Circuits and Signals

Periodic current circuits

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Periodic signal

Definition

Periodic signal is a function $x \colon \mathbb{R} \to \mathbb{R}$, for which there exists a constant $T > 0$, called the period of the signal $x(t)$, such that

$$x(t) = x(t + T).$$

For convenience, we restrict periodic signals to signals that are piecewise smooth and such that their derivatives have one-side limits everywhere.

Fundamental period T: is the smallest non-zero period,
Fundamental frequency f: $f = \frac{1}{T}$ [Hz],
Fundamental pulsation ω: $\omega = 2\pi f = \frac{2\pi}{T}$ [rad/s].

Steady state

Definition

Periodic current circuit is a circuit admitting solutions that consist of periodic signals (of common period). Every such solution is called a steady state.

Superposition rule

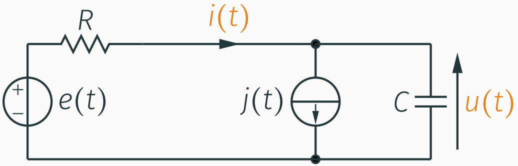
Superposition rule for per. current circuits

Each T-periodic signal x (a voltage or a current) that is a part of a steady state in a periodic current circuit, can be expressed as the sum of a constant signal and signals of pulsations that are (integer) multiples of the fundamental pulsation $\omega = \frac{2\pi}{T}$:

$$x(t) = X_0 + x_1(t) + x_2(t) + \dots$$

The constant term X_0 can be obtained by the means of DC analysis.
Each k-th harmonics x_k (of pulsation $k\omega$) can be computed by the means of AC analysis of the circuit that results from the original if all other harmonics (i.e. all but the k-th) are reduced to zero.

Superposition rule — example



$$e(t) = E_0 + E_{2m}\cos(2\omega t), \quad j(t) = J_{1m}\cos(\omega t) + J_{2m}\cos(2\omega t + \frac{\pi}{2}),$$

$$u(t) = ?, \quad i(t) = ?.$$

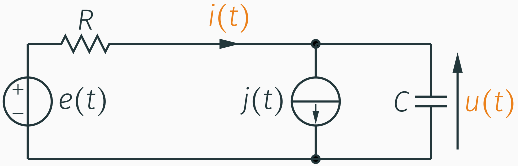
$$u(t) = U_0 + \underbrace{U_{1m}\cos(\omega t + \phi_1)}_{u_1(t)} + \underbrace{U_{2m}\cos(2\omega t + \phi_2)}_{u_2(t)}.$$

$$i(t) = I_0 + \underbrace{I_{1m}\cos(\omega t + \psi_1)}_{i_1(t)} + \underbrace{I_{2m}\cos(2\omega t + \psi_2)}_{i_2(t)}.$$

Superposition rule — example — DC term

$$e(t) = E_0 + E_{2m}\cos(2\omega t),$$

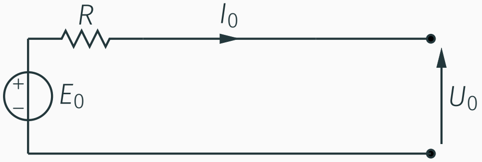
$$j(t) = J_{1m}\cos(\omega t) + J_{2m}\cos\left(2\omega t + \frac{\pi}{2}\right),$$



$$I_0 = 0, \qquad U_0 = E_0.$$

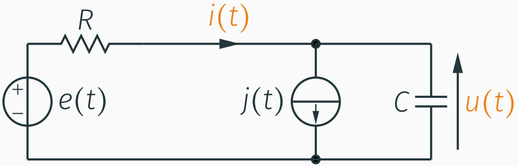
Superposition rule — example — DC term

$$e(t) = E_0 + E_{2m}\cos(2\omega t),$$
$$j(t) = J_{1m}\cos(\omega t) + J_{2m}\cos(2\omega t + \frac{\pi}{2}),$$



$$I_0 = 0, \qquad U_0 = E_0.$$

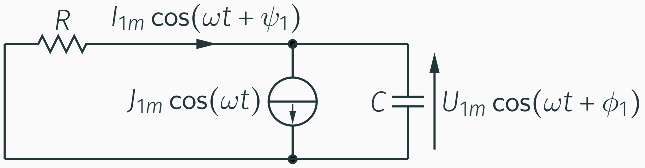
Superposition rule — example — first harmonics



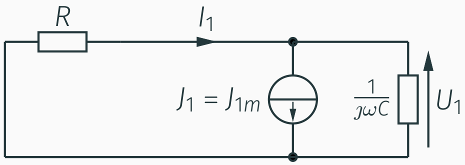
$$e(t) = E_0 + E_{2m} \cos(2\omega t),$$

$$j(t) = J_{1m} \cos(\omega t) + J_{2m} \cos(2\omega t + \frac{\pi}{2}),$$

Superposition rule — example — first harmonics



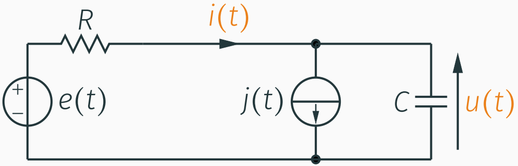
Superposition rule — example — first harmonics



$$I_1 = J_1 \frac{1/\jmath\omega C}{R + 1/\jmath\omega C} = J_1 \frac{1}{1 + \jmath\omega RC}, \qquad U_1 = -J_1 \frac{R}{1 + \jmath\omega RC}.$$

$$\begin{cases} I_{1m} = \dfrac{J_{1m}}{\sqrt{1 + (\omega RC)^2}}, & \psi_1 = -\arctan \omega RC, \\ U_{1m} = \dfrac{J_{1m} R}{\sqrt{1 + (\omega RC)^2}}, & \phi_1 = \pi - \arctan \omega RC, \end{cases}$$

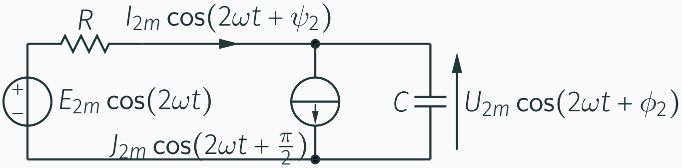
Superposition rule — example — second harmonics



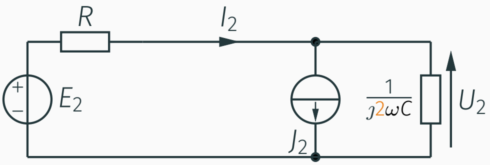
$$e(t) = E_0 + E_{2m}\cos(2\omega t),$$

$$j(t) = J_{1m}\cos(\omega t) + J_{2m}\cos\left(2\omega t + \frac{\pi}{2}\right),$$

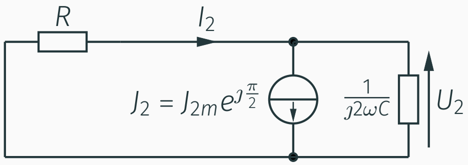
Superposition rule — example — second harmonics



Superposition rule — example — second harmonics



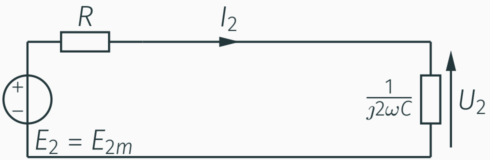
Superposition rule — example — second harmonics



$$I_2 = J_2 \frac{1}{1 + \jmath 2\omega RC} +$$

$$U_2 = -J_2 \frac{R}{1 + \jmath 2\omega RC} +$$

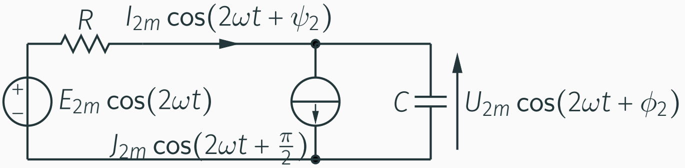
Superposition rule — example — second harmonics



$$I_2 = J_2 \frac{1}{1 + j2\omega RC} + E_2 \frac{j2\omega C}{1 + j2\omega RC},$$

$$U_2 = -J_2 \frac{R}{1 + j2\omega RC} + E_2 \frac{1}{1 + j2\omega RC}.$$

Superposition rule — example — second harmonics



$$I_{2m} = \frac{J_{2m} + 2\omega C E_{2m}}{\sqrt{1 + (2\omega RC)^2}}, \qquad \psi_2 = \frac{\pi}{2} - \arctan 2\omega RC,$$

$$U_{2m} = \sqrt{\frac{E_{2m}^2 + (J_{2m}R)^2}{1 + (2\omega RC)^2}}, \qquad \phi_2 = -\arctan\frac{RJ_{2m}}{E_{2m}} - \arctan 2\omega RC.$$

What if a T-periodic signal x (electromotive force, current generated by a current source and so on) is not a finite sum of alternating signals (of pulsations that are multiples of $\frac{2\pi}{T}$)?

We can approximate the signal by such a finite sum!

How can we get such an approximation?

Fourier series — definition

Definition

Fourier series of a T-periodic signal $x(t)$ is the series

$$\sum_{k=-\infty}^{+\infty} X_k e^{\jmath k\omega t}, \qquad \omega = \frac{2\pi}{T},$$

where

$$X_k = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-\jmath k\omega t}\mathrm{d}t, \qquad T = \frac{2\pi}{\omega}.$$

X_0 is the mean value of signal x!

$$X_{-k} = \overline{X_k}.$$

Fourier series — point-wise convergence

Theorem on point-wise convergence of FS

The partial sums of the Fourier series of a T-periodic signal x:

$$\sum_{k=-N}^{N} X_k e^{\jmath k\omega t},$$

converge point-wise to the mean of one-sided limits of the signal, i.e. for a fixed t, the above sums converge to

$$\frac{x(t^-) + x(t^+)}{2}.$$

That's how we should interpret the equality:

$$x(t) = \sum_{k=-\infty}^{+\infty} X_k e^{\jmath k\omega t}.$$

Fourier series — uniqueness

Theorem on uniqueness of FS

Two T-periodic signals x_1 and x_2 have equal Fourier series if, and only if, they are equal up to their values at non-continuity points (i.e., equality $x_1(t) = x_2(t)$ holds at every point at which both signals are continuous).

Fourier series — expansion into harmonics

$$x(t) = X_0 + \sum_{k=1}^{\infty} \left(X_k e^{\jmath k\omega t} + X_{-k} e^{-\jmath k\omega t}\right) =$$

$$\overset{X_{-k}=\overline{X_k},\; z+\bar{z}=2\operatorname{Re} z}{=} X_0 + \sum_{k=1}^{\infty} 2\operatorname{Re}\left(X_k e^{\jmath k\omega t}\right)$$

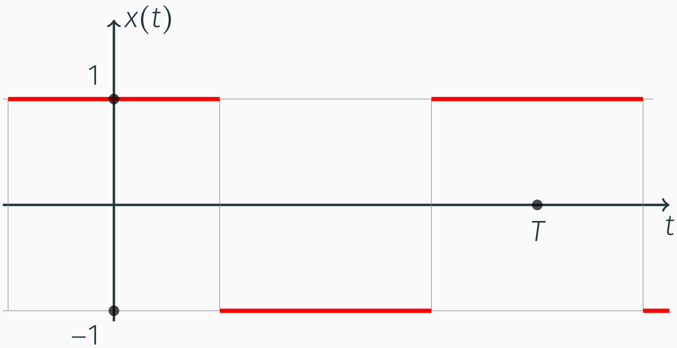
$$\overset{X_k=|X_k|e^{\jmath\phi_k}}{=} X_0 + \sum_{k=1}^{\infty} 2\operatorname{Re}\left(|X_k| e^{\jmath k\omega t + \jmath\phi_k}\right) = \underbrace{X_0 + \sum_{k=1}^{\infty} 2|X_k| \cos\left(k\omega t + \phi_k\right)}_{\text{trigonometric Fourier series}}.$$

k-th harmonics of signal x ($k \geq 1$) is the signal

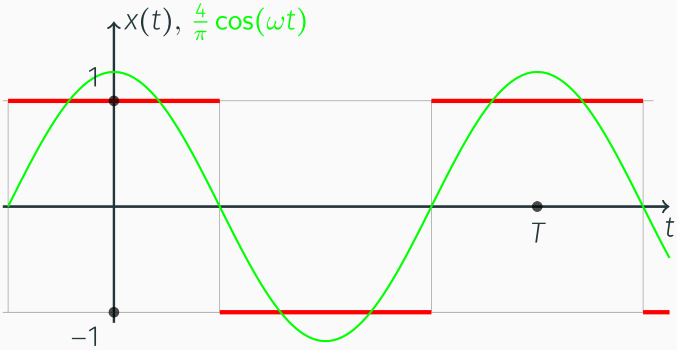
$$x_k(t) = X_{km} \cos\left(k\omega t + \phi_k\right), \qquad X_{km} = 2|X_k|, \quad \phi_k = \arg X_k.$$

$$x(t) = X_0 + x_1(t) + x_2(t) + \dots$$

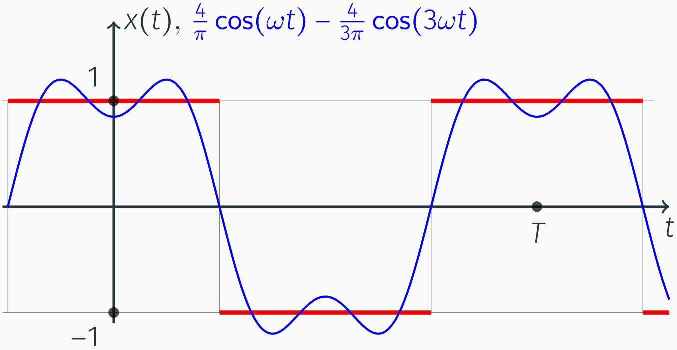
Example



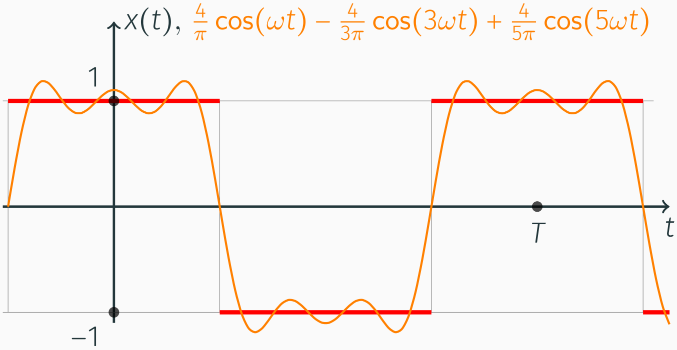
Example



Example



Example


$x(t)$, $\frac{4}{\pi}\cos(\omega t) - \frac{4}{3\pi}\cos(3\omega t) + \frac{4}{5\pi}\cos(5\omega t)$
1
−1
T
t

Example


$x(t)$, $\frac{4}{\pi}\cos(\omega t) - \frac{4}{3\pi}\cos(3\omega t) + \cdots + \frac{4}{17\pi}\cos(17\omega t)$
1
−1
T
t

Parseval's theorem

$$p(t) = i(t)u(t),\ P = \frac{1}{T}\int_{t_0}^{t_0+T} p(t)\mathrm{d}t = \frac{1}{T}\int_{t_0}^{t_0+T} i(t)u(t)\mathrm{d}t.$$

Parseval's theorem

For T-periodic signals x, y

$$\frac{1}{T}\int_{t_0}^{t_0+T} x(t)y(t)\,\mathrm{d}t = \sum_{k=-\infty}^{+\infty} X_k\overline{Y_k},$$

where t_0 is an arbitrary time instant. In particular:

$$\frac{1}{T}\int_{t_0}^{t_0+T} x^2(t)\,\mathrm{d}t = \sum_{k=-\infty}^{+\infty} |X_k|^2, \quad X_{\mathrm{RMS}} = \sqrt{X_0^2 + \frac{1}{2}\sum_{k=1}^{+\infty} X_{km}^2}.$$

Mean power in per. current circuits

$$i(t) = \sum_{k=-\infty}^{+\infty} I_k e^{\jmath k\omega t}, \qquad u(t) = \sum_{k=-\infty}^{+\infty} U_k e^{\jmath k\omega t}.$$

$$P = \sum_{k=-\infty}^{+\infty} U_k \overline{I_k}.$$

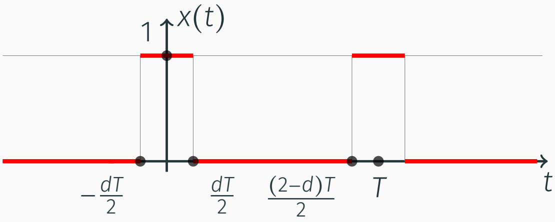
$$U_0 I_0 + 2\sum_{k=1}^{\infty} Re(U_k \overline{I_k}) = U_0 I_0 + \sum_{k=1}^{\infty} \frac{1}{2} U_{km} I_{km} \cos(\arg U_k - \arg I_k).$$

Proposition

The mean power delivered to a device in a per. current circuit is a sum of the DC-power (power due to DC signals only) and the mean (real) powers resulting from each harmonics.

How to compute Fourier series — pulse wave

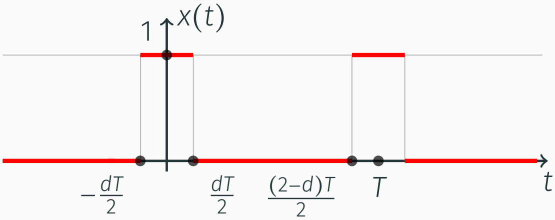
Pulse wave $x(t)$ with duty cycle d and amplitude 1:



$$X_0 = \frac{1}{T}\int_{-\frac{dT}{2}}^{\frac{dT}{2}} 1\mathrm{d}t = d.$$

$$X_k = \frac{1}{T}\int_{-\frac{dT}{2}}^{\frac{dT}{2}} e^{-\jmath k\omega t}\mathrm{d}t = \frac{1}{T}\left.\frac{e^{-\jmath k\omega t}}{-\jmath k\omega}\right|_{-\frac{dT}{2}}^{\frac{dT}{2}} = \frac{1}{-\jmath 2\pi k}\left(e^{-\jmath k\pi d} - e^{\jmath k\pi d}\right) = \frac{\sin(k\pi d)}{k\pi}.$$

How to compute Fourier series — pulse wave, cont.



Fourier series coefficients:

$$X_0 = d, \qquad X_k = \frac{\sin(\pi k d)}{k\pi}, \quad k \neq 0.$$

Parameters of individual harmonics:

$$X_{km} = 2\frac{|\sin(\pi k d)|}{k\pi}, \qquad k \geq 1,$$

$$\phi_k = \begin{cases} 0 & \text{if } \sin(\pi k d) > 0 \\ \pi & \text{if } \sin(\pi k d) < 0 \end{cases}, \qquad k \geq 1.$$

Fourier series — linearity

Linearity

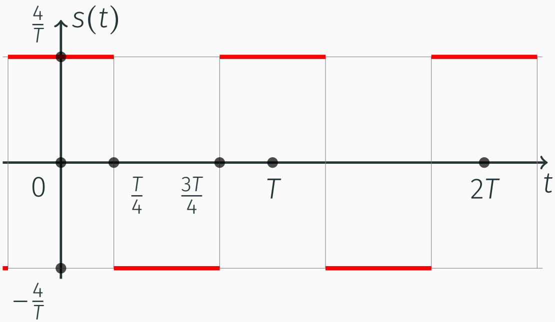
The Fourier series of a signal

$$\alpha x(t) + y(t), \qquad \alpha \in \mathbb{R}$$

is the series

$$\sum_{k=-\infty}^{+\infty} (\alpha X_k + Y_k) e^{\jmath k \omega t}.$$

How to compute Fourier series — square wave



$$s(t) = \frac{8}{T}\left(x(t) - \frac{1}{2}\right), \qquad d = \frac{1}{2}.$$

Fourier coefficients:

$$X_0 = d, \qquad X_k = \frac{\sin(\pi k d)}{k\pi}, \qquad d = \frac{1}{2}.$$

$$S_0 = 0, \qquad S_k = \frac{8}{T}\frac{\sin(k\frac{\pi}{2})}{k\pi}, \quad k \neq 0.$$

Fourier series — derivative of a signal

Theorem (of Fourier series of a derivative)

If x is a T-periodic signal such that

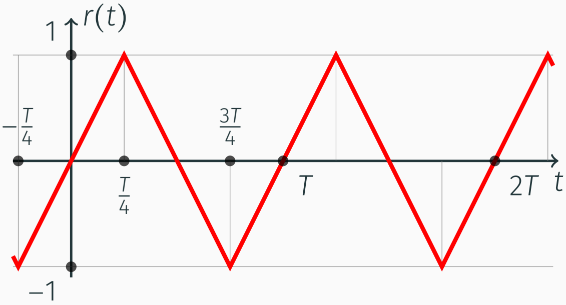
$$x(t) = \sum_{k=-\infty}^{+\infty} X_k e^{\jmath k\omega t},$$

then

$$x'(t) = \sum_{k=-\infty}^{+\infty} (\jmath k\omega) X_k e^{\jmath k\omega t}.$$

In other words we may differentiate FS term-wise.

How to compute Fourier series — sawtooth wave



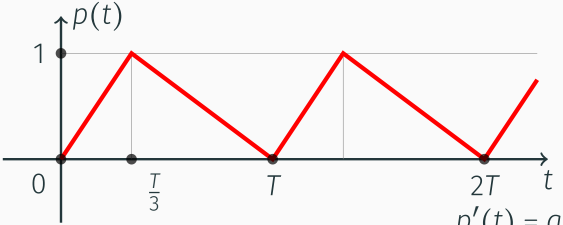
$$r'(t) = s(t).$$

Fourier series coefficients:

$$S_0 = 0, \qquad S_k = \frac{8}{T}\frac{\sin\left(k\frac{\pi}{2}\right)}{k\pi}, \quad k \neq 0.$$

$$R_0 = 0, \qquad R_k = \frac{1}{\jmath k\omega}\frac{8}{T}\frac{\sin\left(k\frac{\pi}{2}\right)}{k\pi} = \frac{4}{\jmath k\pi}\frac{\sin\left(k\frac{\pi}{2}\right)}{k\pi}, \qquad k \neq 0.$$

How to compute Fourier series — another saw



$$p'(t) = q(t).$$

$$q(t) = \frac{9}{2T} x\left(t - \frac{T}{6}\right) - \frac{3}{2T}, \qquad d = \frac{1}{3}.$$

Fourier series — time shifting

Theorem (on shifting in time domain)

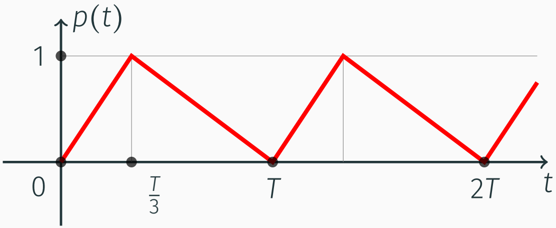
The Fourier series of a T-periodic signal $y(t) = x(t - t_0)$ is the series

$$y(t) = \sum_{k=-\infty}^{+\infty} \left(X_k e^{-\jmath\omega k t_0}\right) e^{\jmath k\omega t},$$

where

$$x(t) = \sum_{k=-\infty}^{+\infty} X_k e^{\jmath k\omega t}.$$

How to compute Fourier series — another saw



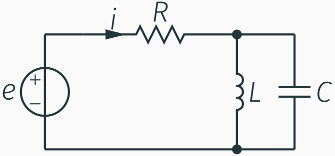
$$X_k = \frac{\sin(\pi k d)}{k\pi}, \qquad d = \frac{1}{3}.$$

$$p'(t) = q(t),\ q(t) = \frac{9}{2T}x(t - \frac{T}{6}) - \frac{3}{2T}, \qquad d = \frac{1}{3}.$$

$$Q_k = \frac{9}{2T}\frac{\sin(k\frac{\pi}{3})}{k\pi}e^{-jk\frac{\pi}{3}}, \quad k \neq 0; \qquad P_0 = \frac{1}{2}, \qquad P_k = \frac{9}{jk4\pi}\frac{\sin(k\frac{\pi}{3})}{k\pi}e^{-jk\frac{\pi}{3}}, \quad k \neq 0.$$

Example

Find $i(t)$ and its RMS value I_{RMS}.



$$e(t) = 10 + 10\cos(\omega t - \frac{\pi}{3}) + 10\cos(2\omega t - \frac{\pi}{6})\,\mathrm{V}.$$

$\omega = 100\,\mathrm{krad/s}$, $R = 1\,\mathrm{k\Omega}$, $L = 20\,\mathrm{mH}$, $C = 5\,\mathrm{nF}$.