EPRST: Probability and Statistics
Problem set 09

1. The joint density of the vector (X, Y) is of the form

$$f(x, y) = \begin{cases} |x|, & -1 < x < y < 1; \\ 0, & \text{otherwise.} \end{cases}$$

Compute $\mathrm{Cov}(X, Y)$. Are X and Y independent?

2. Random vector (X, Y) has the joint PMF

$$p_{X,Y}(x, y) = \begin{cases} c(x^2 + y^2), & \textit{if } x \in \{1, 2, 4\} \textit{ and } y \in \{1, 3\}, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) What is the value of the constant c?
   (b) Find $\mathrm{Cov}(X, Y)$ and $\mathrm{Var}(X + Y)$.

3. On the probability space $(\Omega, \mathbb{P})$, where $\Omega = \{0, 1, \ldots, 9\}$, $\mathbb{P}(\{\omega\}) = 0.1 \ \forall\, \omega \in \Omega$, we define random variables $X(\omega)$ – the remainder from the division of ω by 2, $Y(\omega)$ – the remainder of the division of ω by 3. Compute $\mathbb{E}(XY)$, $\mathbb{E}[\cos \pi(X + Y)]$ and $\mathrm{Cov}(X^2, Y^2)$.

4. We toss a symmetric coin four times – X denotes the number of tails on the first three tosses, and Y – the number of tails on the last three tosses. Find the covariance of (X, Y) and the correlation coefficient $\rho_{(X,Y)} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}}$.

5. Let X and Y be the outcomes when a pair of fair six-sided dice is rolled.
   (a) Compute the covariance of $X + Y$ and $X - Y$.
   (b) Are $X + Y$ and $X - Y$ independent?

6. Random variable (X, Y) is uniformly distributed on $D = \{(x, y) : |x| + |y| \leq 1\}$.
   (a) Compute $\mathbb{E}(X + Y)$.
   (b) Compute $\mathbb{E}(\sqrt{X^2 + Y^2})$.
   (c) Compute $\mathrm{Cov}(X, Y)$.

7. Random vector (X, Y) has a continuous distribution with the density

$$f(x, y) = \begin{cases} -x/8, & x \in [-2, 0],\ y \in [0, 2], \\ y/8, & x \in (0, 2],\ y \in [0, 2], \\ 0, & \text{otherwise} \end{cases}$$

Compute $\mathrm{Var}(X + 2Y)$.

8. Compute $\mathbb{E}(2Y + 1)$, if a random vector (X, Y) has continuous distribution with the joint pdf :

$$f(x, y) = 2x^2 y \cdot \mathbb{1}_{(-1,0)\times(0,1)}(x, y) + 4xy^2 \cdot \mathbb{1}_{(0,1)\times(0,1)}(x, y),$$

where the function $\mathbb{1}_A(\cdot)$, for any set A, is defined as

$$\mathbb{1}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

9. Let X and Y be independent random variables such that $\mathbb{E}X^2 < \infty$ and $\mathbb{E}Y^2 < \infty$. Check, if the following equalities/inequalities are true?
   (a) $\mathrm{Cov}(X, Y + 1) = 0$,
   (b) $\mathrm{Var}(2X + Y) = 4\,(\mathrm{Var}(X) + \mathrm{Var}(Y))$,
   (c) $\mathbb{P}(\max(X, Y) \leq 0) = F_X(0)F_Y(0)$,
   (d) $\rho(X - 1, Y + 1) > \rho(X, Y)$ ($\rho(X, Y)$ is a correlation coefficient of random variables X and Y).

10. A symmetric coin is tossed three times. Detrmine the covariance matrix of the random vector (X, Y), if

   (a) X denotes the number of heads in the first toss, Y - the number of heads in the second toss,

   (b) X denotes the number of heads in the first two tosses, Y - the number of heads in the last two tosses,

   (c) X denotes the number of heads in three tosses, Y - the number of tails in three tosses.

11. Discrete random vector $\mathbf{X} = (X, Y)$ has a uniform distribution on the set $\{(0,0), (\sqrt{2}/2, \sqrt{2}/2), (0, \sqrt{2}), (-\sqrt{2}/2, \sqrt{2}/2)\}$.

   (a) Determine the covariance matrix and the correlation coefficient for (X, Y).

   (b) Determine the covariance matrix and the correlation coefficient for $\mathbf{AX}$, if

$$\mathbf{A} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}.$$