Circuits and Signals

Resonance

Marek Rupniewski

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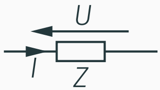
Faculty of Electronics and Information Technology
WARSAW UNIVERSITY OF TECHNOLOGY

Resonance pulsation/frequency



Definition

A resonant pulsation of a one port is any pulsation ω_0 such that $|Z(\omega)|$ attains at ω_0 a proper local minimum or a proper local maximum.

$|Z|$ can be substituted with $|Z|^2$, $\frac{1}{|Z|} = |Y|$ or $|Y|^2$.

Another definition (not equivalent to the previous one):
Resonant pulsation is a pulsation for which impedance becomes real.

Frequency and pulsation:

$$\omega_0 = 2\pi f_0.$$

Q factor

Definition

A Q factor related to resonant pulsation ω_0 is the number

$$Q_{\omega_0} = 2\pi \frac{w_{\max}}{w(0,T)}, \qquad T = \frac{2\pi}{\omega_0},$$

where $w(0,T)$ is the energy transferred to the one-port during the period ($T_0 = 2\pi/\omega_0$), and $w_{\max}$ is the maximal (over interval of length T_0) value of the energy stored in the one-port.

Series resonant circuit (SRC)

$$Z(\omega) = R + \jmath\omega L + \frac{1}{\jmath\omega C} = R + \jmath(\omega L - \frac{1}{\omega C}).$$

$$|Z(\omega)|^2 = R^2 + (\omega L - \frac{1}{\omega C})^2.$$

SRC — resonant pulsation ω_0

$$|Z(\omega)|^2 = R^2 + (\omega L - \frac{1}{\omega C})^2 = R^2 - \frac{2L}{C} + \omega^2 L^2 + \frac{1}{C^2\omega^2}.$$

$$0 = \frac{\mathrm{d}|Z|^2}{\mathrm{d}\omega}(\omega_0),$$

therefore

$$0 = 2L^2\omega_0 - \frac{2}{C^2\omega_0^3} \quad \Leftrightarrow \quad 1 = \omega_0^4 L^2 C^2 \quad \Rightarrow \quad \boxed{\omega_0 = \frac{1}{\sqrt{LC}}}.$$

We have

$$\frac{\mathrm{d}^2|Z|^2}{\mathrm{d}\omega^2}(\omega) = 2L^2 + \frac{6}{C^2\omega^4} > 0.$$

Thus ω_0 is the unique resonant pulsation of the SRC, and $|Z|$ attains a global minimum (in the domain $\omega > 0$) at ω_0.

SRC — resonant pulsation ω_0 cont.

$$Z(\omega) = R + \jmath(\omega L - \frac{1}{\omega C}) = R + \frac{1}{\omega C}\jmath(\omega^2 LC - 1).$$

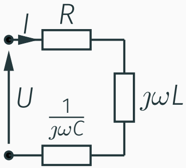
$$\omega_0 = \frac{1}{\sqrt{LC}}.$$

Thus Z becomes real at ω_0:

$$Z(\omega_0) = R.$$

ω_0 is resonant according to both (non-equivalent in general) definitions of a resonant pulsation.

SRC — Q factor



$$Q = 2\pi \frac{W_{\max}}{w(0,T)}, \qquad T = \frac{2\pi}{\omega_0},$$

Real power transferred to the one-port

$$P_\omega = \frac{1}{2}|I|^2 \operatorname{Re} Z(\omega).$$

Thus

$$w(0,T) = \frac{2\pi}{\omega_0} P_{\omega_0} = \frac{\pi}{\omega_0}|I|^2 R.$$

$$W_{\max} = \max_{t \in (t_0, t_0+T]} (w_L(t) + w_C(t)) = \frac{1}{2} \max_{t \in (t_0, t_0+T]} (Li^2(t) + Cu_C^2(t)).$$

SRC — Q factor cont.

$$Q = 2\pi \frac{W_{\mathrm{max}}}{w(0,T)}, \qquad T = \frac{2\pi}{\omega_0}, \qquad w(0,T) = \frac{\pi}{\omega_0}|I|^2 R.$$

$$i(t) = |I| \cos(\omega_0 t + \arg I), \qquad U_C = \frac{I}{\jmath \omega_0 C}$$

$$u_C(t) = \frac{|I|}{\omega_0 C} \cos(\omega_0 t + \arg I - \frac{\pi}{2}) = \frac{|I|}{\omega_0 C} \sin(\omega_0 t + \arg I).$$

$$W_{\mathrm{max}} = \frac{1}{2} \max_{t \in (t_0, t_0 + T]} (L i^2(t) + C u_C^2(t))$$

$$= \frac{|I|^2}{2\omega_0^2 C} \max_{t \in (t_0, t_0 + T]} \Big(\underbrace{\omega_0^2 LC}_{1} \cos^2(\omega_0 t + \arg I) + \sin^2(\omega_0 t + \arg I) \Big) = \frac{|I|^2}{2\omega_0^2 C}.$$

$$Q = 2\pi \frac{|I|^2}{2\omega_0^2 C} \frac{\omega_0}{\pi |I|^2 R} = \frac{1}{\omega_0 R C}.$$

SRC — Q factor cont.

$$Q = 2\pi \frac{W_{\max}}{w(0,T)}, \qquad T = \frac{2\pi}{\omega_0}, \qquad Q = \frac{1}{\omega_0 RC}.$$

Since $\omega_0^2 LC = 1$, we have also

$$Q \overset{\omega_0 C = 1/\omega_0 L}{=} \frac{\omega_0 L}{R}$$

and

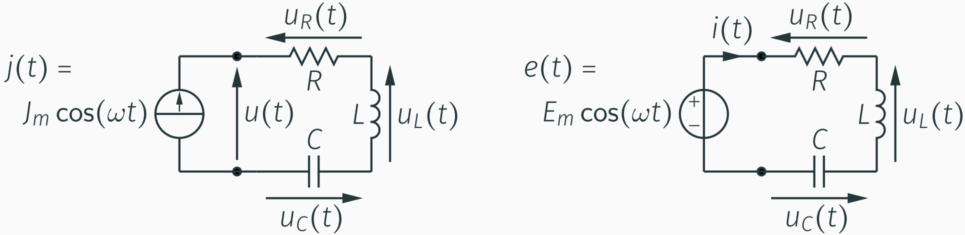
$$Q \overset{\omega_0 = 1/\sqrt{LC}}{=} \boxed{\frac{\sqrt{\frac{L}{C}}}{R}}.$$

The quantity

$$\boxed{\rho = \sqrt{\frac{L}{C}}}$$

is called characteristic resistance of SRC.

Absolute detuning

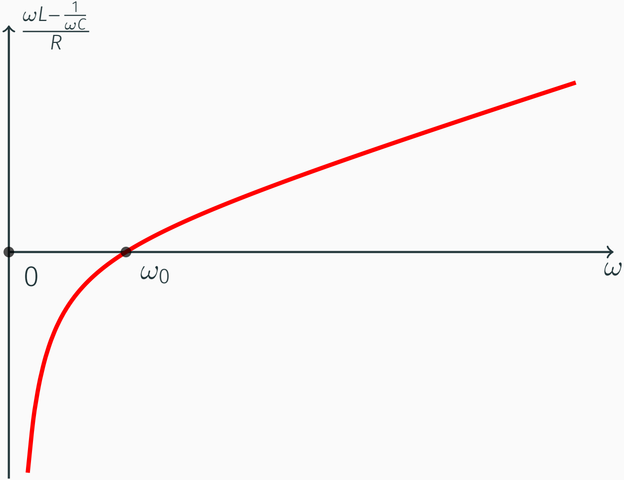


Definition

For SRC, the absolute detuning related to pulsation ω is defined as

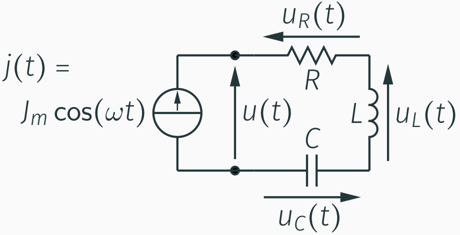
$$\xi_\omega = (\omega L - \frac{1}{\omega C})/R.$$

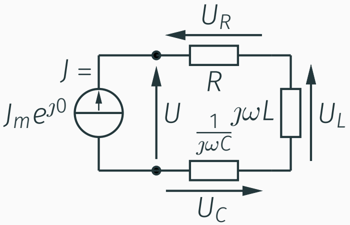
Absolute detuning cont.



$$Z(\omega) = R + \jmath(\omega L - \frac{1}{\omega C}) = R + \jmath R\xi_\omega = R(1 + \jmath\xi_\omega).$$

SRC — filtration





$$U = JZ(\omega), \quad U_R = JR, \quad U_L = J\jmath\omega L, \quad U_C = \frac{J}{\jmath\omega C}, \quad Z(\omega) = R(1 + j\xi_\omega)$$

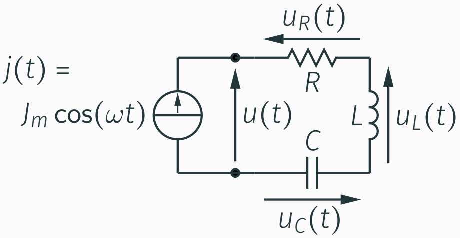
Thus

$$u_R(t) = J_m R\cos(\omega t),\; u(t) = J_m R\sqrt{1+\xi_\omega^2}\cos(\omega t + \arctan\xi_\omega),$$

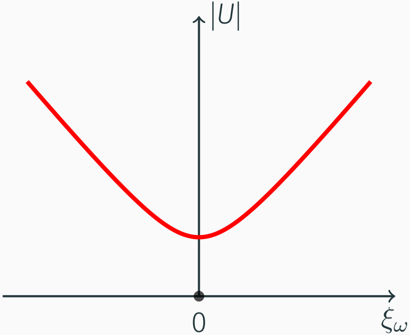
$$u_L(t) = J_m\omega L\cos(\omega t + \frac{\pi}{2}) = J_m R\frac{\omega L}{R}\cos(\omega t + \frac{\pi}{2}),$$

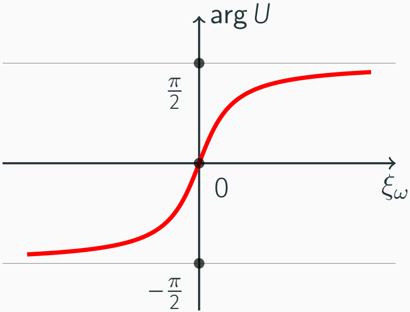
$$u_C(t) = \frac{J_m}{\omega C}\cos(\omega t - \frac{\pi}{2}) = \frac{J_m R}{\omega RC}\cos(\omega t - \frac{\pi}{2}).$$

SRC — filtration cont.

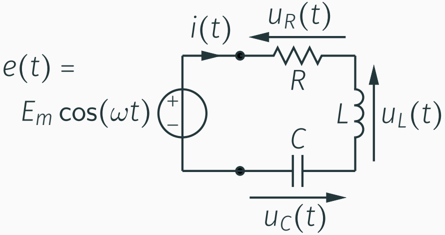


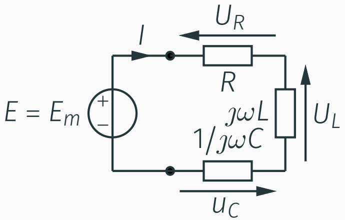
$$u(t) = J_m R \sqrt{1 + \xi_\omega^2} \cos(\omega t + \arctan \xi_\omega).$$





SRC — filtration cont.

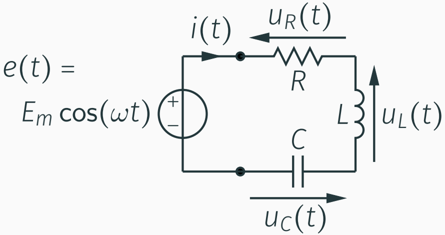




$$I = \frac{E}{Z(\omega)} = \frac{E}{R(1+\jmath\xi_\omega)} \quad \text{thus} \quad i(t) = \underbrace{E_m/(R\sqrt{1+\xi_\omega^2})}_{I_m(\omega)} \cos(\omega t - \arctan \xi_\omega)$$

$$\begin{aligned} u_R(t) &= I_m(\omega) R \cos(\omega t - \arctan \xi_\omega), \\ u_L(t) &= I_m(\omega) \omega L \cos(\omega t + \pi/2 - \arctan \xi_\omega), \\ u_C(t) &= I_m(\omega)/(\omega C) \cos(\omega t - \pi/2 - \arctan \xi_\omega). \end{aligned}$$

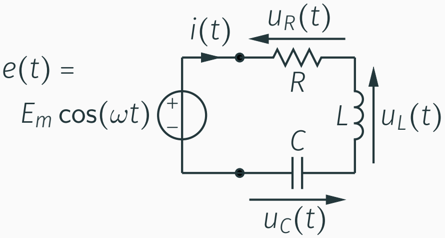
SRC — filtration cont.



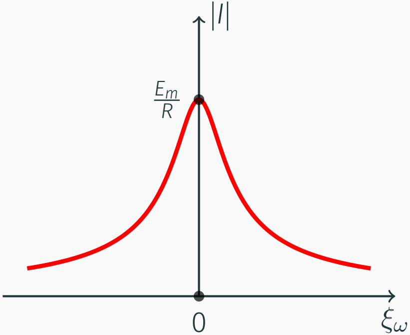
For resonant pulsation $\omega = \omega_0$ (i.e. $\xi = 0$): $i(t) = \frac{E_m}{R}\cos(\omega_0 t)$ and

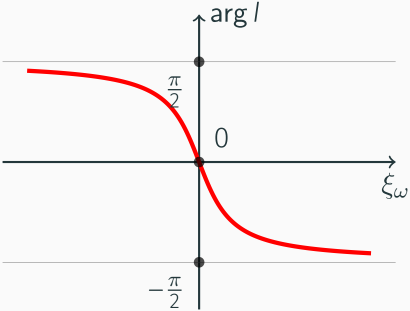
$$
\begin{aligned}
u_R(t) &= E_m \cos(\omega_0 t), \\
u_L(t) &= QE_m \cos\left(\omega_0 t + \frac{\pi}{2}\right), \\
u_C(t) &= QE_m \cos\left(\omega_0 t - \frac{\pi}{2}\right).
\end{aligned}
$$

SRC — filtration cont.



$$i(t) = E_m/(R\sqrt{1+\xi_\omega^2})\cos(\omega t - \arctan \xi_\omega)$$





Relative detuning

Definition

The relative detuning related to pulsation ω is defined as

$$\nu_\omega = \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}.$$

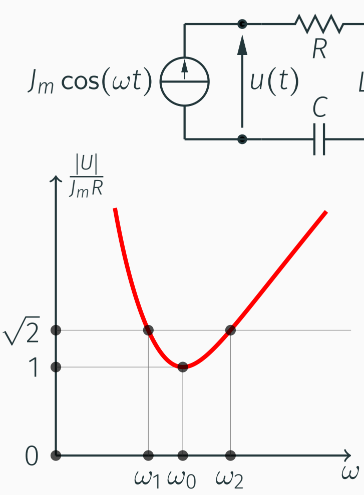
It occurs that

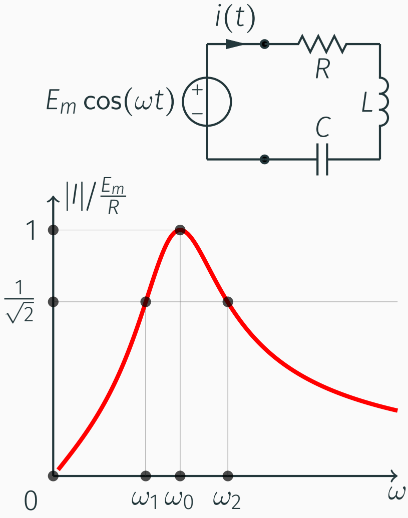
$$\xi_\omega = Q\nu_\omega.$$

Indeed

$$Q\nu_\omega = Q\frac{\omega}{\omega_0} - Q\frac{\omega_0}{\omega} = \frac{\omega_0 L}{R}\frac{\omega}{\omega_0} - \frac{1}{\omega_0 RC}\frac{\omega_0}{\omega} = (\omega L - \frac{1}{\omega C})/R = \xi_\omega.$$

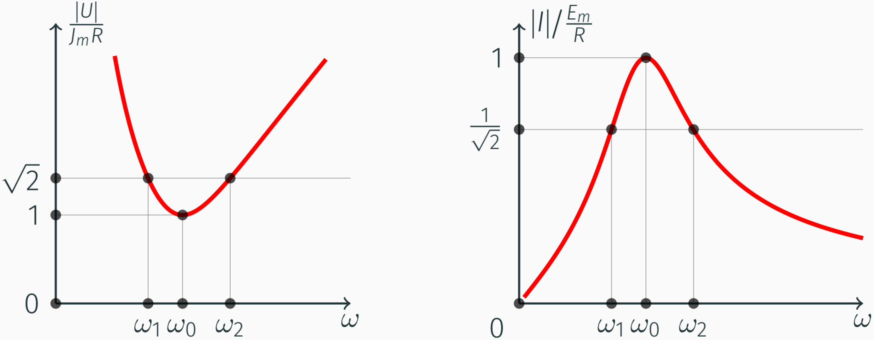
SRC — filtration cont.





$$|U| = J_m R\sqrt{1+\xi_\omega^2} \qquad |I| = E_m/(R\sqrt{1+\xi_\omega^2}), \qquad \xi_{\omega_1} = -1, \quad \xi_{\omega_2} = 1.$$

SRC — filtration cont.

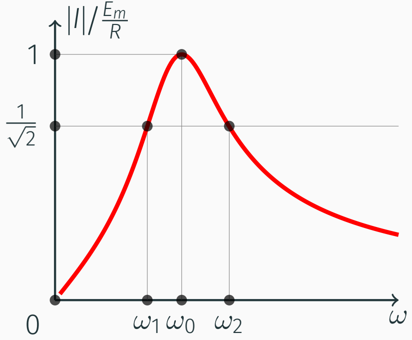


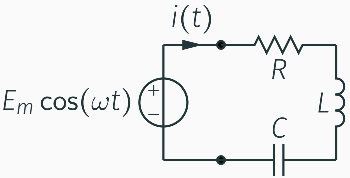
$$\xi_{\omega_1} = -1, \qquad \xi_{\omega_2} = 1, \qquad \xi = Q\nu.$$

$$\frac{-1}{Q} = \frac{\omega_1}{\omega_0} - \frac{\omega_0}{\omega_1} \qquad \boxed{+} \qquad \frac{1}{\omega_0} - \frac{\omega_0}{\omega_1\omega_2} = 0 \Rightarrow \omega_1\omega_2 = \omega_0^2$$

$$\frac{1}{Q} = \frac{\omega_2}{\omega_0} - \frac{\omega_0}{\omega_2} \qquad \boxed{-} \qquad \frac{2(\omega_2 - \omega_1)}{\omega_0} = \frac{2}{Q} \Rightarrow \omega_2 - \omega_1 = \frac{\omega_0}{Q}.$$

SRC — filtration cont.



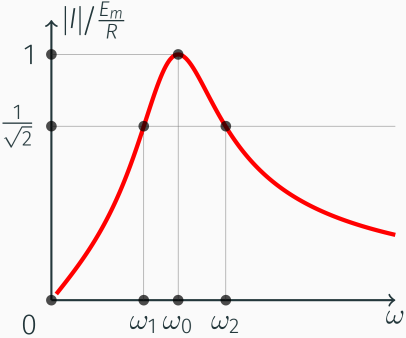


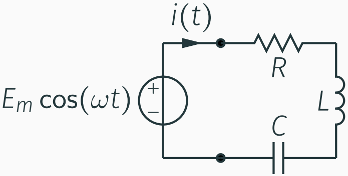
$$\xi_{\omega_1} = -1, \quad \xi_{\omega_2} = 1, \quad \omega_0 = \sqrt{\omega_1\omega_2}, \quad \omega_2 - \omega_1 = \omega_0/Q.$$

$$P = \frac{1}{2}|I|^2 R$$

ω_1, ω_2 are pulsations, for which (real) power transferred to the one port falls to one half of the value attained for ω_0.

SRC — filtration cont.





$$\xi_{\omega_1} = -1, \quad \xi_{\omega_2} = 1,$$

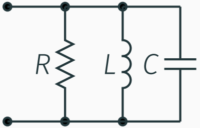
$$\omega_2 - \omega_1 = \omega_0/Q.$$

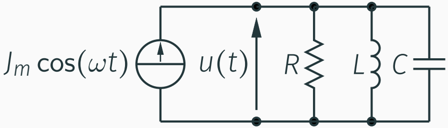
ω_1, ω_2 are pulsations, for which (real) power transferred to the one port falls to one half of the value attained for ω_0 i.e. the power changes by 3 dB.

$$\text{power ratio [dB]} = 10 \log_{10} \frac{P}{P_0},$$

$B = B|_{3\,\mathrm{dB}} = \omega_2 - \omega_1 = \omega_0/Q$ is called the Bandwidth of SRC.

Parallel resonant circuit (PRC)





$$Y(\omega) = \frac{1}{R} + \frac{1}{\jmath\omega L} + \jmath\omega C = \frac{1}{R}\Big(1 + \jmath\,\underbrace{R\big(\omega C - \frac{1}{\omega L}\big)}_{\xi_\omega}\Big).$$

$$Z(\omega) = \frac{R}{1 + \jmath\xi_\omega}.$$

$$\omega_0 = \frac{1}{\sqrt{LC}}.$$

PRC — Q factor

$$Q = 2\pi \frac{W_{\max}}{W(0,T)}, \qquad T = \frac{2\pi}{\omega_0},$$

Analogously to SRC case one can get:

$$Q = \frac{R}{\omega_0 L} = Q = \omega_0 RC = \boxed{\frac{R}{\sqrt{\frac{L}{C}}}}.$$

The quantity

$$\boxed{\rho = \sqrt{\frac{L}{C}}}$$

is called characteristic resistance of PRC.

Detunings

Absolute detuning

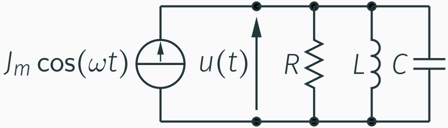
$$\xi_\omega = R(\omega C - \frac{1}{\omega L}).$$

Relative detuning

$$\nu_\omega = \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}.$$

$$\xi = Q\nu.$$

PRC — current resonance



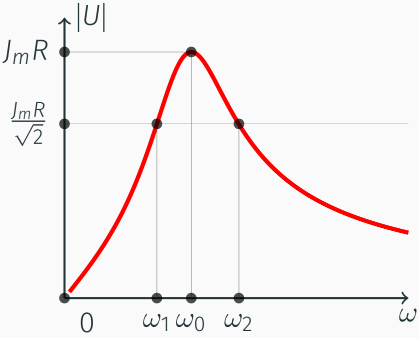
For resonant pulsation $\omega = \omega_0$ (i.e. $\xi = 0$):

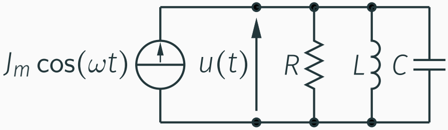
$$u(t) = J_m R \cos(\omega_0 t)$$

and

$$\begin{aligned} i_R(t) &= J_m \cos(\omega_0 t), \\ i_L(t) &= Q J_m \cos(\omega_0 t - \frac{\pi}{2}), \\ i_C(t) &= Q J_m \cos(\omega_0 t + \frac{\pi}{2}). \end{aligned}$$

PRC — filtration





$$\xi_{\omega_1} = -1, \quad \xi_{\omega_2} = 1, \quad \omega_0 = \sqrt{\omega_1\omega_2}, \quad \omega_2 - \omega_1 = \omega_0/Q.$$

ω_1, ω_2 are half-power points ($P = \frac{1}{2}|I|^2R$), i.e. they form the limits of 3 dB-pass-band

$$B = B|_{3\,\mathrm{dB}} = \omega_2 - \omega_1 = \omega_0/Q.$$

SRC and PRC — comparison

	SRC	PRC
ω_0	$1/\sqrt{LC}$	$1/\sqrt{LC}$
ρ	$\sqrt{L/C}$	$\sqrt{L/C}$
Q	$2\pi\frac{W_{\max}}{W(0,T)} = \frac{1}{\omega_0 RC} = \frac{\omega_0 L}{R} = \frac{\rho}{R}$	$2\pi\frac{W_{\max}}{W(0,T)} = \omega_0 RC = \frac{R}{\omega_0 L} = \frac{R}{\rho}$
ν_ω	$\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}$	$\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}$
ξ_ω	$Q\nu_\omega = (\omega L - \frac{1}{\omega C})/R$	$Q\nu_\omega = R(\omega C - \frac{1}{\omega L})$
$Z(\omega)$	$R(1 + \jmath\xi_\omega)$	$R/(1 + \jmath\xi_\omega)$
$Z(\omega_0)$	R	R
B	$\omega_2 - \omega_1 = \omega_0/Q$	$\omega_2 - \omega_1 = \omega_0/Q$
ω_0	$\sqrt{\omega_1\omega_2}$	$\sqrt{\omega_1\omega_2}$