EPRST: Probability and Statistics
Problem set 5

1. Suppose that you buy a lottery ticket in each of 50 lotteries. In every lottery probability of winning is $\frac{1}{100}$. Using Poisson approximation, compute the probability that you win:
   (a) at least once,
   (b) exactly once,
   (c) at least twice.

2. A message is sent over a noisy channel. The message is a sequence x_1, x_2, $\ldots, x_n$ of n bits ($x_i \in \{0, 1\}$). Since the channel is noisy, there is a chance that any bit might be corrupted, resulting in an error (a 0 becomes a 1 or vice versa). Assume that the error events are independent. Let p be the probability that an individual bit has an error ($0 < p < \frac{1}{2}$). Let $y_1, y_2, \ldots, y_n$ be the received message (so $y_i = x_i$ if there is no error in that bit, but $y_i = 1 - x_i$ if there is an error there).

   To help detect errors, the nth bit is reserved for the parity check: x_n is defined to be 0 if $x_1+x_2+\ldots+x_{n-1}$ is even, and 1 if $x_1 + x_2 + \ldots + x_n$ is odd. When the message is received, the recipient checks whether y_n has the same parity as $y_1 + y_2 + \ldots + y_{n-1}$. If the parity is wrong, the recipient knows that at least one error occurred; otherwise, the recipient assumes that there were no errors.
   (a) For $n = 3$ and $p = 0.1$, what is the probability that the received message has errors which go undetected?
   (b) For general n and p, determine the probability that the received message has errors which go undetected.

3. Find the value
   (a) $\mathbb{P}(X \geq 3.5)$ if $X \sim \mathcal{N}(2, 4)$,
   (b) $\mathbb{P}(X \geq -4)$ if $X \sim \mathcal{N}(-5, 1)$.

4. The weight of any person in a group of people is described (in kgs) by the normal distribution $\mathcal{N}(75, 16)$.
   (a) What is the probability that a randomly picked person from the group weighs more than 83 kgs?
   (b) What is the probability that a randomly picked person from the group weighs no more than 79 kgs?
   (c) What is the fraction of people with the weight between 71 and 80 kgs?
   (d) Find such value of weight that is not exceeded by 80% of people from the group.

5. If $X \sim \mathcal{N}(-1, 9)$ then (answer *yes* or *no*):
   (a) $\mathbb{P}(|X + 1| > 3) = 1 - 2\Phi(1)$,
   (b) $\mathbb{P}(X > 2) = \mathbb{P}(X < -4)$,
   (c) $F_X(-6) + F_X(3) < 1$,
   (d) $\mathbb{P}(-2 < X < 0) > \mathbb{P}(10 < X < 12)$.

6. Assume that $\operatorname{supp} X = \{-2, -1, 0, 1, 3\}$ and $\mathbb{P}(X = -2) = \mathbb{P}(X = -1) = \mathbb{P}(X = 0) = \mathbb{P}(X = 1) = \mathbb{P}(X = 3)$. Let $Y = X^4$. Find the distribution of Y.

7. Let X be uniformly distributed on $[-1, 3]$. Determine the distribution of
   (a) $Y = X^2$,
   (b) $Y = \max(0, X)$.

8. Our aim now is to compute the integral
$$I := \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x.$$
   (a) Write I^2 as a double integral over the real plane
   Hint:
$$I^2 = \left(\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x\right)^2 = \left(\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x\right)\left(\int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2}\right) \mathrm{d}y\right)$$

(b) Compute I^2 using the polar coordinates. What is the value of I?

9. Let $g(x) = x^2$. Find the distribution of $Y = g(X)$ if X is uniformly distributed on $[-1, 3]$.

10. Random variable X has a continuous distribution with the density $f(x) = \frac{1}{x^2}\mathbb{1}_{(1,\infty)}(x)$. Find the distribution of

    (a) $Y = 2X + 1$,

    (b) $Z = X^2 + X$.