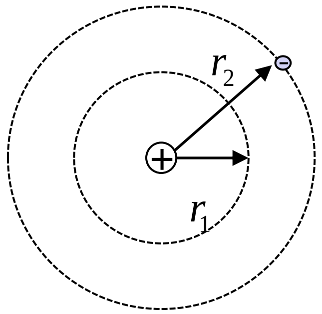
Problem 03: Hydrogen atom - Bohr's model

Derive the expressions and calculate the first three allowed energies of an electron in the hydrogen atom according to Bohr's atom model.



Bohr's atom model postulates:

- circular orbits of electrons around the nucleus, due to the balance of the Coulomb force and centripetal force:

$$\frac{Zq^2}{4\pi\varepsilon_0 r^2} = \frac{m_0 v^2}{r} \qquad (1)$$

- allowed when satisfying the condition:

$$L = mvr = n\frac{h}{2\pi} \qquad (2)$$

- the system is stable; a change of the orbit is associated with an emission/absorption of electromagnetic radiation of a frequency ν:

$$\nu = \frac{E_i - E_j}{h} \qquad (3)$$

Starting from substitution (from (2)):

$$m = n\frac{\hbar}{vr}$$

$$\hbar = \frac{h}{2\pi}$$

and employing it in (1), one obtains the orbital velocity:

$$v_n = \frac{q^2}{4\pi\varepsilon_0\hbar}\frac{Z}{n} \qquad (4)$$

For example, for the 1st orbit of a Hydrogen atom:

$$v_1 = \frac{q^2}{4\pi\varepsilon_0\hbar}\frac{Z}{n} \overset{Z=1,n=1}{=} \frac{(1.602\times10^{-19})^2}{4\cdot3.142\cdot8.854\times10^{-12}\cdot1.055\times10^{-34}} =$$

$$v_1 = \frac{(1.602)^2 \times 10^{-38}}{4 \cdot 3.142 \cdot 8.854 \cdot 1.055 \times 10^{-46}} = 0.02186 \times 10^8 = 2.186 \times 10^6 \left[\tfrac{m}{s}\right]$$

h = 6.626 x 10^{-34} [Js]

$\hbar = \frac{h}{2\pi} = 1.055x10^{-34}\ [Js]$

m_0 = 9.11 x 10^{-31} [kg]

q = 1.602 x 10^{-19} [C]

k_B = 1.381 x 10^{-23} [J/K]

ε_0 = 8.854 x 10^{-12} [F/m]

c = 3 x 10^8 [m/s]

$$v_n = \frac{q^2}{4\pi\varepsilon_0\hbar}\frac{Z}{n} = \frac{v_1}{n}$$

Compare the orbital velocity (in H) with the light speed. $v_1 << c = 3\times 10^8\ [m/s]$

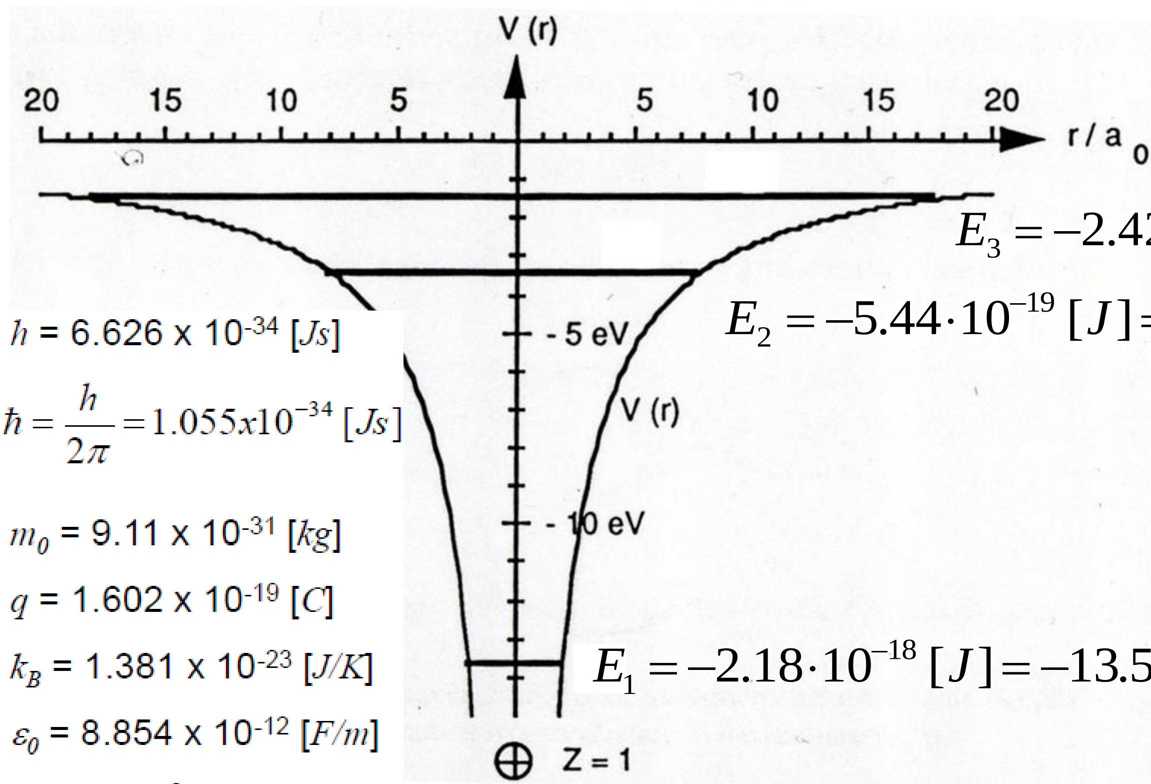
Having (4), and using (1) the orbital radii may be found: $$r_n = \frac{4\pi\varepsilon_0\hbar^2}{m_0 q^2}\frac{n^2}{Z} \overset{Z=1}{=} a_0 n^2 \qquad (5)$$

$$a_0 = \frac{4\pi \cdot 8.854 \cdot 10^{-12} \cdot (1.055 \cdot 10^{-34})^2}{9.11 \cdot 10^{-31} \cdot (1.602 \cdot 10^{-19})^2} = \frac{4\pi \cdot 8.854 \cdot (1.055)^2}{9.11 \cdot (1.602)^2} \cdot 10^{-11} = 0.53 \cdot 10^{-10}\ [m] = 0.53[\text{Å(angstrom)}]$$

The kinetic energy is given by: $$E_{kin} = \frac{m_0 v^2}{2} = \frac{m_0 q^4}{2(4\pi\varepsilon_0)^2 \hbar^2} \frac{1}{n^2} \qquad (6)$$

The potential energy: $$E_{pot} = \int_r^\infty F(r)dr = -\frac{q^2}{4\pi\varepsilon_0 r_n} = -\frac{m_0 q^4}{(4\pi\varepsilon_0)^2 \hbar^2} \frac{1}{n^2} \qquad (7)$$

And the total energy: $$E_n = E_{kin} + E_{pot} = -\frac{m_0 q^4}{2(4\pi\varepsilon_0)^2 \hbar^2} \frac{1}{n^2} = E_1 \frac{1}{n^2} \qquad (8)$$



$$E_3 = -2.42 \cdot 10^{-19} [J] = -1.51 [eV]$$

$$E_2 = -5.44 \cdot 10^{-19} [J] = -3.40 [eV]$$

$$\boxed{E[eV] = \frac{E[J]}{q}}$$

h = 6.626 x 10^{-34} [Js]

$$\hbar = \frac{h}{2\pi} = 1.055 x 10^{-34} [Js]$$

m_0 = 9.11 x 10^{-31} [kg]

q = 1.602 x 10^{-19} [C]

k_B = 1.381 x 10^{-23} [J/K]

ε_0 = 8.854 x 10^{-12} [F/m]

c = 3 x 10^8 [m/s]

$$E_1 = -2.18 \cdot 10^{-18} [J] = -13.59 [eV]$$

$$E_n = E_{kin} + E_{pot} = -\frac{m_0 q^4}{2(4\pi\varepsilon_0)^2 \hbar^2}\frac{1}{n^2} = E_1\frac{1}{n^2} \qquad E[eV] = \frac{E[J]}{q}$$

? Starting from which orbit, the abs. value of the total energy, **|E_n|**, becomes smaller than **0.1 eV**?

$$|E_n| < 0.1 \qquad |E_1|\frac{1}{n^2} < 0.1 \qquad \frac{1}{n^2} < \frac{0.1}{|E_1|} \qquad n^2 > \frac{0.1}{|E_1|}$$

$$n > \sqrt{\frac{0.1}{13.59}} \qquad n > 11.66, so\ n > 12$$

$\hbar$ = 6.626 x 10^{-34} [Js]

$$\hbar = \frac{h}{2\pi} = 1.055x10^{-34}\,[Js]$$

m_0 = 9.11 x 10^{-31} [kg]

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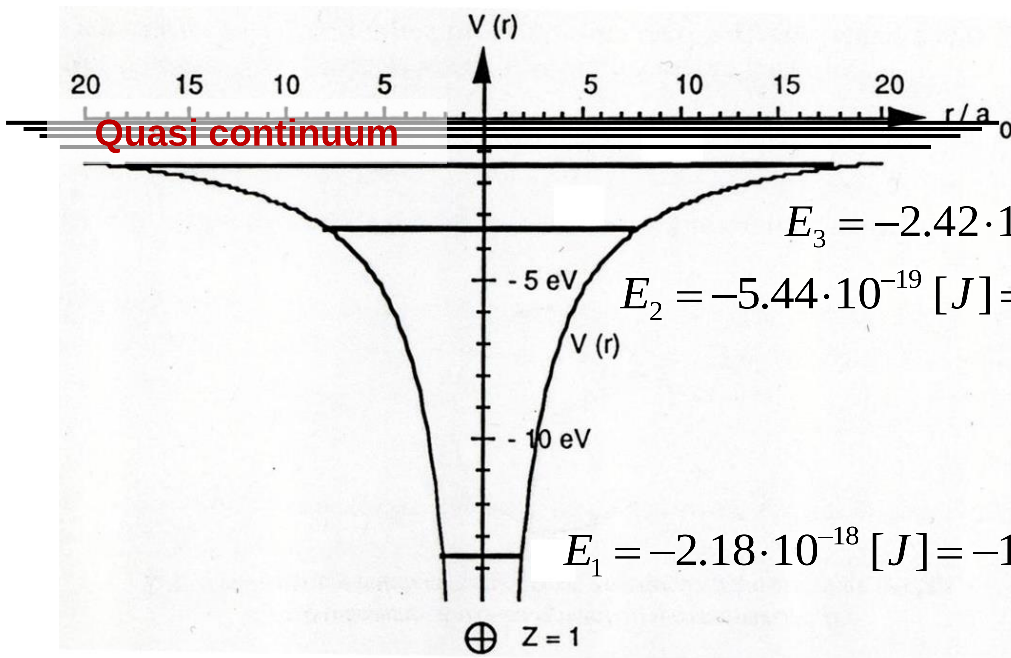


Free electron

$$E_3 = -2.42\cdot10^{-19}\,[J] = -1.51[eV]$$

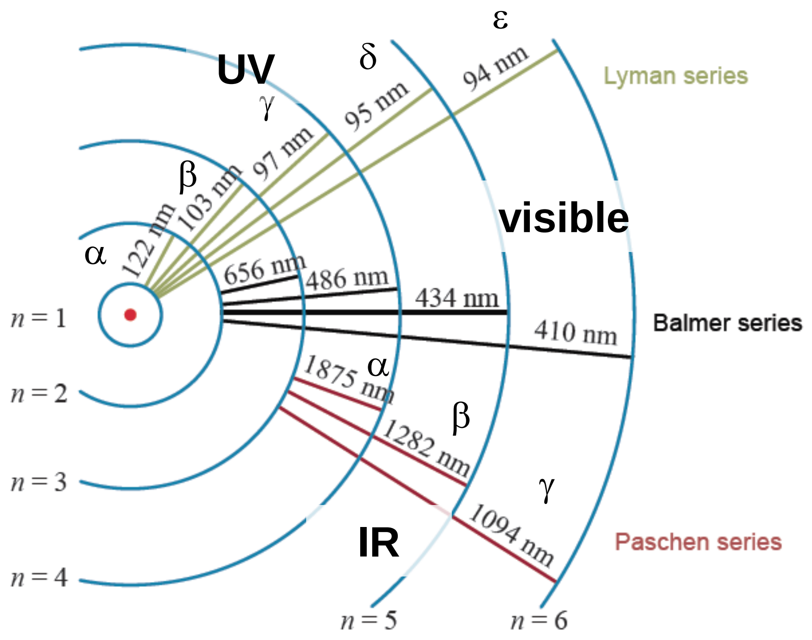
$$E_2 = -5.44\cdot10^{-19}\,[J] = -3.40[eV]$$

$$E_1 = -2.18\cdot10^{-18}\,[J] = -13.59[eV]$$

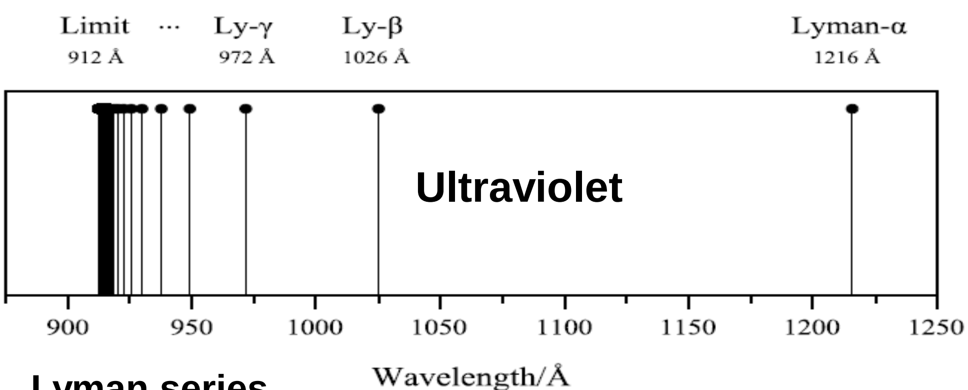
H atom spectral lines

$$\lambda_{n\to m} = \frac{hc}{\Delta E_{n\to m}} \qquad E_n = E_1 \frac{1}{n^2}$$

$$\Delta E_{n\to m} = E_m - E_n = E_1\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$



Source: WIKI; not to scale



Lyman series

Visible light

Balmer series