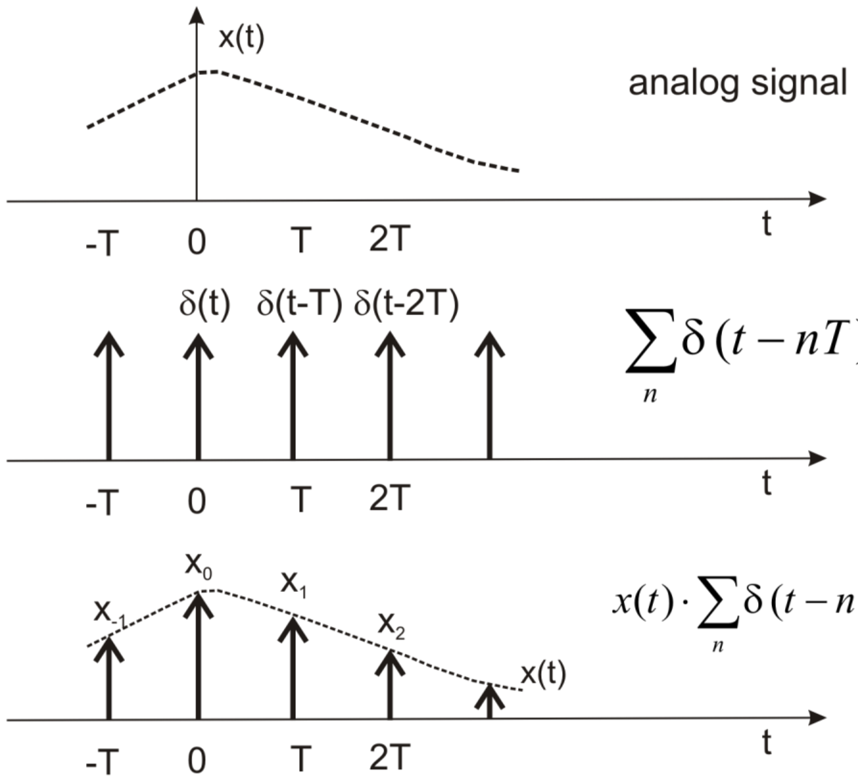
SAMPLING

Ideal sampling of an analog signal x(t):



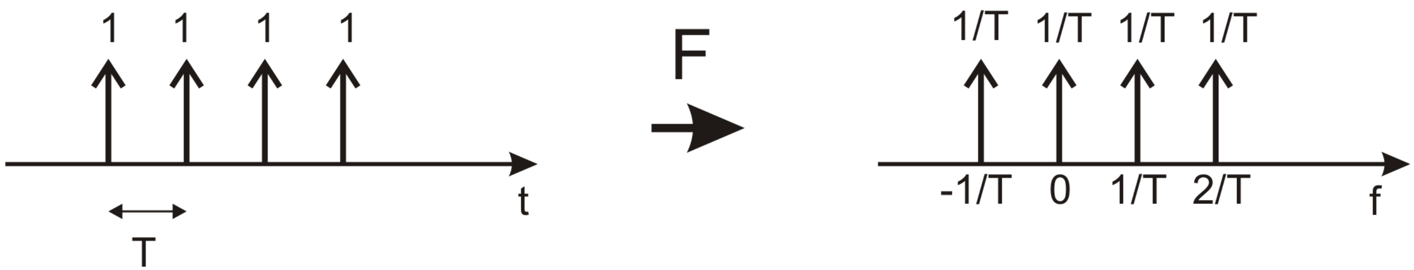
$$\sum_n \delta(t - nT)$$

T - sampling period
f_s=1/T - sampling frequency

$$x(t) \cdot \sum_n \delta(t - nT)$$

$$x(t) \cdot \sum_n \delta(t - nT) = \sum_n x(nT)\,\delta(t - nT) = \sum_n x_n\,\delta(t - nT)$$

Fourier Transform of a series of pulses:



$$F\left\{\sum_n \delta(t-nT)\right\} = \frac{1}{T}\sum_n \delta\left(f-\frac{n}{T}\right)$$

Fourier Transform of a series of pulses:

$$F\left\{\sum_n \delta(t-nT)\right\} = \frac{1}{T}\sum_n \delta\left(f-\frac{n}{T}\right)$$

Proof (using Fourier series):

Periodic signal having period T may be described using Fourier series:

$x(t)=\sum_n X_n \exp(j2\pi \tfrac{n}{T} t)$, where the n-th Fourier coefficient equals:

$$X_n = \tfrac{1}{T}\int_0^T x(t)\exp(-j2\pi \tfrac{n}{T} t)dt$$

For series of Dirac pulses $x(t)=\sum_n \delta(t-nT)$ we obtain:

$$X_n = \tfrac{1}{T}\int_0^T \delta(t)\exp(-j2\pi \tfrac{n}{T} t)dt = \tfrac{1}{T}\int_0^\varepsilon \delta(t)\exp(0)dt = \tfrac{1}{T}$$

finally $$x(t)=\frac{1}{T}\sum_n \exp(j2\pi \tfrac{n}{T} t)$$

Spectrum (Fourier transform) of this signal equals:

$$X(f)=\frac{1}{T}\sum_n F\left\{\exp(j2\pi \tfrac{n}{T} t)\right\} = \frac{1}{T}\sum_n \delta(f-\tfrac{n}{T}),$$ because

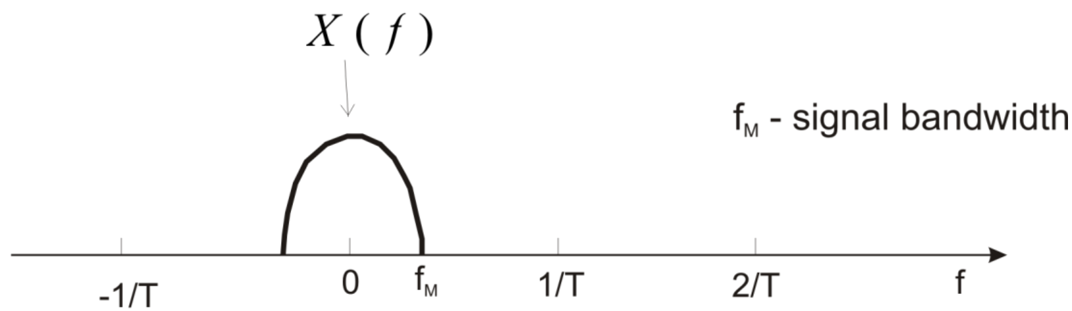
(from shift theorem in frequency domain)

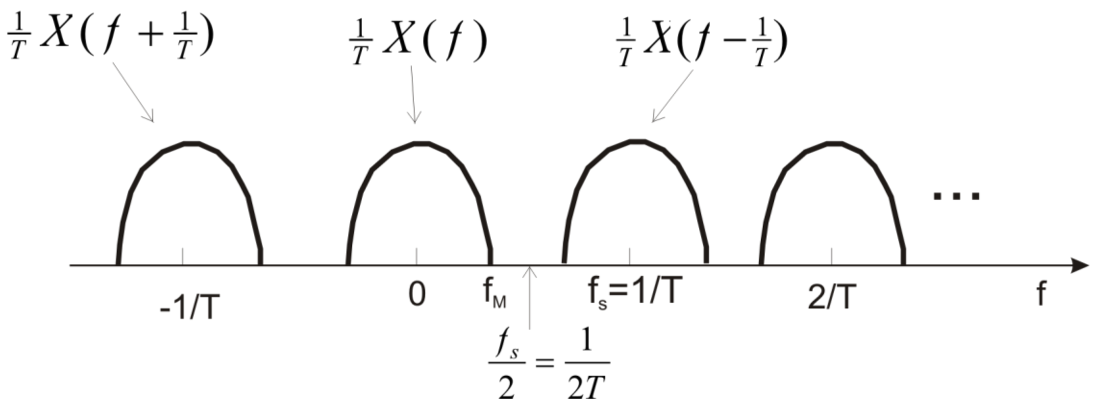
$$F^{-1}[\delta(f)]=1 \quad --> \quad F^{-1}[\delta(f-f_0)]=1\cdot\exp(j2\pi f_0 t), \quad f_0=\tfrac{n}{T}$$

Fourier Transform of sampled signal

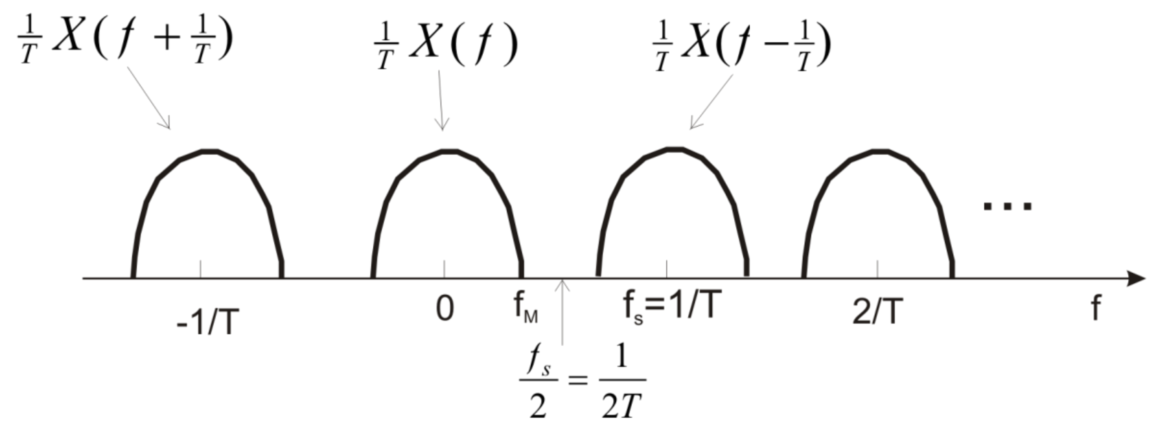
$$F[x(t)\cdot\sum_n \delta(t-nT)] = X(f) * \frac{1}{T}\sum_n \delta(f-\frac{n}{T}) =$$

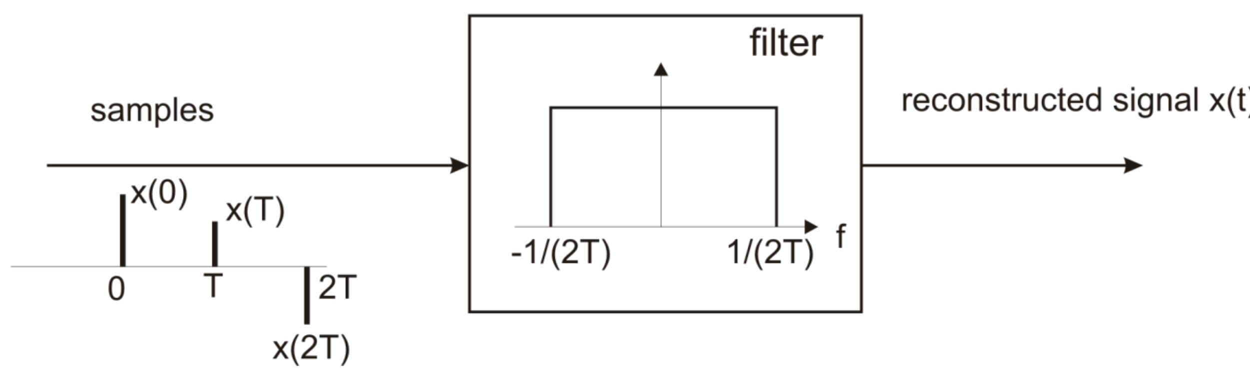
$$= \frac{1}{T}\sum_n X(f-\frac{n}{T})$$





Reconstruction of the continuous signal





Sampling Theorem

Sampling theorem (Shannon, Nyquist, Kotelnikov):

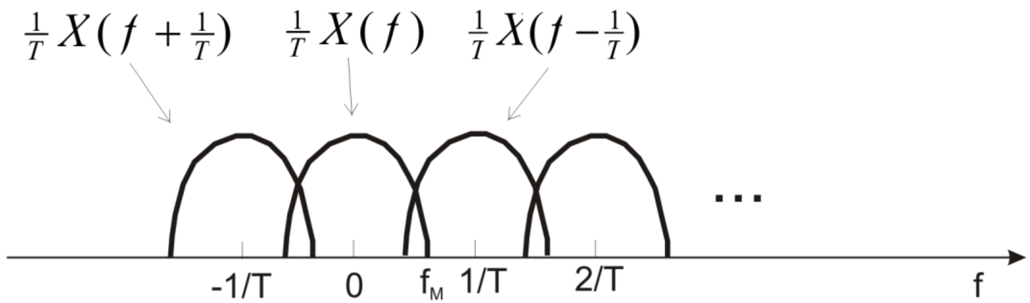
Perfect reconstruction is possible if the sampling frequency f_s=1/T is greater than 2 f_M

Claude Shannon
1916 - 2001

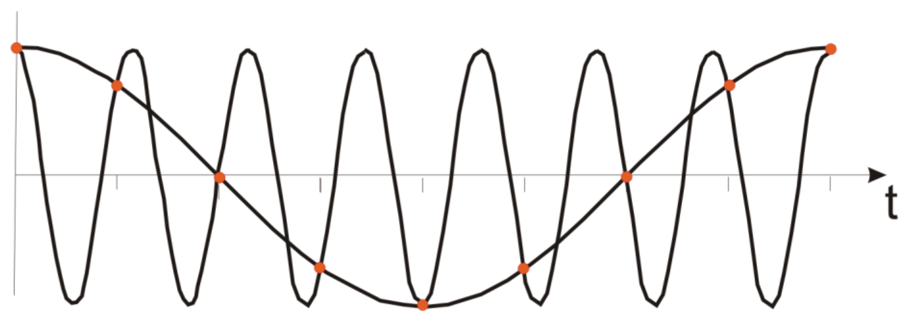
Harry Nyquist
1889 - 1976

Vladimir Kotelnikov
1908 - 2005

Aliasing if sampling frequency is less than 2 f_M

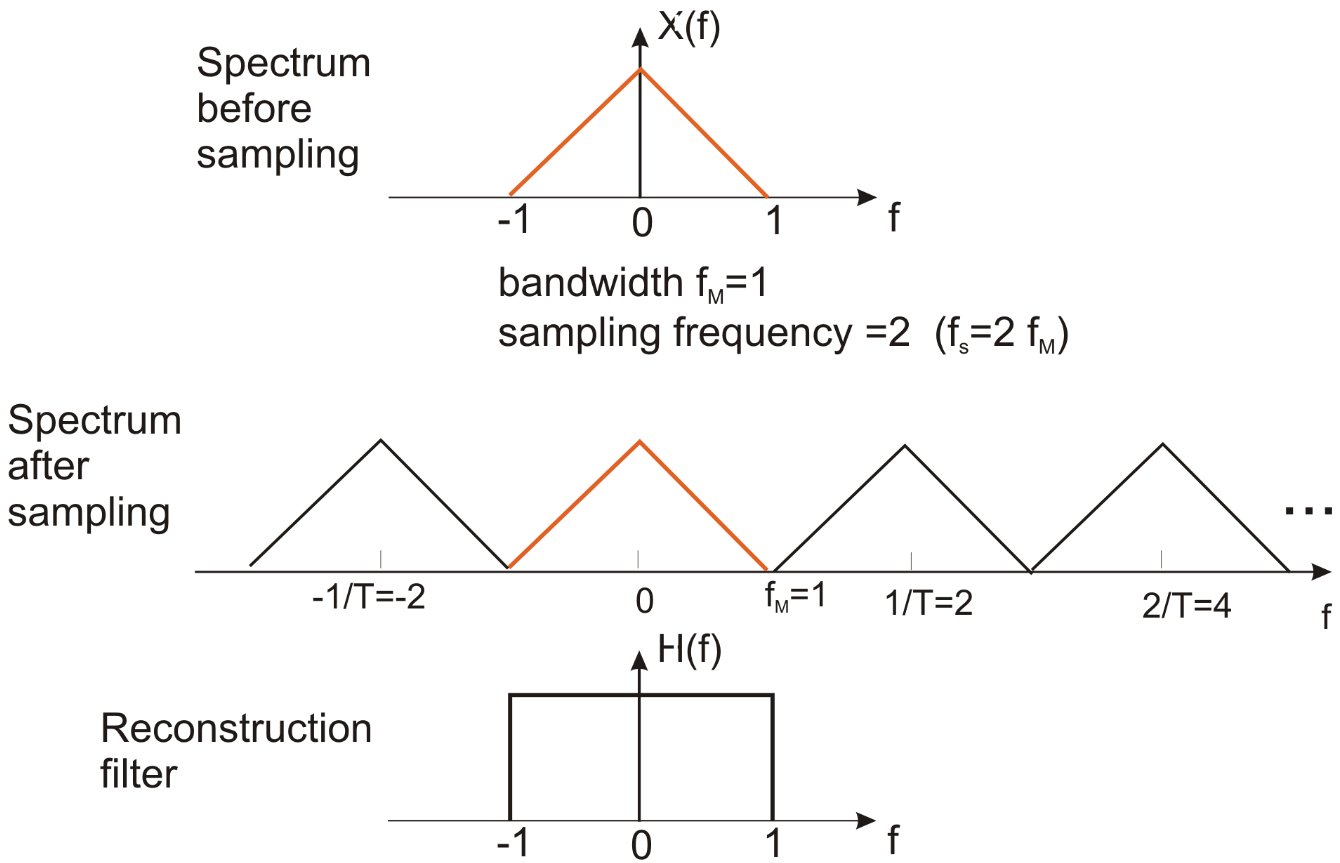


If sampling frequency is less than 2 times bandwidth, the spectral copies overlap and the original signal x(t) cannot be obtained from its samples



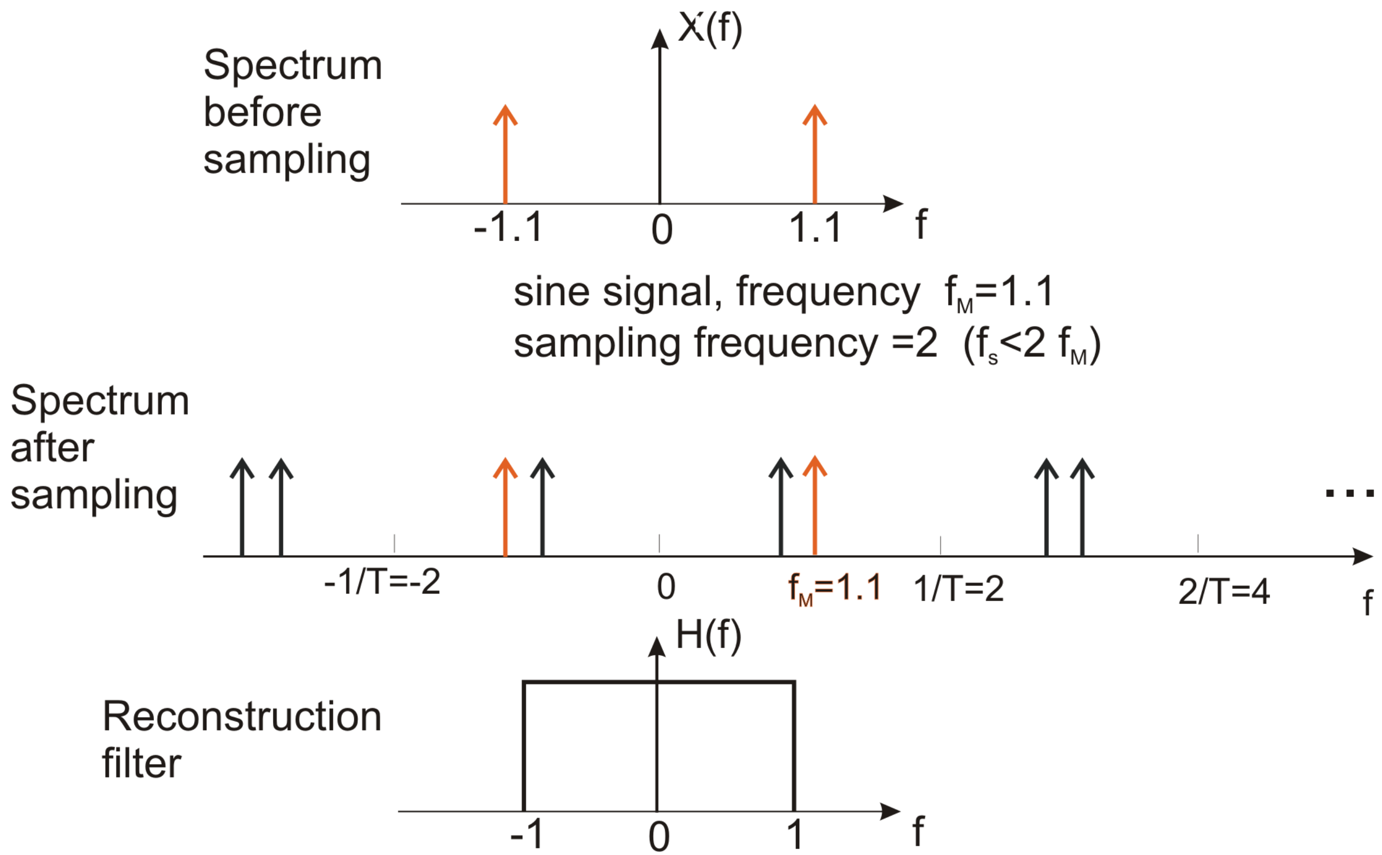
Signals of frequency higher than half of the sampling frequency may have the same samples as signal of frequency lower than half of sampling frequency. These signals are aliases of the low frequency signal.

Signal reconstruction – example 1



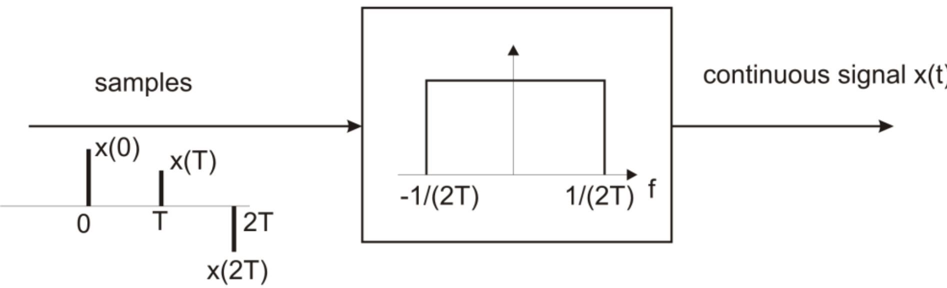
It is **possible** to reconstruct signal from its samples

Signal reconstruction – example 2

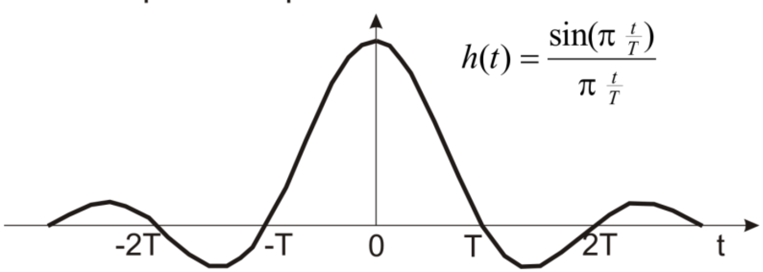


It is **not possible** to reconstruct signal from its samples

Signal reconstruction in time domain

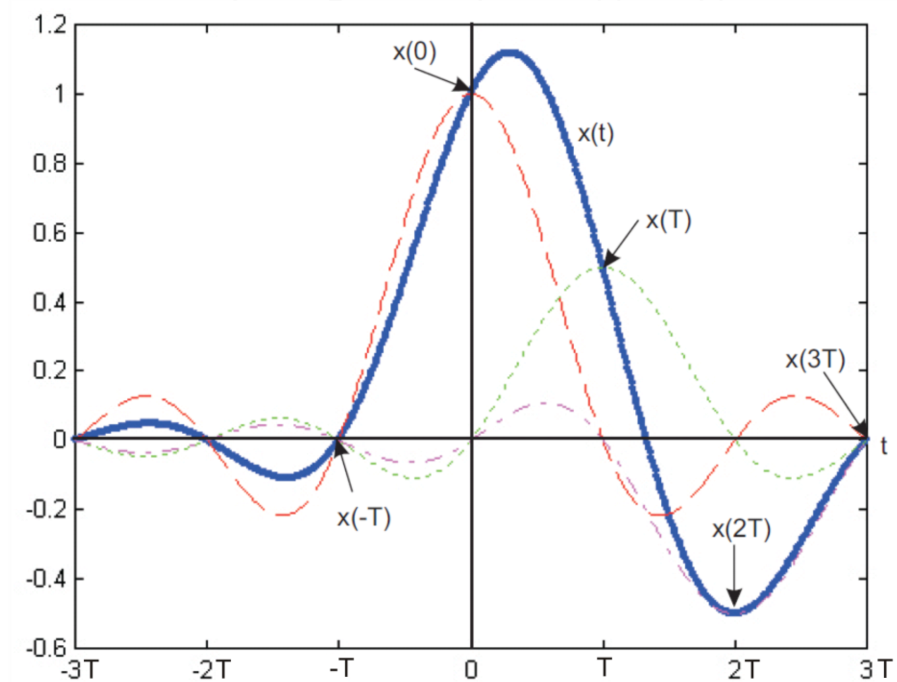


Impulse response of this filter:

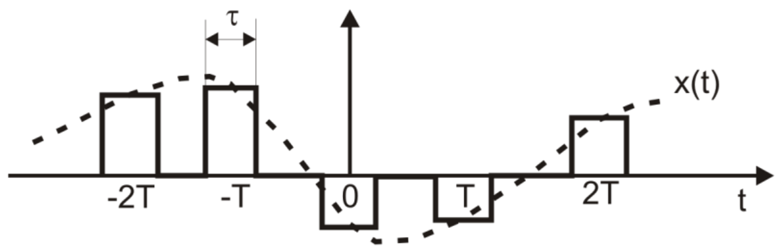


$$h(t) = \frac{\sin(\pi \frac{t}{T})}{\pi \frac{t}{T}}$$

output signal = input * h(t) = x(t)

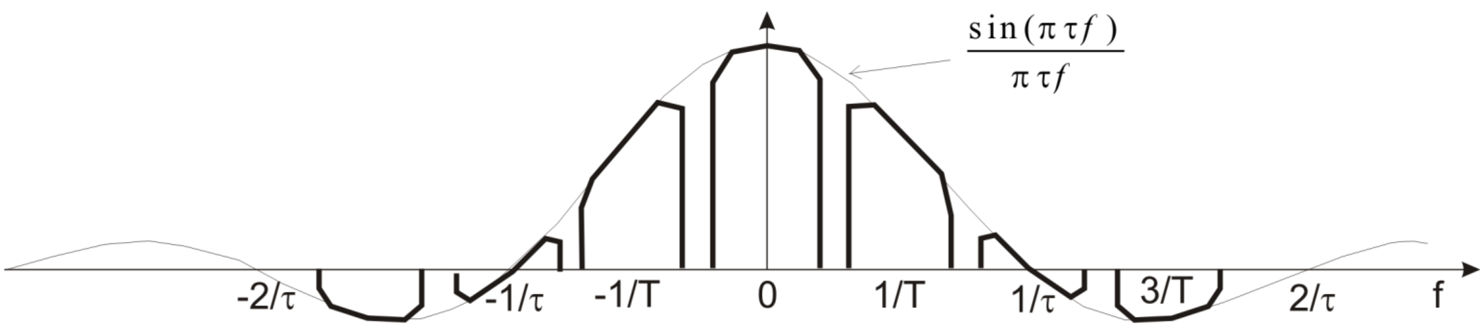


Instantaneous sampling

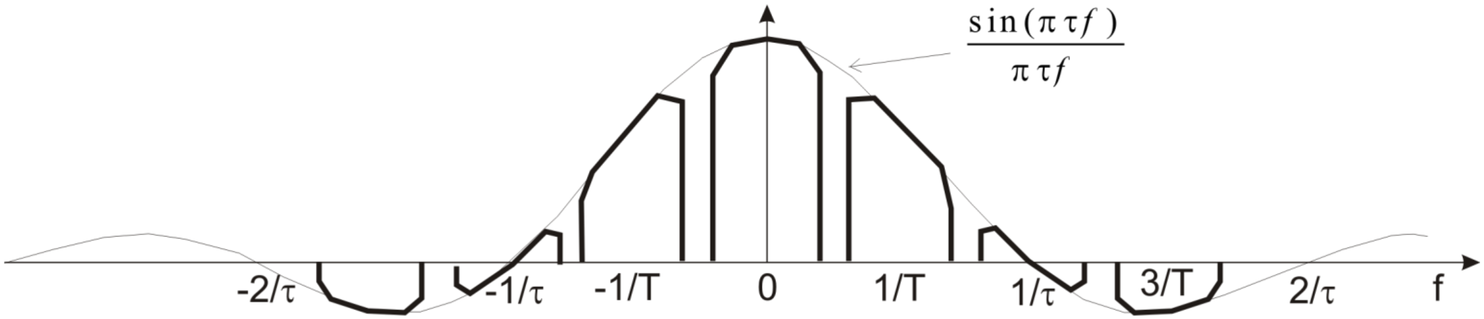


formal description: $\underbrace{x(t)\cdot\sum_n \delta(t-nT)}_{\text{ideal sampling}} * \sqcap$

spectrum: $F\left\{[x(t)\cdot\sum_n \delta(t-nT)] * \sqcap\right\} = \frac{1}{T}\sum_n X(f-\frac{n}{T})\cdot\tau\,\frac{\sin(\pi\tau f)}{\pi\tau f}$

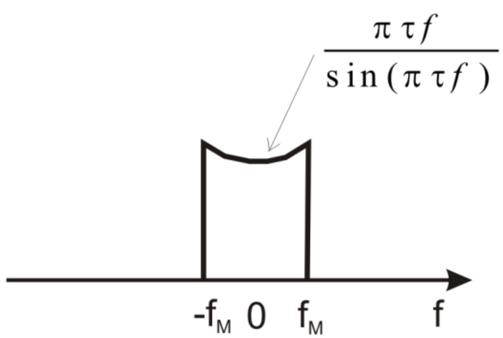


Instantaneous sampling – signal reconstruction

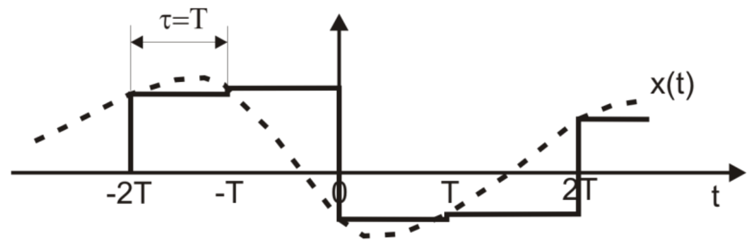


Conclusion:

Sampling theorem is valid for instantaneous sampling,
but the reconstruction filter should have the following frequency response:

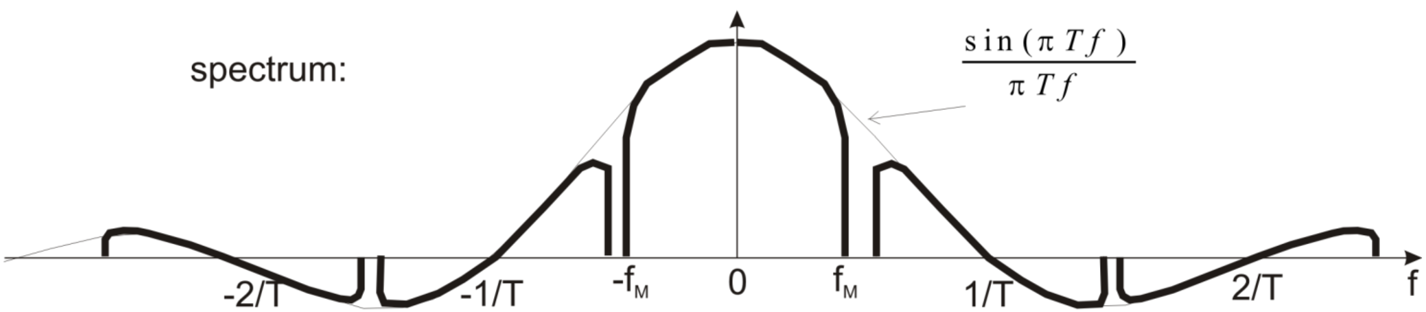


Sampling and hold

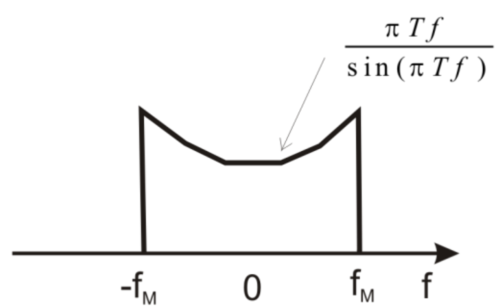


Sampling and hold is a very popular method of sampling, especially for reconstruction of continuous signal from its samples.
I is a special case of instantaneous sampling.
Duration of sampling pulse is equal to sampling interval.

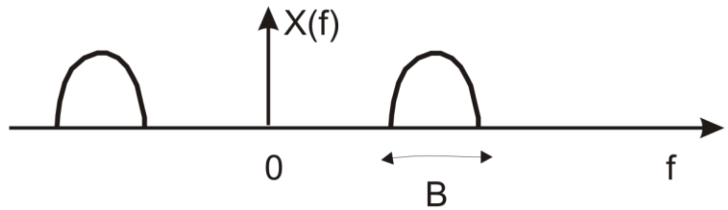
spectrum:



reconstruction filter:

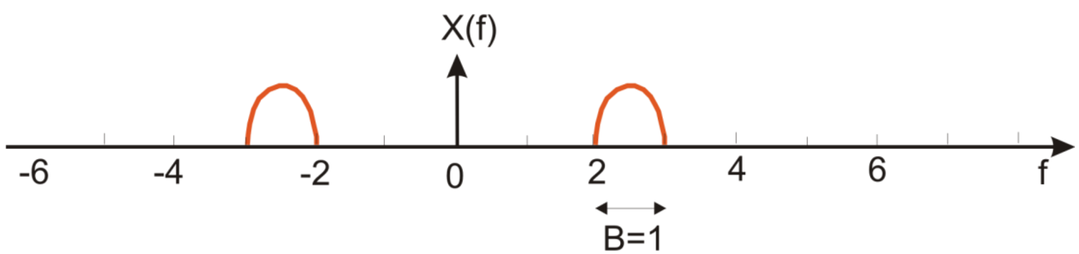


Sampling of bandpass signals



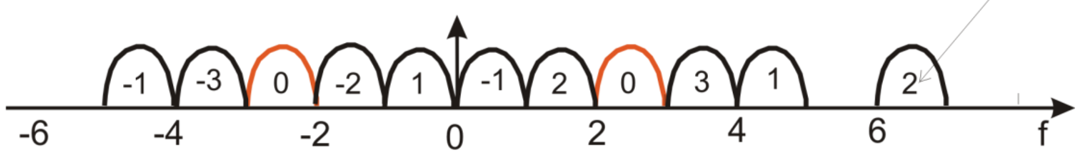
Genaralized sampling theorem:
2B samples per second yield perfect reconstruction

Example: integer sampling



sampling frequency 1/T=2B=2

spectrum of a sampled signal $$\frac{1}{T}\sum_{n} X(f-\frac{n}{T}) = 2\sum_{n} X(f-2n)$$



reconstruction filter: