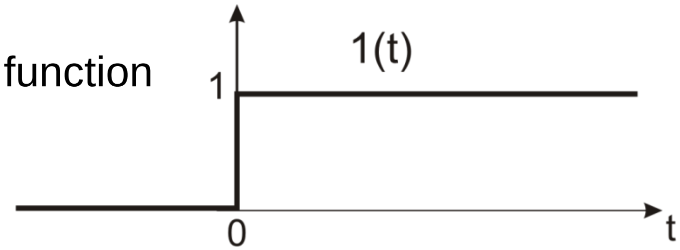
Proof: $$R_x(0) = \int_{-\infty}^{\infty} x^2(t)\,dt = E$$

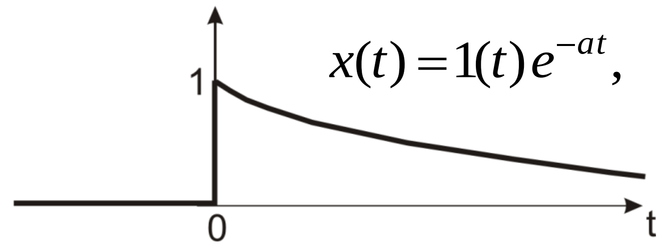
$$R_x(\tau) = F^{-1}[X(f)\,X^*(f)] = \int_{-\infty}^{\infty} X(f)\,X^*(f)\,e^{j2\pi f\tau}\,df$$

$$R_x(0) = \int_{-\infty}^{\infty} X(f)\,X^*(f)\,e^{j2\pi f0}\,df = \int_{-\infty}^{\infty} X(f)\,X^*(f)\,df = \int_{-\infty}^{\infty} |X(f)|^2\,df = E$$

Calculation of energy in time domain and frequency domain – an example

Unit step function





$$x(t)=1(t)e^{-at}, \qquad a>0$$

In time domain

$$E=\int_{-\infty}^{\infty} x^2(t)\,dt=\int_0^{\infty}(e^{-at})^2\,dt=\int_0^{\infty} e^{-2at}\,dt=\tfrac{1}{-2a}e^{-2at}\Big|_0^{\infty}=\tfrac{1}{-2a}[0-1]=\tfrac{1}{2a}$$

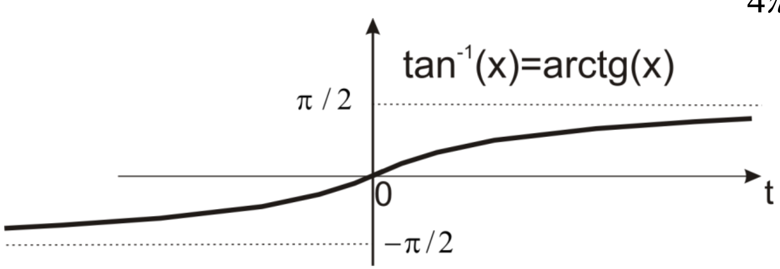
Spectrum

$$X(f)=F[1(t)e^{-at}]=\int_0^{\infty} e^{-at}\,e^{-j2\pi ft}dt=\int_0^{\infty} e^{(-a-j2\pi f)t}dt=\tfrac{1}{-a-j2\pi f}e^{(-a-j2\pi f)t}\Big|_0^{\infty}=\tfrac{1}{a+j2\pi f}$$

Energy spectral density $\quad |X(f)|^2=X(f)X^*(f)=\dfrac{1}{a+j2\pi f}\;\dfrac{1}{a-j2\pi f}=\dfrac{1}{a^2+4\pi^2 f^2}$

Energy in frequency domain

$$\int_{-\infty}^{\infty}|X(f)|^2\,df=\int_{-\infty}^{\infty}\frac{df}{a^2+4\pi^2f^2}=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\frac{df}{\frac{a^2}{4\pi^2}+f^2}=\frac{1}{4\pi^2}\,\frac{1}{\frac{a}{2\pi}}\,arctg\Big(\frac{f}{\frac{a}{2\pi}}\Big)\Big|_{-\infty}^{\infty}=\frac{1}{2\pi a}\Big[\frac{\pi}{2}+\frac{\pi}{2}\Big]=\frac{1}{2a}$$



$$E=\int_{-\infty}^{\infty} x^2(t)\,dt=\int_{-\infty}^{\infty}|X(f)|^2\,df=\tfrac{1}{2a}$$

Power spectral density (psd)

m(t) - signal of infinite duration, infinite energy but finite power P

In a window of duration T we have finite energy: $E_T = \int_{-T/2}^{T/2} m^2(t)dt \qquad P = \lim_{T\to\infty} \frac{1}{T} E_T$

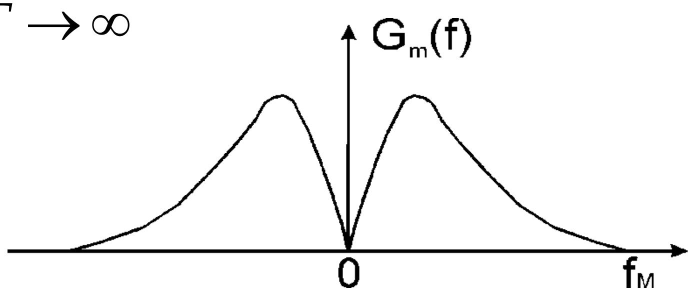
Spectrum (Fourier transform) $M_T(f) = \int_{-T/2}^{T/2} m(t) e^{-j2\pi f t} dt$

Amplitude spectrum $|M_T(f)|$ Energy spectrum $|M_T(f)|^2$

Parseval: $E_T = \int_{-T/2}^{T/2} m^2(t)dt = \int_{-\infty}^{\infty} |M_T(f)|^2\, df \longrightarrow \frac{E_T}{T} = \frac{1}{T}\int_{-T/2}^{T/2} m^2(t)dt = \int_{-\infty}^{\infty} \frac{1}{T} |M_T(f)|^2\, df$

Power spectrum (psd) $\frac{1}{T}|M_T(f)|^2, \qquad T \to \infty$

Psd $G_m(f) = \lim_{T\to\infty} \frac{1}{T} |M_T(f)|^2$

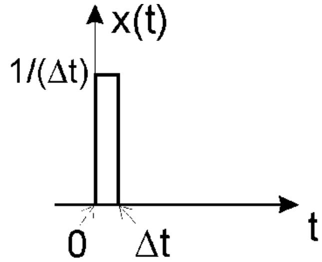


Power
$$\boxed{P = \int_{-\infty}^{\infty} G_m(f)\, df = \int_{-\infty}^{\infty} x^2 p_m(x)\, dx = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} m^2(t)dt}$$

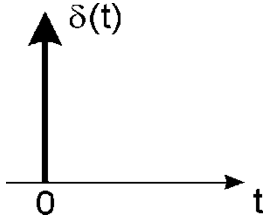
Paul Dirac

The Dirac pulse

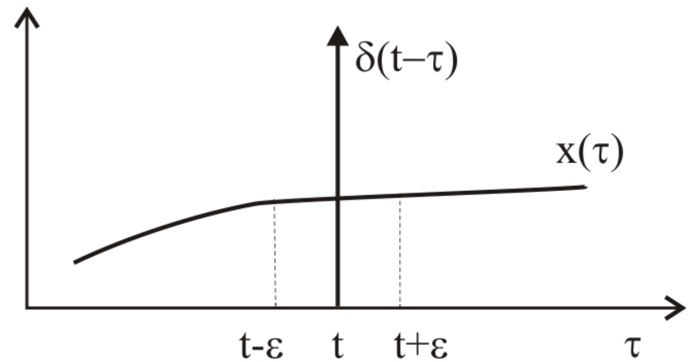
Let us define a short rectangular pulse of unit area:



If $\Delta t \to 0$, we obtain the **Dirac pulse δ(t)**:



$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$



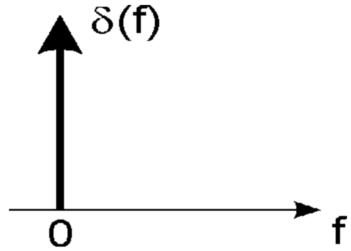
Convolution with δ(t) $x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t)$ **does not change the signal**.

Convolution with $\delta(t-t_0)$ yields a shift by t_0: $x(t) * \delta(t-t_0) = \int_{-\infty}^{\infty} x(\tau)\delta(t-t_0-\tau)d\tau = x(t-t_0)$

Fourier transform of δ(t): $F\left[\delta(t)\right] = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t} dt = 1$

From the shift theorem: $F\left[\delta(t-t_0)\right] = e^{-j2\pi f t_0}$

The Dirac pulse in frequency domain



The inverse Fourier Transform of δ(f): $$F^{-1}\left[\delta(f)\right]=\int_{-\infty}^{\infty}\delta(f)e^{j2\pi ft}df=1$$

Thus **F[1]=δ(f)** (despite of the fact that the function x(t)=1 is not integrable).

From the shift theorem: $$F^{-1}[\delta(f-f_0)]=\int_{-\infty}^{\infty}\delta(f-f_0)e^{j2\pi ft}df=e^{j2\pi f_0 t} \longrightarrow F[e^{j2\pi f_0 t}]=\delta(f-f_0)$$

and $$F^{-1}[\delta(f+f_0)]=\int_{-\infty}^{\infty}\delta(f+f_0)e^{j2\pi ft}df=e^{-j2\pi f_0 t} \longrightarrow F[e^{-j2\pi f_0 t}]=\delta(f+f_0)$$

Thus Fourier transforms of sine and cosine functions are obtained:

$$\boxed{\begin{aligned} F\left[\cos(2\pi f_0 t)\right]&=F\left[\tfrac{e^{j2\pi f_0 t}+e^{-j2\pi f_0 t}}{2}\right]=\tfrac{1}{2}\left[\delta(f-f_0)+\delta(f+f_0)\right]\\ F\left[\sin(2\pi f_0 t)\right]&=F\left[\tfrac{e^{j2\pi f_0 t}-e^{-j2\pi f_0 t}}{2j}\right]=\tfrac{1}{2j}\left[\delta(f-f_0)-\delta(f+f_0)\right]\end{aligned}}$$

Some examples of Fourier Transforms

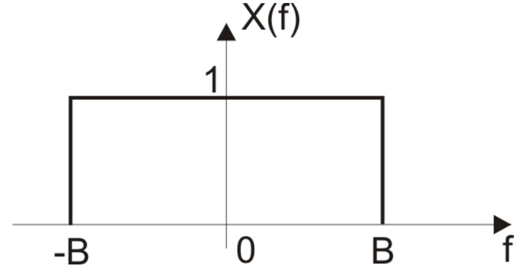
time: $x(t)=F^{-1}[X(f)]$

frequency: $X(f)=F[x(t)]$

$x(t) = rect_\tau(t)$	$X(f) = F\left[rect_\tau(t)\right] = \tau \dfrac{\sin(\pi\tau f)}{\pi\tau f}$
$x(t) = F^{-1}\left[X(f)\right] = 2B\dfrac{\sin(2\pi Bt)}{2\pi Bt}$	$X(f) = rect_{2B}(f)$
$x(t)=\delta(t)$	$X(f)=1$
$x(t)=1$	$X(f)=\delta(f)$
$x(t)=\delta(t-t_0)$	$X(f)= \exp(-j2\pi ft_0) = \cos(2\pi ft_0) - j \sin(2\pi ft_0)$
$x(t)=\exp(j2\pi f_0t)$	$X(f) = \delta(f-f_0)$
$x(t) = \cos(2\pi f_0 t)$	$X(f) = \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right]$
$x(t) = \sin(2\pi f_0 t)$	$X(f) = \frac{1}{2j}\left[\delta(f - f_0) - \delta(f + f_0)\right]$

Some examples of Fourier Transforms

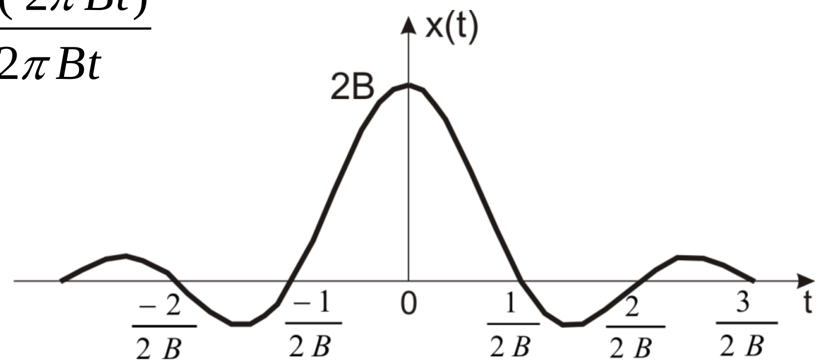
$$X(f) = rect_{2B}(f)$$



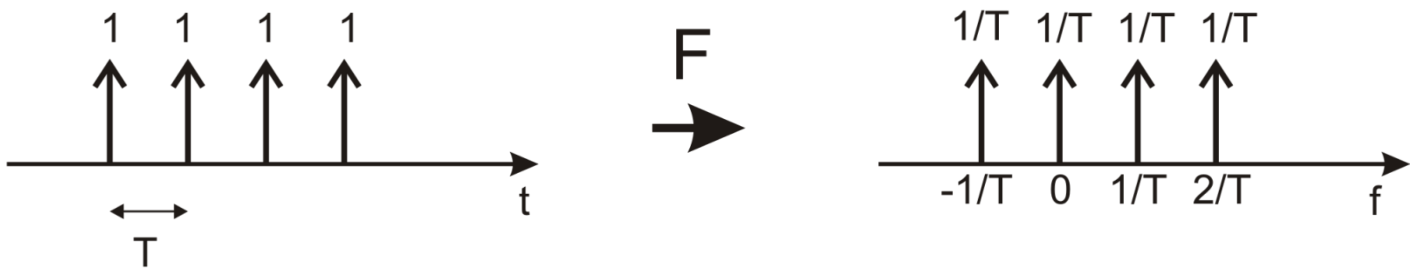
$$x(t) = F^{-1}\left[X(f)\right] = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df = \int_{-B}^{B} e^{j2\pi f t} df =$$

$$= \tfrac{1}{j2\pi t}\left[e^{j2\pi Bt} - e^{-j2\pi Bt}\right] = \tfrac{1}{\pi t}\,\frac{e^{j2\pi Bt} - e^{-j2\pi Bt}}{2j} = \tfrac{1}{\pi t}\sin(2\pi Bt) =$$

$$= 2B\,\frac{\sin(2\pi Bt)}{2\pi Bt}$$



Fourier Transform of a series of pulses:



$$F\left\{\sum_{n}\delta\,(t-nT)\right\}=\frac{1}{T}\sum_{n}\delta\,(f-\frac{n}{T})$$

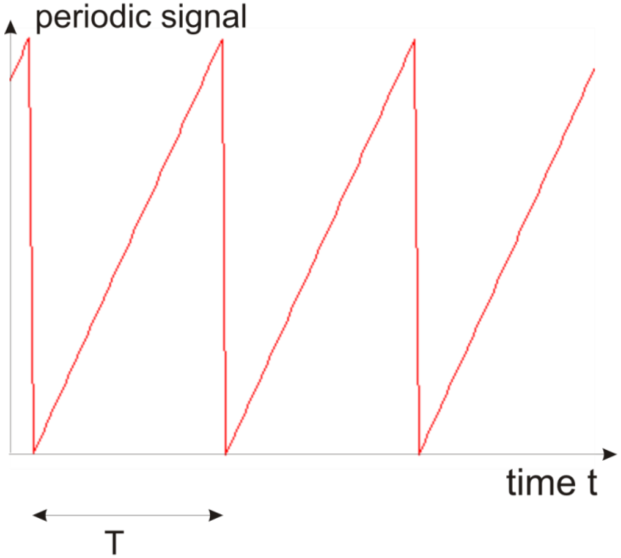
Fourier Transforms of periodic signals

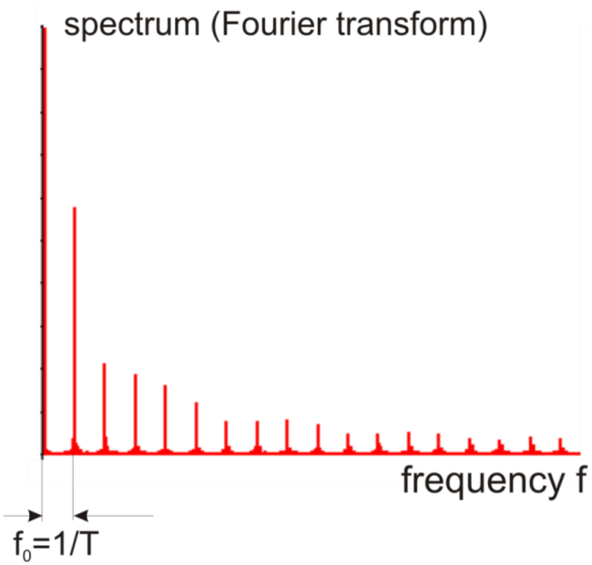
x(t) – periodic signal, period T, finite energy within one period, $f_0 = \frac{1}{T}$

One period: $x_T(t)$ $$x(t) = \sum_n x_T(t - nT) = x_T(t) * \sum_n \delta(t - nT)$$

Fourier transform of $\sum_n \delta(t - nT)$: $$F[\sum_n \delta(t - nT)] = \frac{1}{T} \sum_n \delta(f - \frac{n}{T})$$

Fourier transform of x(t): $$\boxed{X(f) = X_T(f) \cdot \frac{1}{T} \sum_n \delta(f - \frac{n}{T}) = \frac{1}{T} \sum_n \delta(f - \frac{n}{T}) X_T(\frac{n}{T})}$$





Fourier Transforms of periodic signals

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One period: $x_T(t)$ $$x(t) = \sum_n x_T(t - nT) = x_T(t) * \sum_n \delta(t - nT)$$

Fourier transform of x(t):

$$X(f) = X_T(f) \cdot \tfrac{1}{T} \sum_n \delta(f - \tfrac{n}{T}) = \tfrac{1}{T} \sum_n \delta(f - \tfrac{n}{T}) X_T(\tfrac{n}{T})$$

Fourier series

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} \tfrac{1}{T} X_T(nf_0) e^{j2\pi n f_0 t} = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t}}$$

because $F^{-1}[\delta(f - nf_0)] = e^{j2\pi n f_0 t}$

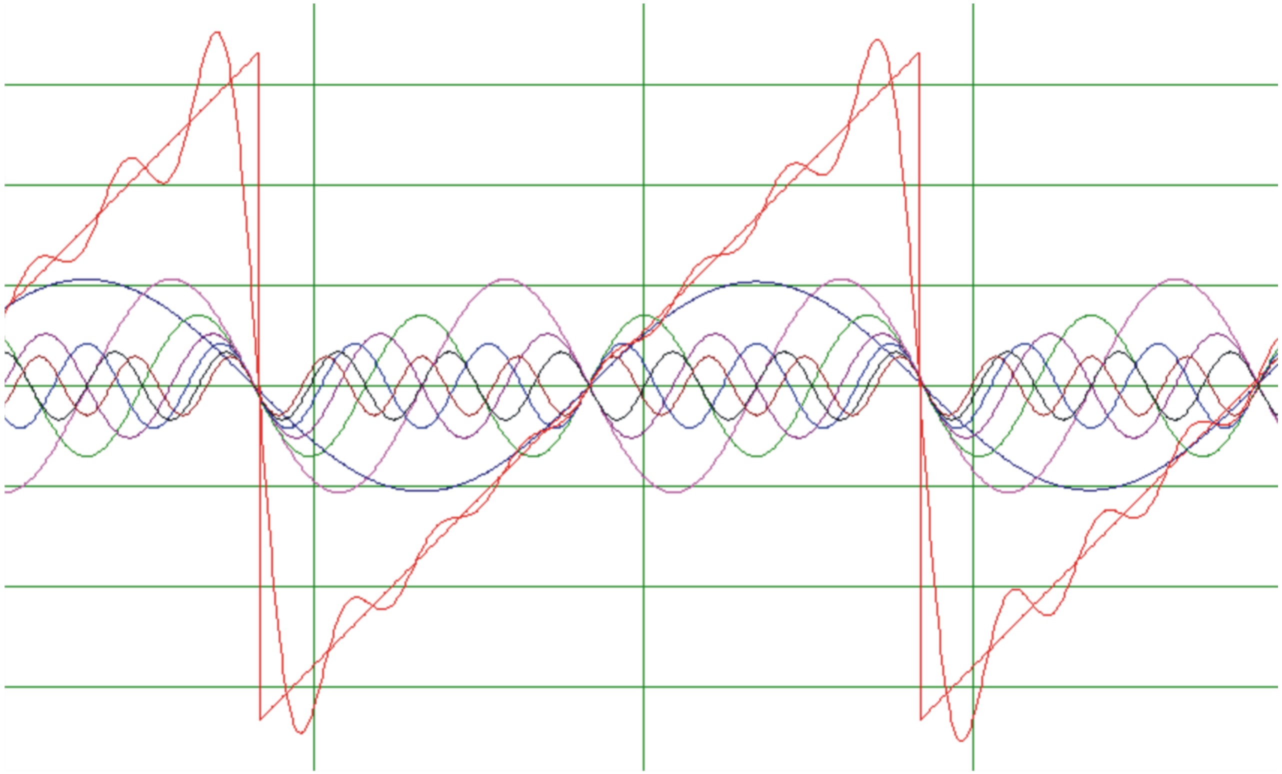
Fourier series coefficients:

$$X_n = \tfrac{1}{T} X_T(nf_0) = \tfrac{1}{T} \int_0^T x_T(t) e^{-j2\pi n f_0 t} dt = \tfrac{1}{T} \int_0^T x(t) e^{-j2\pi n f_0 t} dt$$

Fourier series without complex functions

x(t) - periodic signal, period T,

Fourier series: $$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(2\pi n f_0 t + \phi_n), \qquad f_0 = \tfrac{1}{T}$$



A_0 - a constant, A_n - amplitude, φ_n-phase of the n^{th} harmonic of x(t)

Fourier series without complex functions

$$x(t)=\sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t} = X_0 + \sum_{n=1}^{\infty}(X_n e^{j2\pi n f_0 t} + X_{-n} e^{-j2\pi n f_0 t})$$

where $X_n = |X_n| e^{j\varphi_n}, \quad X_{-n} = X_n^* = |X_n| e^{-j\varphi_n}$ are Fourier coefficients

$$x(t) = X_0 + \sum_{n=1}^{\infty}(|X_n| e^{j\varphi_n} e^{j2\pi n f_0 t} + |X_n| e^{-j\varphi_n} e^{-j2\pi n f_0 t}) =$$

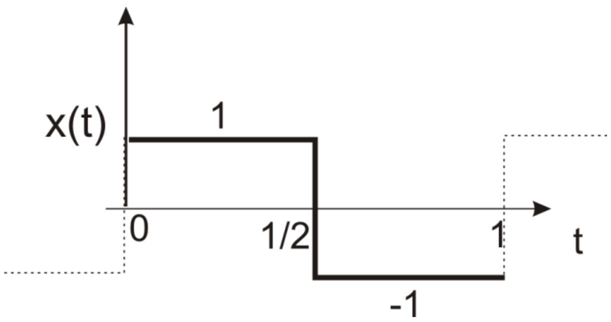
$$= X_0 + \sum_{n=1}^{\infty} 2\,\mathrm{Re}\,(|X_n| e^{j\varphi_n} e^{j2\pi n f_0 t}) = X_0 + \sum_{n=1}^{\infty} 2\,\mathrm{Re}\,(|X_n| e^{j(2\pi n f_0 t + \varphi_n)}) =$$

$$= X_0 + \sum_{n=1}^{\infty} 2|X_n| \cos(2\pi n f_0 t + \varphi_n) = A_0 + \sum_{n=1}^{\infty} A_n \cos(2\pi n f_0 t + \varphi_n)$$

where $A_0 = X_0$ is mean value of x(t) (DC component)

$A_n = 2|X_n|$ is the amplitude of n[th] harmonic component

Calculation of Fourier coefficients – an example



Rectangular periodic signal x(t) has a period T=1
Fundamental frequency (first harmonic frequency) is equal to f_0=1/T=1.

Fourier series: $x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t}$ Fourier coefficients: $X_n = \frac{1}{T}\int_0^T x(t) e^{-j2\pi n f_0 t} dt$

$$X_n = \frac{1}{T}\int_0^T x(t) e^{-j2\pi n f_0 t} dt = \int_0^{1/2} e^{-j2\pi nt} dt - \int_{1/2}^{1} e^{-j2\pi nt} dt =$$

$$= \frac{1}{-j2\pi n} e^{-j2\pi nt}\Big|_0^{1/2} - \frac{1}{-j2\pi n} e^{-j2\pi nt}\Big|_{1/2}^{1} =$$

$$= \frac{1}{-j2\pi n}[e^{-j\pi n} - 1 - e^{-j2\pi n} + e^{-j\pi n}] = \frac{1}{j2\pi n}[2 - 2e^{-j\pi n}] = \frac{1}{j\pi n}[1 - e^{-j\pi n}]$$

For $n = 0, \pm 2, \pm 4, \pm 6, \ldots$ $e^{-j\pi n} = 1$ and $X_n = 0$ $\longrightarrow$ $A_n = 0$

For $n = \pm 1, \pm 3, \pm 5, \ldots$ $e^{-j\pi n} = -1$ and $X_n = \frac{2}{j\pi n}$ $\longrightarrow$ $A_n = \frac{4}{\pi n}$