Circuits and Signals

Power and Energy

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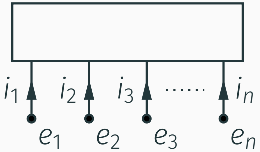
Instantaneous power



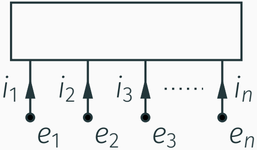
Instantaneous power delivered to an n-terminal device:

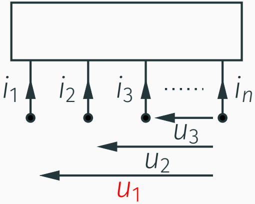
$$p(t) = e_1(t)i_1(t) + \cdots + e_n(t)i_n(t).$$

Instantaneous power generated by a device:

$$p_{\text{gen}}(t) = -p(t).$$

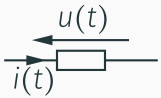
Instantaneous power — current-voltage definition





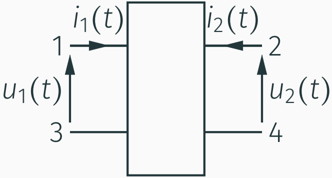
$$
\begin{aligned}
p(t) &= e_1(t)i_1(t) + \cdots + e_{n-1}(t)i_{n-1}(t) + e_n(t)i_n(t) \\
&= e_1(t)i_1(t) + \cdots + e_{n-1}(t)i_{n-1}(t) + e_n(t)(-i_1 - i_2 - \cdots - i_{n-1})(t) \\
&= i_1(t)\,(e_1{-}e_n)\,\underbrace{(e_1 - e_n)}_{u_1}(t) + i_2(t)\,(e_2{-}e_n)\,\underbrace{(e_2 - e_n)}_{u_2}(t) + \cdots + i_{n-1}(t)\,(e_{n-1}{-}e_n)\,\underbrace{(e_{n-1} - e_n)}_{u_{n-1}}(t) \\
&= i_1(t)u_1(t) + i_2(t)u_2(t) + \cdots + i_{n-1}(t)u_{n-1}(t).
\end{aligned}
$$

Instantaneous power delivered to one-ports



$$p(t) = u(t)i(t).$$

Instantaneous power delivered to two-ports



$$p(t) = \big(e_1 i_1 + e_3(-i_1) + e_2 i_2 + e_4(-i_2)\big)(t)$$

$$= u_1(t) i_1(t) + u_2(t) i_2(t).$$

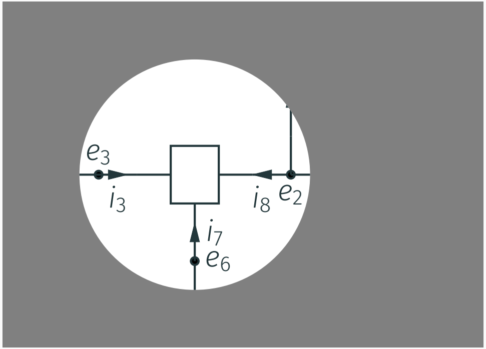
Tellegen's Theorem

Tellegen's Theorem

The total sum of instantaneous powers delivered to all the device comprising a circuit eqauls zero.

Proof of Tellegen's Theorem

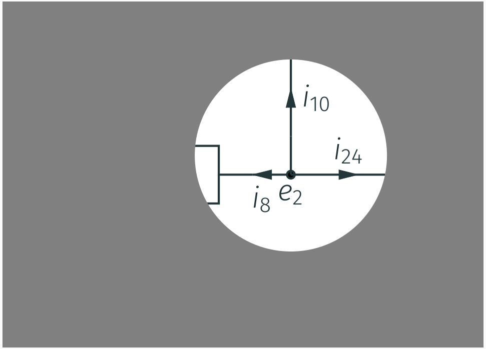
a complicated circuit:



The sum of instantaneous powers is the sum of all the products $e_k i_l$, where e_k is a node's potential and i_l is a current diverging from that node (and flowing into a terminal of some device). Every potential of a node is multiplied by the sum of all the currents diverging from that node

Proof of Tellegen's Theorem

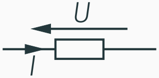
a complicated circuit:



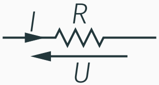
Every potential of a node is multiplied by the sum of all the currents diverging from that nodewhich (the sum) is zero by KCL.

$$p = \dots + e_2(i_8 + i_{10} + i_{24}) + \dots$$

Power delivered to one-port in DC circuits — examples



$$P = UI.$$

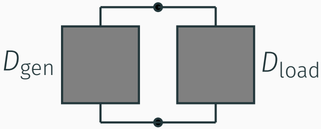


$$P = UI \overset{U=RI}{=} I^2 R = \frac{U^2}{R}.$$

$$P = 0.$$

$$P = 0.$$

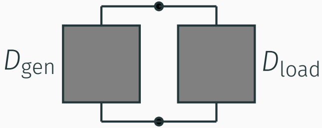
Maximum Power Transfer (MPT) Theorem — introduction

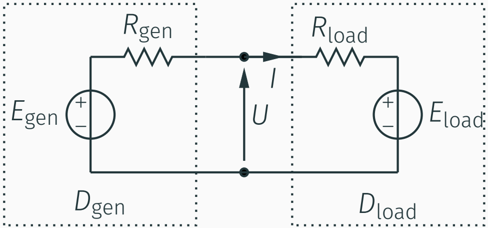


assumption: D_{gen} is fixed (we cannot change it).

For what one-port D_{load} the power delivered to D_{load} is maximal?

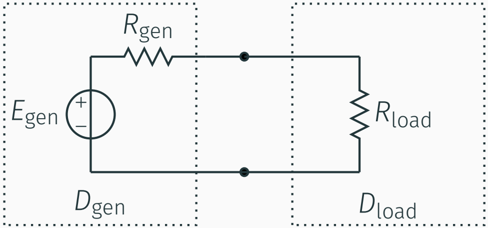
MPT Theorem — restrictions



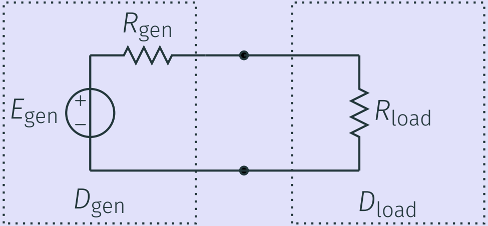


We may restrict the problem to:

From an engineer's viewpoint, the most important case is:



Maximum Power Transfer Theorem (DC case)



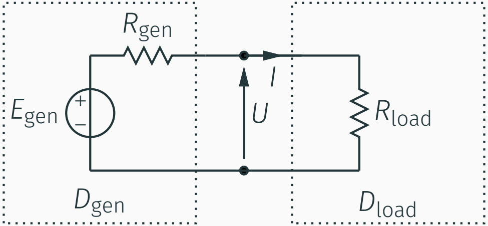
If $E_{\text{gen}} \neq 0$ and $R_{\text{gen}} > 0$ are fixed parameters, then the maximal power that can be delivered (transferred) to R_{load} equals

$$P_{\max} = \frac{E_{\text{gen}}^2}{4R_{\text{gen}}}.$$

Such power is delivered to R_{load} if and only if

$$R_{\text{load}} = R_{\text{gen}}.$$

Proof of MPT Theorem



$$P = \left(\frac{E_{\text{gen}}}{R_{\text{gen}} + R_{\text{load}}}\right)^2 R_{\text{load}} = \frac{E_{\text{gen}}^2}{4R_{\text{gen}}} \frac{R_{\text{load}}}{R_{\text{gen}} + R_{\text{load}}} \frac{4R_{\text{gen}}}{R_{\text{gen}} + R_{\text{load}}}$$

$$4\frac{R_{\text{load}}}{R_{\text{gen}} + R_{\text{load}}} \frac{R_{\text{gen}}}{R_{\text{gen}} + R_{\text{load}}} \overset{4ab \leq (a+b)^2}{\leq} \left(\frac{R_{\text{load}}}{R_{\text{gen}} + R_{\text{load}}} + \frac{R_{\text{gen}}}{R_{\text{gen}} + R_{\text{load}}}\right)^2 = 1.$$

Thus $P \leq \frac{E_{\text{gen}}^2}{4R_{\text{gen}}}$ and „ = ” $\overset{a=b}{\iff} R_{\text{load}} = R_{\text{gen}}$.

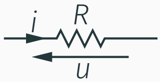
Electric Energy

Energy delivered to a device in the period (t_0, t_1) is the quantity

$$w(t_0, t_1) = \int_{t_0}^{t_1} p(t)\mathrm{d}t,$$

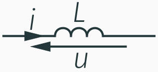
where $p(t)$ is the instantaneous power delivered to the device in time instant $t \in (t_0, t_1)$.

Energy delivered to a resistor



$$w(t_0, t_1) = \int_{t_0}^{t_1} p(t)\mathrm{d}t = \int_{t_0}^{t_1} i^2(t)R\mathrm{d}t \geq 0.$$

Energy delivered to an inductor



$$w(t_0, t_1) = \int_{t_0}^{t_1} p(t)\mathrm{d}t = \int_{t_0}^{t_1} \underbrace{Li'(t)}_{u(t)} i(t)\mathrm{d}t = \int_{t_0}^{t_1} \left(\frac{1}{2}Li^2(t)\right)' \mathrm{d}t$$

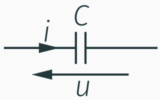
$$= \frac{1}{2}Li^2(t_1) - \frac{1}{2}Li^2(t_0).$$

$w(t_0, t_1)$ may be negative, positive or zero.

Energy stored in the inductor: $w_L(i) = \frac{1}{2}Li^2$.

$w(t_0, t_1) = w_L(i(t_1)) - w_L(i(t_0))$.

Energy delivered to a capacitor



$$w(t_0, t_1) = \int_{t_0}^{t_1} p(t)\mathrm{d}t = \int_{t_0}^{t_1} u(t)\underbrace{Cu'(t)}_{i(t)}\mathrm{d}t = \int_{t_0}^{t_1} \left(\frac{1}{2}Cu^2(t)\right)' \mathrm{d}t$$

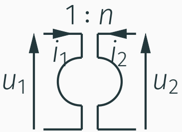
$$= \frac{1}{2}Cu^2(t_1) - \frac{1}{2}Cu^2(t_0).$$

$w(t_0, t_1)$ may be negative, positive or zero.

Energy stored in the capacitor: $w_C(u) = \frac{1}{2}Cu^2$.

$w(t_0, t_1) = w_C(u(t_1)) - w_C(u(t_0))$.

Energy delivered to a transformer

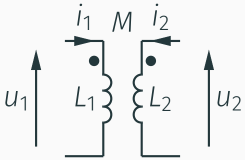


$$p(t) = (u_1 i_1 + u_2 i_2)(t) = \left(u_1 i_1 + \underbrace{(n u_1)\left(-\frac{1}{n} i_1\right)}_{-u_1 i_1} \right)(t) = 0.$$

$$w(t_0, t_1) = \int_{t_0}^{t_1} p(t)\mathrm{d}t = 0.$$

Coupled Inductors

Coupled inductors is a 2-port:



$$u_1 = L_1 i_1' + M i_2',$$
$$u_2 = L_2 i_2' + M i_1',$$

where $0 \leq M \leq \sqrt{L_1 L_2}$ is called mutual inductance [H]
($k = M/\sqrt{L_1 L_2}$ is called the coupling coefficient).

Energy delivered to coupled inductors

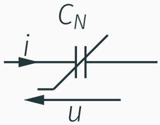
$$
\begin{aligned}
u_1 &= L_1 i_1' + M i_2', \\
u_2 &= L_2 i_2' + M i_1',
\end{aligned}
$$

$$
\begin{aligned}
w(t_0, t_1) = \int_{t_0}^{t_1} \underbrace{(u_1 i_1 + u_2 i_2)}_{p}(t)\mathrm{d}t = \\
= \frac{1}{2} L_1 \left(i_1^2(t_1) - i_1^2(t_0) \right) + \frac{1}{2} L_2 \left(i_2^2(t_1) - i_2^2(t_0) \right) + \\
+ M \left(i_1(t_1) i_2(t_1) - i_1(t_0) i_2(t_0) \right).
\end{aligned}
$$

Energy stored in coupled inductors:

$$
w_M(i_1, i_2) = \frac{1}{2} L_1 i_1^2 + \frac{1}{2} L_2 i_2^2 + M i_1 i_2.
$$

Nonlinear capacitor



$$\boxed{q = q(u)}$$

linear case: $q = Cu$

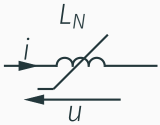
$$i(t) = q'(t) = \frac{\mathrm{d}q}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}t}(t) = \frac{\mathrm{d}q(u(t))}{\mathrm{d}u}u'(t).$$

$$w(t_0, t_1) = \int_{t_0}^{t_1} p(t)\mathrm{d}t = \int_{t_0}^{t_1} u(t)\frac{\mathrm{d}q(u(t))}{\mathrm{d}u}u'(t)\mathrm{d}t =$$

$$\dots \left| \begin{array}{c} u = u(t) \\ \mathrm{d}u = u'(t)\mathrm{d}t \end{array} \right| \dots = \int_{u(t_0)}^{u(t_1)} u\frac{\mathrm{d}q(u)}{\mathrm{d}u}\mathrm{d}u.$$

Energy stored: $w_{C_N}(u) = \int_0^u u\frac{\mathrm{d}q(u)}{\mathrm{d}u}\mathrm{d}u.$

Nonlinear inductor



$$\boxed{\psi = \psi(i)}$$ linear case: $\psi = Li$

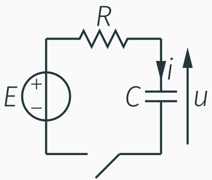
$$u(t) = \psi'(t) = \frac{\mathrm{d}\psi}{\mathrm{d}i}\frac{\mathrm{d}i}{\mathrm{d}t}(t) = \frac{\mathrm{d}\psi(i(t))}{\mathrm{d}i}i'(t).$$

$$w(t_0, t_1) = \int_{t_0}^{t_1} p(t)\mathrm{d}t = \int_{t_0}^{t_1} i(t)\frac{\mathrm{d}\psi(i(t))}{\mathrm{d}i}i'(t)\mathrm{d}t =$$

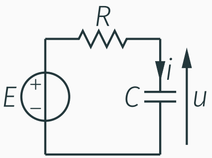
$$\dots \left| \begin{array}{c} i = i(t) \\ \mathrm{d}i = i'(t)\mathrm{d}t \end{array} \right| \dots = \int_{i(t_0)}^{i(t_1)} i\frac{\mathrm{d}\psi(i)}{\mathrm{d}i}\mathrm{d}i.$$

Energy stored: $w_{L_N}(i) = \int_0^i i\frac{\mathrm{d}\psi(i)}{\mathrm{d}i}\mathrm{d}i.$

How is the energy delivered to elements in DC circuits



$$t < t_0,$$
$$i(t) = 0,$$
$$u(t) = 0.$$



$$t \geq t_0$$