Probability and Statistics (EPRST)

Lecture 5

Transformations of random variables

Let X - a random variable, and $g : \mathbb{R} \to \mathbb{R}$ - a function. We can define a new random variable $Y = g(X)$ by

$$Y(\omega) = g(X(\omega)),\ \omega \in \Omega.$$

We will show in a few examples how to determine the distribution of the new random variable Y (knowing the distribution of X and the function g).

Transformations of random variables - an example

Example

Suppose the distribution of X is given by

$$\mathbb{P}(X = -1) = \mathbb{P}(X = 1) = \frac{1}{4},\ \mathbb{P}(X = 0) = \frac{1}{2}.$$

Find the distribution of $Y = X^2$.

Transformations of random variables - an example

Example

Let $X \sim geom(p)$, and $Y = X - 1$. What is the distribution of Y?

Transformations of random variables - an example

Example

Let $X \sim U(-5, 5)$*. Find the distribution of* $Y = sign(X)$*, where*

$$sign(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

Transformations of random variables - an example

Example

Say $X \sim U(0,1)$. What is the distribution of $Y = X^2$? Is it still uniform?

Transformations of random variables - an example

Example

Let $X \sim \mathcal{N}(0, 1)$. What is the distribution of $Y = X^2$?

Expectation

Definition

The expected value *(**the expectation** or **mean**) of a random variable X is a number (denoted $\mathbb{E}X$) given by*

$$\mathbb{E}X = \begin{cases} \sum_{x_i \in S} x_i \mathbb{P}(X = x_i), & \textit{if } X \textit{ is a discrete random variable,} \\ \int_{\mathbb{R}} x \cdot f(x) dx, & \textit{if } X \textit{ is continous random variable} \end{cases}$$

(S denotes the set of values of X in the discrete case, and f stands for the density in the continuous case).
The expectation is well-defined if the series (or the integral) are absolutely convergent. If $X \geq 0$ (that is, $\mathbb{P}(X \geq 0) = 1$), then if the series (or the integral) diverge to ∞ - then we define $\mathbb{E}X := \infty$.

The expectation - some examples

Example

- *If* $\mathbb{P}(X = -1) = 0.2$, $\mathbb{P}(X = 0) = 0.4$, $\mathbb{P}(X = 1) = 0.4$, *then* $\mathbb{E}X =$
- *If* $\mathbb{P}(X = -1) = 0.4$, $\mathbb{P}(X = 0) = 0.2$, $\mathbb{P}(X = 1) = 0.4$, *then* $\mathbb{E}X =$
- *If* $\mathbb{P}(X = -1) = 0.2$, $\mathbb{P}(X = 0) = 0.4$, $\mathbb{P}(X = 100) = 0.4$, *then* $\mathbb{E}X =$

Example

Compute the mean of $X \sim \mathcal{U}[0,1]$.

The expectation - some examples

The expected value is not always finite:

Example

Let $\mathbb{P}(X = k) = c/k^2$ for $k \in \mathbb{N}$. What is the value of c? Does $\mathbb{E}X$ exist?

The expected value does not always exist:

Example

Let X - a random variable with a Cauchy distribution, that is, with the density

$$f(x) = \frac{1}{\pi(1 + x^2)}, \quad x \in \mathbb{R}.$$

Does $\mathbb{E}X$ exist?

The expectation - some properties

- $\mathbb{E}c = c$ for every $c \in \mathbb{R}$,
- $\mathbb{E}(c \cdot X) = c \cdot \mathbb{E}X$ for every $c \in \mathbb{R}$,
- $\mathbb{E}\left(X_1 + \ldots + X_n\right) = \mathbb{E}X_1 + \ldots + \mathbb{E}X_n$
- if $X \geq 0$, then $\mathbb{E}X \geq 0$.

In particular, for all $a, b \in \mathbb{R}$

$$\mathbb{E}(aX + b) = a\mathbb{E}X + b.$$

The expectations of some important distributions

- if $X \sim \mathrm{bin}(n, p)$, then $\mathbb{E}X = ?$,
- if $X \sim \mathrm{geom}(p)$, then $\mathbb{E}X = ?$,
- if $X \sim \mathrm{Poiss}(\lambda)$, then $\mathbb{E}X = ?$,
- if $X \sim \mathrm{U}(a, b)$, then $\mathbb{E}X = ?$,
- if $X \sim \mathrm{Exp}\,(\lambda)$, then $\mathbb{E}X = ?$,
- if $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$, then $\mathbb{E}X = ?$