Z Transform

A series of samples $\{x_n\}$, ideal sampling:

$$x_s(t) = \sum_n x_n \delta(t - nT)$$

T – sampling period

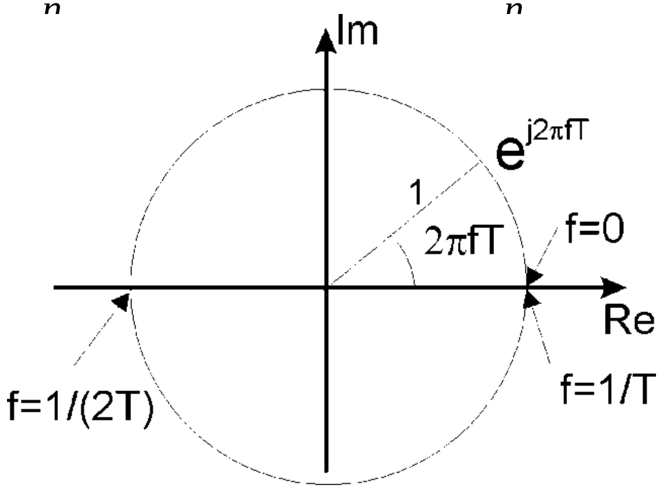
Spectrum (DTFT)

$$X_s(f) = \sum_n x_n F[\delta(t - nT)] = \sum_n x_n e^{-j2\pi fnT}$$

substitution: $\boxed{z = e^{j2\pi fT}}$

$$z = e^{j2\pi fT} = \cos(2\pi fT) + j\sin(2\pi fT)$$

$$|e^{j2\pi fT}| = \sqrt{\cos^2(2\pi fT) + \sin^2(2\pi fT)} = 1$$



f:	0	1/(4T)	1/(2T)	3/(4T)	f_s=1/T
z:	1	j	-1	-j	1

Z Transform

A series of samples $\{x_n\}$, ideal sampling:

$$x_s(t) = \sum_n x_n \delta(t - nT)$$

T – sampling period

Spectrum (DTFT)

$$X_s(f) = \sum_n x_n F[\delta(t - nT)] = \sum_n x_n e^{-j2\pi fnT}$$

substitution:

$$\boxed{z = e^{j2\pi fT}}$$

$$X_s(f) = \sum_n x_n e^{-j2\pi fnT} = \sum_n x_n z^{-n}$$

Z Transform:

$$\boxed{X(z) = Z[\{x_n\}] = \sum_n x_n z^{-n}}$$

Z transform is defined for any complex variable z.

Reading X(z) on the unit circle we obtain Fourier Transform of sampled signal (DTFT)

Properties of Z transform

Linearity $Z[\{ax_n + by_n\}] = aX(z) + bY(z)$

Shift (translation) $Z[\{x_{n+k}\}] = \sum_n x_{n+k} z^{-n} = z^k \sum_n x_{n+k} z^{-(n+k)} = z^k X(z)$

z^{-1} - delay by 1 sample (time interval T):

Attenuation (modulation) $Z[\{x_n a^n\}] = \sum_n x_n a^n z^{-n} = \sum_n x_n \left(z/a\right)^{-n} = X[\frac{z}{a}]$

Convolution $y_n = \sum_{k=-\infty}^{\infty} x_k h_{n-k} \quad \Rightarrow \quad y_n = x_n * h_n$

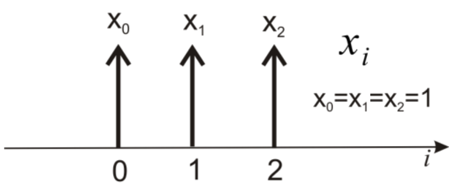
$$Y(z) = \sum_{n=-\infty}^{\infty} y_n z^{-n} = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_k h_{n-k} z^{-n} = \sum_{k=-\infty}^{\infty} x_k \sum_{n=-\infty}^{\infty} h_{n-k} z^{-n} =$$

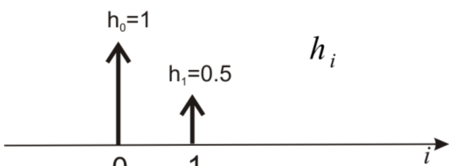
$$= \sum_{k=-\infty}^{\infty} x_k z^{-k} \sum_{n=-\infty}^{\infty} h_{n-k} z^{-(n-k)} = X(z) \cdot H(z)$$

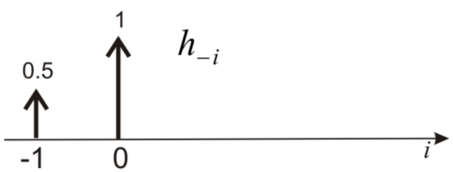
Calculation of convolution

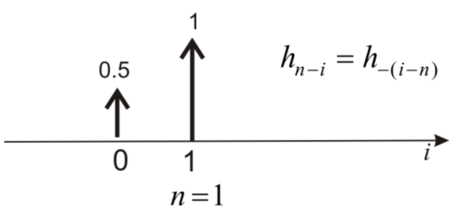
Calculation in time domain

$$y_n = \sum_{i=-\infty}^{\infty} x_i h_{n-i}$$









n=0, y_0=1x1=1
n=1, y_1=1x0.5 + 1x1=1.5
n=2, y_2=1x0.5 + 1x1=1.5
n=3, y_3=1x0.5 =0.5

Calculation in transform domain

$$Y(z) = X(z) \cdot H(z)$$

$$X(z) = 1 + z^{-1} + z^{-2}$$

$$H(z) = 1 + \tfrac{1}{2} z^{-1}$$

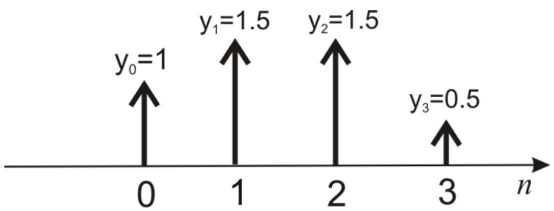
$$Y(z) = X(z)H(z) =$$

$$= 1 + \tfrac{1}{2} z^{-1} + z^{-1} + \tfrac{1}{2} z^{-2} + z^{-2} + \tfrac{1}{2} z^{-3} =$$

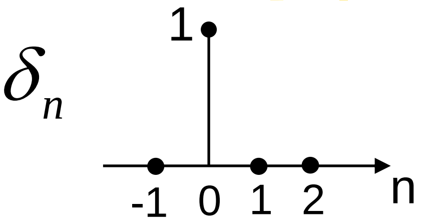
$$= 1 + \tfrac{3}{2} z^{-1} + \tfrac{3}{2} z^{-2} + \tfrac{1}{2} z^{-3} =$$

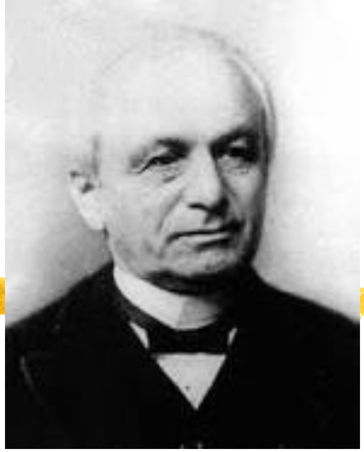
$$= y_0 + y_1 z^{-1} + y_2 z^{-2} + y_3 z^{-3}$$

Both methods give the same result



Kronecker delta





Leopold Kronecker
1823 - 1891

Convolution with δ_n:

$$y_n = x_n * \delta_n = \sum_{k=-\infty}^{\infty} x_k \delta_{n-k} = x_n$$

Convolution with shifted δ_n :

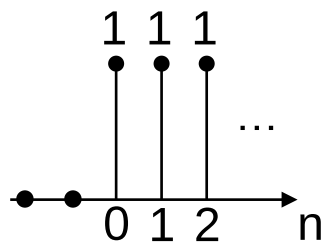
$$y_n = x_n * \delta_{n-m} = \sum_{k=-\infty}^{\infty} x_k \delta_{n-m-k} = x_{n-m}$$

Z transform of Kronecker delta and shifted Kronecker delta:

$$Z[\delta_n] = \sum_n \delta_n z^{-n} = 1 \qquad Z[\delta_{n-m}] = \sum_n \delta_{n-m} z^{-n} = z^{-m}$$

Step function

Discrete step function 1_n



$$Z[1_n] = \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} (\tfrac{1}{z})^n = \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1}$$

Convolution with the step function: $$y_n = x_n * 1_n = \sum_{k=-\infty}^{\infty} x_k 1_{n-k} \xrightarrow{k \le n} \sum_{k=-\infty}^{n} x_k$$

= cumulative summation of x_k

$$Y(z) = Z[x_n * 1_n] = X(z)\frac{z}{z-1}$$

Causal series

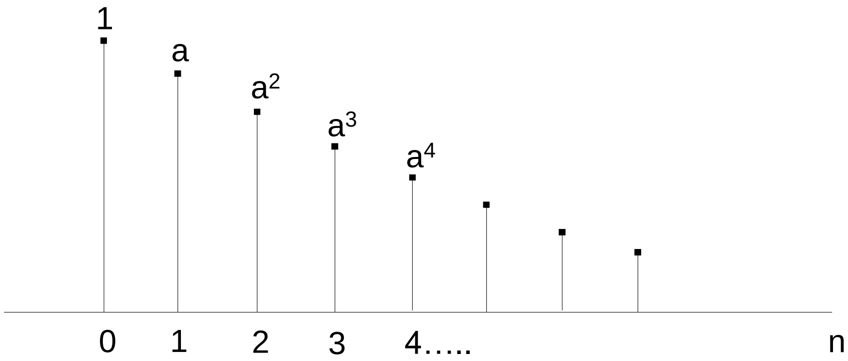
$$x_n = 0, \quad n < 0$$

Right sided Z transform:

$$X(z) = \sum_{n=0}^{\infty} x_n z^{-n}$$

Attenuation property

$$y_n = a^n 1_n \xrightarrow{|a|<1} Y(z) = \frac{\frac{z}{a}}{\frac{z}{a} - 1} = \frac{z}{z-a}$$



(exponentially decaying signal)

Geometric series - revision

Calculation of Z transform of signals like $a^n 1_n$ is based on geometric series:

$$x_n = q^n, \quad n = 0,1,2,\ldots$$

Sum of N+1 terms equals: $\displaystyle\sum_{n=0}^{N} x_n = \sum_{n=0}^{N} q^n = \frac{1-q^{N+1}}{1-q}$

Proof:

$$(1-q)\sum_{n=0}^{N} q^n = \sum_{n=0}^{N} q^n - \sum_{n=0}^{N} q^{n+1} = 1 + q + q^2 + \cdots + q^N - q - q^2 - \cdots - q^N - q^{N+1} = 1 - q^{N+1}$$

If $N \to \infty$ and $|q| < 1$ then $q^{N+1} \to 0$ and $\displaystyle\sum_{n=0}^{N} q^n \to \frac{1}{1-q}$

Finally $\displaystyle\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$

Calculation of Z transform

For finite number of samples we use straight method: $$X(z) = \sum_{n=M}^{N} x_n z^{-n}$$

for example $x_1 = 2, x_2 = -1,$ $$X(z) = \sum_{n=1}^{2} x_n z^{-n} = 2z^{-1} - z^{-2}$$

For infinite number of samples we use series theory, for example

$$y_n = a^n 1_n \longrightarrow Y(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (\tfrac{a}{z})^n = \frac{1}{1 - \frac{a}{z}} = \frac{z}{z-a}$$

We may use the properties of Z transform, like shift property or convolution property

Calculation of Inverse Z transform

Inverse Z transform: calculation of time series (samples) knowing X(z): $x_n = Z^{-1}[X(z)]$

If X(z) is a polynomial $X(z) = \sum_{n=M}^{N} x_n z^{-n}$ we use straight method:

for example $X(z) = 2z - 1 + z^{-2} \rightarrow x_{-1} = 2, x_0 = -1, x_2 = 1$

The following result maybe obtained directly or by shifting the previous one (z^{-1} is a shift to the right)

$$X(z) = \frac{2z - 1 + z^{-2}}{z} \rightarrow x_0 = 2, x_1 = -1, x_3 = 1$$

Calculation of Inverse Z transform

$$x_n = Z^{-1}[X(z)]$$

Often we use known Z transforms, like these:

$$Y(z) = \frac{z}{z-a} \quad \rightarrow \quad y_n = a^n 1_n$$

$$Y(z) = \frac{za}{(z-a)^2} \quad \rightarrow \quad y_n = n a^n 1_n$$

We also use properties of Z transform, for example:

$$Z^{-1}[\tfrac{b}{z-a}] = b Z^{-1}[\tfrac{1}{z-a}] = b Z^{-1}[\tfrac{1}{z}\ \tfrac{z}{z-a}] = b\, 1_{n-1} a^{n-1}$$

Above we use the inverse Z transform for $\frac{z}{z-a}$ and shift to the right side, because z^{-1} is a delay of one sample.
Multiplier b is put before the Z^{-1} operation, because this operation is linear.

The inverse Z transform: Z^{-1} calculation by partial fraction expansion

If X(z) is a rational function, the number of poles is equal to the number of zeros and all poles are single, then the following **partial fraction expansion** may be applied:

$$X(z) = \frac{b_m z^m + b_{m-1} z^{m-1} + \ldots + b_0}{a_m z^m + a_{m-1} z^{m-1} + \ldots + a_0} = \frac{(z-\hat{z}_1)(z-\hat{z}_2)\cdots(z-\hat{z}_m)}{(z-z_1)(z-z_2)\cdots(z-z_m)} = \sum_{i=1}^{m} \frac{r_i\, z}{(z-z_i)}$$

Because $Z^{-1}(\frac{z}{z-a}) = a^n 1_n$, then $x_n = \sum_{i=1}^{m} r_i\,(z_i)^n 1_n$

Example:

$$X(z) = \frac{z^2}{(z-0.5)(z-0.6)} = \frac{r_1\, z}{(z-0.5)} + \frac{r_2\, z}{(z-0.6)} \qquad r_1 = -5, \quad r_2 = 6$$

$$x_n = \sum_{i=1}^{2} r_i\,(z_i)^n 1_n = r_1 (z_1)^n 1_n + r_2 (z_2)^n 1_n = (-5 \cdot 0.5^n + 6 \cdot 0.6^n) 1_n$$

The inverse Z transform: Z^{-1} calculation by partial fraction expansion

$$X(z) = \frac{b_m z^m + b_{m-1} z^{m-1} + \ldots + b_0}{a_m z^m + a_{m-1} z^{m-1} + \ldots + a_0} = \frac{(z-\hat{z}_1)(z-\hat{z}_2)\cdots(z-\hat{z}_m)}{(z-z_1)(z-z_2)\cdots(z-z_m)} =$$

$$= \frac{r_1 z}{(z-z_1)} + \frac{r_2 z}{(z-z_2)} + \cdots + \frac{r_m z}{(z-z_m)}$$

How to calculate the coefficients $r_1, r_2, \ldots, r_m$?

Let us multiply both sides by $\frac{z-z_1}{z}$: $\quad X(z)\frac{z-z_1}{z} = r_1 + \frac{r_2(z-z_1)}{(z-z_2)} + \cdots + \frac{r_m(z-z_1)}{(z-z_m)}$

And substitute $z = z_1$. We obtain r_1 . In similar way we obtain $r_2, \ldots, r_m$

$$\boxed{r_i = \lim_{z \to z_i} X(z) \frac{z-z_i}{z}}$$

For the rational function $X(z) = \frac{z^2}{(z-0.5)(z-0.6)} = \frac{r_1 z}{(z-0.5)} + \frac{r_2 z}{(z-0.6)}$

we obtain $r_1 = \lim_{z \to 0.5} X(z)\frac{z-0.5}{z} = \left.\frac{z}{z-0.6}\right|_{z=0.5} = -5 \qquad r_2 = \lim_{z \to 0.6} X(z)\frac{z-0.6}{z} = \left.\frac{z}{z-0.5}\right|_{z=0.6} = 6$

Z^{-1} calculation by partial fraction expansion an example

$$X(z) = \frac{z^2}{z^2 - 1.3z + 0.4}$$

Firstly we calculate poles of rational function X(z): poles of X(z) are zeros of the polynomial in denominator of X(z).

Thus we solve $z^2 - 1.3z + 0.4 = 0$ and we obtain $z_1 = 0.5 \qquad z_2 = 0.8$

Now we rewrite X(z) using partial fractions: $X(z) = \dfrac{z^2}{(z-0.5)(z-0.8)} = \dfrac{r_1\, z}{(z-0.5)} + \dfrac{r_2\, z}{(z-0.8)}$

If we multiply both sides by $\dfrac{z - z_1}{z} = \dfrac{z-0.5}{z}$ and substitute $z = z_1 = 0.5$ we obtain r_1

$$X(z)\frac{z-0.5}{z} = r_1 + \frac{r_2(z-0.5)}{(z-0.8)} \xrightarrow{z=0.5} r_1 \qquad X(z)\frac{z-0.5}{z} = \frac{z^2}{(z-0.5)(z-0.8)}\frac{z-0.5}{z} = \frac{z}{(z-0.8)} \xrightarrow{z=0.5} r_1 = -\frac{5}{3}$$

In similar way we obtain r_2

$$X(z)\frac{z-0.8}{z} = \frac{r_1(z-0.8)}{(z-0.5)} + r_2 \xrightarrow{z=0.8} r_2 \qquad X(z)\frac{z-0.8}{z} = \frac{z^2}{(z-0.5)(z-0.8)}\frac{z-0.8}{z} = \frac{z}{(z-0.5)} \xrightarrow{z=0.8} r_2 = \frac{8}{3}$$

Finally $X(z) = \dfrac{-\frac{5}{3}z}{(z-0.5)} + \dfrac{\frac{8}{3}z}{(z-0.8)}$ and $x_n = r_1(z_1)^n 1_n + r_2(z_2)^n 1_n = (-\frac{5}{3}\cdot 0.5^n + \frac{8}{3}\cdot 0.8^n)1_n$

What happens with x_n if $n \to \infty$?

If $z_i \neq 0$ – a single pole of $X(z)$

then a component $r_i (z_i)^n 1_n$ appears in x_n

If for all z_i $|z_i| < 1 \Rightarrow x_n \to 0$

If for only one z_i $|z_i| > 1 \Rightarrow x_n \to \infty$