Circuits and Signals

Transient states. Analog filters

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Transient states

Definition

A solution to a given circuit is called a transient state if it is not a steady state (of periodic current).

May be caused by:

- non-stationarity of excitations, e.g. $e(t) = \mathbb{1}(t)$,
- particular choice of initial conditions imposed to the circuit

Switching law

Switching law

Voltages across capacitors and currents through inductors are continuous functions of time.

Extra assumption 1

We restrict to circuits for which there exist time instants

$$t_0 < t_1 < t_2 < \cdots < t_N$$

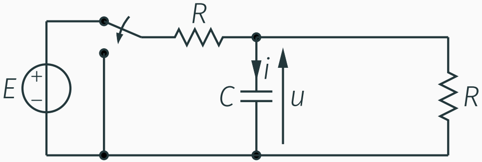
such that for each interval

$$(t_k, t_{k+1}), \quad k = 0, 1, \ldots, N-1$$

the circuit consists of resistors, capacitors, inductors and sources and it (the circuit) admits unique DC solution for each of the intervals (solutions can be different for each interval).

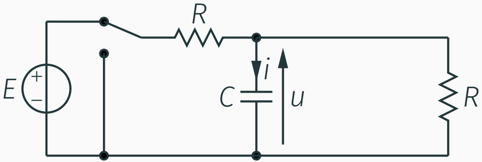
Remark: The concatenation of the DC solutions does rarely form a valid solution to the original problem (for $t \in [t_0, t_N]$) because it usually contradicts the switching law.

Example



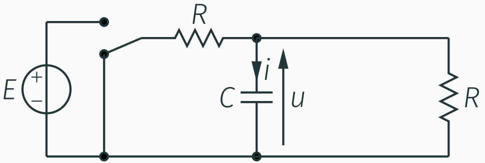
Before $t_0 = 0\,\mathrm{s}$ there was a (DC) steady state. Then (at t_0) the key position is changed. How do the waveforms $u(t)$ and $i(t)$ look like?

Example — DC state before the commutation ($t < t_0$)



$$u(t) = \frac{E}{2},$$
$$i(t) = 0\,\mathrm{A}.$$

Example — after the commutation $t > t_0$



$$i = Cu', \qquad u = -\frac{i}{2}R, \qquad \text{hence} \qquad i = Cu', \qquad \frac{1}{2}RCi' + i = 0.$$

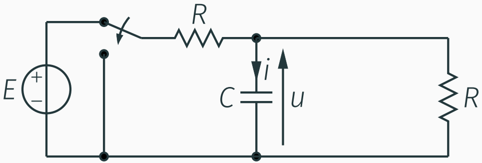
Thus

$$i(t) = \alpha e^{-\frac{2t}{RC}}, \qquad u(t) = -\frac{i}{2}R = -\frac{\alpha R}{2}e^{-\frac{2t}{RC}}.$$

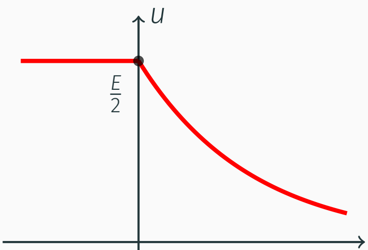
Due to the commutation law: $\alpha = -\frac{E}{R}$. Eventually

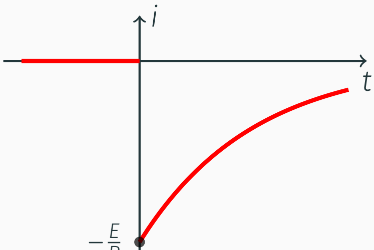
$$u(t) = \begin{cases} \frac{E}{2} & \text{for } t \leq 0\,\text{s}, \\ \frac{E}{2}e^{-\frac{2t}{RC}} & \text{for } t > 0\,\text{s} \end{cases} \quad \text{and} \quad i(t) = \begin{cases} 0\,\text{A} & \text{for } t < 0\,\text{s}, \\ -\frac{E}{R}e^{-\frac{2t}{RC}} & \text{for } t > 0\,\text{s}. \end{cases}$$

Example — summary



$$u(t) = \begin{cases} \frac{E}{2} & \text{for } t \leq 0\,\text{s}, \\ \frac{E}{2} e^{-\frac{2t}{RC}} & \text{for } t > 0\,\text{s} \end{cases} \quad \text{and} \quad i(t) = \begin{cases} 0\,\text{A} & \text{for } t < 0\,\text{s}, \\ -\frac{E}{R} e^{-\frac{2t}{RC}} & \text{for } t > 0\,\text{s}. \end{cases}$$





Simplified analysis of 1st order circuits

Extra assumption 2:

$$t_0 < t_1 < t_2 < \cdots < t_N,$$

such that in each interval

$$(t_k, t_{k+1}), \quad k = 0, 1 \ldots, N-1$$

the circuit is described by a set of equations among which there is only one ordinary differential equation and it is either the capacitor's or the inductor's equation:

$$i = Cu', \quad \text{or} \quad u = Li'.$$

Simplified analysis — formula

In a circuit satisfying assumptions 1 and 2, every signal (voltage or a current) for $t \in (t_k, t_{k+1})$ can be written as

$$x(t) = x_{k,\infty} + \left(x(t_k^+) - x_{k,\infty}\right) e^{-\frac{t-t_k}{\tau_k}} \qquad t \in (t_k, t_{k+1}),$$

Simplified analysis — $x_{k,\infty}$ calculation

$$x(t) = x_{k,\infty} + \left(x(t_k^+) - x_{k,\infty}\right) e^{-\frac{t-t_k}{\tau_k}} \qquad t \in (t_k, t_{k+1}),$$

$x_{k,\infty}$ can be obtained by means of DC analysis applied to the circuit for $t \in (t_k,\, t_{k+1})$.

Simplified analysis — τ_k calculation

$$x(t) = x_{k,\infty} + \left(x(t_k^+) - x_{k,\infty}\right) e^{-\frac{t-t_k}{\tau_k}} \qquad t \in (t_k, t_{k+1}),$$

$$\tau_k = R_k C \qquad \text{or} \qquad \tau_k = \frac{L}{R_k},$$

where R_k is the internal resistance of the one-port attached to the capacitor C or the inductor L for $t \in (t_k, t_{k+1})$.

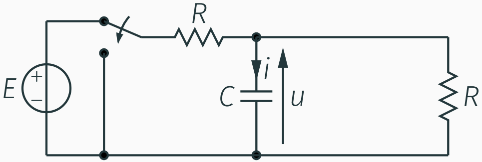
Simplified analysis — $x(t_k^+)$ calculation

$$x(t) = x_{k,\infty} + \left(x(t_k^+) - x_{k,\infty}\right) e^{-\frac{t-t_k}{\tau_k}} \qquad t \in (t_k, t_{k+1}),$$

$x(t_k^+)$ can be obtained by means of DC analysis applied to the circuit that results from the original one considered for $t \in (t_k, t_{k+1})$, by replacing the capacitor C with the voltage source of the electromotive force equal to $u_C(t_k^-)$ or by replacing the inductor L with the current source of the current $i_L(t_k^-)$.

If there was a DC steady state before t_k then $u_C(t_k^-)$ and $i_L(t_k^-)$ can be obtained by means of DC analysis (for $t < t_k$).

Example

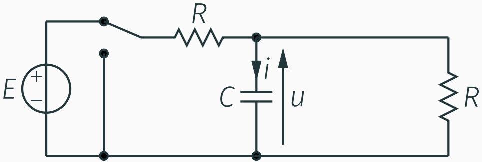


Before $t_0 = 0\,\mathrm{s}$ there was a steady state in the circuit. Determine the waveforms $u(t)$ and $i(t)$.

$$u(t) = u_\infty + \left(u(0^+) - u_\infty\right) e^{-\frac{t-0}{\tau}},$$

$$i(t) = i_\infty + \left(i(0^+) - i_\infty\right) e^{-\frac{t-0}{\tau}},$$

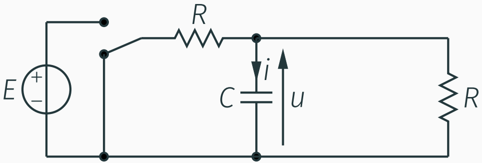
Example — steady state before t_0



$$U = \frac{E}{2},$$

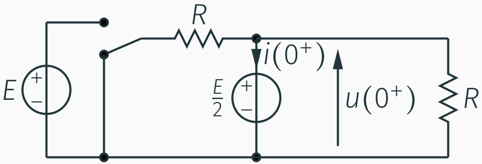
$$I = 0\,\mathrm{A}.$$

Example — u_∞



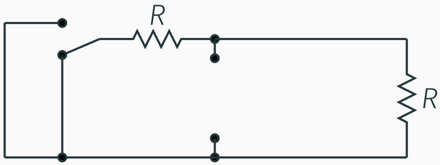
$$u_\infty = 0\,\mathrm{V},$$
$$i_\infty = 0\,\mathrm{A}.$$

Example — $u_C(0^+)$



$$u(0^+) = \frac{E}{2},$$
$$i(0^+) = -\frac{E}{2}\Big/\frac{R}{2} = -\frac{E}{R}.$$

Example — time constant τ

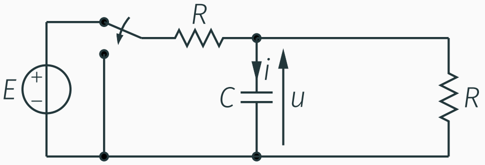


$$\tau = R_W C$$

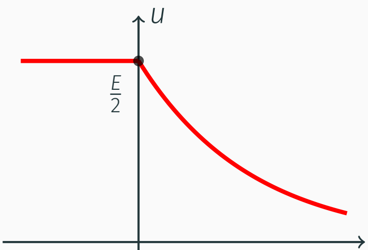
$$R_W = \frac{R}{2}$$

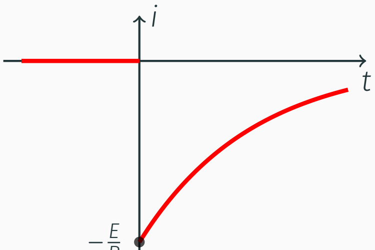
$$\tau = \frac{RC}{2}$$

Example — summary

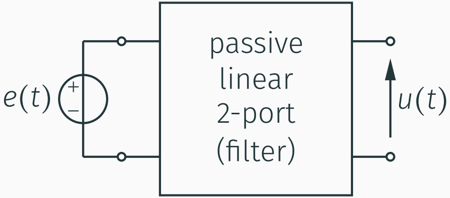


$$u(t) = \begin{cases} \frac{E}{2} & \text{for } t \le 0\,\text{s}, \\ \frac{E}{2} e^{-\frac{2t}{RC}} & \text{for } t > 0\,\text{s} \end{cases} \quad \text{and} \quad i(t) = \begin{cases} 0\,\text{A} & \text{for } t < 0\,\text{s}, \\ -\frac{E}{R} e^{-\frac{2t}{RC}} & \text{for } t > 0\,\text{s}. \end{cases}$$

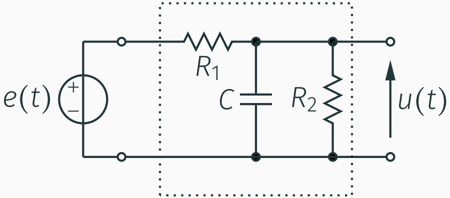




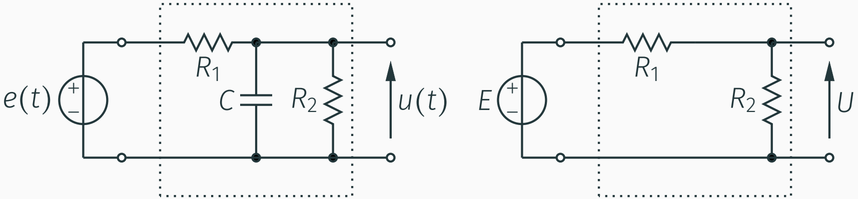
Analog filters — introduction



e.g.



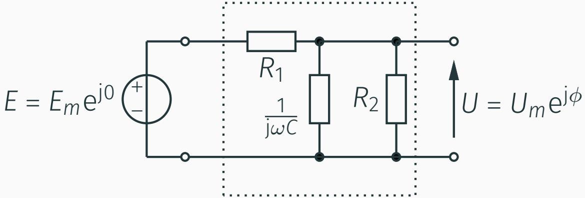
Analog filters — DC gain



$$U = H_0 E,$$

$$H_0 = \frac{R_2}{R_1 + R_2}.$$

Analog filters — transfer function



$$e(t) = E_m \cos \omega t, \qquad u(t) = U_m \cos(\omega t + \phi).$$

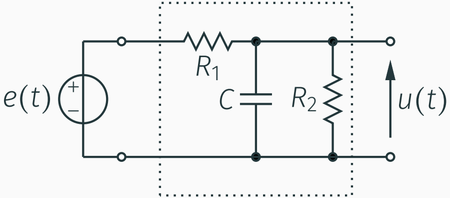
$$U = H_\omega E$$

$$U = \frac{\frac{1}{R_1}}{j\omega C + \frac{1}{R_1} + \frac{1}{R_2}} E = \frac{R_2}{R_1 + R_2 + j\omega R_1 R_2 C} E,$$

$$H_\omega = \frac{R_2}{R_1 + R_2 + j\omega R_1 R_2 C}.$$

H_ω is called transfer function of the filter.

Analog filters — transfer function

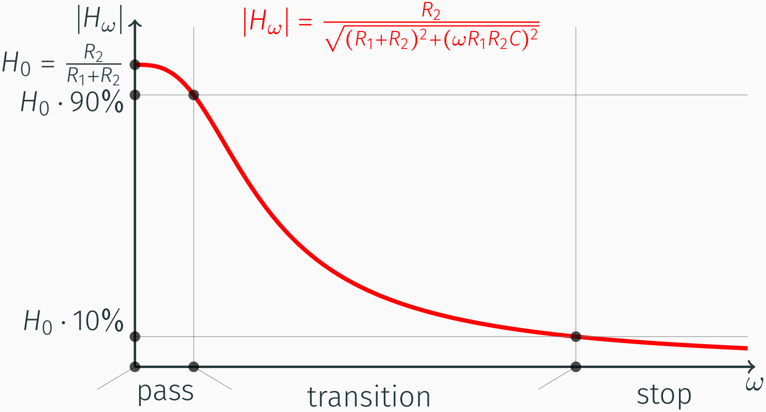


$$e(t) = E_0 + \sum_{k=1}^{\infty} E_{km} \cos(k\omega t + \varphi_{e_k}),$$

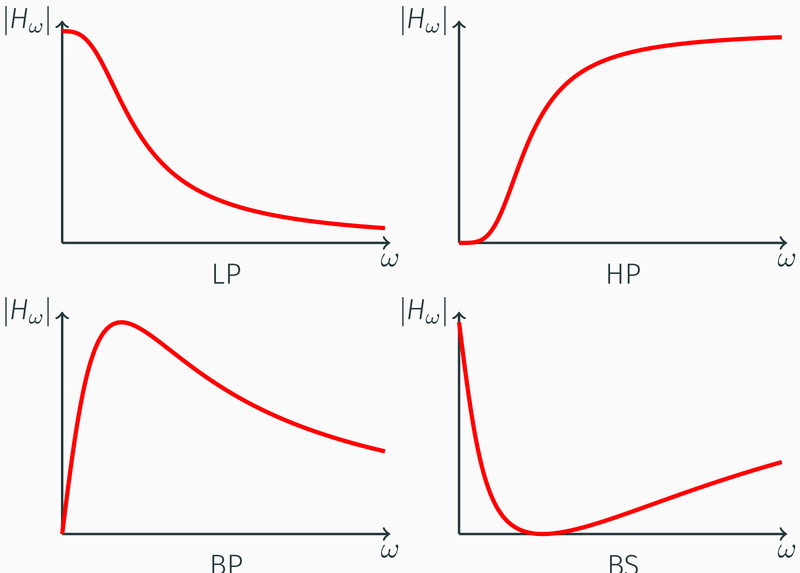
$$u(t) = \underbrace{H_0 E_0}_{U_0} + \sum_{k=1}^{\infty} \underbrace{|H_{k\omega}| E_{km}}_{U_{km}} \cos(k\omega t + \underbrace{\varphi_{e_k} + \arg H_{k\omega}}_{\varphi_{u_k}}).$$

Analog filters — gain

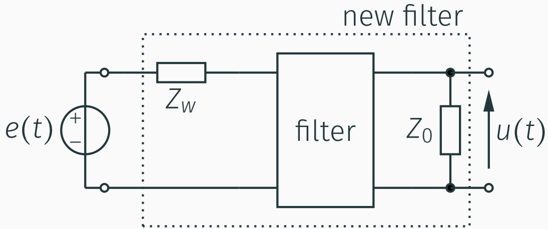
$$H_\omega = \frac{R_2}{R_1 + R_2 + j\omega R_1 R_2 C}.$$



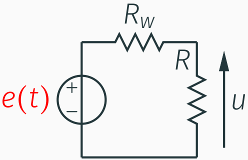
Types of analog filters

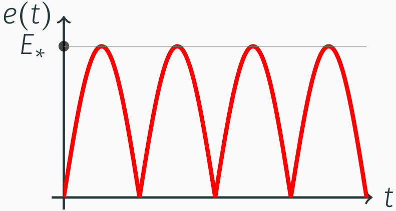


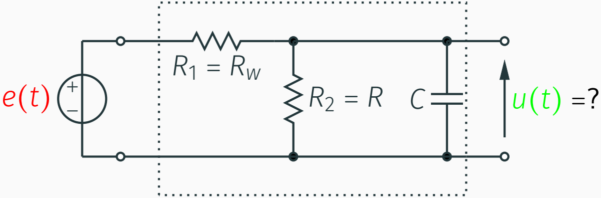
Analog filters — operating state



Analog filtration — example



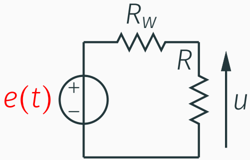


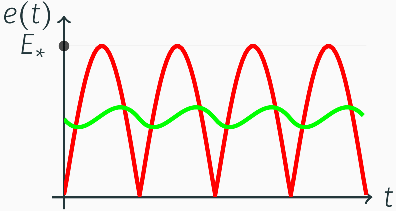


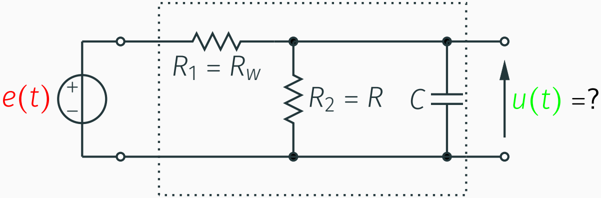
$$e(t) = E_0 + \sum_{k=1}^{\infty} E_{km} \cos(k\omega t + \phi_k),$$

$$u(t) = \underbrace{H_0 E_0}_{U_0} + \sum_{k=1}^{\infty} \underbrace{|H_{k\omega}| E_{km}}_{U_{km}} \cos(k\omega t + \underbrace{\varphi_{e_k} + \arg H_{k\omega}}_{\varphi_{u_k}}).$$

Analog filtration — example







$$e(t) = E_0 + \sum_{k=1}^{\infty} E_{km} \cos(k\omega t + \phi_k),$$

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