Circuits and Signals

AC analysis

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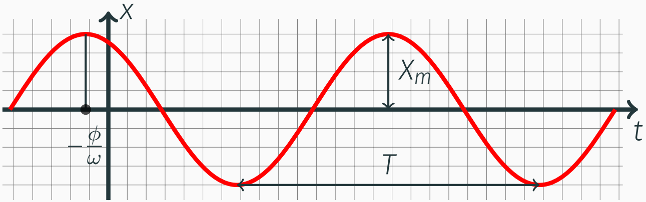
Alternating Signal



A signal (time function) x is called an alternating signal if it is of the form

$$x(t) = X_m \cos(\omega t + \phi), \quad X_m \geq 0, \omega > 0.$$

X_m is called the amplitude of the signal,
ω is called the pulsation of the signal,
ϕ is called the phase shift of the signal.

$f = \frac{\omega}{2\pi}$ is the frequency,
$T = \frac{1}{f} = \frac{2\pi}{\omega}$ is the period of the signal x.

Examples

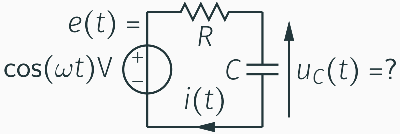
- $x(t) = 1\,\mathrm{V}\cos\omega t$ is alternating $X_m = 1\,\mathrm{V}$, $\phi = 0$,
- $y(t) = 0\,\mathrm{A}$ is alternating $Y_m = 0$,
- $z(t) = 1\,\mathrm{V}$ is not alternating,
- $x(t) = -2\,\mathrm{V}\cos(\omega t + \frac{\pi}{4})$ is alt., $X_m = 2\,\mathrm{V}$, $\phi = -\frac{3\pi}{4}$,
- $i(t) = 1\,\mathrm{mA}\sin(\omega t)$ is alt., $I_m = 1\,\mathrm{mA}$, $\phi = -\frac{\pi}{2}$,
- $u(t) = 2\,\mathrm{V}\cos(\omega t) + 3\,\mathrm{V}\sin(\omega t + \frac{\pi}{3})$ is alt. $U_m = \sqrt{13 + 6\sqrt{3}}\mathrm{V}$, $\quad \phi = -\arctan\frac{3}{4+3\sqrt{3}}$.

AC circuits

An AC circuit is a circuit for which there exist solutions consisting of alternating signals of the same pulsation.

Steady state in an AC circuit is a solution to an AC circuit consisting of alternating signals of the same pulsation.

AC analysis — an example



$$i(t) = Cu_C'(t),$$
$$e(t) = u_C(t) + RCu_C'(t).$$

$$u_C(t) = U_{Cm}\cos(\omega t + \phi) \Rightarrow u_C'(t) = -\omega U_{Cm}\sin(\omega t + \phi).$$

$$\begin{aligned} 1\,\mathrm{V}\cos\omega t &= -\omega RCU_{Cm}\sin(\omega t + \phi) + U_{Cm}\cos(\omega t + \phi) \\ &= U_{Cm}\underbrace{\sqrt{1 + (\omega RC)^2}}_{\bigstar}\Big(\underbrace{\frac{1}{\bigstar}}_{\cos\theta}\cos(\omega t + \phi) - \underbrace{\frac{\omega RC}{\bigstar}}_{\sin\theta}\sin(\omega t + \phi)\Big) = U_{Cm}(\bigstar)\cos(\omega t + \phi + \theta). \end{aligned}$$

$$U_{Cm} = \frac{1\,\mathrm{V}}{\sqrt{1 + (\omega RC)^2}}, \quad \phi = -\theta = -\arctan(\omega RC).$$

AC signals — notations

Steady state in an AC circuit is the solution consisting of alternating signals of the same pulsation ω.

e.g.

$u_1(t) = 2\,\mathrm{V}\cos(\omega t)$, $u_2(t) = 3\,\mathrm{V}\cos(\omega t + \frac{\pi}{3})$, $i(t) = 1\,\mathrm{mA}\cos(\omega t - \frac{\pi}{4})$,

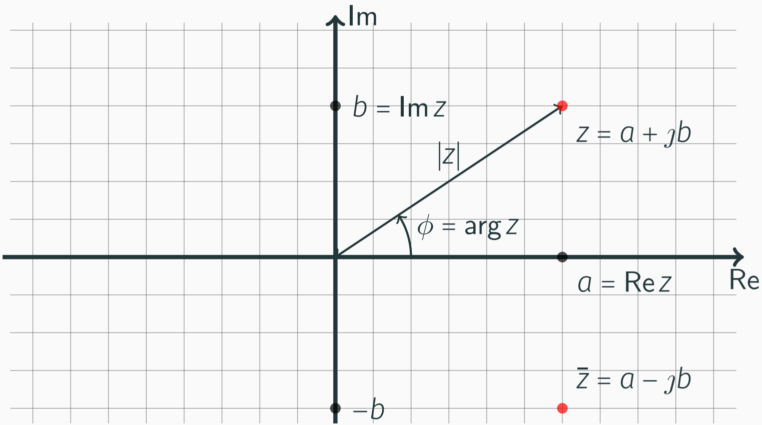
For the brevity's sake (pulsation ω is fixed!):

$2\,\mathrm{V}\underline{/0}$, $3\,\mathrm{V}\underline{/60^\circ}$, $1\,\mathrm{mA}\underline{/-45^\circ}$,

Another approach:

$2\,\mathrm{V}\,e^{\jmath 0}$, $3\,\mathrm{V}\,e^{\jmath\frac{\pi}{3}}$, $1\,\mathrm{mA}\,e^{-\jmath\frac{\pi}{4}}$.

Complex Numbers



algebraic form: $z = a + \jmath b$, $a = \operatorname{Re} z$, $b = \operatorname{Im} z$,

polar form: $z = re^{\jmath\phi}$, $r = |z|$, $\phi = \arg z$.

Euler's formula

$$e^{\jmath\phi} = \cos\phi + \jmath\sin\phi$$

in particular: $$e^{\jmath\pi} + 1 = 0,$$

and also:

$$\cos\phi = \frac{e^{\jmath\phi} + e^{-\jmath\phi}}{2},$$
$$\sin\phi = \frac{e^{\jmath\phi} - e^{-\jmath\phi}}{2\jmath},$$
$$\left|e^{\jmath\phi}\right| = 1.$$

Phasor of an alternating signal

The phasor of an alternating signal

$$x(t) = X_m \cos(\omega t + \phi), \quad X_m \geq 0, \omega > 0.$$

is the following complex number

$$X = X_m e^{\jmath\phi} = X_m \cos\phi + \jmath X_m \sin\phi.$$

Alternating signals and their phasors

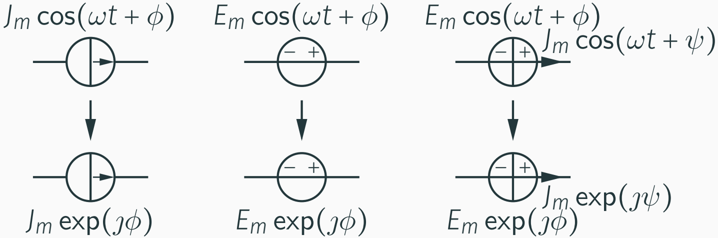
Proposition

A mapping that maps each a. s. to its phasor is a linear isomorphism between the space of a. s. of a fixed pulsation ω and the space $\mathbb{C}$ of the complex numbers.

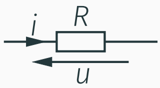
$$x(t) = X_m \cos(\omega t + \phi) \longrightarrow X = X_m e^{\jmath\phi},$$
$$x(t) = |X| \cos(\omega t + \arg X) \longleftarrow X$$

$$x(t) \leftrightarrow X, \qquad ax(t) \leftrightarrow aX,$$
$$y(t) \leftrightarrow Y, \qquad x(t) + y(t) \leftrightarrow X + Y.$$

Sources in the phasor domain

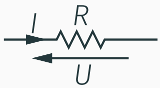


Resistors in the phasor domain



$$u(t) = U_m \cos(\omega t + \phi_u) = Ri(t) = \underbrace{RI_m}_{U_m} \cos(\omega t + \underbrace{\phi_i}_{\phi_u})$$

In the phasor domain:



$$U = U_m e^{\jmath\phi_u}, \quad I = I_m e^{\jmath\phi_i}.$$

Thus

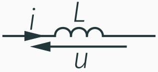
$$U = RI.$$

Linear algebraic devices in the phasor domain

Proposition

Devices that are described (in time domain) by linear algebraic equations (resistors, controlled sources, transformers, ...) are governed by "the same" equations in the phasor domain.

Inductors in the phasor domain



$$u(t) = U_m \cos(\omega t + \phi_u) = Li'(t) = L\big(I_m \cos(\omega t + \phi_i)\big)' =$$

$$= -L\omega I_m \sin(\omega t + \phi_i) = \underbrace{\omega L I_m}_{U_m} \cos\Big(\omega t + \underbrace{\phi_i + \frac{\pi}{2}}_{\phi_u}\Big).$$

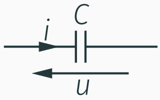
In terms of phasors:

$$U = U_m e^{\jmath \phi_u}, \quad I = I_m e^{\jmath \phi_i}.$$

Thus

$$U = \omega L I e^{\jmath \frac{\pi}{2}} = \jmath \omega L I.$$

Capacitors in the phasor domain



$$i(t) = I_m \cos(\omega t + \phi_i) = Cu'(t) = C\big(U_m \cos(\omega t + \phi_u)\big)' =$$
$$= -C\omega U_m \sin(\omega t + \phi_u) = \underbrace{\omega C U_m}_{I_m} \cos\Big(\omega t + \underbrace{\phi_u + \frac{\pi}{2}}_{\phi_i}\Big).$$

In the “phasor language”:

$$U = U_m e^{\jmath \phi_u}, \quad I = I_m e^{\jmath \phi_i}.$$

Thus

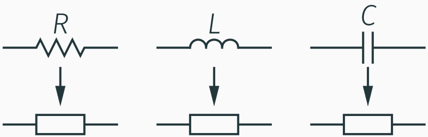
$$I = \omega C U e^{\jmath \frac{\pi}{2}} = \jmath \omega C U, \quad \text{in other words: } U = \frac{1}{\jmath \omega C} I.$$

Linear devices described by differential equations

Proposition

Devices that are described (in time domain) by linear differential equations (inductors, capacitors, coupled inductors, ...) are governed by linear algebraic equations in the phasor domain.

Linear homogeneous devices and their Impedance



$Z_R = R$ $\quad$ $Z_L = \jmath\omega L$ $\quad$ $Z_C = \frac{1}{\jmath\omega C}$

$$U = ZI$$

Impedance, Admittance, ...

$$\underbrace{Z}_{\text{impedance}} = \underbrace{R}_{\text{resistance}} + \jmath \underbrace{X}_{\text{reactance}} .$$

$$\frac{1}{Z} = \underbrace{Y}_{\text{admittance}} = \underbrace{G}_{\text{conductance}} + \jmath \underbrace{B}_{\text{susceptance}} .$$

device	$Z(\omega)$	$R(\omega)$	$X(\omega)$	$Y(\omega)$	$G(\omega)$	$B(\omega)$
resistor R	R	R	0	$\frac{1}{R}$	$\frac{1}{R}$	0
inductor L	$j\omega L$	0	ωL	$\frac{1}{j\omega L}$	0	$-\frac{1}{\omega L}$
capacitor C	$\frac{1}{j\omega C}$	0	$-\frac{1}{\omega C}$	$j\omega C$	0	ωC

Impedance, Admittance, ... – example

A series connection of an inductor L, capacitor C and a resistor R has impedance

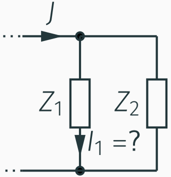
$$Z_{RLC} = R + \jmath\omega L + \frac{1}{\jmath\omega C} = R + \jmath\frac{\omega^2 LC - 1}{\omega C}.$$

AC analysis tools

- KCL, KVL,
- nodal method,
- superposition rule (for linear circuits only!),
- current and voltage dividers,
- Thévenin's and Norton's theorems,
- ...

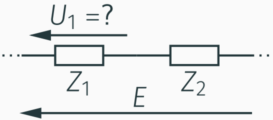
Current and voltage dividers

- Current divider



$$I_1 = J\frac{Z_2}{Z_1 + Z_2}$$

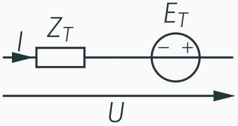
- Voltage divider



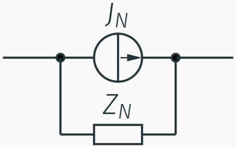
$$U_1 = E\frac{Z_1}{Z_1 + Z_2}$$

Thévenin's and Norton's theorems

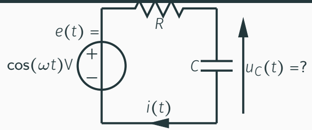
- Thévenin's equivalent



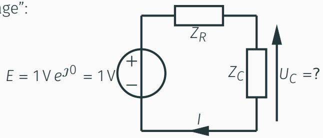
- Norton's equivalent



An example of AC analysis supported by phasor tools



reformulation of the problem in the "phasor language":



$$U_C = E\frac{Z_C}{Z_R + Z_C} = 1\,\mathrm{V}\frac{\frac{1}{\jmath\omega C}}{R + \frac{1}{\jmath\omega C}} = \frac{1\,\mathrm{V}}{1 + \jmath\omega RC}.$$

return to the time domain:

$$u_C(t) = |U_C|\cos(\omega t + \arg U_C)$$

$$|U_C| = \frac{1\,\mathrm{V}}{\sqrt{1 + (\omega RC)^2}}, \quad \arg U_C = 0 - \arg(1 + \jmath\omega RC) = -\arctan(\omega RC).$$

Some useful formulas

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$
$$|z_1 z_2| = |z_1||z_2|$$
$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$
$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$
$$\left|X_m e^{\jmath\phi}\right| = X_m \qquad \arg(X_m e^{\jmath\phi}) = \phi.$$
$$|a + \jmath b| = \sqrt{a^2 + b^2}.$$

$$\arg(a + \jmath b) = \begin{cases} \frac{\pi}{2} & \text{if } a = 0 \text{ and } b > 0, \\ -\frac{\pi}{2} & \text{if } a = 0 \text{ and } b < 0, \\ \arctan(\frac{b}{a}) & \text{if } a > 0, \\ \arctan(\frac{b}{a}) + \pi & \text{if } a < 0. \end{cases}$$