EPRST: Probability and Statistics
Problem set 3

1. Let X be a random variable with the following probability mass function:

$$p_X(x) = \frac{x^2}{a} \text{ for } x \in \{-2,-1,1,2\} \text{ and } p_X(x) = 0 \text{ for } x \notin \{-2,-1,1,2\},$$

where a is a real parameter. Find a.

2. Let $\Omega = \{0,1,2,3\}$, $\mathbb{P}(\{k\}) = \frac{1}{4}$ for $k = 0,1,2,3$. Define two random variables $X(\omega) = \sin\frac{\pi\omega}{2}$ and $Y(\omega) = \cos\frac{\pi\omega}{2}$. Find the probability mass functions (PMFs) of the random variables X and Y. Compute $\mathbb{P}(\{\omega \in \Omega : X(\omega) = Y(\omega)\})$.

3. From a deck of 52 cards we draw 6 and we consider the number of spades among the 6 cards. Determine the distribution of the random variable defined this way.

4. There are 10 coins, and 2 of them have tails on both sides. We pick one coin at random and start to toss it. Find the distribution of the number of tosses if:

   (a) we toss the picked coin until we get tail,

   (b) we toss the picked coin until we get two tails (not necessarily in a row),

   (c) we toss the picked coin until we get tail but not more than 4 times.

5. A fair die is thrown until the sum of the results of the throws exceeds 6. The random variable X is the number of throws needed for this. Find $\mathbb{P}(X \leq 1)$, $\mathbb{P}(X \leq 2)$ and $\mathbb{P}(X \leq 7)$.

6. (a) A random variable X is defined as the larger of the scores obtained in two throws of a fair six-sided die. Find the distribution of the random variable X.

   (b) A random variable Y is defined as the highest score obtained in k throws of a fair six-sided die. Determine the probability mass function of Y.

7. A white die and a red die are thrown at the same time and the difference $W - R$ is observed, where R is the number on top of the red die and W is that on top of the white. Find the probability mass function of this difference $W - R$.

8. You just rented a large house and the realtor gave you 5 keys, one for each of the 5 doors of the house. Unfortunately, all keys look identical. so to open the front door, you try them at random. (a) Find the PMF of the number of trials you will need to open the door, under the following alternative assumptions: (1) after an unsuccessful trial. you mark the corresponding key. so that you never try it again. and (2) at each trial you are equally likely to choose any key. (b) Repeat part (a) for the case where the realtor gave you an extra duplicate key for each of the 5 doors.

9. Two girls want to meet at a specified place between 4 and 5 p.m.. Let T denote the random variable describing the waiting time of the girl who arrived first. Find the distribution of the random variable T.

10. The cumulative distribution function of a discrete random variable X is given by

$$F_X(t) = \begin{cases} 0, & t < 0, \\ \frac{1}{3}, & 0 \leq t < \frac{1}{2}, \\ a, & \frac{1}{2} \leq t < \frac{3}{4}, \\ 1, & t \geq \frac{3}{4}. \end{cases}$$

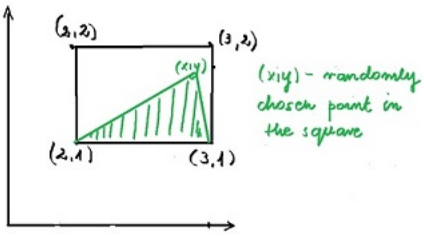
Determine the constant a, if $\mathbb{P}\left(X > \frac{5}{8}\right) = \frac{1}{2}$ and the probability mass function of X.

11. Let X be a result of a student on the exam, this can be any number between $[0,1]$. The probability density function of X is given by

$$f(x) = \begin{cases} 4x, & 0 \leq x \leq \frac{1}{2}, \\ 4 - 4x, & \frac{1}{2} \leq x \leq 1, \\ 0, & \text{otherwise}. \end{cases}$$

Suppose that the student passes the exam if his result is at least 0.6. Compute the probability that the studnet pass the exam.

12. We choose randomly a point from the square with corners at $(2,1)$, $(3,1)$, $(2,2)$, $(3,2)$. The random variable X is the area of the triangle with the corners at $(2,1)$, $(3,1)$ and the chosen point:



(a) What is the largest value that X can take on?

(b) Determine the cumulative distribution function of X.