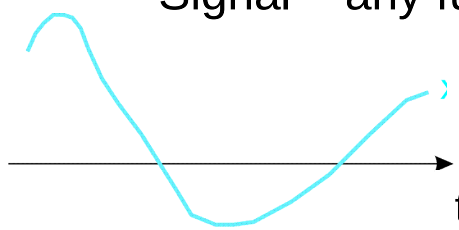
Continuous time signals

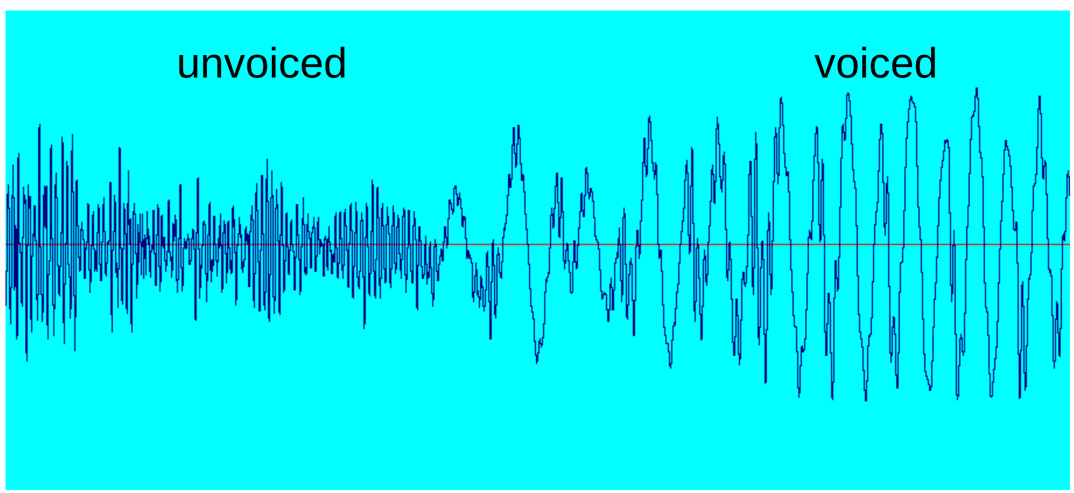
Signal – any function of time



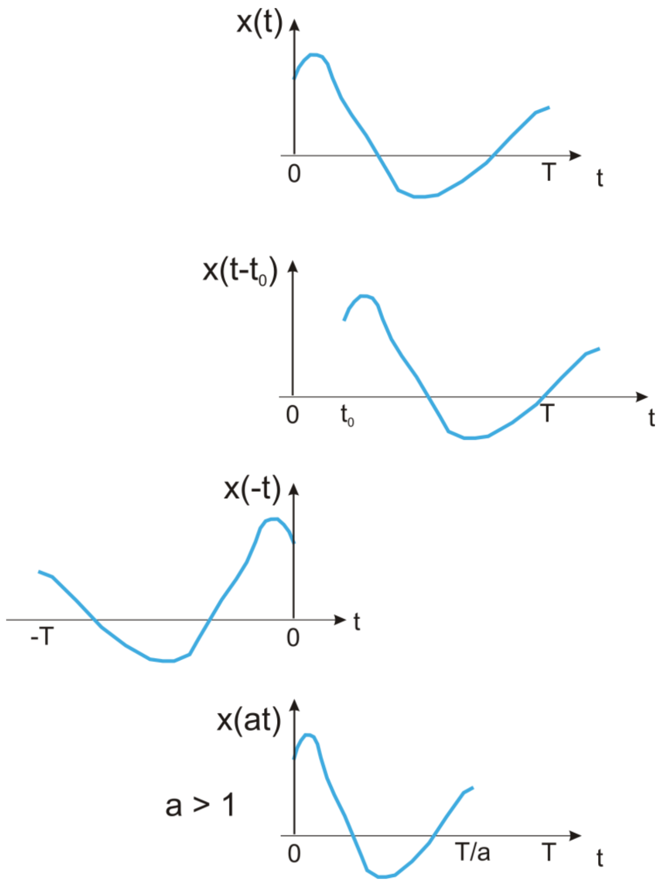
x(t) - signal in time domain

t - time

Example :
speech signal



Basic transformations

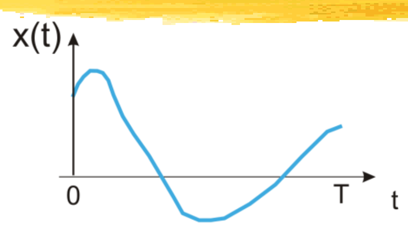


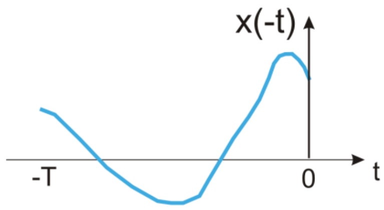
Time shift (delay)

Mirror image

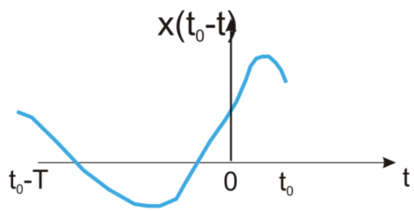
Time scaling

Basic transformations

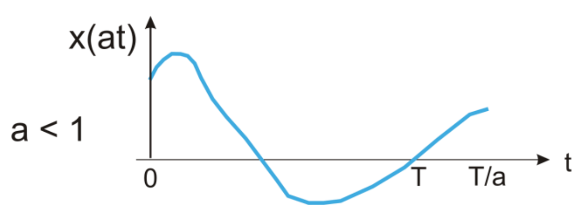




Mirror image

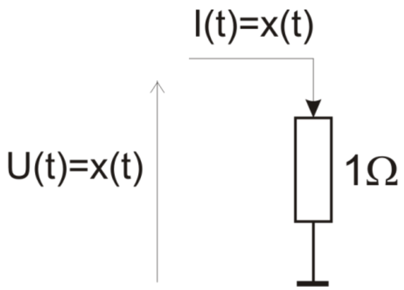


Mirror image and shift



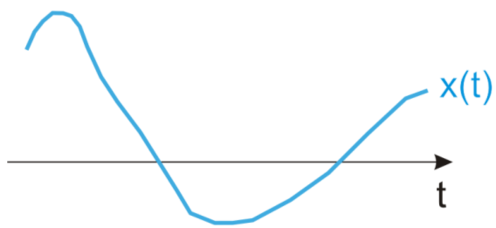
Time scaling

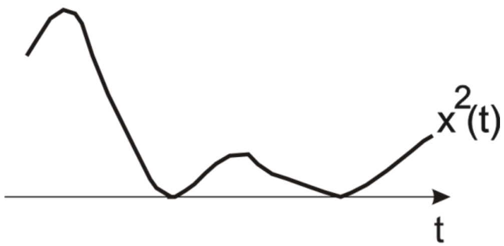
Signal energy and power

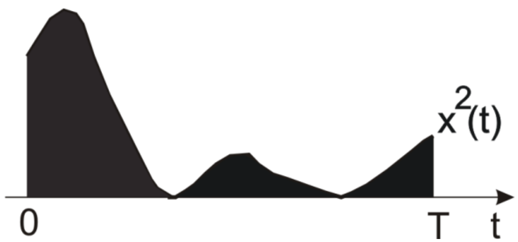


Instantaneous power

$P(t)=x^2(t)$







Energy $$E=\int_0^T x^2(t)\,dt$$

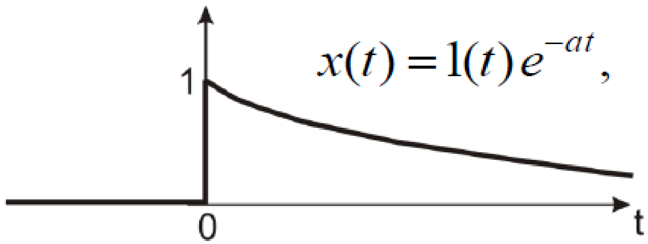
Average power $$P=\frac{1}{T}\int_0^T x^2(t)\,dt$$

Signals of infinite duration

Signals of infinite duration may have finite or infinite energy:

e.g. $x(t) = \exp(-at), \qquad t \in (0, \infty), \quad a > 0$

has energy equal to $\dfrac{1}{2a}$



$$x(t) = 1(t)\, e^{-at}, \qquad a > 0$$

$$E = \int_{-\infty}^{\infty} x^2(t)\, dt = \int_{0}^{\infty} (e^{-at})^2\, dt = \int_{0}^{\infty} e^{-2at}\, dt = \tfrac{1}{-2a} e^{-2at} \Big|_0^{\infty} = \tfrac{1}{-2a}[0-1] = \tfrac{1}{2a}$$

Signals of infinite duration

Signals $x(t)=A$, $x(t) = A\cos(2\pi f_0 t)$, have infinite energy but finite power

E.g. for $x(t)=A$, $P=A^2$,

for $x(t) = A\cos(2\pi f_0 t)$, $P=A^2/2$

Energy may be calculated in a window od duration T, $E_T = \int_{-T/2}^{T/2} x^2(t)\,dt$

Then average power is estimated for $T \to \infty$ $\quad P = \lim_{T\to\infty} \frac{E_T}{T}$

For $x(t)=A$: $E_T = \int_{-T/2}^{T/2} A^2\,dt = A^2 T$ and $P = A^2$

Signals of infinite duration

For $x(t) = A\cos(2\pi f_0 t)$,

$$x^2(t) = A^2\cos^2(2\pi f_0 t) = A^2[\tfrac{1}{2} + \tfrac{1}{2}\cos(4\pi f_0 t)] = \tfrac{A^2}{2} + \tfrac{A^2}{2}\cos(4\pi f_0 t)$$

$$E_T = \int_{-T/2}^{T/2} x^2(t)\,dt = \int_{-T/2}^{T/2} \tfrac{A^2}{2}\,dt + \tfrac{A^2}{2}\int_{-T/2}^{T/2}\cos(4\pi f_0 t)\,dt =$$

$$= \tfrac{A^2 T}{2} + \tfrac{A^2}{8\pi f_0}\left[\sin(4\pi f_0 t)\right]_{-T/2}^{T/2} = \tfrac{A^2 T}{2} + \tfrac{A^2}{8\pi f_0}[\sin(2\pi f_0 T) + \sin(2\pi f_0 T)] =$$

$$= \tfrac{A^2 T}{2} + \tfrac{A^2}{4\pi f_0}\sin(2\pi f_0 T)$$

$$P = \lim_{T\to\infty} \tfrac{E_T}{T} = \lim_{T\to\infty}[\tfrac{A^2}{2} + \tfrac{A^2}{4\pi f_0 T}\sin(2\pi f_0 T)] = \tfrac{A^2}{2}$$

Similarity of signals. Correlation

Similarity of real signals x(t) and y(t) may be characterized by the scalar product:

$$< x, y > = \int x(t)\, y(t)\, dt$$

The scalar product $< x, x > = \int x^2(t)\, dt = \| x \|^2 = E$

is a squared norm of the signal x(t) and is equal to its energy .

Scalar product $< x, y >$ is also called a correlation of x(t) and y(t),
but correlation coefficient of x and y is usually calculated using the normalized

signals $\rho(x, y) = < \frac{x}{||x||}, \frac{y}{||y||} > = \frac{< x, y >}{\| x \| \| y \|}$

If x(t)=y(t), then $\rho(x, y) = 1$. If $< x, y > = 0$, signals x and y are orthogonal

Correlation function and convolution

Scalar product of x(t) and shifted signal $y(t-t_0)$ is a function of delay t_0 and is called the correlation function of signals x and y.

$$R_{xy}(t_0) = \int x(t)\, y(t - t_0)\, dt$$

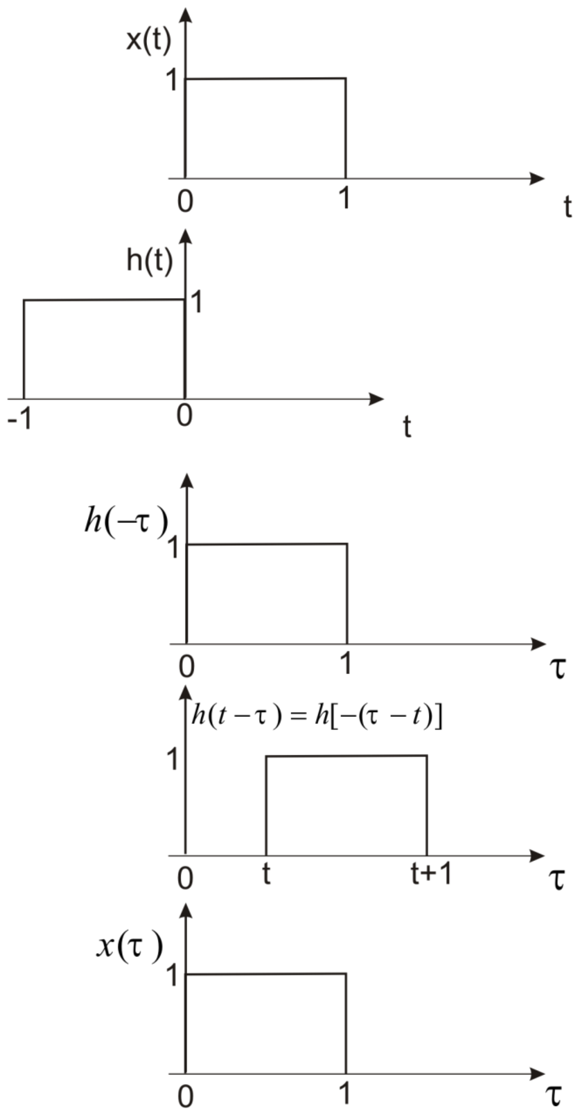
The function $R_x(t_0) = \int x(t)\, x(t - t_0)\, dt$ is the autocorrelation function of x.

At $t_0 = 0$ the autocorrelation is equal to energy: $R_x(0) = \int x^2(t)\, dt = E$

The function $y(t) = x(t) * h(t) = \int x(\tau)\, h(t - \tau)\, d\tau$

is called a convolution of x(t) and h(t).

Calculation of convolution – an example



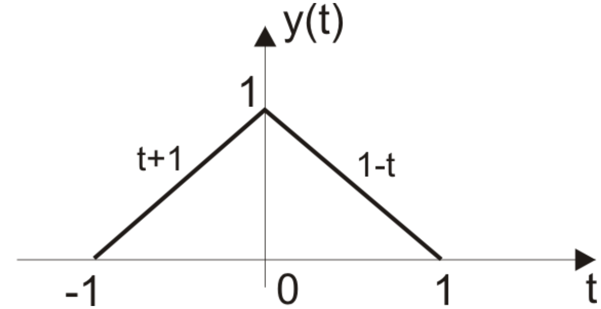
$$y(t) = x(t) * h(t) = \int x(\tau)\,h(t-\tau)\,d\tau$$

For t > 1 and for t < -1 pulses $x(\tau)$ and $h(t-\tau)$ do not overlap , $x(\tau)\,h(t-\tau) = 0$ and y(t)=0

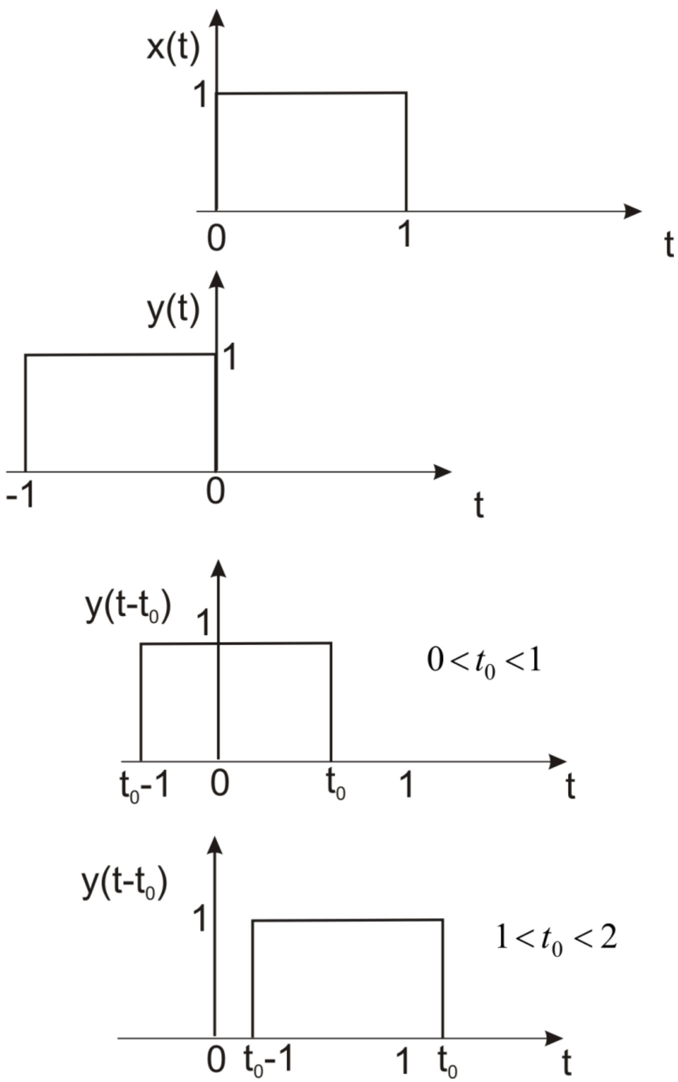
For 0 < t < 1 $\quad x(\tau)\,h(t-\tau) = 1, \quad t < \tau < 1$
and y(t)=1-t

For -1 < t < 0 $\quad x(\tau)\,h(t-\tau) = 1, \quad 0 < \tau < t+1$
and y(t)=t+1

Finally
the convolution
equals:



Calculation of correlation function – an example



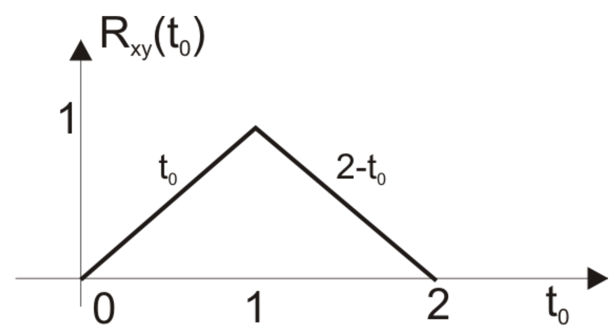
$$R_{xy}(t_0) = \int x(t)\, y(t-t_0)\, dt$$

For $t_0 > 2$ and for $t_0 < 0$ pulses $x(t)$ and $y(t-t_0)$ do not overlap , $x(t)\, y(t-t_0) = 0$ and $R_{xy}(t_0)=0$

For $0 < t_0 < 1$ $x(t)\, y(t-t_0) = 1,\ 0 < t < t_0$ and $y(t)=t_0$

For $1 < t_0 < 2$ $x(t)\, y(t-t_0) = 1,\ t_0-1 < t < 1$ and $y(t)=2-t_0$

Finally the correlation function equals:



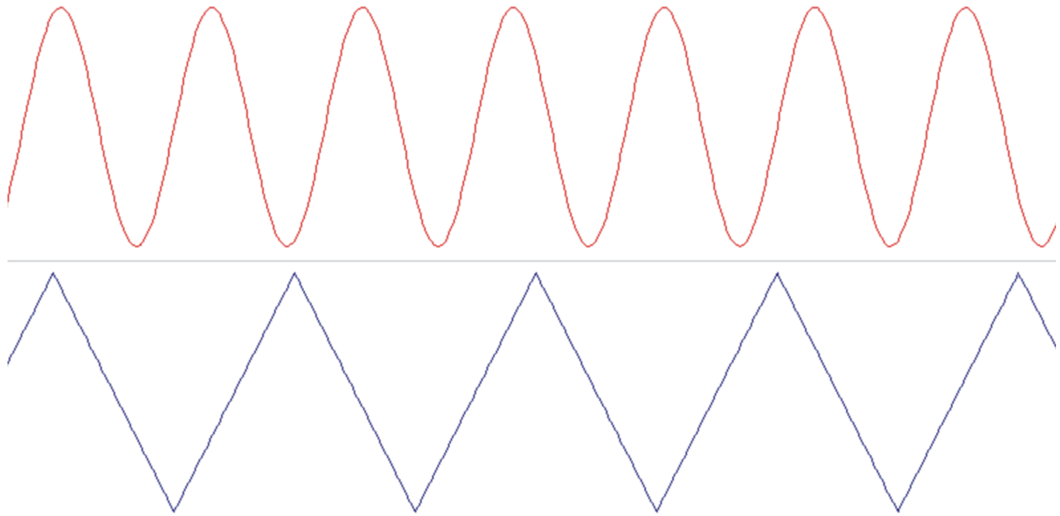
Periodic signals

Periodic signal: $\forall_t \; x(t+T) = x(t)$

T > 0 – period

Minimum value of *T* - fundamental period or just period

e.g. For x(t) = cos(2πft), fundamental period *T*=1/f



Are these signals periodic?

$x(t) = 1 + \sin(2\pi t)$ Periodic signal, period T=1

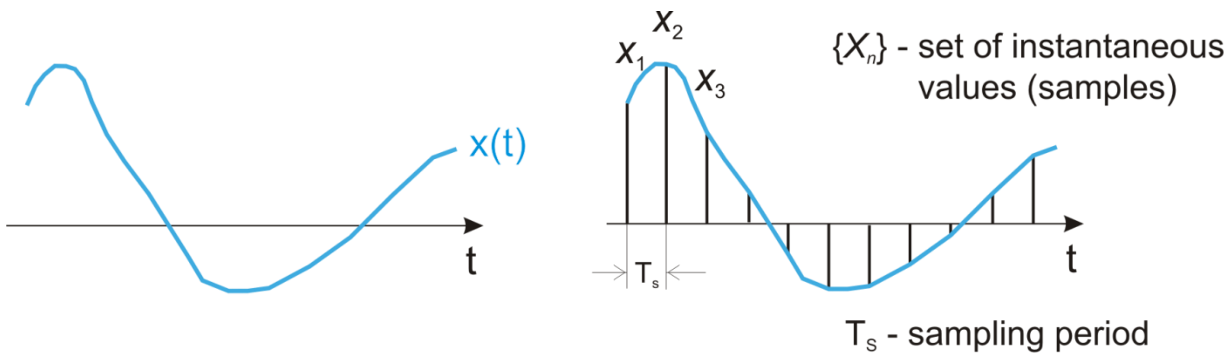
$x(t) = \cos^2(2\pi t)$ Periodic, period T=0.5, because $\cos^2(2\pi t) = \frac{1}{2} + \frac{1}{2}\cos(4\pi t)$

$x(t) = \cos(2\pi t) + \cos(\pi t)$ periodic, $T_1=1$, $T_2=2$, common period T=2

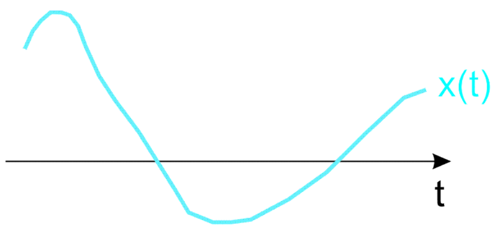
$x(t) = \cos(2\pi t) + \cos(\frac{2\pi t}{\sqrt{2}})$ first period $T_1=1$, second $T_2 = \sqrt{2}$

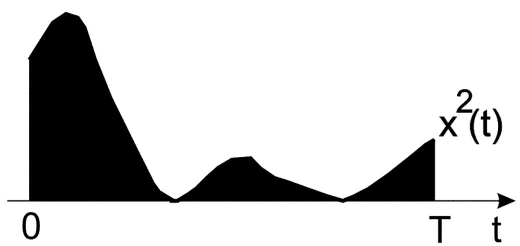
There is no common period, because T_1/T_2 is not a rational number. x(0)=2, but this value appears only once. The signal x(t) is not periodic.

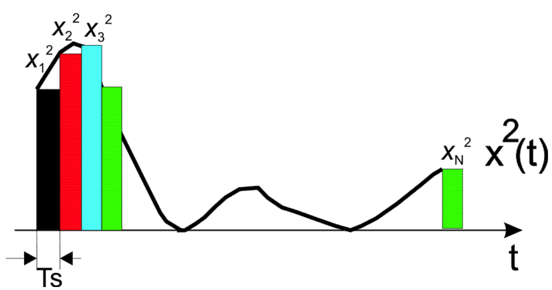
Instantaneous values (samples)



Calculation of energy and power using signal samples







Energy:

$$\mathrm{E} = \int_0^T x^2(t)\,dt$$

$$\mathrm{E} = \sum_n x_n^2 T_s$$

T_s – sampling period
x_n – n-th sample

Mean power:

$$P = \frac{1}{T}\int_0^T x^2(t)\,dt$$

$$P = \frac{1}{T}\sum_n x_n^2 T_s =$$

$$= \frac{1}{NT_s}\sum_{n=1}^{N} x_n^2 T_s = \frac{1}{N}\sum_n x_n^2$$

Calculation of signal parameters using signal samples

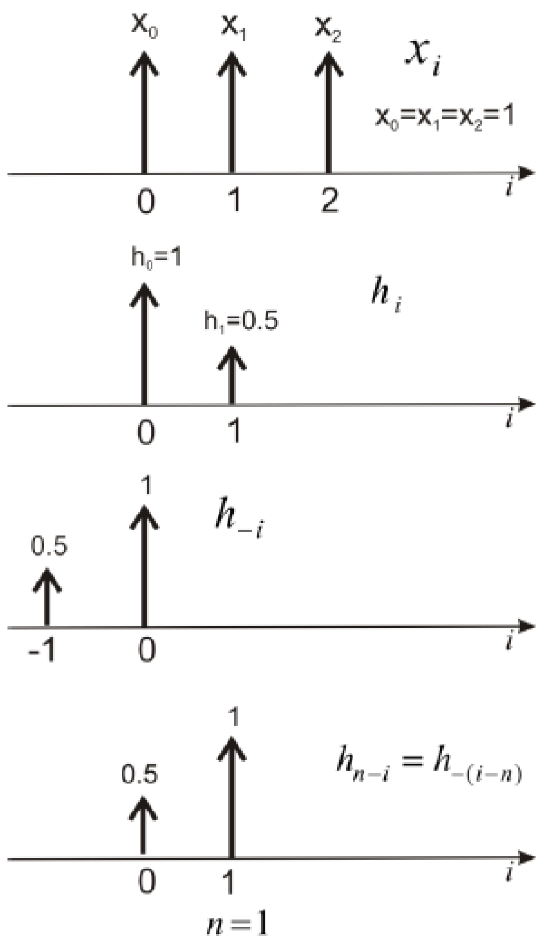
	Continuous time	Discrete time	
Energy:	$E = \int_0^T x^2(t)\,dt$	$E = \sum_{n=1}^{N} x_n^2$	T_s is set to 1
Power:	$P = \frac{1}{T}\int_0^T x^2(t)\,dt$	$P = \frac{1}{N}\sum_{n=1}^{N} x_n^2$	
Scalar product:	$< x, y > = \int_0^T x(t)\,y(t)\,dt$	$< x, y > = \sum_{n=1}^{N} x_n y_n = \mathbf{x}^T \mathbf{y}$	

x, **y** - column vectors of samples, T - transpose

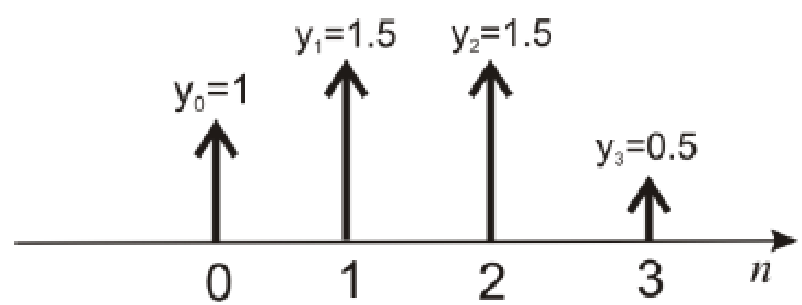
Squared norm:	$\|x\|^2 = \int_0^T x^2(t)\,dt$	$\|x\|^2 = \sum_{n=1}^{N} x_n^2 = \mathbf{x}^T \mathbf{x}$
Correlation:	$R_{xy}(t_0) = \int x(t)\,y(t-t_0)\,dt$	$R_{xy}(m) = \sum_n x_n\, y_{n-m}$
Convolution:	$x(t) * h(t) = \int x(\tau)\,h(t-\tau)\,d\tau$	$x_n * h_n = \sum_m x_m\, h_{n-m}$

Calculation of convolution in discrete time

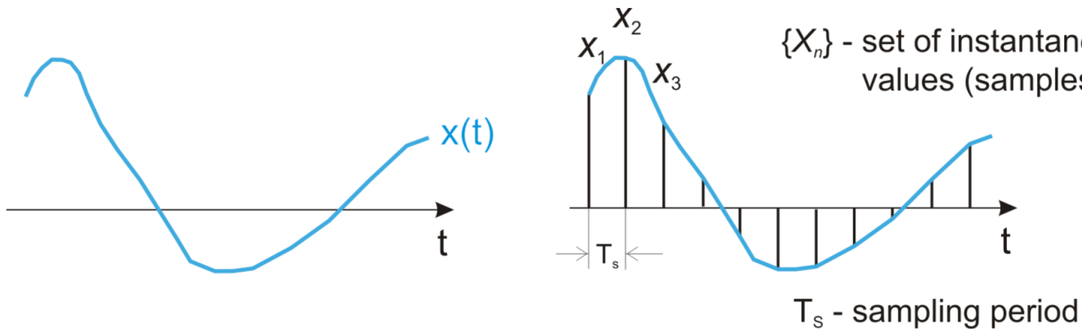
$$y_n = \sum_{i=-\infty}^{\infty} x_i h_{n-i} \qquad h_{n-i} = h_{-(i-n)}$$



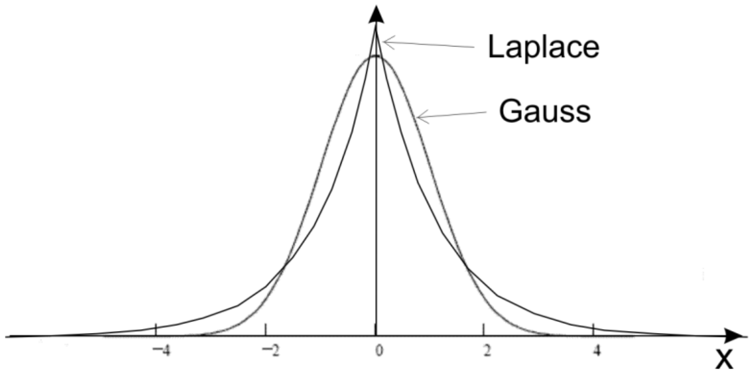
n=0, y_0=1x1=1
n=1, y_1=1x0.5 + 1x1=1.5
n=2, y_2=1x0.5 + 1x1=1.5
n=3, y_3=1x0.5 =0.5



Statistical description – random signals



Samples are random variables described with probability density function p(x)



Gauss

$$p(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(-\frac{x^2}{2\sigma_x^2}\right)$$

Laplace

$$p(x) = \frac{1}{\sqrt{2\sigma_x^2}} \exp\left(-\sqrt{2}\frac{|x|}{\sigma_x}\right)$$

σ_x^2 – variance

Calculation of signal parameters using probability density function (pdf)

Given p(x) - pdf of samples x – we may calculate:

Mean value of signal samples $m_x = E[x] = \int x\, p(x)\, dx$ (here E – statistical average)

Instatnateous power: x^2

Mean power $P = E[x^2] = \int x^2 p(x)\, dx$

Variance of signal samples : $\sigma_x^2 = E[(x - m_x)^2] = \int (x - m_x)^2 p(x)\, dx$

$$\sigma_x^2 = E[(x - m_x)^2] = E[x^2 - 2xm_x + m_x^2] = E[x^2] - 2m_x E[x] + m_x^2 =$$

$$= E[x^2] - 2m_x^2 + m_x^2 = E[x^2] - m_x^2$$

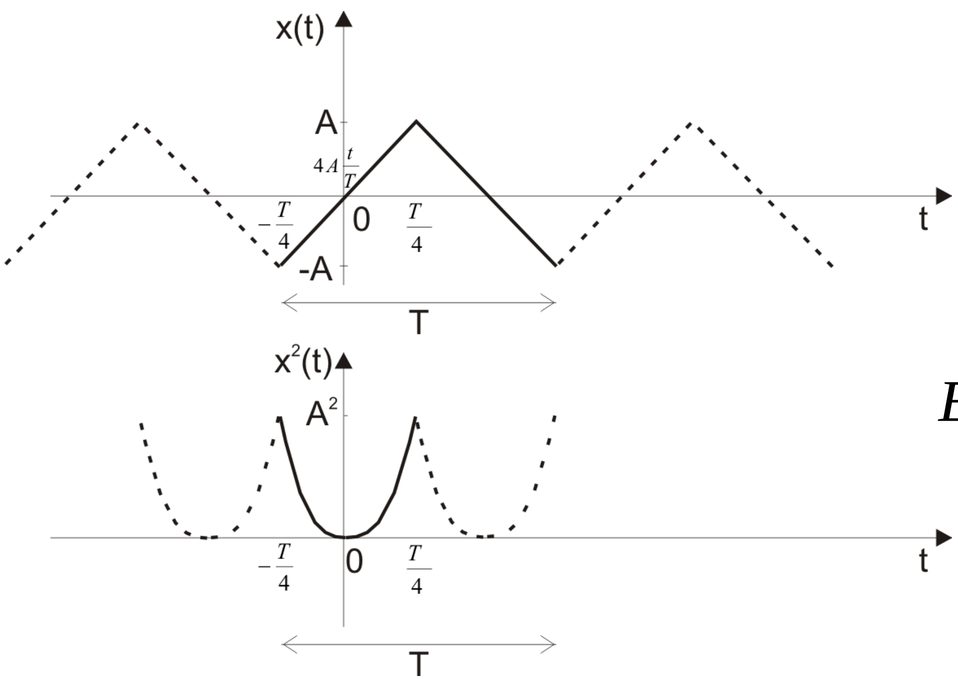
$$\longrightarrow P = E[x^2] = \sigma_x^2 + m_x^2 \longleftarrow$$

Power of constant component (direct current DC)

Power of variable component (alternating current AC)

Calculation of signal parameters using probability density function (pdf)

Example – sawtooth signal, amplitude A, period T



For –T/4 < t < T/4 $\quad x(t) = \frac{4At}{T}$

$$x^2(t) = \frac{16A^2t^2}{T^2}$$

Energy for –T/4 < t < T/4

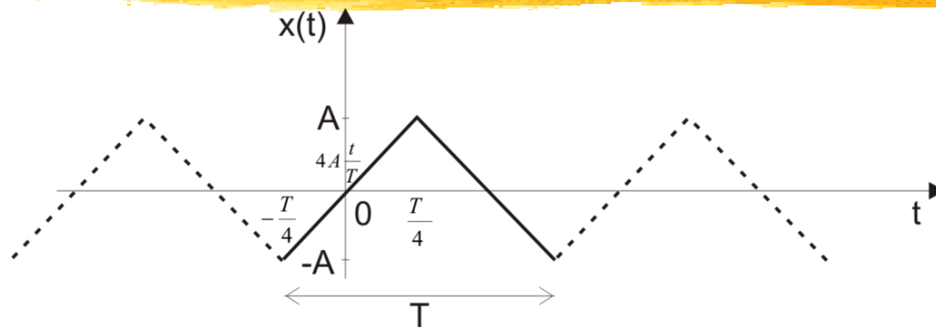
$$E_{T/2} = \int_{-T/4}^{T/4} x^2(t)dt = 2\int_0^{T/4} \frac{16A^2t^2}{T^2}dt = \frac{A^2T}{6}$$

Mean power $\quad P = \frac{E_{T/2}}{T/2} = \frac{A^2}{3}$

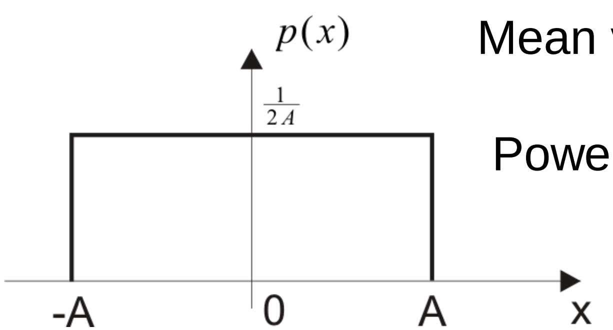
Mean power was calculated in time domain. On the next slide it will be calculated using probability density function of signal values.

Calculation of signal parameters using probability density function (pdf)

Example :
sawtooth signal,
amplitude A, period T



Instantaneous values (samples x) of this signal vary from –A to +A. Probability density function of these samples is equal to zero for x < -A and for x > A. Sawtooth function is linear, therefore probability density is the same for all the values between –A and A . Pdf of this signal is presented below.



Mean value $$m_x = \int_{-\infty}^{\infty} x\, p(x)dx = \int_{-A}^{A} \frac{x}{2A}dx = \frac{1}{2A}\left[\frac{A^2}{2} - \frac{A^2}{2}\right] = 0$$

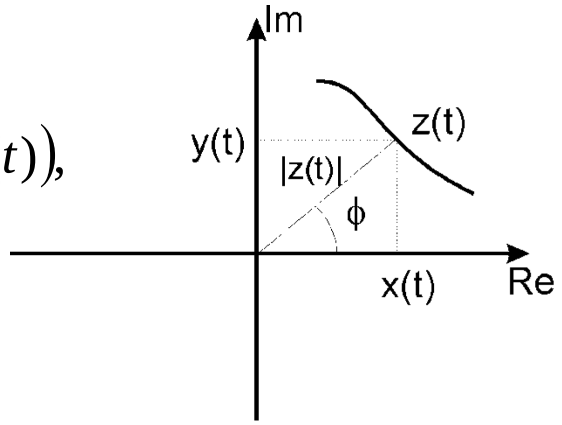
Power $$P = \int_{-\infty}^{\infty} x^2\, p(x)dx = \int_{-A}^{A} \frac{x^2}{2A}dx = \frac{1}{2A}\left[\frac{A^3}{3} + \frac{A^3}{3}\right] = \frac{A^2}{3}$$

The same power is obtained in time domain (see last slide)

Complex functions (revision)

Complex number: $z = x + jy, \quad j^2 = -1, \quad x = \mathrm{Re}(z), \quad y = \mathrm{Im}(z)$

Complex signal: $z(t) = x(t) + jy(t)$



In polar coordinates: $z(t) = |z(t)|\left(\cos\phi(t) + j\sin\phi(t)\right),$

$$|z(t)| = \sqrt{x^2(t) + y^2(t)}$$

$$\phi(t) = arctg\left({}^{y(t)}\!\big/_{x(t)}\right)$$

Complex conjugate: $z^*(t) = x(t) - jy(t)$

$$z(t)\,z^*(t) = [x(t) + jy(t)][x(t) - jy(t)] = x^2(t) + y^2(t) = |z(t)|^2$$

arctg(x) = tg^{-1}(x)

Complex functions

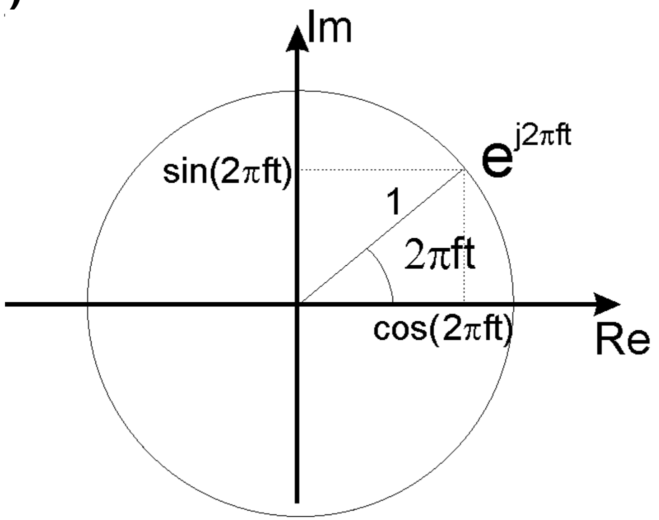
From Euler's formula: $\cos\phi + j\sin\phi = e^{j\phi}$ $\cos\varphi - j\sin\varphi = e^{-j\varphi}$

we obtain: $\cos\phi = \frac{1}{2}(e^{j\phi} + e^{-j\phi})$

$\sin\phi = \frac{1}{2j}(e^{j\phi} - e^{-j\phi})$

A function $e^{j2\pi ft} = \exp(j2\pi ft) =$

$= \cos(2\pi ft) + j\sin(2\pi ft)$



→ periodic, period T=1/f

Euler's formula

Leonhard Euler

$$\cos\phi + j\sin\phi = e^{j\phi}$$

Proof:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$e^{j\phi} = 1 + j\phi + \frac{(j\phi)^2}{2!} + \frac{(j\phi)^3}{3!} + \frac{(j\phi)^4}{4!} + \frac{(j\phi)^5}{5!} + \dots$$

$$= 1 + j\phi - \frac{\phi^2}{2!} - \frac{j\phi^3}{3!} + \frac{\phi^4}{4!} + \frac{j\phi^5}{5!} - \dots$$

$$= 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \dots + j\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \dots\right) =$$

$$= \cos(\phi) + j\sin(\phi)$$

Calculation of complex signal parameters

	Continuous time	Discrete time
Energy:	$E = \int_0^T \lvert x(t) \rvert^2 \, dt$	$E = \sum_{n=1}^{N} \lvert x_n \rvert^2$ T_s is set to 1
Power:	$P = \frac{1}{T} \int_0^T \lvert x(t) \rvert^2 \, dt$	$P = \frac{1}{N} \sum_{n=1}^{N} \lvert x_n \rvert^2$
Scalar product (inner product):	$< x, y > = \int_0^T x(t)\, y^*(t)\, dt$	$< x, y > = \sum_{n=1}^{N} x_n y_n^* = \mathbf{y}^H \mathbf{x}$ H – transpose and complex conjugate
Squared norm:	$\lVert x \rVert^2 = \int_0^T \lvert x(t) \rvert^2 \, dt$	$\lVert x \rVert^2 = \sum_{n=1}^{N} \lvert x_n \rvert^2 = \mathbf{x}^H \mathbf{x}$
Correlation:	$R_{xy}(t_0) = \int x(t)\, y^*(t - t_0)\, dt$	$R_{xy}(m) = \sum_n x_n\, y_{n-m}^*$
Convolution:	$x(t) * h(t) = \int x(\tau)\, h^*(t - \tau)\, d\tau$	$x_n * h_n = \sum_m x_m\, h_{n-m}^*$