Probability and Statistics (EPRST)

Lecture 2

Probability of an event - the impact of available information

Example

A person in the next room rolls a regular die.

1. *What is the probability that 4 showed up on the die?*
2. *The person reports that an even number showed up. What is the probability the number is 4?*
3. *Now the person reports that an odd number showed up. What is now the probability the number is 4?*

Conditional probability - definition

Definition

Let A, B - random events. Assume that $\mathbb{P}(B) \neq 0$. ***The conditional probability of*** *A* ***given*** *B (denoted: $\mathbb{P}(A|B)$) is*

$$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Let's get back to the first example. We have

$$\Omega = \{1, 2, 3, 4, 5, 6\}, \ \mathbb{P}(\{i\}) = \frac{1}{6}, \ i = 1, \dots, 6.$$

We consider events:

- $A = \{4\}$,
- $B = \{2, 4, 6\}$,
- $C = \{1, 3, 5\}$.

Then:

- $\mathbb{P}(A) = 1/6$ (**unconditional probability**)
- $\mathbb{P}(B) = 1/2$, $A \cap B = \{4\}$, so $\mathbb{P}(A \cap B) = 1/6$ and

$$\mathbb{P}\left(A \middle| B\right) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{1/6}{1/2} = \frac{1}{3},$$

- $\mathbb{P}(C) = 1/2$, $A \cap C = \emptyset$, so $\mathbb{P}(A \cap C) = 0$ and

$$\mathbb{P}\left(A \middle| C\right) = \frac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)} = \frac{0}{1/2} = 0.$$

Conditional probability - another example

Example

We flip two coins. Consider random events:

- A *- „heads on the 1st coin”,*
- B *- „heads on the 2nd coin”.*

Compute $\mathbb{P}(A)$ *and* $\mathbb{P}(A|B)$*.*

If

$$\mathbb{P}(A|B) = \mathbb{P}(A),$$

for some random events A, B, such that $\mathbb{P}(B) \neq 0$, we interpret

the occurrence of event B had no effect on the occurrence of event A,

or, more precisely,

knowledge of the occurrence of event B does not affect the assessment of the odds for event A

Observe that, if A does not depend on B, then also B does not depend on A, meaning that

$$\mathbb{P}(A|B) = \mathbb{P}(A) \Rightarrow \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) \Rightarrow \mathbb{P}(B|A) = \mathbb{P}(B).$$

Independence of two events

Definition

Let A and B be some random events. We say that A and B are ***independent***, *if*

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

If this equality does not hold, we call A and B ***dependent***.

Remark

If A and B are independent, then the following events are also independent

- A *and* B',
- A' *and* B,
- A' *and* B'.

Independence of many events

Definition

Random events $A_1, \ldots, A_n$ are ***independent****, if for any $k = 2, \ldots, n$ and for arbitrary $i_1, \ldots, i_k \in \{1, \ldots, n\}$,*

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = \mathbb{P}(A_{i_1}) \cdot \mathbb{P}(A_{i_2}) \cdot \ldots \cdot \mathbb{P}(A_{i_k}).$$

Independence of many events - caution

It is not difficult to find examples of random events A_1, A_2, A_3 such that:

- $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j)$ for all i, j, but

  $$\mathbb{P}(A_1 \cap A_2 \cap A_3) \neq \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3);$$

  (**pairwise independence** does not imply independence - learning about A_1 or learning about A_2 is of no use in predicting whether A_3 occurred, but learning that both A_1 and A_2 occurred could be highly relevant for A_3);
- $\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3)$ but

  $$\mathbb{P}(A_i \cap A_j) \neq \mathbb{P}(A_i)\mathbb{P}(A_j)$$

  for some i, j.

It is also not difficult to find analogous examples for an arbitrary number of events (not necessarily three).

Independent repetitions of the same random experiment

An experiment that can result in either a *success* or a *failure* (but not both) is called a **Bernoulli trial**. We perform **independently** a number of Bernoulli trials, each with the same **success probability** p (in a single trial). (So **failure probability** in any single trial is $1 - p$.) What can we compute?

- the probability of exactly k successes in n trials (**Bernoulli scheme**),
- the probability of getting the first success in the k-th trial.

Independent repetitions of the same random experiment - examples

Example

We roll an ordinary die 10 *times. What is the probability that even numbers will be rolled exactly* 5 *times?*

Example

We keep rolling a symmetric die until the first 4 shows up. What is the probability that we will make three rolls?

Law of total probability - an example

Example

There are 9 symmetric coins and one that always shows heads. We pick one coin randomly and toss it five times. What is the probability of getting exactly 5 heads this way?

Law of total probability - formulation

Theorem

Let $A_1, \dots, A_n$ be a partition of the sample space Ω, that is:

- $\bigcup_n A_n = \Omega$,
- $A_i \cap A_j = \emptyset$ *for* $i \neq j$ *(pairwise disjoint events),*
- $\mathbb{P}(A_n) > 0$ *for all* n.

Then for any random event $A \subset \Omega$

$$\mathbb{P}(A) = \sum_n \mathbb{P}\left(A \middle| A_n\right) \mathbb{P}(A_n).$$

Bayes' rule

Theorem

Under the assumptions of the law of total probability, and if additionally $\mathbb{P}(A) > 0$*, then for* $k = 1, 2, \ldots$

$$\mathbb{P}\left(A_k \middle| A\right) = \frac{\mathbb{P}\left(A \middle| A_k\right)\mathbb{P}(A_k)}{\sum_n \mathbb{P}\left(A \middle| A_n\right)\mathbb{P}(A_n)}.$$

Example

There are 9 symmetric coins and one that always shows heads. We picked one coin randomly and tossed it five times. We have got exactly 5 heads this way. What is the probability that the we had picked the non-symmetric coin?