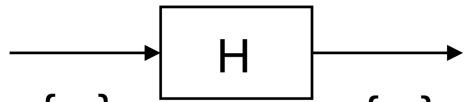
Discrete – time systems
(digital filters)



$\{x_n\}$ $\{y_n\} = H\,[\{x_n\}]$

Linear system: superposition property

Input $\{x_n\} = x_0, x_1, x_2\ldots$ $\longrightarrow$ output $\{y_n\}= H[\{x_n\}]$
Input $\{u_n\} = u_0, u_1, u_2\ldots$ $\longrightarrow$ output $\{v_n\}= H[\{u_n\}]$
Input $a\,\{x_n\} + b\,\{u_n\}$ $\longrightarrow$ output $a\,\{y_n\} + b\,\{v_n\}$

Time invariant system:

Input $\{x_n\}$ $\longrightarrow$ output $\{y_n\}$
Input $\{x_{n-m}\}$ $\longrightarrow$ output $\{y_{n-m}\}$

LTI – linear and time invariant

LTI systems

$$\{x_n\} \longrightarrow \boxed{H} \longrightarrow \{y_n\} = H\,[\{x_n\}]$$

LTI - description in time domain

Input δ_n (Kronecker delta) $\longrightarrow$ output $y_n = h_n$ (impulse response)
Input δ_{n-k} (delayed delta) $\longrightarrow$ output $y_n = h_{n-k}$
Input $x_k\,\delta_{n-k}$ $\longrightarrow$ output $y_n = x_k\,h_{n-k}$

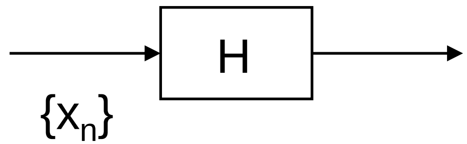
Input $\{x_n\} = \sum_k x_k\,\delta_{n-k}$ (series of samples) $\longrightarrow$ output $y_n = \sum_{k=-\infty}^{\infty} x_k\,h_{n-k} = x_n * h_n$

convolution

Discrete – time LTI system is described with convolution

For **causal** systems: $h_n = 0$, $n < 0$ $\qquad y_n = \sum_{k=-\infty}^{n} x_k\,h_{n-k}$

LTI systems



$$y_n = \sum_{k=-\infty}^{\infty} x_k h_{n-k} = x_n * h_n$$

$$Y(z) = X(z)H(z), \qquad H(z) = Z[\{h_n\}] \longrightarrow$$ **transfer function** of the system

The output signal: $$y_n = Z^{-1}[Y(z)] = Z^{-1}[X(z)H(z)]$$

Stability / instability: Let us assume, that X(z) has all its poles within the unit circle. It means that the input signal $x_n \to 0$ as $n \to \infty$ (see last slide „Z transform”) For stable systems the output signal $y_n \to 0$ as $n \to \infty$. That is every pole of X(z) H(z) must be within the unit circle.

If at least one pole of H(z) stems out of this circle, then $y_n \to \infty$ i.e. the system H(z) is unstable.

Stable filter is a **BIBO** (Bounded Input – Bounded Output) filter. If the input signal is bounded, then the output signal is also bounded.

Difference equations

Linear filters are usually described with difference equations

$$\xrightarrow{x_n} \boxed{H(z)} \longrightarrow \quad y_n = \sum_{i=0}^{N} b_i\, x_{n-i} - \sum_{i=1}^{M} a_i\, y_{n-i}$$

Introducing a_0=1, we obtain:

$$\sum_{i=0}^{M} a_i\, y_{n-i} = \sum_{i=0}^{N} b_i\, x_{n-i}$$

Z transform:

$$\sum_{i=0}^{M} a_i\, Z[y_{n-i}] = \sum_{i=0}^{N} b_i\, Z[x_{n-i}] \longrightarrow \sum_{i=0}^{M} a_i\, z^{-i} Y(z) = \sum_{i=0}^{N} b_i\, z^{-i} X(z)$$

$$\longrightarrow Y(z) \sum_{i=0}^{M} a_i\, z^{-i} = X(z) \sum_{i=0}^{N} b_i\, z^{-i}$$

Polynomial of degree N has N zeros (roots)

Transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{N} b_i\, z^{-i}}{\sum_{i=0}^{M} a_i\, z^{-i}} = \frac{z^{-N} \sum_{i=0}^{N} b_i\, z^{N-i}}{z^{-M} \sum_{i=0}^{M} a_i\, z^{M-i}} = \frac{B(z)}{A(z)}$$

Rational function,
has N zeros and M poles

Values z^{-M} and z^{-N} represent only time shift

Polynomial of degree M has M zeros (roots)

FIR and IIR filters

Nonrecursive (FIR): *M*=0

FIR – finite impulse response

$$y_n = \sum_{i=0}^{N} b_i \, x_{n-i}$$

If the input signal is the Kronecker delta

$$x_n = \delta_n$$

then the output signal equals:

$$y_n = \sum_{i=0}^{N} b_i \, \delta_{n-i} = b_n$$

The impulse response:

$$b_0, b_1, \ldots, b_N, 0, 0, 0, \ldots$$

Transfer function $H(z) = \sum_{i=0}^{N} b_i \, z^{-i}$ $\longrightarrow$ **FIR filter is always stable!**

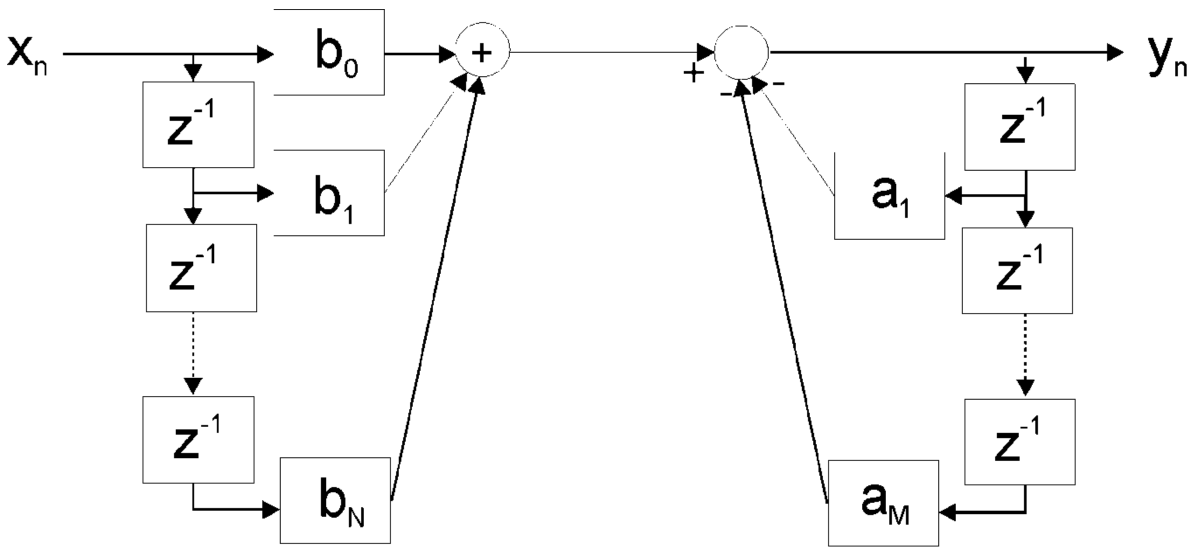
Recursive (IIR): *M*≠0

IIR – infinite impulse response

If *N*=0, we obtain the **all pole filter**

Transversal structure of digital filter

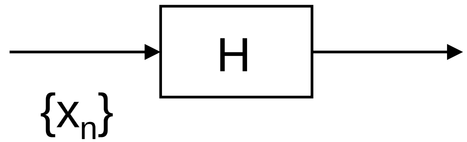
$$y_n = \sum_{i=0}^{N} b_i \, x_{n-i} - \sum_{i=1}^{M} a_i \, y_{n-i}$$



FIR part is always stable

IIR part may be stable or not

LTI systems - stability analysis



$$y_n = \sum_{k=-\infty}^{\infty} x_k h_{n-k} = x_n * h_n$$

An example:

Transfer function of a filter is described with the formula: $H(z) = \dfrac{z^2}{z^2 - z + 0.5}$

Is this filter stable?

There is a polynomial in denominator of H(z), so it is a IIR filter.
Its stability depends on position of the poles of H(z).

These poles are the roots of the polynomial $z^2 - z + 0.5$

We solve the equation $z^2 - z + 0.5 = 0$ and obtain two roots:

$z_1 = 0.5 + j0.5$ and $z_2 = 0.5 - j0.5$

$|z_1| = |z_2| = \dfrac{1}{\sqrt{2}} < 1$, poles lie in the circle of radius =1 and the filter is stable.

Difference equations – an example

A filter is described with the following difference equation

$$x_n \longrightarrow \boxed{H(z)} \longrightarrow \qquad y_n = x_n + \tfrac{1}{4} y_{n-1} + \tfrac{1}{8} y_{n-2}$$

Is it FIR or IIR filter? Calculate the transfer function H(z) . Is this filter stable?

y is put to the left side: $y_n - \tfrac{1}{4} y_{n-1} - \tfrac{1}{8} y_{n-2} = x_n$

Z transform: $Y(z) - \tfrac{1}{4} z^{-1} Y(z) - \tfrac{1}{8} z^{-2} Y(z) = X(z)$ because $Z[y_{n-i}] = z^{-i} Y(z)$

$$\longrightarrow \qquad Y(z)[1 - \tfrac{1}{4} z^{-1} - \tfrac{1}{8} z^{-2}] = X(z)$$

Transfer function $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1}{1 - \tfrac{1}{4} z^{-1} - \tfrac{1}{8} z^{-2}} = \dfrac{z^2}{z^2 - \tfrac{1}{4} z - \tfrac{1}{8}}$

Rational function,
has 2 zeros and 2 poles

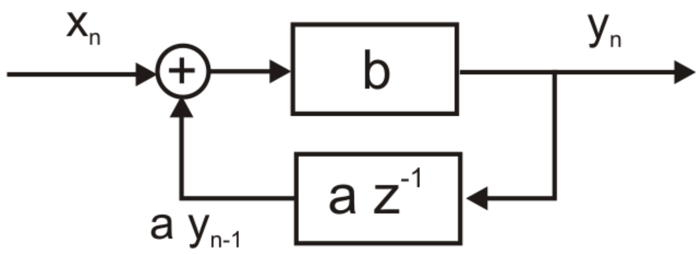
This is an IIR filter, because the current output sample y_n depends on previous output samples y_{n-1}, y_{n-2} (see difference equation).

Moreover, the transfer function has poles (roots of the polynomial in denominator). These roots are $z_1 = \tfrac{1}{2}$ and $z_2 = -\tfrac{1}{4}$. They lie in the unit circle, so the filter is stable.

Block diagrams

Filter may be described with the block diagram. Block diagram (block scheme) contains sufficient information to obtain difference equation or transfer function.

Example:



Operator z^{-1} represents delay of one sample (T seconds, where T is sampling interval).

From the block diagram we read: $y_n = b(x_n + a\,y_{n-1})$

y is put to the left side: $y_n - ab\,y_{n-1} = b\,x_n$

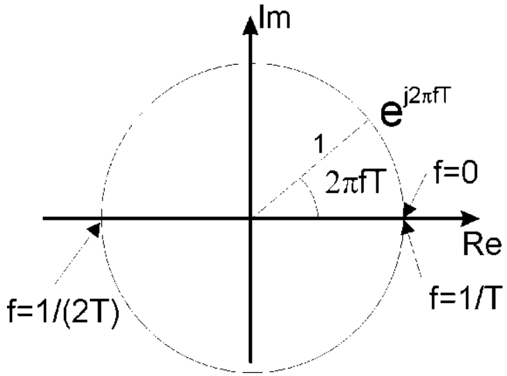
Z transform: $Y(z) - abz^{-1}Y(z) = bX(z)$ because $Z[y_{n-i}] = z^{-i}Y(z)$

$$\longrightarrow \quad Y(z)[1 - abz^{-1}] = bX(z) \quad \rightarrow \quad H(z) = \frac{Y(z)}{X(z)} = \frac{b}{1 - abz^{-1}} = \frac{b\,z}{z - ab}$$

Transfer function may be obrained directly from the block diagram, by reading

$$Y(z) = b[X(z) + a\,z^{-1}Y(z)]$$

Frequency response



The substitution $z = e^{j2\pi fT}$ leads us to frequency domain (DTFT)

T – sampling period
1/T – sampling frequency

$$H(z) = H(e^{j2\pi fT}) = H_s(f)$$

frequency response of discrete time filter
(a complex function!)

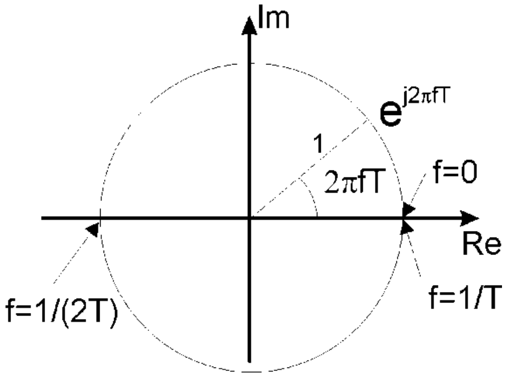
Frequency response tells us about filter's reaction to input signal components of different frequencies. Are these components amplified or attenuated?
It is the steady state analysis like in analog systems, but we use sampled sine or cosine signals:
Let this sampled cosine of frequency f_0 be the input to LTI system: $x_n = A\cos(2\pi f_0 nT + \varphi)$

Frequency response of this system equals $H_s(f)$, but we are interested only in its value at

frequency f_0: $H_s(f_0) = |H_s(f_0)| e^{j\psi(f_0)}$ where $|H_s(f_0)|$ = frequency magnitude response

At the filter output we obtain $y_n = A|H_s(f_0)|\cos(2\pi f_0 nT + \varphi + \psi(f_0))$

Zeros and poles of the transfer function and their influence on frequency response

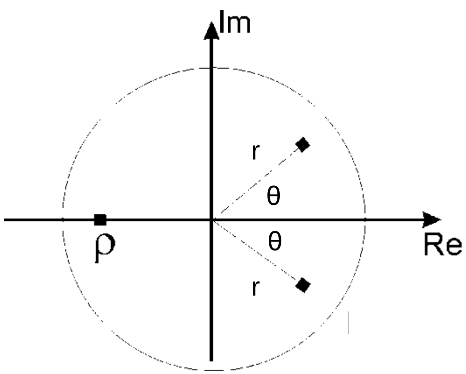


T – sampling period
1/T – sampling frequency

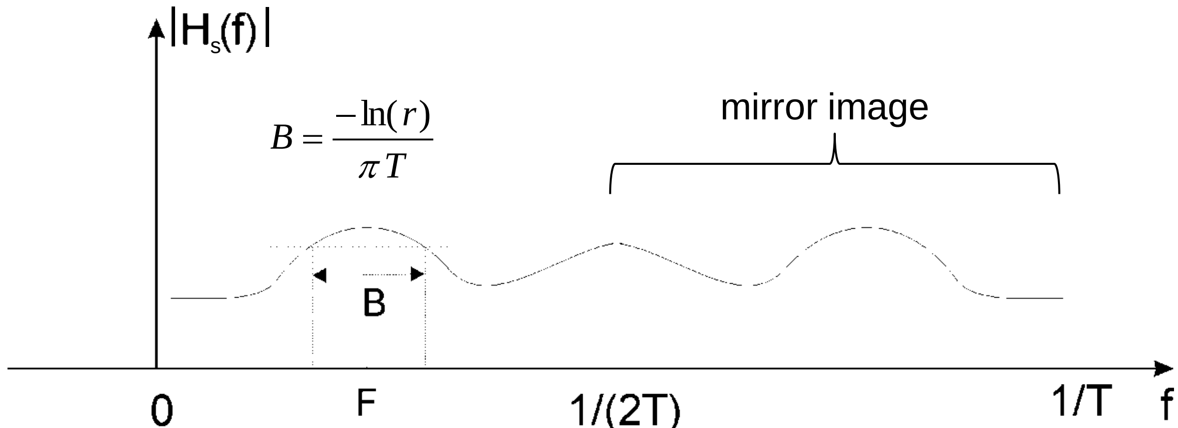
frequency response

$$H(z) = H(e^{j2\pi fT}) = H_s(f)$$

We read the frequency response along the unit circle from frequency f=0 (that is z=1) to the frequency equal to the half of sampling frequency f=1/(2T), that is z= -1. Further we obtain the mirror image. If we pass nearby a zero then we observe decrease of the frequency magnitude response. If we pass nearby a pole then the we observe increase of the frequency magnitude response.

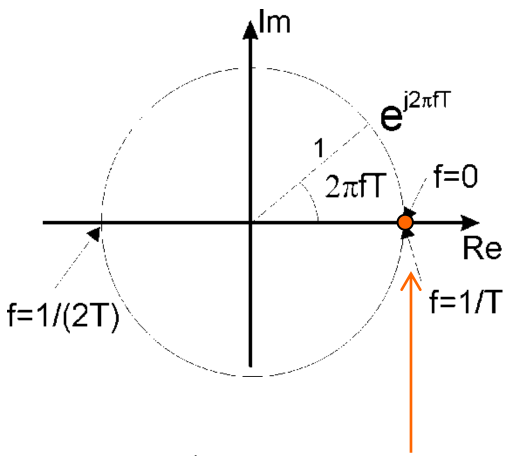


Example: M=3 poles



$$F = \frac{\theta}{2\pi T}$$

Zeros and poles of the transfer function and their influence on frequency response - example



Transfer function of a filter equals $H(z) = 1 - z^{-1}$

It is a FIR filter, B(z)= 1- z $^{-1}$, A(z) = 1.
The coefficients b_0=1 and b_1= -1 are the samples of the impulse response of this filter: h_0=1 and h_1= -1
As a FIR filter it is a stable filter.

$H(z) = 1 - z^{-1} = \frac{z-1}{z}$ has one zero at z=1.
Pole at z=0 has no influence on the frequency response and stability (it is only a time shift)

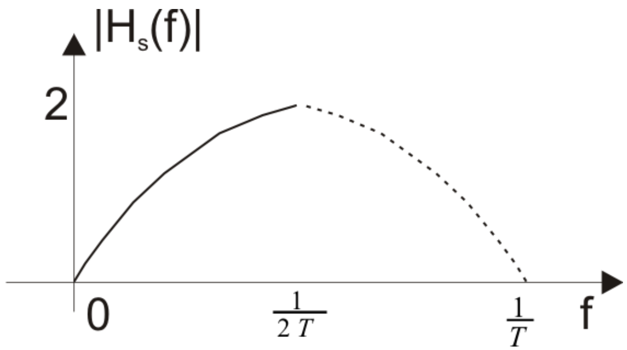
Using the substitution $z = e^{j2\pi fT}$ we read the frequency response of our filter on the unit circle:

$H_s(f) = H(e^{j2\pi fT})$ At f=0, z=1 and H(z) for z=1 is equal to zero. Our filter stops frequency equal to zero (signals of constant values): $H_s(0)=0$. It is due to zero of the polynomial (see red point).
At frequency equal to half of the sampling frequency $f = \frac{1}{2T}$ z = -1 and H(z) = 2 . Our filter is a highpass filter.

Zeros and poles of the transfer function and their influence on frequency response - example

Without detailed calculation we may draw the approximate frequency magnitude response

of the filter $H(z) = 1 - z^{-1}$

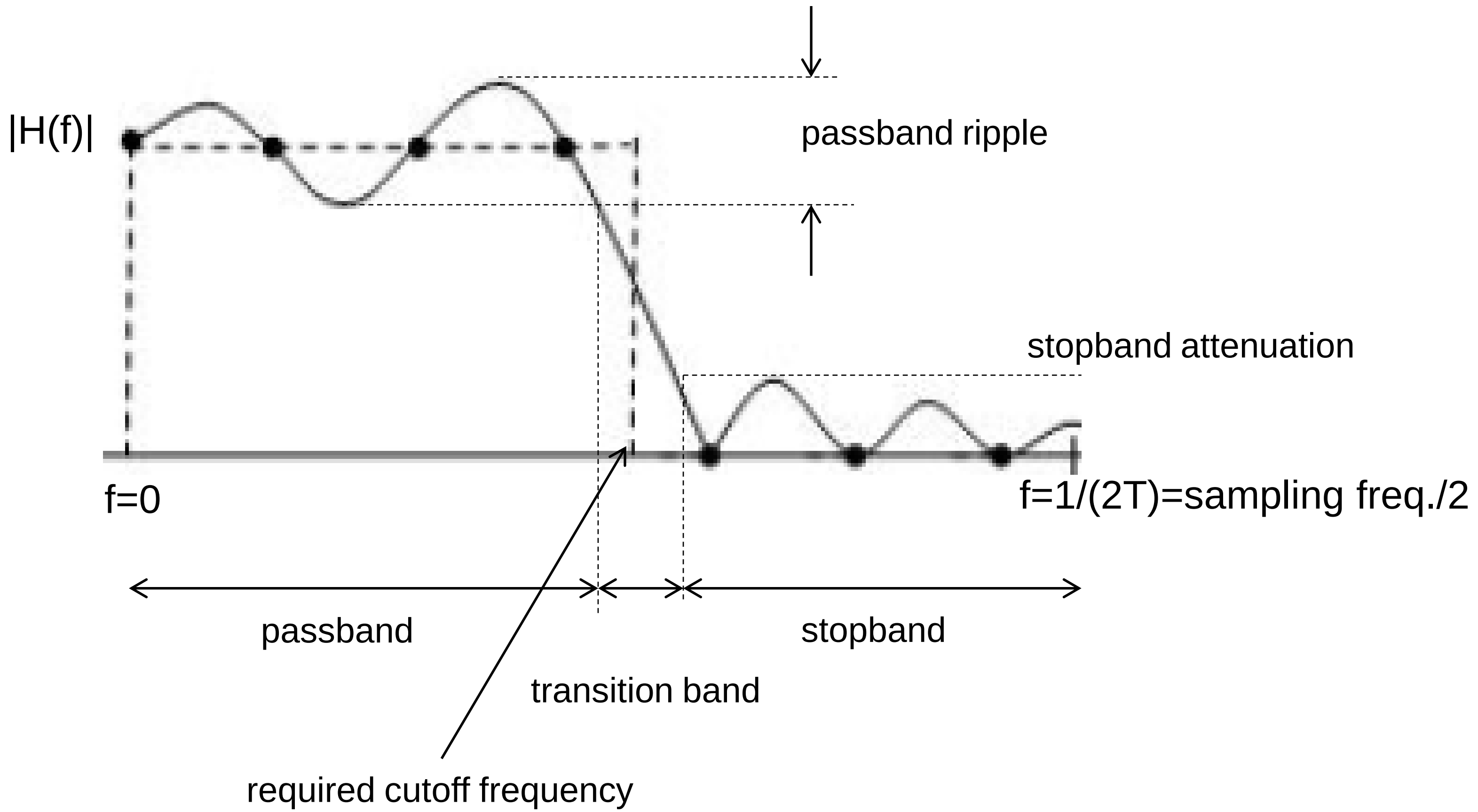


We may substitute $z = e^{j2\pi fT}$ to H(z): $H_s(f) = H[e^{j2\pi fT}] = 1 - e^{-j2\pi fT}$

and calculate frequency magnitude response $|H_s(f)|$ for each frequency value.

Parameters of a lowpass filter

Frequency magnitude response |H(f)|



Design of finite impulse response filters (FIR)

M=0, N zeros on the complex plane $$y_n = \sum_{i=0}^{N} b_i \, x_{n-i}$$

Transfer function $$H(z) = \sum_{i=0}^{N} b_i \, z^{-i} \longrightarrow b_i = h_i$$ impulse response

1. Design by sampling in frequency

Given: $H_0, H_1, \ldots, H_{L-1}$ – values of frequency response at frequencies
0, 1/(LT), 2/(LT), (L-1)/(LT)
where $f_s = 1/T$ – sampling frequency.
Searched: $h_0, h_1, \ldots, h_{L-1}$ – samples of impulse response.

$$\overline{h} = \overline{W}^{-1} \, \overline{H} = IDFT\,(\overline{H})$$

The impulse response is obtained using the inverse discrete Fourier transform (IDFT) of the frequency response values.

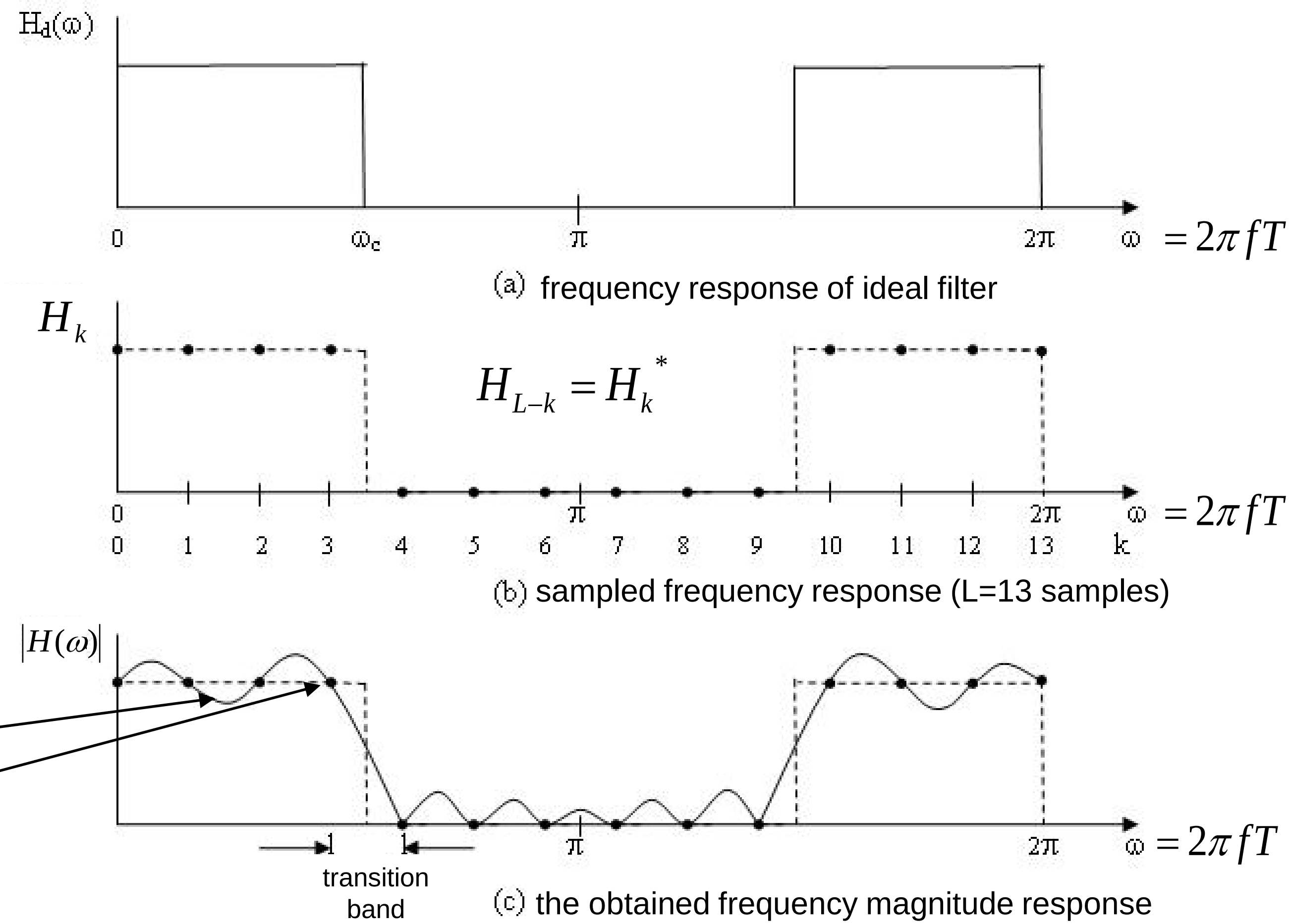
FIR filters
design by sampling in frequency

$$h_0, h_1, \ldots, h_{12} =$$
$$= \mathrm{IDFT}[H_0, \ldots, H_{12}]$$

$$H(z) = \sum_{n=0}^{12} h_n z^{-n}$$

$$H_f(f) = H(z)\Big|_{z=e^{j2\pi fT}}$$

$$H_f\left(\tfrac{k}{LT}\right) = H_k$$

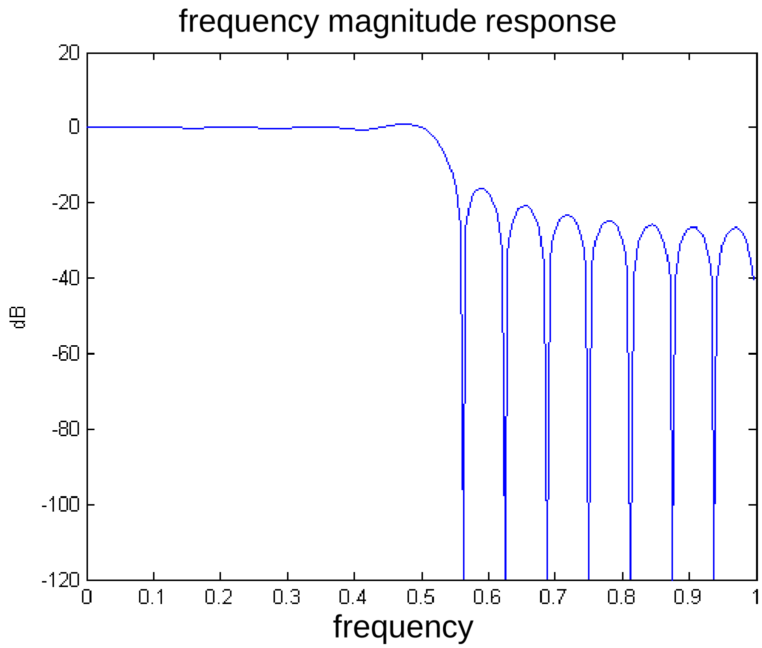


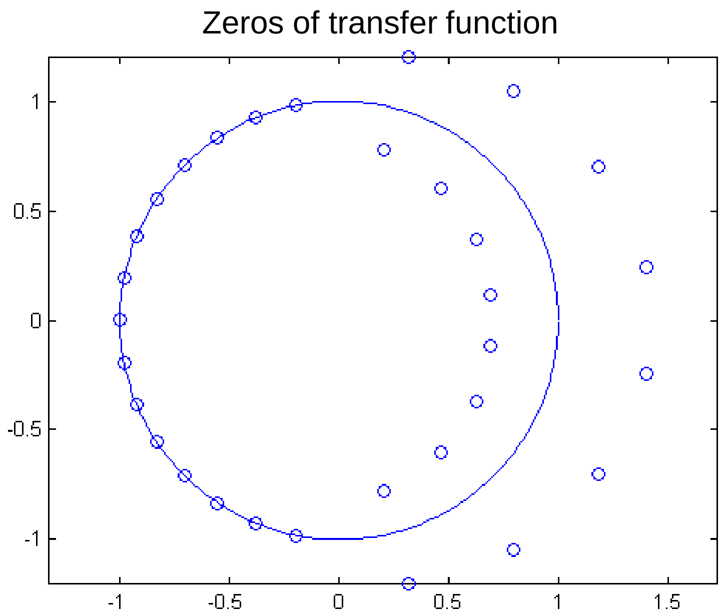
(a) frequency response of ideal filter

(b) sampled frequency response (L=13 samples)

(c) the obtained frequency magnitude response

FIR filters
design by sampling in frequency





Design of a lowpass filter:
sampling frequency=2, passband width = 0.5, L=32 samples

FIR filters
design by Chebyshev approximation

Required frequency magnitude response: $H_d(\omega)$, $\omega=2\pi fT$

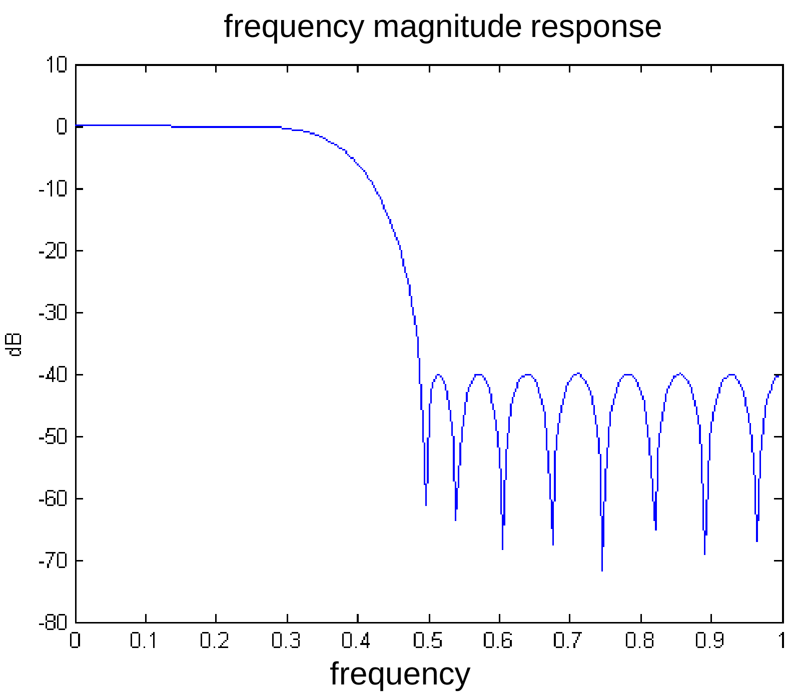
Chebyshev approximation $H(\omega)=\sum_{n=0}^{M} c_n \cos(n\omega)$

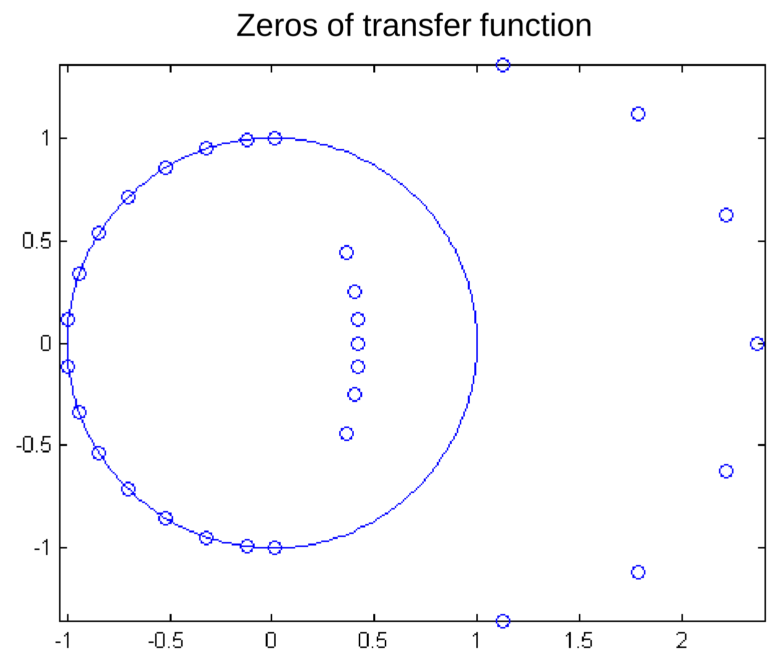
Error function $E(\omega)=W(\omega)\left|H(\omega)-H_d(\omega)\right|$
(W(ω) – weighting function)

Minimization criterion: mini-max (minimization of the greatest value of E(ω))

Minimization procedure: Remez algorithm (looking for M+2 frequencies, at which the error function E(ω) exhibits the greatest values)

FIR filters
design by Chebyshev approximation





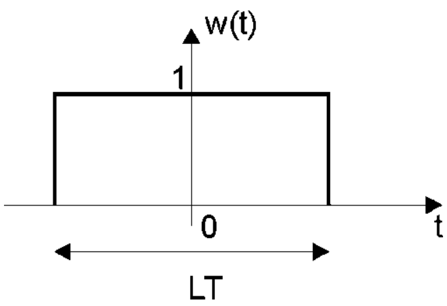
Design of a lowpass filter:
sampling frequency=2, passband width = 0.5, L=31 samples

FIR filters
design by windowing in time domain

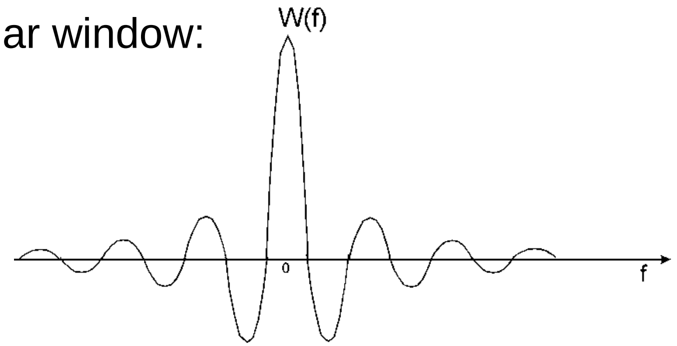
time	frequency
Impulse response of the ideal filter: h(t)	Ideal frequency response: $H_d(f)$
Limiting duration of h(t) to LT seconds, using a window w(t): $\bar{h}(t)=h(t)w(t)$	Convolution of $H_d(f)$ with Fourier transform of the window: $\bar{H}(f)=H_d(f)*W(f)$
Sampling: L samples, sampling interval T, sampling frequency 1/T	Spectral copies of $\bar{H}(f)$ appear every 1/T [Hz] on frequency axis

FIR filters
design by windowing in time domain

Design of a lowpass filter:
sampling frequency=2,
passband width = 0.5,
L=32 samples

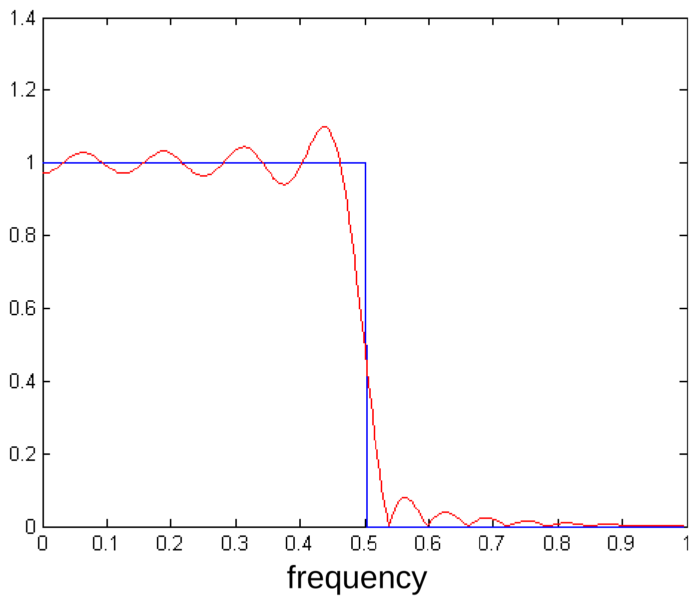


Rectangular window:

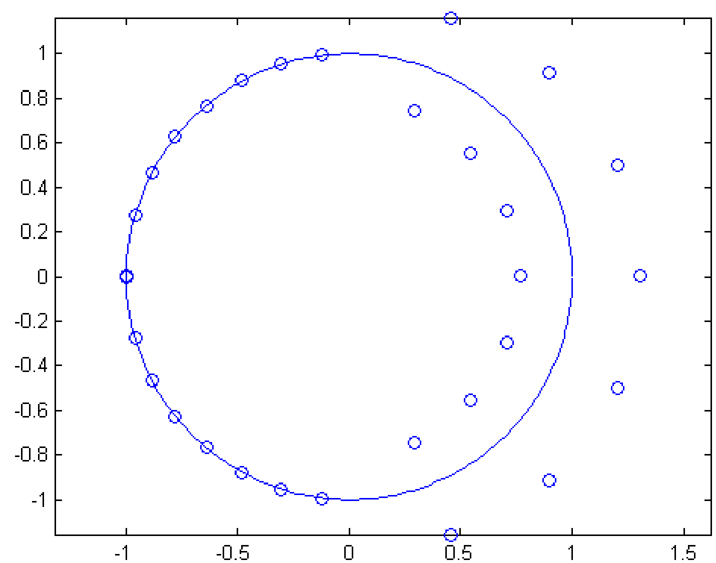


zero crossings: f=1/(LT), -1/(LT), 2/(LT), -2/(LT),...

frequency magnitude response

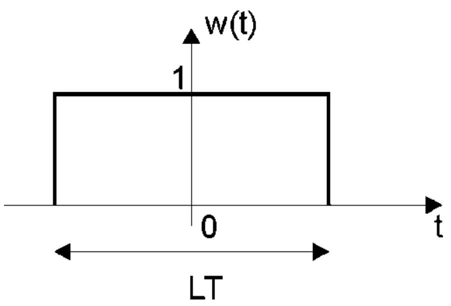


Zeros of transfer function

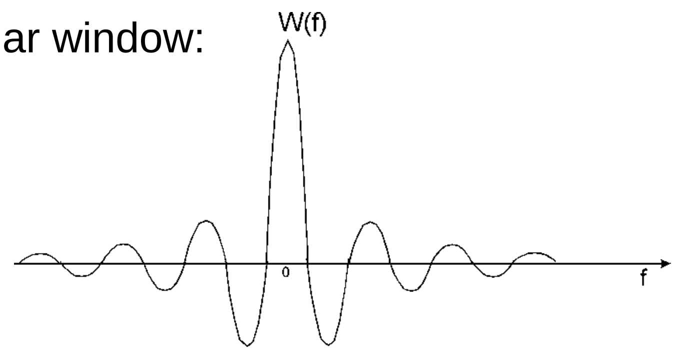


FIR filters
design by windowing in time domain

Design of a lowpass filter:
sampling frequency=2,
passband width = 0.5,
L=32 samples

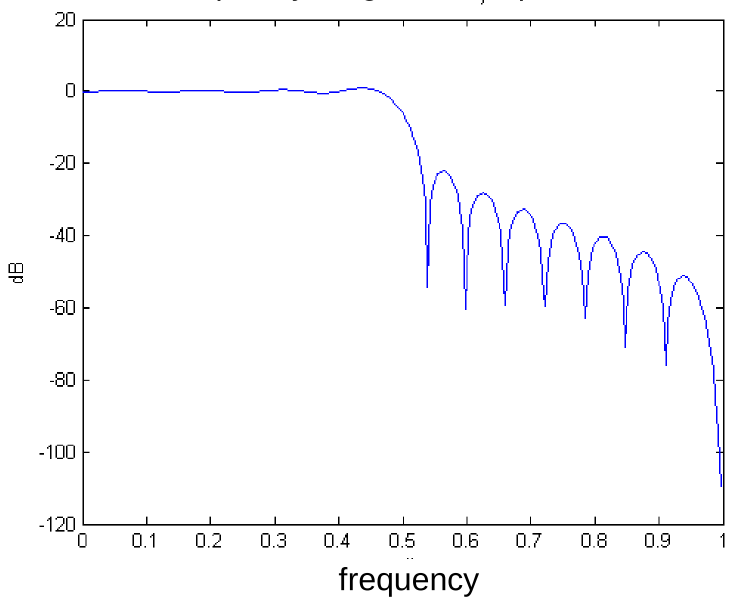


Rectangular window:

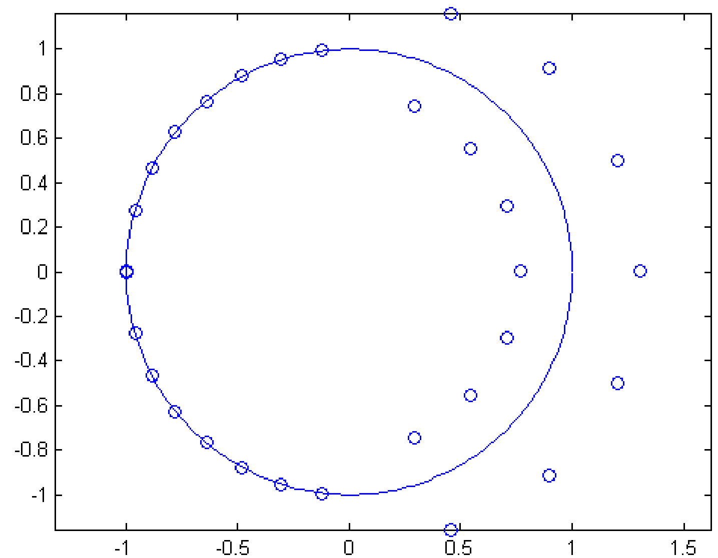


zero crossings: f=1/(LT), -1/(LT), 2/(LT), -2/(LT),...

frequency magnitude response



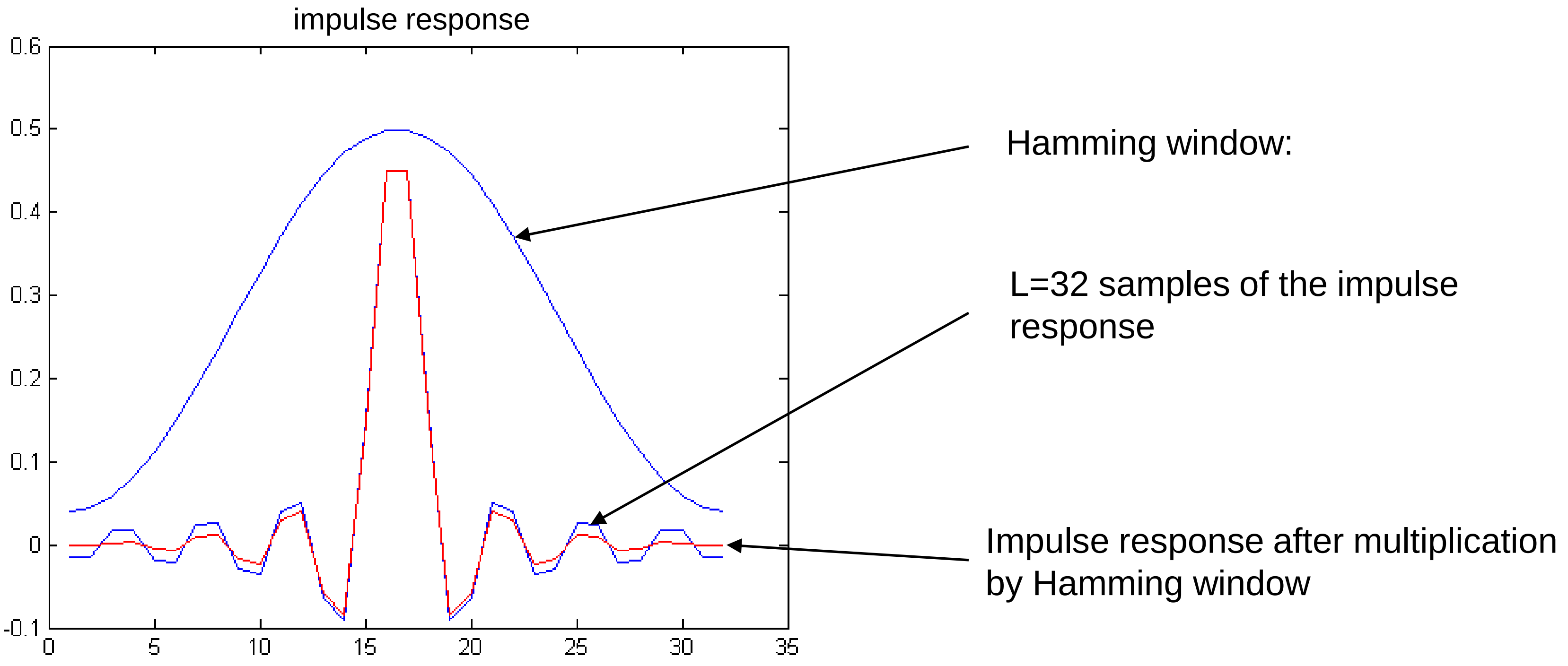
Zeros of transfer function



FIR filters
design by windowing in time domain

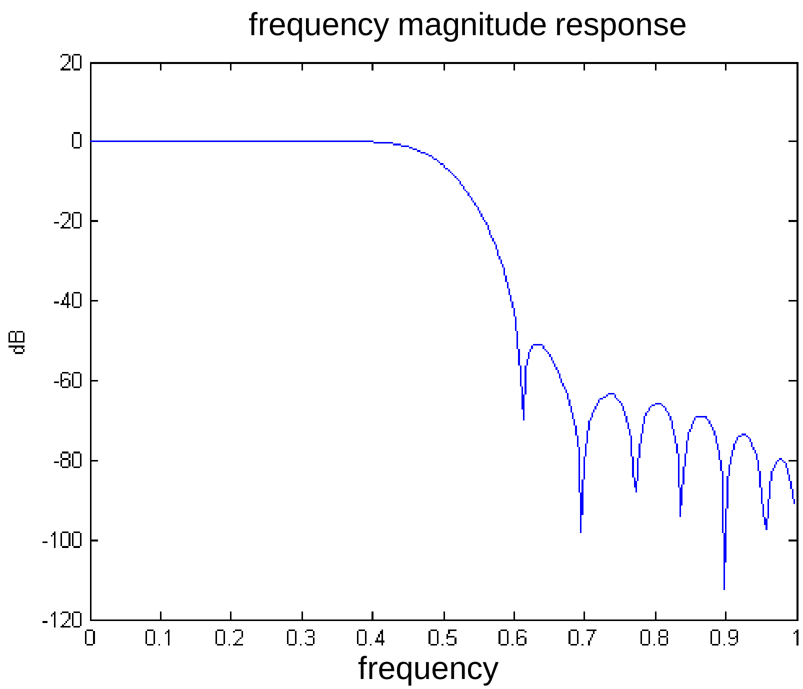
Hamming window:

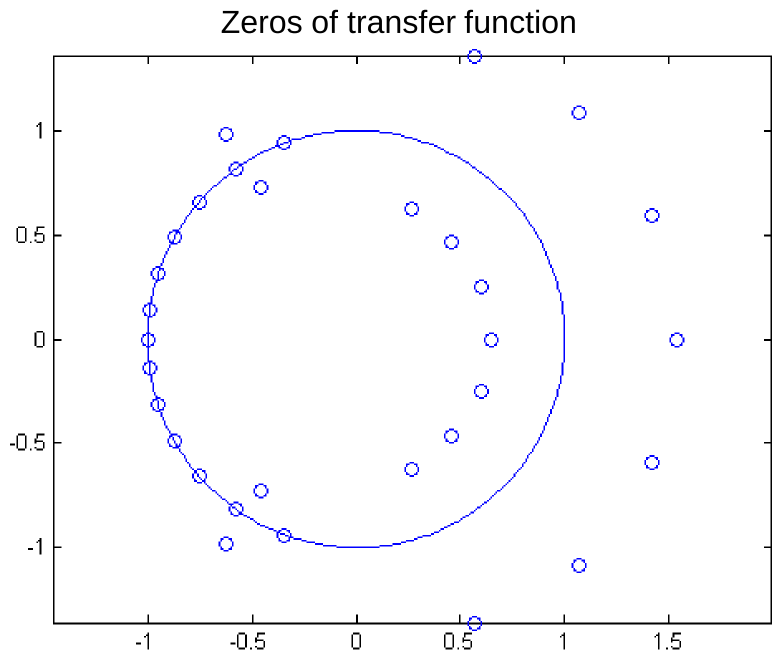
Design of a lowpass filter:
sampling frequency=2, passband width = 0.5,
L=32 samples



FIR filters
design by windowing in time domain

Design of a lowpass filter:
sampling frequency=2, passband width = 0.5,
L=32 samples, Hamming window





Infinite impulse response filters (IIR)

Butterworth filters (number of zeros N = number of poles M)

Lowpass Butterworth filter

$$H(z) = \frac{B(z)}{A(z)} = \frac{g(z+1)^M}{\prod_{i=1}^{M}(z - z_i)}$$

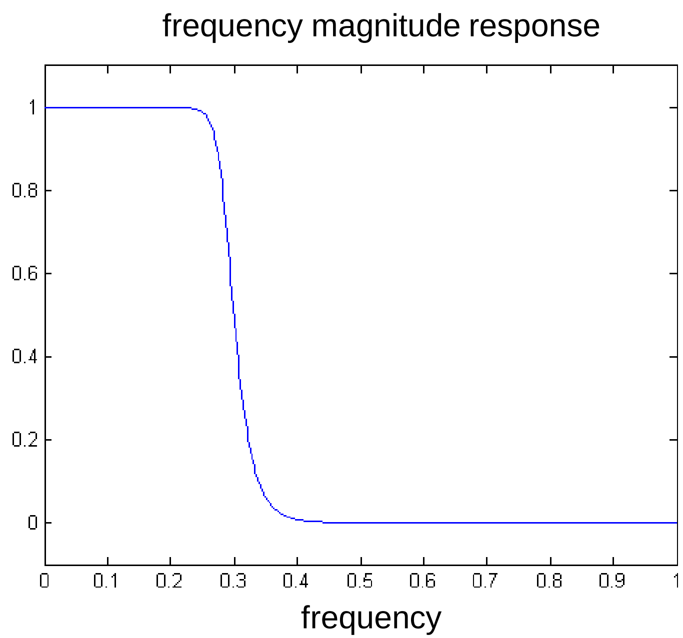
where $z_1,...z_M$ – poles (real or complex conjugates)

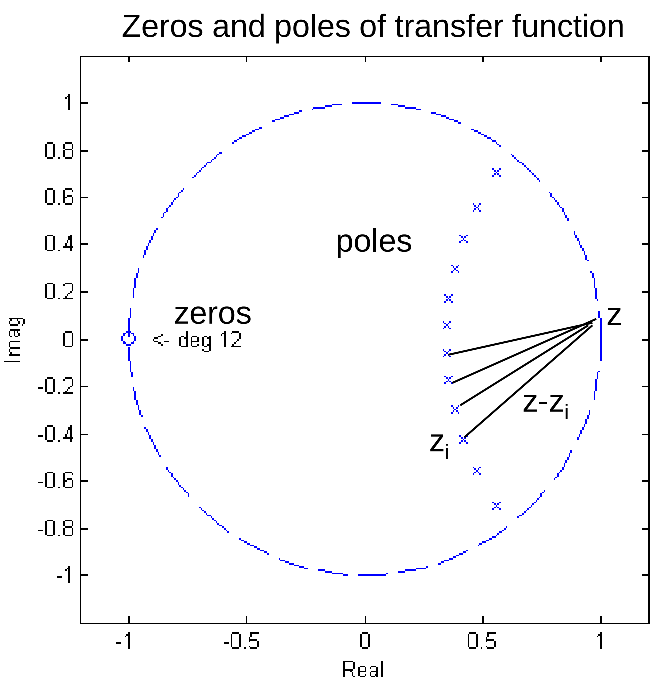
Butterworth filter has no passband and stopband ripples.

Butterworth IIR filters

Design of a lowpass filter:
sampling frequency=2, passband width = 0.3,
M=12

$$H(z) = \frac{B(z)}{A(z)} = \frac{g(z+1)^M}{\prod_{i=1}^{M}(z - z_i)}$$



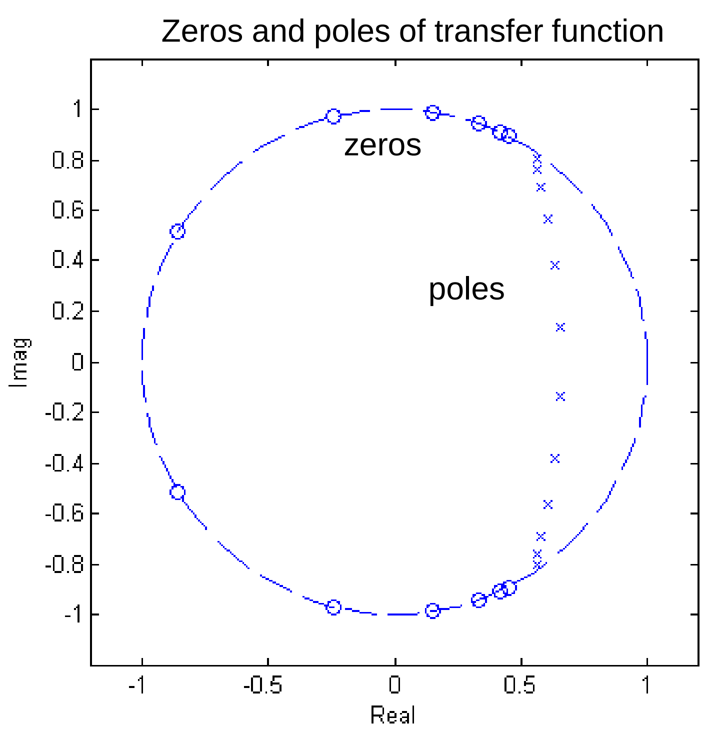


In passband $\prod_{i=1}^{M}\left|(z - z_i)\right| = const$

Elliptic IIR filters

Elliptic (Cauer) filters (number of zeros N = number of poles M)

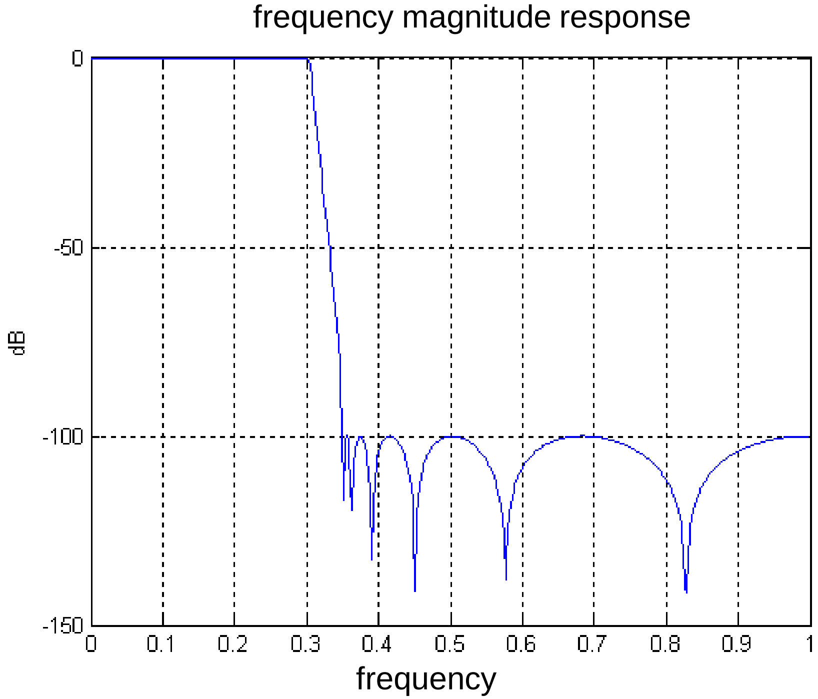
Properties: narrow transition band, small passband ripples,
high stopband attenuation

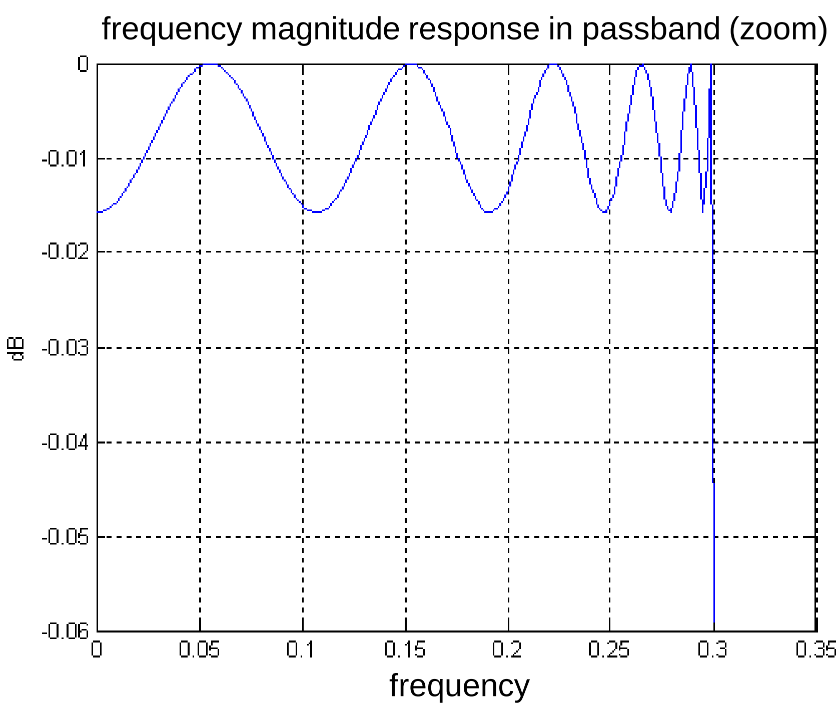


lowpass filter

Elliptic IIR filters

Design of a lowpass filter:
sampling frequency=2, passband width = 0.3,
M=12





Highpass, bandpass and stopband filters

Having a lowpass filter $H_{LP}(z)$

we may obtain highpass filter with a substitution: $H_{HP}(z) = H_{LP}(-z)$

Frequency response of highpass filter will be a mirror image of frequency response of lowpass filter. Stability will be preserved.

Bandpass filter may be obtained by subtraction of transfer functions of two lowpass filters.

Stopband filter for suppression of narrowband distortions , e.g. hum of power network 50 Hz, may be obtained by zeros of transfer function at

z= $\exp(\pm j\varphi)$, where $\dfrac{\varphi}{2\pi} = \dfrac{50}{f_s}$ (f_s is sampling frequency)

In order to reduce attenuation at the other frequencies, poles may be inserted ,

at z = $\rho \exp(\pm j\varphi),\qquad \rho < 1$

Decibels (revision)

In the last slides, frequency magnitude response is given in decibels [dB].
Decibels are used for comparison of two signal powers or amplitudes.
Ratio of a measured power P to the reference power P_r, expressed in dB, equals

$$P_{dB} = 10\log_{10}\frac{P}{P_r}$$

Instead of powers we may compare amplitudes od two signals of the same kind.
Because power is proportional to the squared amplitude (for example $P=A^2/2$ for sine signals), we may rewrite the formula as follows:

$$P_{dB} = 10\log_{10}\frac{A^2}{A_r^2} = 10\log_{10}\left(\frac{A}{A_r}\right)^2 = 20\log_{10}\left(\frac{A}{A_r}\right)$$

In the last slides, reference power and reference amplitude are equal to one.
Frequency magnitude response equal to 0 dB means that $|H_s(f)| = 1$.
Frequency magnitude response equal to -100 dB means that $|H_s(f)| = 10^{-5}$.
See table in the following slide

Decibels

P/P_r	A/A_r	P_{dB}
1	1	0 dB
2	$\sqrt{2}$	3 dB
4	2	6 dB
10	$\sqrt{10}$	10 dB
100	10	20 dB
1000	$10\sqrt{10}$	30 dB
10^n	$\sqrt{10^n}$	10 n dB
$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	-3 dB
$\frac{1}{4}$	$\frac{1}{2}$	-6 dB
0.1	$\sqrt{0.1}$	-10 dB
0.01	0.1	-20 dB
0.001	$\sqrt{0.001}$	-30 dB
10^{-n}	$10^{-n/2}$	-10 n dB