LTI (linear time invariant) systems

$$\{x_n\} \longrightarrow \boxed{H} \longrightarrow \{y_n\} = H[\{x_n\}]$$

Input δ_n (Kronecker delta) $\longrightarrow$ output $y_n = h_n$ (impulse response)
Input δ_{n-k} (delayed delta) $\longrightarrow$ output $y_n = h_{n-k}$
Input $x_k\,\delta_{n-k}$ $\longrightarrow$ output $y_n = x_k\,h_{n-k}$

Input $\{x_n\} = \sum_k x_k\,\delta_{n-k}$ (series of samples) $\longrightarrow$ output $\boxed{y_n = \sum_{k=-\infty}^{\infty} x_k\,h_{n-k} = x_n * h_n}$

convolution

Discrete – time LTI system is described with convolution

For **causal** systems: $h_n = 0,\ n < 0$

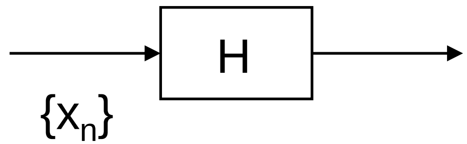
$$y_n = \sum_{k=-\infty}^{n} x_k\,h_{n-k}$$

In Z transform domain:

$\boxed{Y(z) = X(z)\,H(z)}$, $\quad H(z) = Z[\{h_n\}]$ $\longrightarrow$ **transfer function** of the system

The output signal: $\quad y_n = Z^{-1}[Y(z)] = Z^{-1}[X(z)H(z)]$
(see the problem: calculation of convolution)

LTI systems - stability



$$y_n = \sum_{k=-\infty}^{\infty} x_k h_{n-k} = x_n * h_n$$

$$Y(z) = X(z)H(z), \qquad H(z) = Z[\{h_n\}] \longrightarrow$$ **transfer function** of the system

The output signal: $$y_n = Z^{-1}[Y(z)] = Z^{-1}[X(z)H(z)]$$

Stability / instability: Stable filter is a **BIBO** (Bounded Input – Bounded Output) filter. If the input signal is bounded, then the output signal is also bounded.

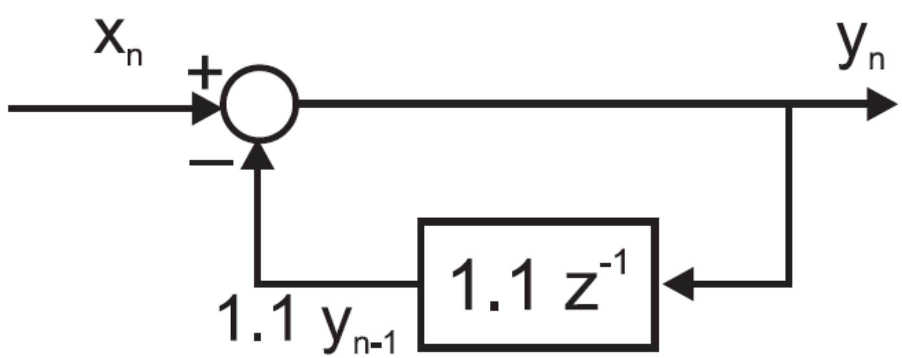
> If at least one pole of H(z) stems out of this circle, then $y_n \to \infty$
> i.e. the system H(z) is unstable.

An example:
Transfer function $$H(z) = \frac{z^2}{z^2 - z + 0.5}$$ Is this filter stable?

Poles are the roots of the polynomial $z^2 - z + 0.5$
We solve the equation $z^2 - z + 0.5 = 0$ and obtain two roots: $z_1 = 0.5 + j0.5$
and $z_2 = 0.5 - j0.5$
Poles lie in the circle of radius =1 so the filter is stable. $$|z_1| = |z_2| = \frac{1}{\sqrt{2}} < 1$$

LTI systems - stability

A discrete time system (digital filter) is described with the block diagram
Is it a FIR or IIR system?
Calculate its transfer function $H(z) = \frac{Y(z)}{X(z)}$
Is this filter stable or unstable?



Solution:
From the block diagram we may obtain difference equation describing this filter:
We read $y_n = x_n - 1.1y_{n-1}$, then we put y to the left side: $y_n + 1.1y_{n-1} = x_n$.
Calculating Z transform of this equation we obtain: $Y(z) + 1.1z^{-1}Y(z) = X(z)$, finally we calculate $H(z)$:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 + 1.1z^{-1}} = \frac{z}{z + 1.1}$$

Transfer function may be obtained directly from the block diagram, by reading: $Y(z) = X(z) - 1.1z^{-1}Y(z)$.
This filter is an IIR filter, because the current output value y_n depends on a previous value y_{n-1}.
Transfer function has a pole at $z_1 = -1.1$. Because $|z_1| > 1$, this filter is unstable.

LTI systems - stability

Transfer function of a LTI filter equals: $H(z) = 1 - 2z^{-2} + 0.5z^{-4}$.
Is it FIR or IIR filter?
Is it stable?
Calculate the impulse response h_n of this filter.

Solution:
It is a FIR filter, because there is no polynomial in denominator of $H(z)$. We may rewrite $H(z)$ as follows

$$H(z) = 1 - 2z^{-2} + 0.5z^{-4} = \frac{z^4 - 2z^2 + 0.5}{z^4}$$

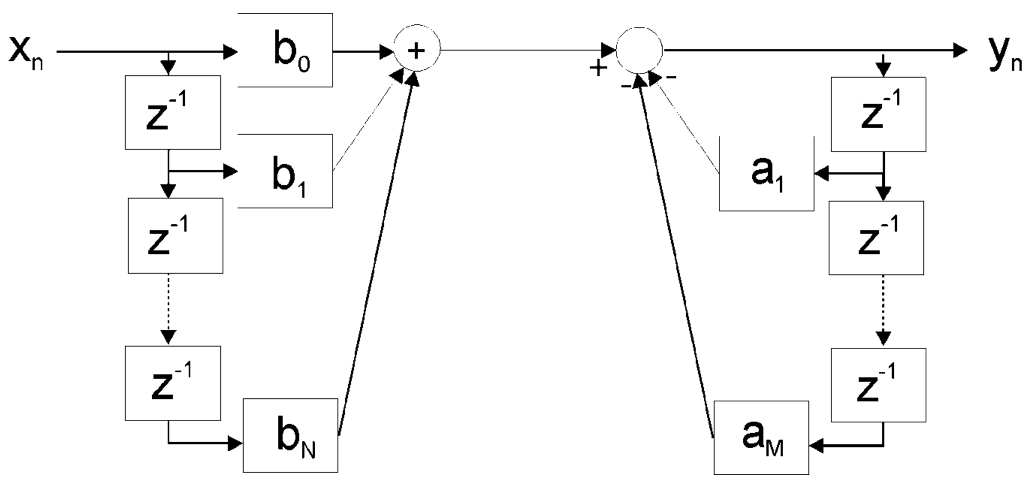
but $\frac{1}{z^4} = z^{-4}$ (a pole at $z = 0$) represents only time delay.
As a FIR filter, this filter is stable.
The impulse response may be read directly from $H(z)$:

$$H(z) = \sum_n h_n z^{-n} = 1 - 2z^{-2} + 0.5z^{-4}$$

We see that $h_0 = 1$, $h_1 = 0$, $h_2 = -2$, $h_3 = 0$, $h_4 = 0.5$, $h_5 = 0$, $h_6 = 0$, ...

Transversal structure of digital filter

$$y_n = \sum_{i=0}^{N} b_i\, x_{n-i} - \sum_{i=1}^{M} a_i\, y_{n-i}$$



FIR part is always stable IIR part may be stable or not

Transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{N} b_i\, z^{-i}}{\sum_{i=0}^{M} a_i\, z^{-i}} = \frac{z^{-N} \sum_{i=0}^{N} b_i\, z^{N-i}}{z^{-M} \sum_{i=0}^{M} a_i\, z^{M-i}} = \frac{B(z)}{A(z)}$$

Difference equations – an example

A filter is described with the following difference equation

$$x_n \longrightarrow \boxed{H(z)} \longrightarrow \qquad y_n = x_n + \tfrac{1}{4} y_{n-1} + \tfrac{1}{8} y_{n-2}$$

Is it FIR or IIR filter? Calculate the transfer function H(z) . Is this filter stable?

y is put to the left side: $y_n - \tfrac{1}{4} y_{n-1} - \tfrac{1}{8} y_{n-2} = x_n$

Z transform: $Y(z) - \tfrac{1}{4} z^{-1} Y(z) - \tfrac{1}{8} z^{-2} Y(z) = X(z)$ because $Z[y_{n-i}] = z^{-i} Y(z)$

$$\longrightarrow \qquad Y(z)[1 - \tfrac{1}{4} z^{-1} - \tfrac{1}{8} z^{-2}] = X(z)$$

Transfer function $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1}{1 - \frac{1}{4} z^{-1} - \frac{1}{8} z^{-2}} = \dfrac{z^2}{z^2 - \frac{1}{4} z - \frac{1}{8}}$

Rational function,
has 2 zeros and 2 poles

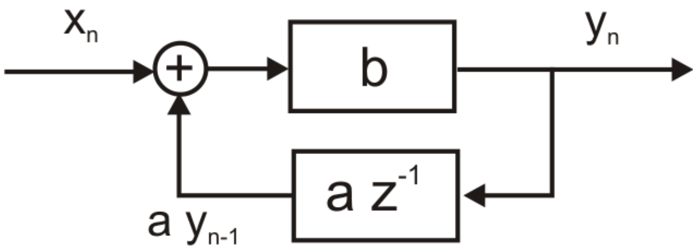
This is an IIR filter, because the current output sample y_n depends on previous output samples y_{n-1}, y_{n-2} (see difference equation).

Moreover, the transfer function has poles (roots of the polynomial in denominator). These roots are $z_1 = \tfrac{1}{2}$ and $z_2 = -\tfrac{1}{4}$. They lie in the unit circle, so the filter is stable.

Block diagrams

Filter may be described with the block diagram. Block diagram (block scheme) contains sufficient information to obtain difference equation or transfer function.

Example:



Operator z^{-1} represents delay of one sample (T seconds, where T is sampling interval).

From the block diagram we read: $y_n = b(x_n + a\, y_{n-1})$

y is put to the left side: $y_n - ab\, y_{n-1} = b\, x_n$

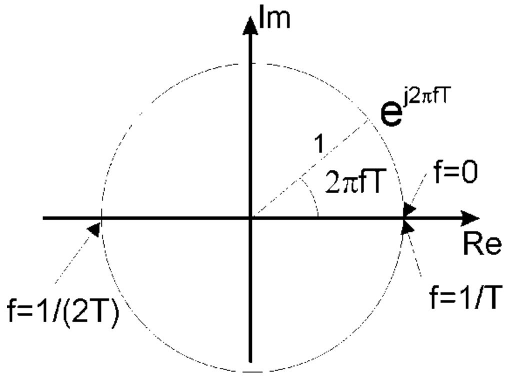
Z transform: $Y(z) - abz^{-1}Y(z) = bX(z)$ because $Z[y_{n-i}] = z^{-i}Y(z)$

$$\longrightarrow \quad Y(z)[1 - abz^{-1}] = bX(z) \quad \rightarrow \quad H(z) = \frac{Y(z)}{X(z)} = \frac{b}{1 - abz^{-1}} = \frac{b\,z}{z - ab}$$

Transfer function may be obrained directly from the block diagram: $Y(z) = b[X(z) + az^{-1}Y(z)]$

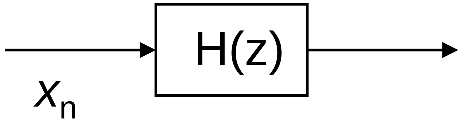
This filter is stable, if |ab| < 1 (z=ab is a pole)
If |ab| =1 , it is on the edge on stability
If |ab| > 1 it is unstable

Frequency response



The substitution $z = e^{j2\pi fT}$ leads us to frequency domain (DTFT)

T – sampling period
1/T – sampling frequency



$$y_n = \sum_{k=-\infty}^{\infty} x_k h_{n-k} = x_n * h_n$$

$$Y(z) = X(z)H(z)$$

$$Y\left(e^{j2\pi fT}\right) = X\left(e^{j2\pi fT}\right) \mathrm{H}\left(e^{j2\pi fT}\right)$$

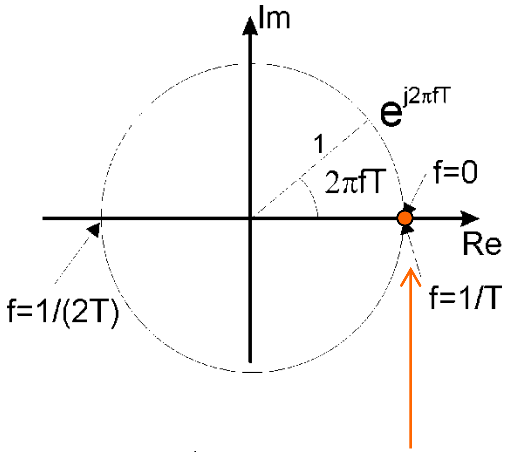
$$Y_S(f) = X_S(f)\, H_S(f)$$

$$H(z) = H(e^{j2\pi fT}) = H_s(f)$$

frequency response of
discrete time filter
(a complex function!)

Attention!
This is valid only for stable systems.
Unstable system has no frequency response

Zeros and poles of the transfer function and their influence on frequency response - example



Transfer function of a filter equals $H(z) = 1 - z^{-1}$

It is a FIR filter, B(z)= 1- z $^{-1}$, A(z) = 1.
The coefficients b_0=1 and b_1= -1 are the samples of the impulse response of this filter: h_0=1 and h_1= -1
As a FIR filter it is a stable filter.

$H(z) = 1 - z^{-1} = \frac{z-1}{z}$ has one zero at z=1.
Pole at z=0 has no influence on the frequency response and stability (it is only a time shift)

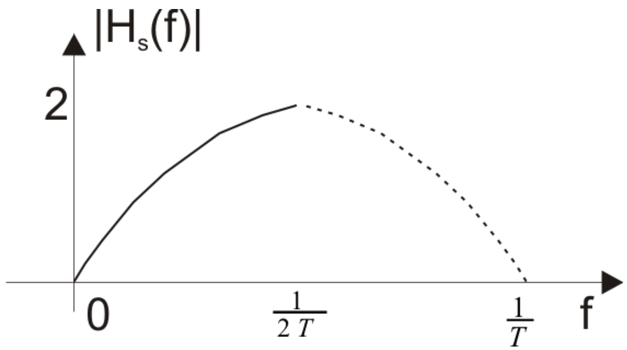
Using the substitution $z = e^{j2\pi fT}$ we read the frequency response of our filter on the unit circle:

$H_s(f) = H(e^{j2\pi fT})$ At f=0, z=1 and H(z) for z=1 is equal to zero. Our filter stops frequency equal to zero (signals of constant values): $H_s(0)=0$. It is due to zero of the polynomial (see red point).
At frequency equal to half of the sampling frequency $f = \frac{1}{2T}$ z = -1 and H(z) = 2 . Our filter is a highpass filter.

Zeros and poles of the transfer function and their influence on frequency response - example

Without detailed calculation we may draw the approximate frequency magnitude response

of the filter $H(z) = 1 - z^{-1}$



We may substitute $z = e^{j2\pi fT}$ to H(z): $H_s(f) = H[e^{j2\pi fT}] = 1 - e^{-j2\pi fT}$

and calculate frequency magnitude response $|H_s(f)|$ for each frequency value.

Zeros and poles of the transfer function and their influence on frequency response - example

Two filters are described with their transfer functions: $H_1(z) = 1 + 0.9z^{-1}$ and $H_2(z) = \frac{1}{1+0.9z^{-1}}$. Which one is a lowpass filter, and which one the highpass filter?

Solution:
The first filter is a FIR filter and the second one - IIR filter. The first filter has zero at $z_1 = -0.9$ and the second one a pole at $z_2 = -0.9$.
Zero on the left side of z plane yields attenuation of high frequencies (that is to say, the first filter is a lowpass filter). Pole on the left side of z plane yields amplification of high frequencies (that is to say, the second filter is a highpass filter).
We may easily calculate the frequency response at frequency $f = 0$, by substitution $z = 1$ to $H(z)$. Frequency response at frequency $f = \frac{f_s}{2}$ (half of the sampling frequency) is obtained by substitution $z = -1$.
For the first filter we obtain $H_1(1) = 1 + 0.9 = 1.9$, $H_1(-1) = 1 - 0.9 = 0.1$. Indeed, it is a lowpass filter.
For the second filter we obtain $H_2(1) = \frac{1}{1+0.9} = \frac{1}{1.9} \simeq 0.53$, $H_2(-1) = \frac{1}{1-0.9} = 10$. It is a highpass filter.