Homework 1 (Separation of variables)**.** Solve the following differential equations.

1) $e^y(1+x^2)y' - x(1+e^y) = 0$, ANSWER: $y = \ln\left(C\sqrt{1+x^2} - 1\right)$.
2) $y' = y - y^2$, ANSWER: $y = \dfrac{Ce^x}{Ce^x + 1}$, $y = 1$.
3) $y'(x^2+1) = x(y-2)$, ANSWER: $y = 2 + C\sqrt{x^2+1}$.
4) $e^{-y}(1+y') = 1$, ANSWER: $\ln|e^y - 1| - y - x = C$, $y = 0$.
5) $\sqrt{x}y' = 1 + y^2$, ANSWER: $y = \tan(2\sqrt{x} + C)$.

Homework 2 (Equations of the form $y' = f(y/x)$)**.** Solve the following differential equations.

1) $(y-x)y' = -y$, ANSWER: $\frac{x}{y} + \ln|y| = C$, $y = x$, $y = 0$.
2) $(3x^2 - y^2)y' - 2xy = 0$, ANSWER: $Cy^3 + x^2 - y^2 = 0$, $y = 0$.
3) $x^2 + y^2 = 2xy \cdot y'$, ANSWER: $y^2 - x^2 - Cx = 0$.
4) $xye^{\frac{x}{y}} + y^2 - x^2e^{\frac{x}{y}}y' = 0$, ANSWER: $e^{\frac{x}{y}} - \ln|x| = C$.
5) $8y + 10x + (5y + 7x)y' = 0$, ANSWER: $(y+x)^2(y+2x)^3 = C$.
6) $2\sqrt{xy} - y + xy' = 0$, ANSWER: $y = x\left(-\ln x + C\right)^2$ for $x > 0$, $y = x\left(\ln(-x) + C\right)^2$ for $x < 0$.

Homework 3 (First order linear non-homogeneous equations)**.** Find all solutions of the following differential equations.

1) $y' - 2xy = x - x^3$, ANSWER: $Ce^{x^2} + \dfrac{1}{2}x^2$.
2) $xy' + y = x\sin x$, ANSWER: $y = -\cos x + \dfrac{\sin x + C}{x}$.
3) $(1+x^2)y' + y = \arctan x$, ANSWER: $y = Ce^{-\arctan x} + \arctan x - 1$.
4) $\cos x \cdot y' + y\sin x = 1$, ANSWER: $y = \sin x + C\cos x$.
5) $y' - \dfrac{2y}{x+1} = (x+1)^3$, ANSWER: $y = (x+1)^2\left[\dfrac{1}{2}(x+1)^2 + C\right]$.

Homework 4 (Bernoulli equations)**.** Find all solutions of the following differential equations.

1) $y' + \dfrac{xy}{1-x^2} = x\sqrt{y}$, ANSWER: $y = C\sqrt[4]{1-x^2} - \frac{1}{3}\left(1-x^2\right)$.
2) $y' + 2xy = 2x^3y^3$, ANSWER: $y^{-2} = Ce^{2x^2} + x^2 + \dfrac{1}{2}$.
3) $\dfrac{y'}{\sqrt{y}} + 4\sqrt{y}x = 2xe^{-x^2}$, ANSWER: $y = \dfrac{(C+x^2)^2}{4e^{2x^2}}$.
4) $y' = \frac{1}{3}y\sin x - y^4\sin x$. ANSWER: $y = \dfrac{1}{\sqrt[3]{3 + Ce^{\cos x}}}$.

Homework 5 (Exact equations)**.** Solve the following differential equations.

1) $4x^3 + 6xy^3 + (9x^2y^2 + 3)y' = 0$, ANSWER: $x^4 + 3x^2y^3 + 3y = A$.

2) $e^x(1+e^y)+e^y(1+e^x)y'=0$, ANSWER: $e^x+e^y+e^{x+y}=A$.
3) $\dfrac{x-y}{x^2+y^2}+\dfrac{x+y}{x^2+y^2}y'=0$, ANSWER: $\arctan\dfrac{y}{x}+\ln\sqrt{x^2+y^2}=A$.

Homework 6 (Non-exact equations)**.** Find the integrating factor and solve the following differential equations.

1) $y\left(2+xy^2\right)+x\left(1+xy^2\right)y'=0$, ANSWER: $\mu(x)=x,\ \ x^2(3y+xy^3)=A$.
2) $x^2-y+xy'=0$, ANSWER: $\mu(x)=\dfrac{1}{x^2},\ \ x+\dfrac{y}{x}=A$.
3) $1+3x^2\sin y-\dfrac{x}{\tan y}y'=0$, ANSWER: $\mu(y)=\dfrac{1}{\sin y},\quad \dfrac{x}{\sin y}+x^3=A$.
4) $y^2+(xy-1)y'=0$, ANSWER: $\mu(y)=\dfrac{1}{y},\quad xy-\ln y=A$.

Homework 7 (Second order linear homogeneous equations with constant coefficients)**.** Find all solutions of the following differential equations.

1) $y''+y'+y=0$, ANSWER: $y=e^{-\frac{1}{2}x}\left(C_1\cos\left(\frac{\sqrt{3}}{2}x\right)+C_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right)$.
2) $y''+2y'+5y=0$, ANSWER: $y=e^{-x}\left(C_1\cos(2x)+C_2\sin(2x)\right)$.
3) $y''+6y'+9y=0$, ANSWER: $y=C_1e^{-3x}+C_2xe^{-3x}$.
4) $y''+4y'+3y=0$, ANSWER: $y=C_1e^{-3x}+C_2e^{-x}$.
5) $y''+6y'+10y=0$, ANSWER: $y=e^{-3x}\left(C_1\cos x+C_2\sin x\right)$.

Homework 8 (Second order linear non-homogeneous equations - part 1)**.** Find all solutions of the following differential equations.

1) $y''-7y'+12y=x$, ANSWER: $y=C_1e^{3x}+C_2e^{4x}+\dfrac{12x+7}{144}$.
2) $y''-y=2\sin x$ with the initial conditions $y(0)=0$, $y'(0)=1$, ANSWER: $y=e^x-e^{-x}-\sin x$.
3) $y''-2y=4x^2e^{2x}$, ANSWER: $y=C_1e^{\sqrt{2}x}+C_2e^{-\sqrt{2}x}+e^{x^2}$.
4) $y''-3y'+2y=x^3+\sin x$, ANSWER: $y=C_1e^x+C_2e^{2x}+\dfrac{1}{10}\left(3\cos x+\sin x\right)+\dfrac{1}{2}x^3+\dfrac{9}{4}x^2+\dfrac{21}{4}x+\dfrac{45}{8}$.
5) $y''+2y'+x=8e^x$ with the initial conditions $y(0)=1$, $y'(0)=0$, ANSWER: $y=-3xe^{-x}-e^{-x}+2e^x$.
6) $y''+6y'+10y=1-x$, ANSWER: $y=e^{-3x}\left(C_1\cos x+C_2\sin x\right)-\dfrac{1}{10}x+\dfrac{4}{25}$.
7) $y''+4y=\dfrac{1}{\cos 2x}$, ANSWER: $y=C_1\cos(2x)+C_2\sin(2x)+\frac{1}{4}\cos(2x)\ln\left(\cos(2x)\right)+\frac{1}{2}x\sin(2x)$.

Homework 9 (Second order linear non-homogeneous equations - part 2)**.** Find the general solution of the following differential equations.

1) $y''+3y'+2y=e^{2x}+3$, ANSWER: $y(x)=C_1e^{-2x}+C_2e^{-x}+\dfrac{e^{2x}}{12}+\dfrac{3}{2}$.

2) $y'' + 9y = \tan 3x$, ANSWER: $y = C_1 \cos 3x + C_2 \sin 3x + \frac{1}{18} \ln \left| \frac{1 - \sin 3x}{1 + \sin 3x} \right|$,

3) $y'' - y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, ANSWER: $y = C_1 e^x + C_2 e^{-x} + (e^x + e^{-x}) \arctan e^x$,

4) $2y'' - y' - y = 3\cos 2x - \sin 2x$, ANSWER: $y = -\frac{29}{85} \cos 2x + \frac{3}{85} \sin 2x + C_1 e^x + C_2 e^{\frac{-x}{2}}$,

5) $y'' - y' + 12y = 3x - \sin 2x$, ANSWER: $y = -\frac{1}{4}x + \frac{1}{48} - \frac{1}{130} \cos 2x + \frac{4}{65} \sin 2x + C_1 e^{4x} + C_2 e^{-3x}$.

Homework 10 (Systems of linear differential equations)**.** Solve the following systems differential equations.

1)

$$\begin{cases} x'(t) = x(t) + y(t), \\ y'(t) = x(t) + y(t) + t, \end{cases}$$

ANSWER:

$$\begin{cases} x(t) = C_1 + C_2 e^{2x} - \frac{1}{4}x^2 - \frac{1}{4}x, \\ y(t) = -C_1 + C_2 e^{2x} + \frac{1}{4}x^2 - \frac{1}{4}x + \frac{1}{4}, \end{cases}$$

2)

$$\begin{cases} x'(t) = x(t) + 5y(t), \\ y'(t) = -x(t) + -3y(t), \\ x(0) = 1, \\ y(0) = 1, \end{cases}$$

ANSWER:

$$\begin{cases} x(t) = e^{-t}(\cos t + 7\cos t), \\ y(t) = \frac{1}{5} e^{-t} \left(5\cos t - 15 \sin t\right), \end{cases}$$

3)

$$\begin{cases} x'(t) = x(t) + y(t), \\ y'(t) = 4y(t) - 2x(t), \\ x(0) = 0, \\ y(0) = -1, \end{cases}$$

ANSWER:

$$\begin{cases} x(t) = e^{2t} - e^{3t}, \\ y(t) = e^{2t} - 2e^{3t}. \end{cases}$$