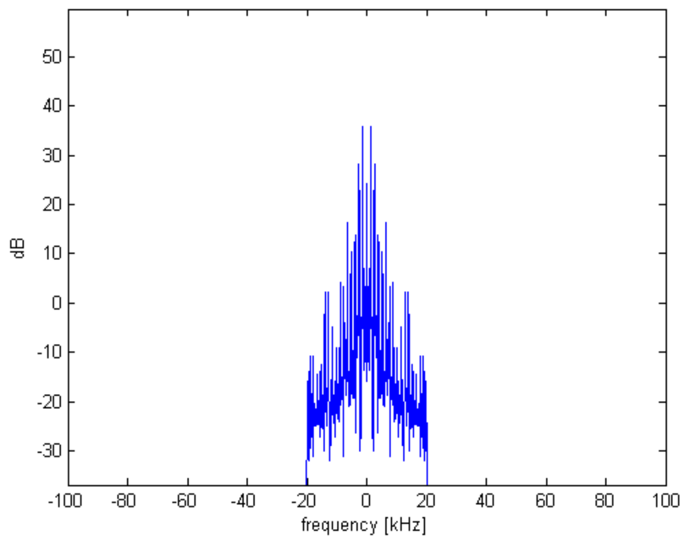
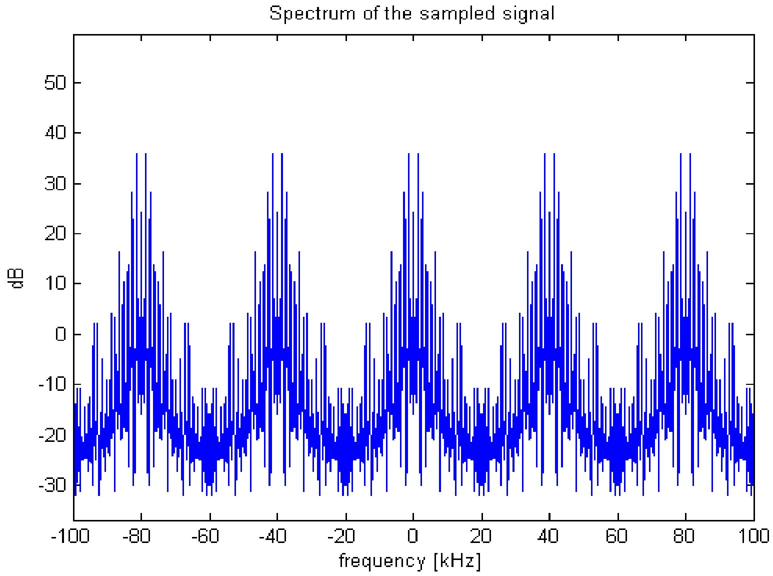
Sampling theorem

Given: spectrum (Fourier transform) of audio signal (on the left).

Find: minimum sampling frequency, enabling perfect reconstruction of audio signal

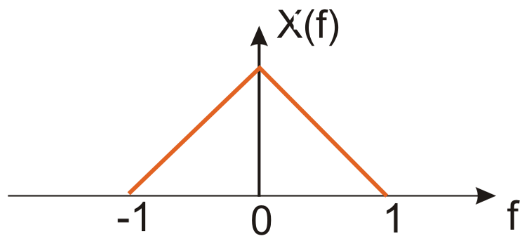
Solution: Bandwidth B=20 kHz, so sampling frequency fs = 1/T = 2B = 40 kHz
spectra copies don't overlap, so perfect reconstruction is possible





Sampling

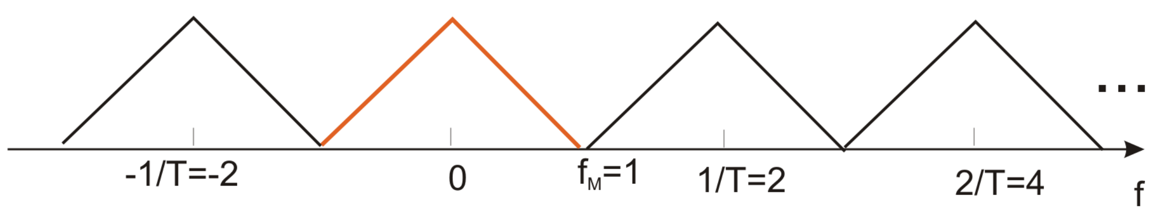
Spectrum before sampling



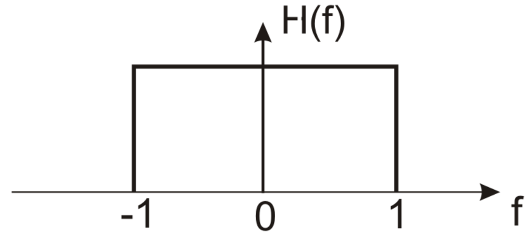
bandwidth $f_M=1$

sampling frequency =2 ($f_s=2\ f_M$)

Spectrum after sampling

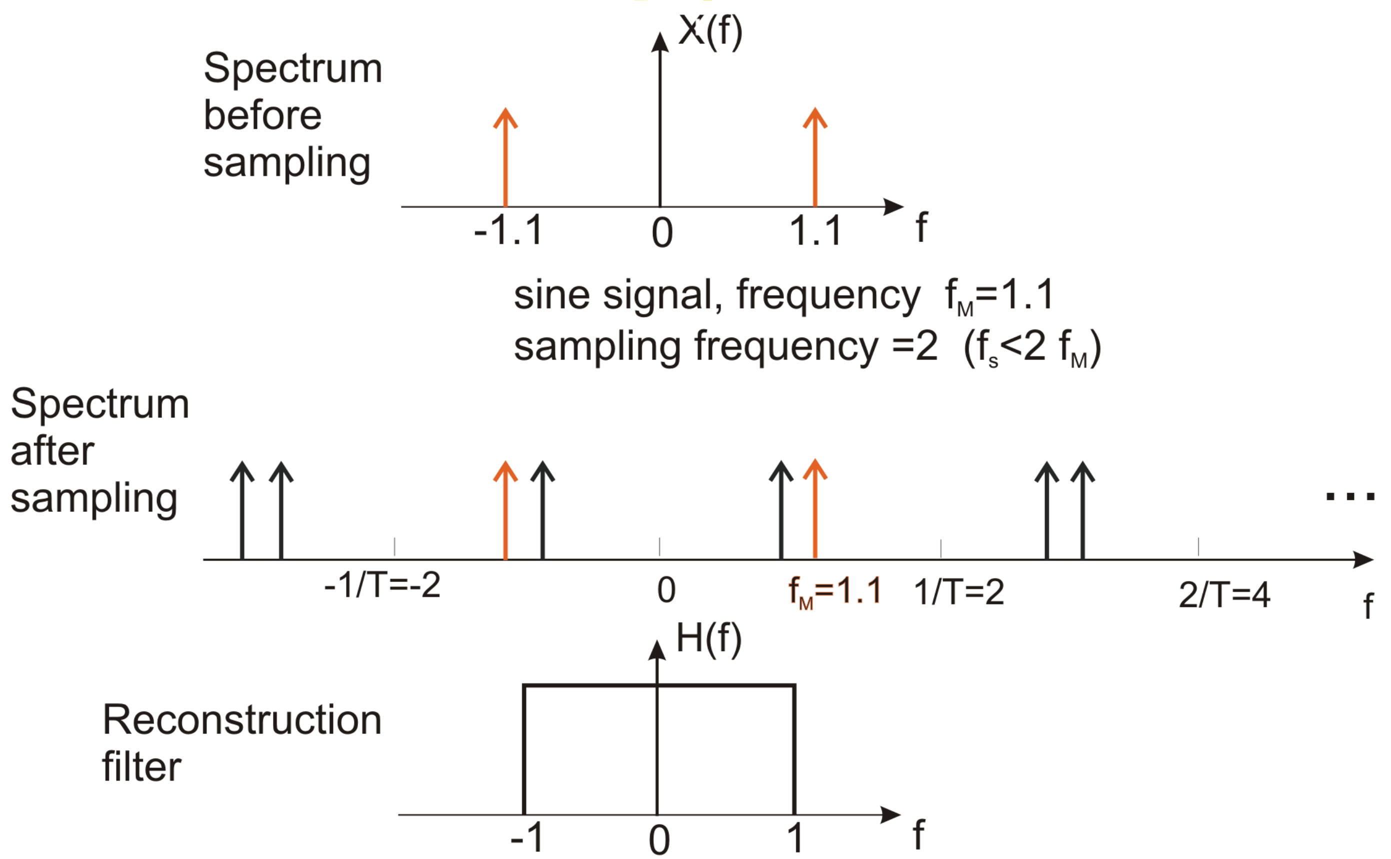


Reconstruction filter



It is **possible** to reconstruct signal from its samples

Sampling

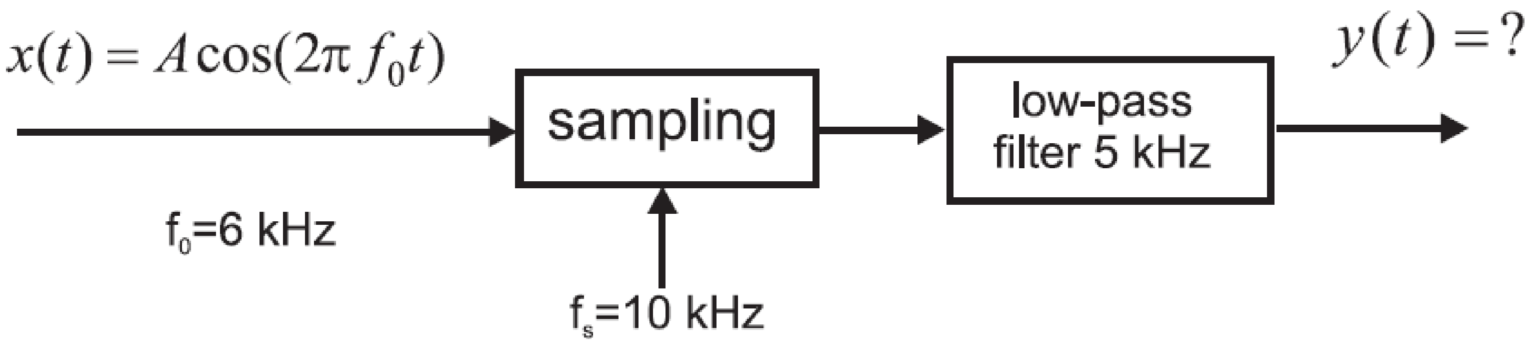


It is **not possible** to reconstruct signal from its samples

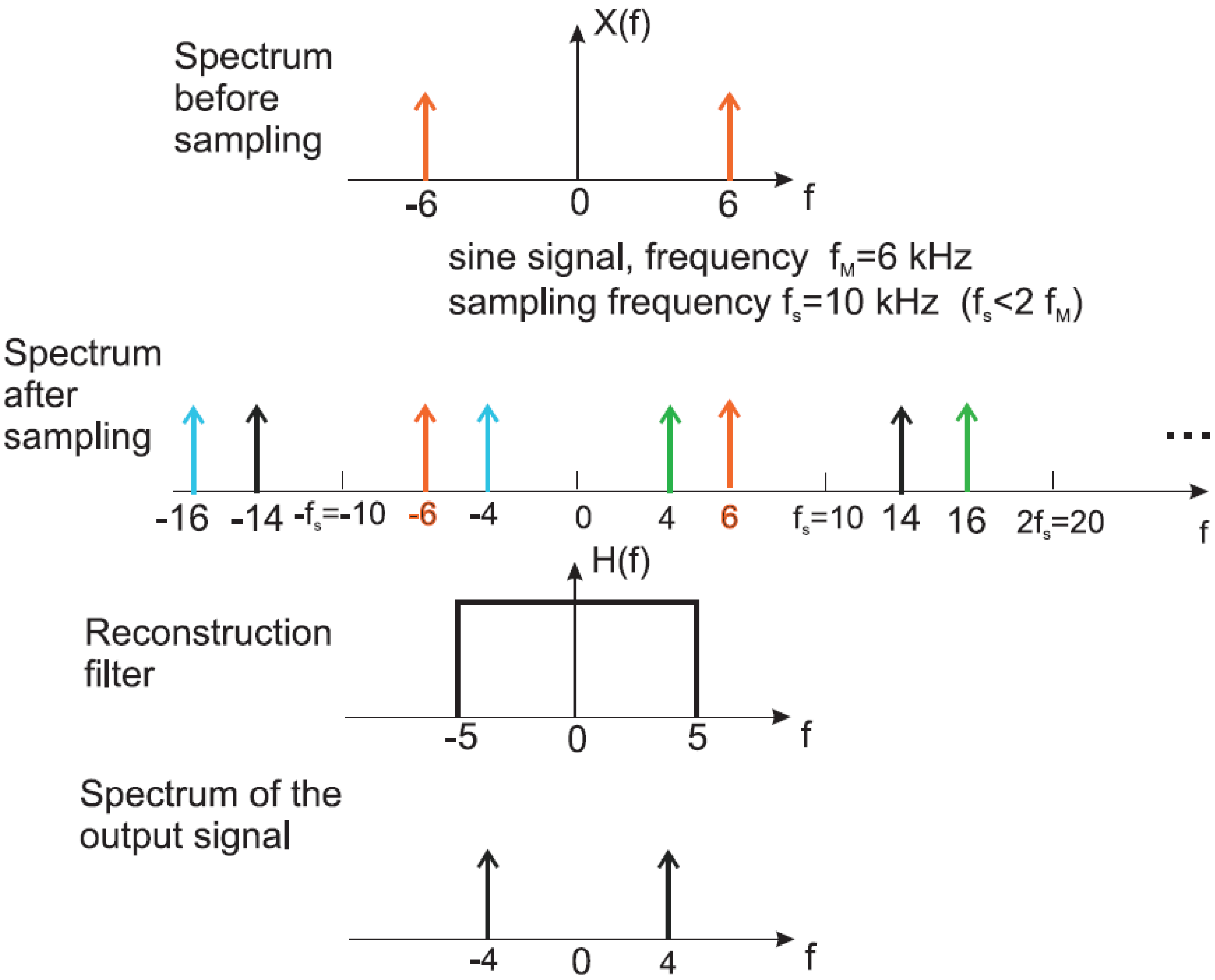
At the output of the lowpass filter we obtain cosine of frequency 0.9 kHz

Sampling

Cosine signal of frequency $f_0 = 6$ kHz is sampled at a sampling frequency $f_s = 1/T = 10$ kHz. Draw the spectrum of the sampled signal. Is it possible to obtain an exact copy of the continuous signal from the sampled signal? What is the signal at the output of the lowpass filter?



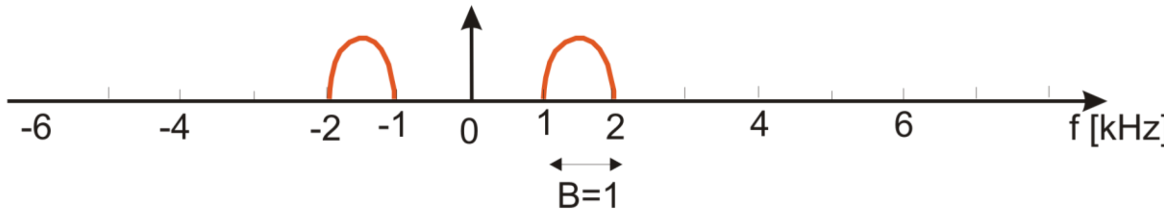
Solution in the next slide:



At the output of the lowpass filter we obtain cosine of frequency 4 kHz

Integer sampling

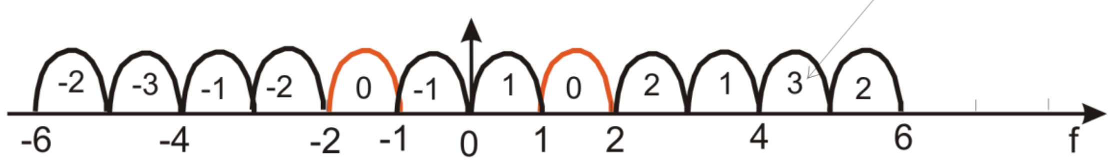
spectrum of input signal



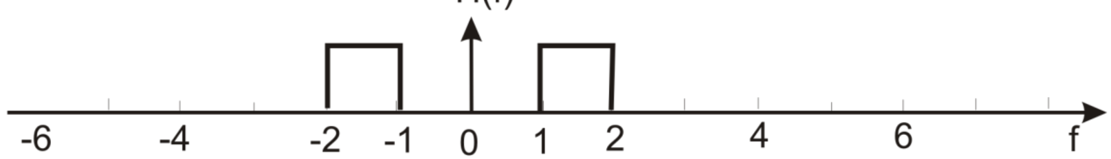
sampling frequency 1/T=2B=2

spectrum of samples

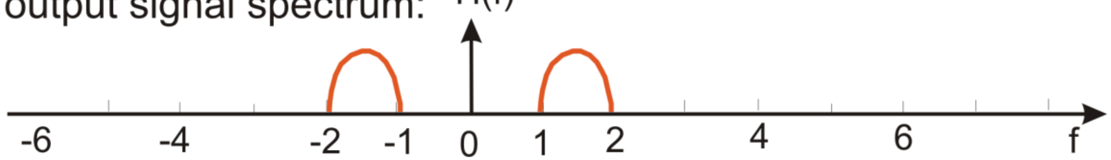
$$\frac{1}{T}\sum_{n} X(f-\frac{n}{T}) = 2\sum_{n} X(f-2n)$$



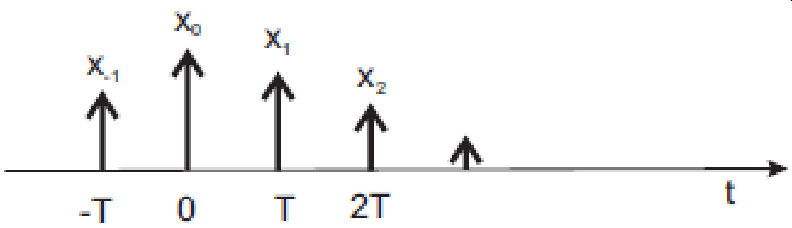
reconstructing filter:



output signal spectrum:



Discrete Time Fourier Transform



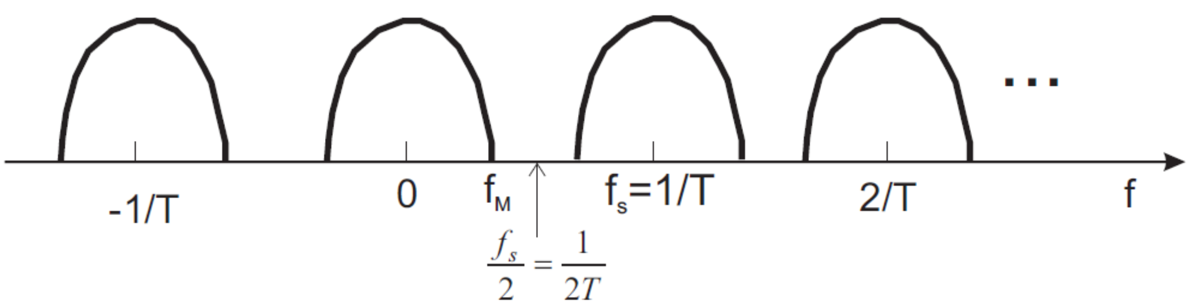
T – sampling period
1/T – sampling frequency

$$x_s(t) = \sum_n x_n\, \delta(t - nT)$$

Spectrum

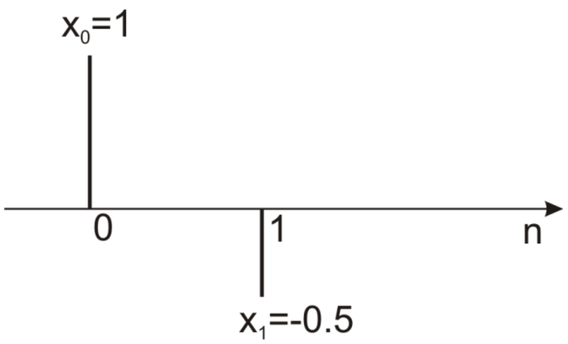
$$X_s(f) = F[x_s(t)] = \sum_n x_n e^{-j2\pi f n T}$$

$X_s(f)$ a periodic function (period f_s=1/T), called the Discrete Time Fourier Transform



Discrete Time Fourier Transform (DTFT) an example

$$X_s(f) = \sum_n x_n e^{-j2\pi f nT}$$



Discrete time signal $\{x_n\}$ has only 2 nonzero values:

$x_0 = 1,\ x_1 = -0.5$

DTFT equals: $$X_s(f) = x_0 e^{-j2\pi f \cdot 0 \cdot T} + x_1 e^{-j2\pi f \cdot 1 \cdot T} = 1 - 0.5\, e^{-j2\pi f T}$$

We may rewrite it: $$X_s(f) = 1 - 0.5\cos(2\pi fT) + 0.5\, j \sin(2\pi fT) = |X_s(f)|\, e^{j\arg(X_s(f))}$$

Amplitude spectrum $$|X_s(f)| = \sqrt{(1 - 0.5\cos(2\pi fT))^2 + (0.5\ \ \sin(2\pi fT))^2} =$$

$$= \sqrt{1 - \cos(2\pi fT) + 0.25} = \sqrt{\tfrac{5}{4} - \cos(2\pi fT)}$$

where T =1/f_s - sampling period, f_s – sampling frequency

Discrete Fourier Transform (DFT)

DFT= sampled DTFT at frequencies $f_k = \frac{k}{TL}, \qquad k = 0, 1, \ldots, L-1$

DTFT is calculated for L signal samples

$$X_k = \sum_{n=0}^{L-1} x_n \, e^{-j2\pi f_k nT} = \sum_{n=0}^{L-1} x_n \, e^{-j2\pi \frac{k}{TL} nT} = \sum_{n=0}^{L-1} x_n \, e^{-j2\pi \frac{kn}{L}}$$

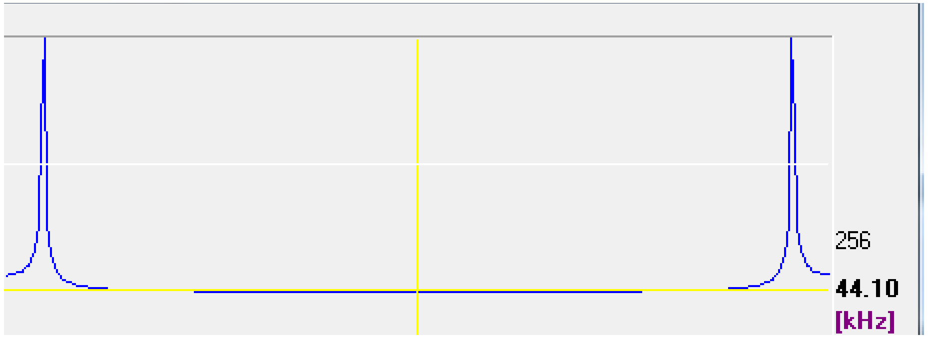
Let's substitute $W_L = e^{-j\frac{2\pi}{L}}$

DFT: $$X_k = \sum_{n=0}^{L-1} x_n W_L^{kn}, \qquad k = 0, 1, \cdots, L-1$$

Example: sine signal of frequency f_0 = 2 kHz is sampled at sampling frequecny f_s=1/T=10 kHz. Then DFT is calculated for L=100 samples (DFT spectrum X_0, X_1, …, X_{L-1} is obtained). Which DFT values have the greatest amplitude?

Solution: DFT delivers L values at frequences $f_k = \frac{k}{TL}, \qquad k = 0, 1, \ldots, L-1$, 1/(TL)=100 Hz, so f_k= k 100 Hz . There will be two DFT values of maximum amplitudes, at frequencies 2 kHz and 10-2=8 kHz (mirror image). Thus DFT coefficients X_{20} and X_{80} will have maximum values.

DFT – frequency analysis



|DFT| of $\sin(2\pi f_0 nT)$, sampling frequency fs=1/T = 44100 Hz, L=256 samples
DFT coeffocients: $X_0, X_1, \ldots, X_{255}$. Maximum value has coefficient X_{12}

Task: Estimate frequency f_0

Solution: DFT coefficient X_k corresponds to frequency $f_k = k / (TL) = k f_s / L$
f_{12}= 12 x 44100 / 256 = 2065 Hz.
Note that our result is not exact. DFT yields 256 samples of spectrum , fk+1 – fk = 1/(TL)= 44100 / 256 = 172 Hz. Frequency of observed sine signal may be between 2065 – 172 / 2 Hz and 2065 + 172 / 2 Hz, that is between 1979 and 2151 Hz. (in fact it was 2 kHz)

Z transform

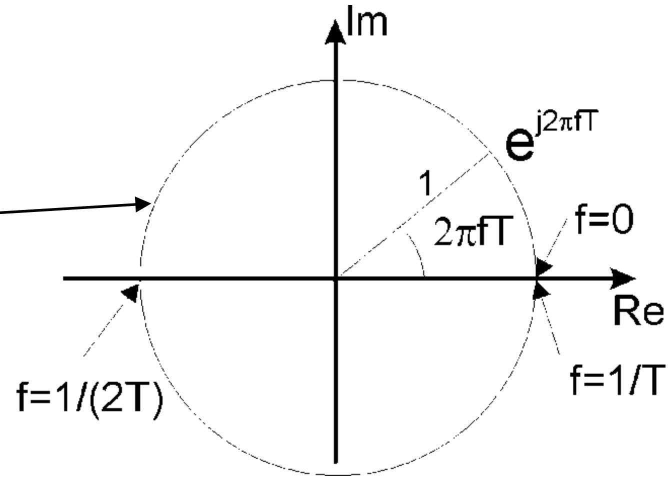
Discrete signal $x_s(t) = \sum_n x_n \delta(t - nT)$

DTFT $X_s(f) = \sum_n x_n F[\delta(t - nT)] = \sum_n x_n e^{-j2\pi fnT}$

Substitution $z = e^{j2\pi fT}$

Z Transform $X(z) = Z[\{x_n\}] = \sum_n x_n z^{-n}$

Generalization: any value of „z” for which the sum converges



Properties: linearity: $Z[a\{x_n\}+b\{y_n\}]=a\ Z[\{x_n\}]+b\ Z[\{y_n\}]= a\ X(z) + b\ Y(z)$,

shift : $Z[\{x_{n-k}\}]=z^{-k}\ X(z)$

convolution $y_n = \sum_{k=-\infty}^{\infty} x_k h_{n-k} \quad \Rightarrow \quad y_n = x_n * h_n$ $\qquad Y(z) = X(z)\ H(z)$

Calculation of Z transform

For finite number of samples we use straight method: $$X(z) = \sum_{n=M}^{N} x_n z^{-n}$$

for example $x_1 = 2, x_2 = -1,$ $$X(z) = \sum_{n=1}^{2} x_n z^{-n} = 2z^{-1} - z^{-2}$$

For infinite number of samples we use series theory, for example

$$y_n = a^n 1_n \xrightarrow{|a|<1} Y(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (\tfrac{a}{z})^n = \frac{1}{1 - \frac{a}{z}} = \frac{z}{z-a}$$

We may use the properties of Z transform, like shift property or convolution property

Calculation of Inverse Z transform

Inverse Z transform: calculation of time series (samples) knowing X(z): $x_n = Z^{-1}[X(z)]$

If X(z) is a polynomial $X(z) = \sum_{n=M}^{N} x_n z^{-n}$ we use straight method:

for example $X(z) = 2z - 1 + z^{-2} \rightarrow x_{-1} = 2, x_0 = -1, x_2 = 1$

Often we use known Z transforms, like these: $Y(z) = \dfrac{z}{z-a} \quad \rightarrow \quad y_n = a^n 1_n$

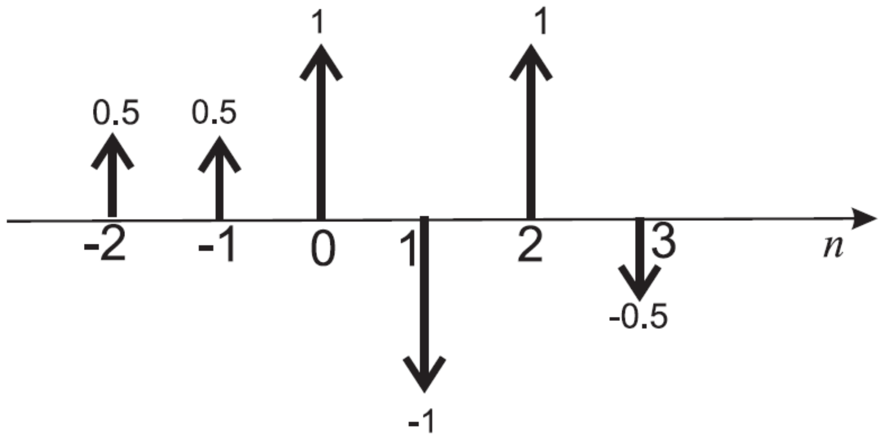
$$Y(z) = \frac{za}{(z-a)^2} \quad \rightarrow \quad y_n = n a^n 1_n$$

We also use properties of Z transform, for example:

$$Z^{-1}[\tfrac{b}{z-a}] = b Z^{-1}[\tfrac{1}{z-a}] = b Z^{-1}[\tfrac{1}{z} \, \tfrac{z}{z-a}] = b 1_{n-1} a^{n-1}$$

Calculation of Straight and Inverse Z transform - examples

For a series of samples x_n (see figure) calculate Z transform.



Solution

$$X(z) = \sum_{n=-\infty}^{\infty} x_n z^{-n} = 0.5z^2 + 0.5z + 1 - z^{-1} + z^{-2} - 0.5z^{-3}$$

Calculation of Straight and Inverse Z transform - examples

For a given Z transform:

$$X(z) = \frac{5z}{z+0.8} + \frac{3}{z-0.5}$$

calculate the inverse Z transform, that is a series of samples x_n.

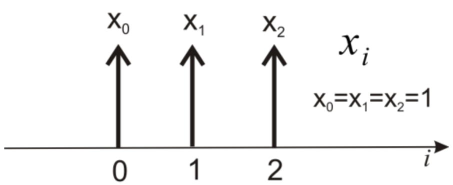
$$X(z) = 5\frac{z}{z+0.8} + 3z^{-1}\frac{z}{z-0.5}$$

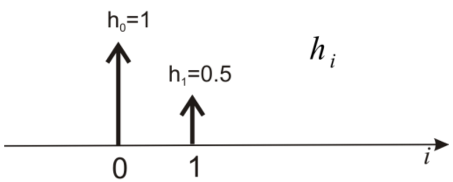
$$x_n = 5 \cdot 1_n \cdot (-0.8)^n + 3 \cdot 1_{n-1} \cdot (0.5)^{n-1}$$

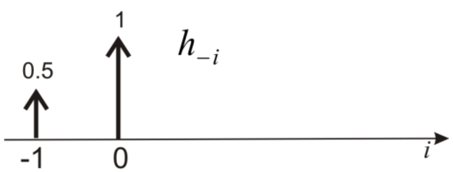
Calculation of convolution

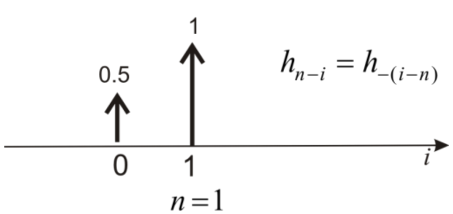
Calculation in time domain

$$y_n = \sum_{i=-\infty}^{\infty} x_i h_{n-i} \qquad h_{n-i} = h_{-(i-n)}$$









n=0, y_0=1x1=1
n=1, y_1=1x0.5 + 1x1=1.5
n=2, y_2=1x0.5 + 1x1=1.5
n=3, y_3=1x0.5 =0.5

Calculation in transform domain

$$Y(z) = X(z) \cdot H(z)$$

$$X(z) = 1 + z^{-1} + z^{-2}$$

$$H(z) = 1 + \tfrac{1}{2} z^{-1}$$

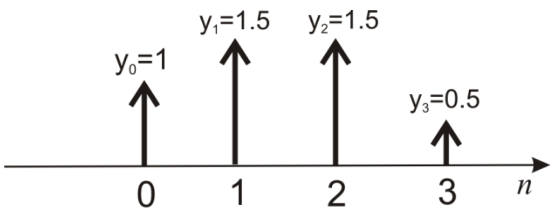
$$Y(z) = X(z)H(z) =$$

$$= 1 + \tfrac{1}{2} z^{-1} + z^{-1} + \tfrac{1}{2} z^{-2} + z^{-2} + \tfrac{1}{2} z^{-3} =$$

$$= 1 + \tfrac{3}{2} z^{-1} + \tfrac{3}{2} z^{-2} + \tfrac{1}{2} z^{-3} =$$

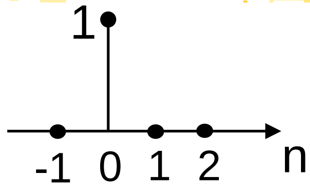
$$= y_0 + y_1 z^{-1} + y_2 z^{-2} + y_3 z^{-3}$$

Both methods give the same result



Some signals and their Z transforms

Kronecker delta δ_n

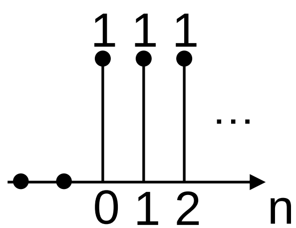


$$Z[\delta_n] = \sum_n \delta_n z^{-n} = 1$$

$$Z[\delta_{n-m}] = \sum_n \delta_{n-m} z^{-n} = z^{-m}$$

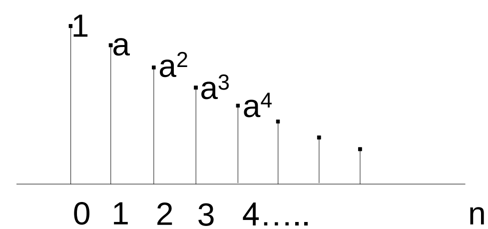
Convolution with Kronecker delta $y_n = x_n * \delta_{n-m} = \sum_{k=-\infty}^{\infty} x_k \delta_{n-m-k} = x_{n-m}$

Unit step function 1_n



$$Z[1_n] = \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} (\tfrac{1}{z})^n = \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1}$$

Exponential function $x_n = 1_n a^n$, $X(z) = \dfrac{z}{z-a}$



a - pole of the rational function X(z)

Important observation: If $|a| < 1$ (pole within unit circle), then $x_n \longrightarrow 0$
If $|a| > 1$ (pole outside unit circle), then $x_n \longrightarrow$ infinity