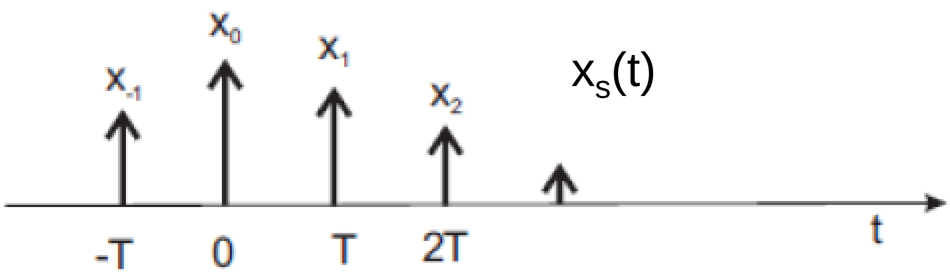
Sampled signal

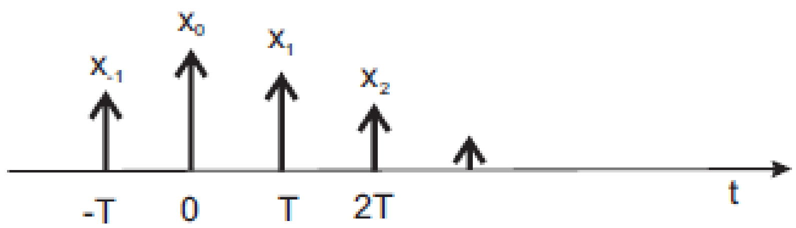
Ideal sampling of an analog signal x(t):

$$x(t) \cdot \sum_{n} \delta(t - nT) = \sum_{n} x(nT)\,\delta(t - nT) = \sum_{n} x_n\, \delta(t - nT)$$



T - sampling period
1/T - sampling frequency

Discrete Time Fourier Transform = Fourier transform of sampled signal



T –sampling period
1/T – sampling frequency

$$x_s(t) = \sum_n x_n \, \delta(t - nT)$$

$$F[\delta(t-nT)] = e^{-j2\pi fnT}$$

Spectrum

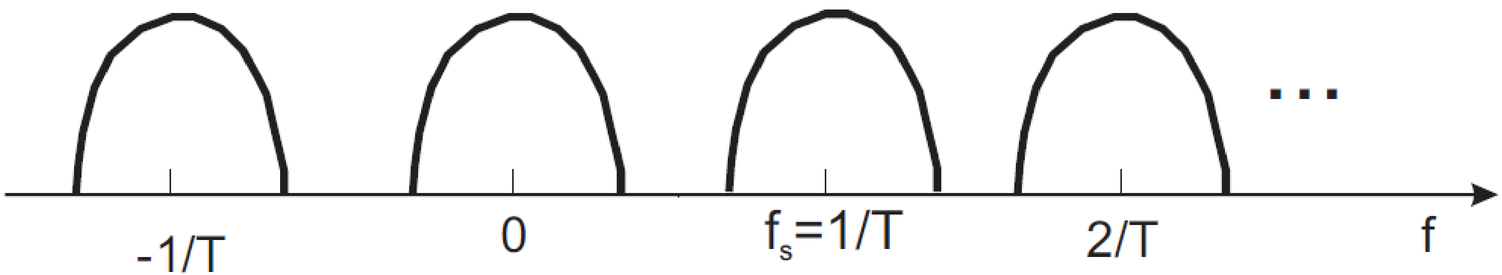
$$X_s(f) = F[x_s(t)] = \sum_n x_n e^{-j2\pi fnT}$$

periodic function of frequency f,
period 1/(nT), common period 1/T

$X_s(f)$ a periodic function (period f_s=1/T), called the **Discrete Time Fourier Transform**

Discrete Time Fourier Transform (DTFT)

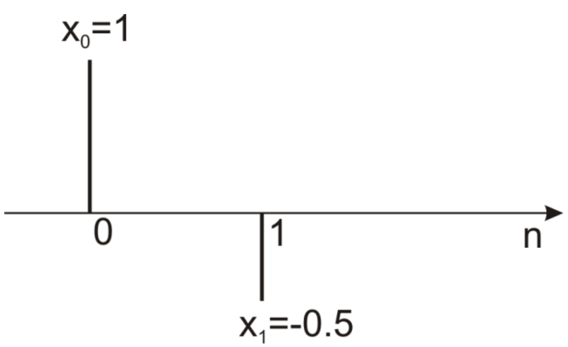
$$X_s(f) = \sum_n x_n e^{-j2\pi f n T}$$



DTFT may be calculated for any value of frequency f

Discrete Time Fourier Transform (DTFT) an example

$$X_s(f) = \sum_n x_n e^{-j2\pi f nT}$$



Discrete time signal $\{x_n\}$ has only 2 nonzero values:

$x_0 = 1,\ x_1 = -0.5$

DTFT equals: $X_s(f) = x_0 e^{-j2\pi f \cdot 0 \cdot T} + x_1 e^{-j2\pi f \cdot 1 \cdot T} = 1 - 0.5\, e^{-j2\pi f T}$

We may rewrite it: $X_s(f) = 1 - 0.5\cos(2\pi fT) + 0.5\, j\sin(2\pi fT) = |X_s(f)|\, e^{j\arg(X_s(f))}$

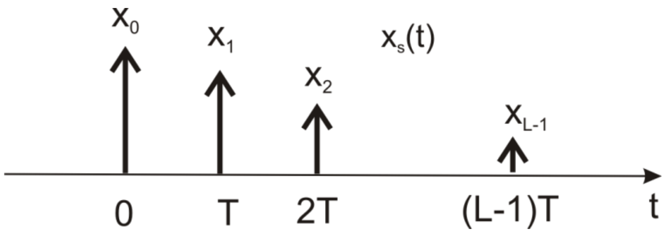
Amplitude spectrum $|X_s(f)| = \sqrt{(1-0.5\cos(2\pi fT))^2 + (0.5\ \ \sin(2\pi fT))^2} =$

$$= \sqrt{1-\cos(2\pi fT) + 0.25} = \sqrt{\tfrac{5}{4} - \cos(2\pi fT)}$$

where T =1/f_s - sampling period, f_s – sampling frequency

Fourier transform of L samples

Let's take L signal samples: $x_0, x_1, \ldots, x_{L-1}$



The Fourier Transform of this signal:

$$X_s(f) = \sum_{n=0}^{L-1} x_n \, e^{-j2\pi f nT}$$

$X_s(f)$ – periodic function of frequency f, period 1/T = sampling freq.

Mirror image property of $X_s(f)$:

$$X_s\left(\tfrac{1}{T} - f\right) = \sum_{n=0}^{L-1} x_n \, e^{-j2\pi\left(\frac{1}{T} - f\right)nT} = \sum_{n=0}^{L-1} x_n \, \underbrace{e^{-j2\pi n}}_{=1} \, e^{j2\pi fnT} = \sum_{n=0}^{L-1} x_n \, e^{j2\pi fnT} = X_s^*(f)$$

=1

complex conjugate

Discrete Fourier Transform (DFT)

Let's take L samples of the spectrum in frequency range <0,1/T), at frequencies:

$$f_k = \frac{k}{TL}, \qquad k = 0, 1, \ldots, L-1$$

$$X_k = \sum_{n=0}^{L-1} x_n e^{-j2\pi f_k nT} = \sum_{n=0}^{L-1} x_n e^{-j2\pi \frac{k}{TL} nT} = \sum_{n=0}^{L-1} x_n e^{-j2\pi \frac{kn}{L}}$$

Let's substitute $\quad W_L = e^{-j\frac{2\pi}{L}}$

DFT: $$X_k = \sum_{n=0}^{L-1} x_n W_L^{kn}, \qquad k = 0, 1, \cdots, L-1$$

DFT in matrix form

$$\overline{X} = \overline{W}\,\overline{x}$$

$$\begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{L-1} \end{bmatrix} = \begin{bmatrix} W_L^{0\cdot 0} & W_L^{0\cdot 1} & \cdots & W_L^{0\cdot (L-1)} \\ W_L^{1\cdot 0} & W_L^{1\cdot 1} & \cdots & W_L^{1\cdot (L-1)} \\ \vdots & \vdots & & \vdots \\ W_L^{(L-1)\cdot 0} & W_L^{(L-1)\cdot 1} & \cdots & W_L^{(L-1)\cdot (L-1)} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{L-1} \end{bmatrix}$$

Matrix $\overline{W}$ is symmetric: $W_L^{kn} = W_L^{nk}$

Its rows (and columns) are $e^{-j\frac{2\pi}{L}kn} = \cos(\frac{2\pi}{L}kn) - j\sin(\frac{2\pi}{L}kn)$

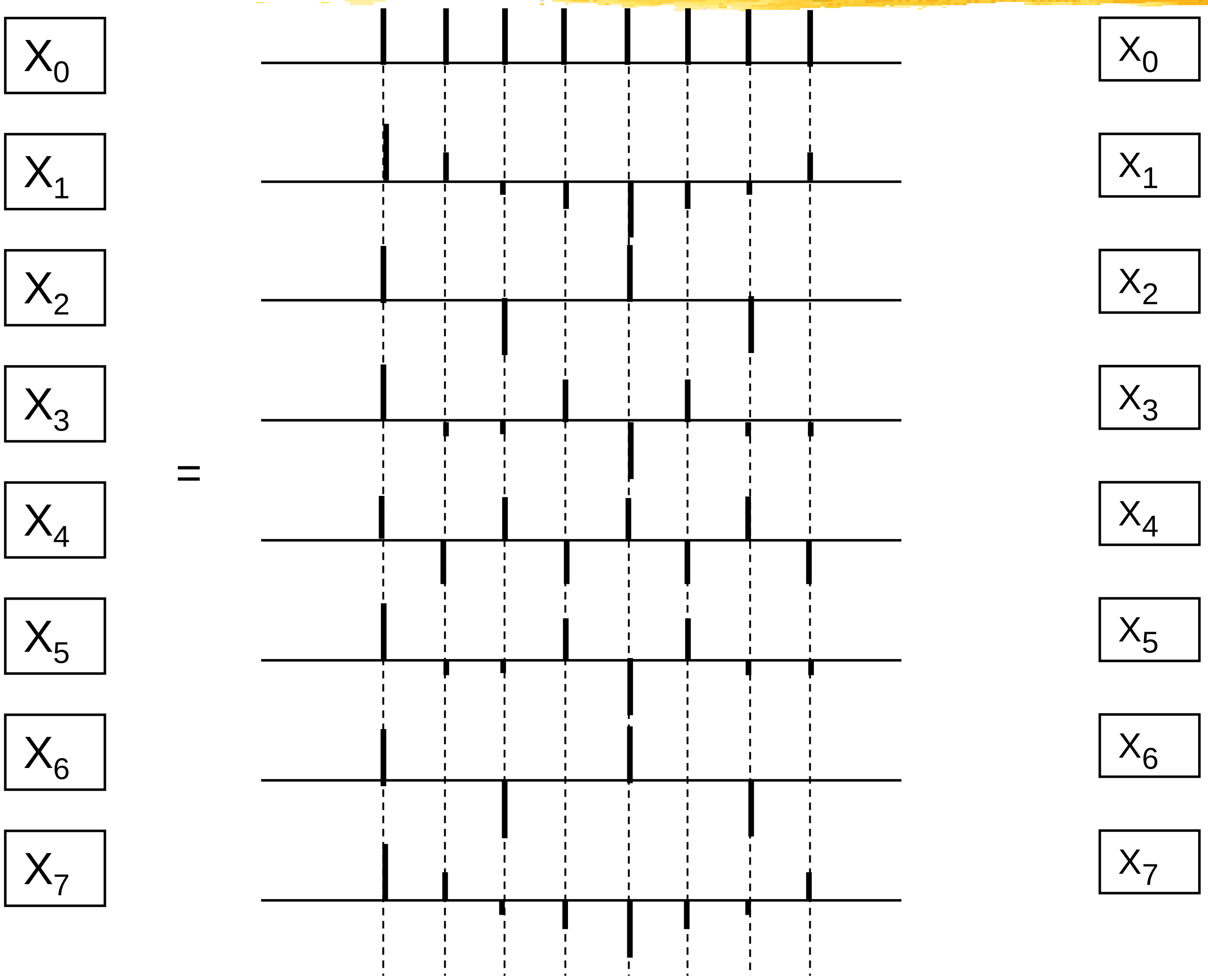
k-th row (and column) includes k periods of the above function (k = 0,1,…,L-1)

rows number k and (L-k) are complex conjugates

Fast Fourier Transform (FFT) = fast algorithm of DFT calculation

DFT (L=8)

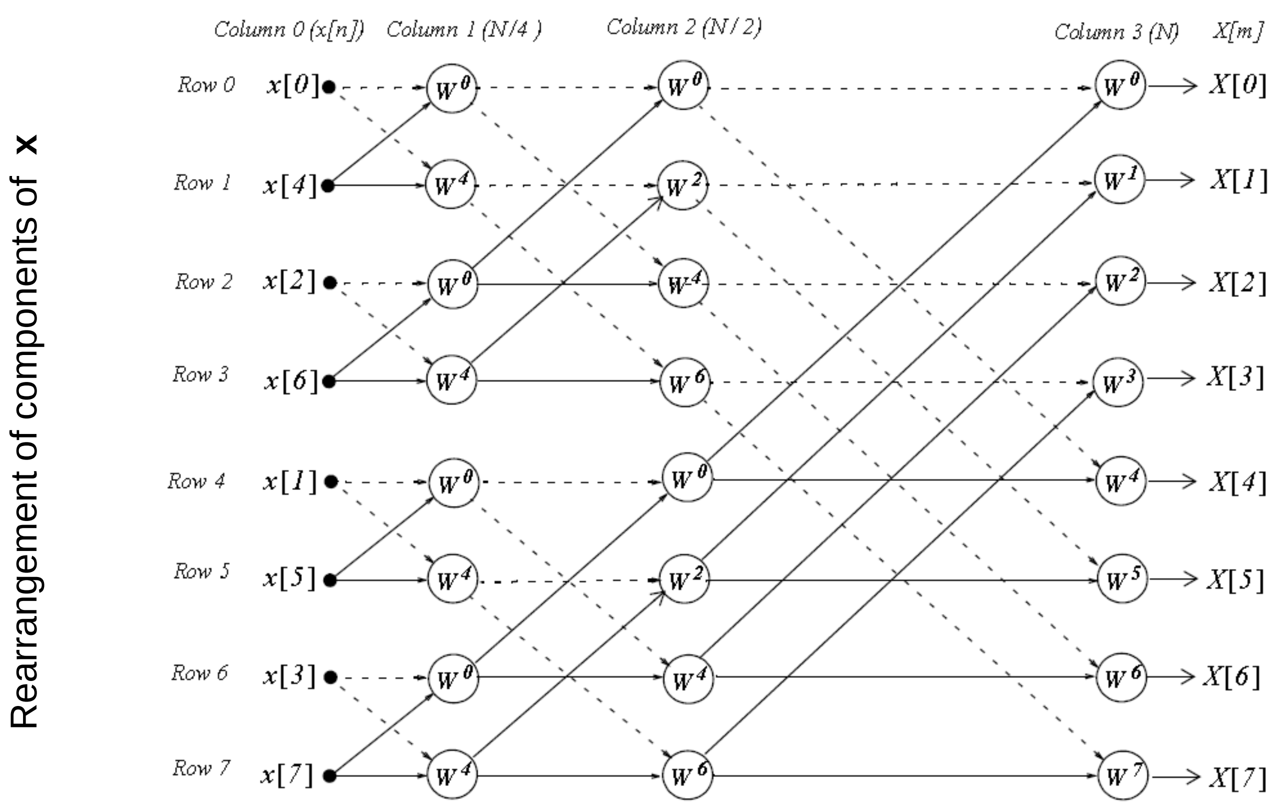
X = **W x** (only the real part of **W** is shown)



Fast Fourier Transform (FFT)

FFT is a fast algorithm of DFT calculation

Calculation of **X** = **W x** requires L^2 multiplications (L is the number of rows = number of columns of **W**). In 1965 James Cooley and John Tukey remarked, that the same operations are repeated. Their Fast Fourier Transform algorithm avoids these repeating multiplications. It requires only $L \log_2(L)$ multiplications.



Inverse DFT (IDFT)

Rows (and columns) w(k) i w(l) are orthogonal:

$$w(k)\,w^t(l) = \sum_{n=0}^{L-1} W_L^{kn}\,W_L^{-nl} = \sum_{n=0}^{L-1} e^{-j2\pi\frac{kn}{L}}\,e^{j2\pi\frac{nl}{L}} = \sum_{n=0}^{L-1} e^{-j2\pi\frac{(k-l)n}{L}} =$$

$$= \sum_{n=0}^{L-1}\cos[2\pi\frac{(k-l)n}{L}] - j\sum_{n=0}^{L-1}\sin[2\pi\frac{(k-l)n}{L}] = \begin{cases} L, & k = l \\ 0, & k \neq l \end{cases}$$

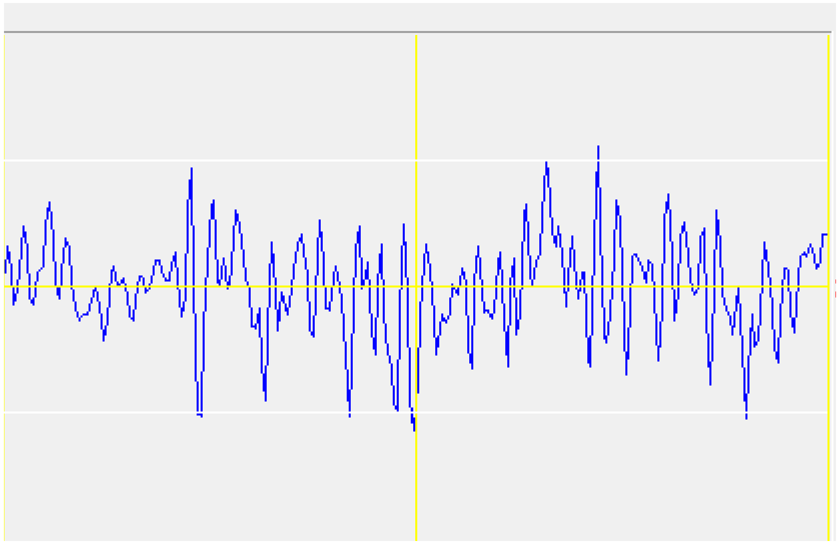
(transposition evokes also a complex conjugate)

Conclusion: $\frac{1}{L}\overline{W}\,\overline{W}^{*} = I$ (*I* – unit matrix, identity matrix)

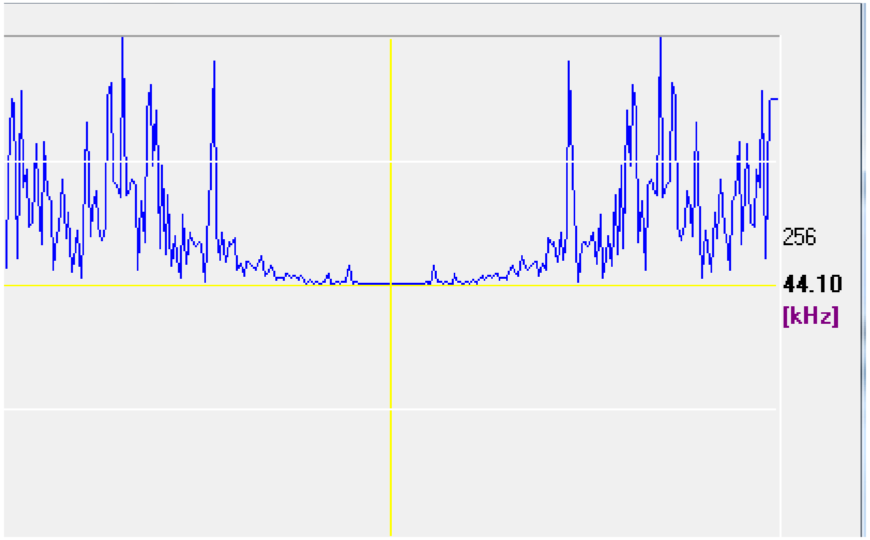
$$\longrightarrow \qquad \overline{W}^{-1} = \frac{1}{L}\overline{W}^{*}$$

IDFT $\boxed{\overline{x} = \overline{W}^{-1}\,\overline{X} = \frac{1}{L}\overline{W}^{*}\,\overline{X}}$

DFT of audio signal – an example



256 samples of audio signal, fs=1/T = 44100 Hz



|DFT| of audio signal, fs=1/T = 44100 Hz, L=256 samples

Discrete cosine transform (DCT)

$$\overline{X} = \overline{W}\,\overline{x}$$

$$W_{k,n} = \begin{cases} \frac{1}{\sqrt{L}}, & k = 0 \\ \sqrt{\frac{2}{L}}\cos(\frac{\pi}{2L}(2n+1)k), & k = 1, \cdots, L-1 \end{cases}$$

k-th row includes about k/2 periods of the cosine function

The rows of **W** are orthonormal: $\overline{W}^t\,\overline{W} = I$ $\quad\quad$ $\overline{W}^{-1} = \overline{W}^t$

DCT

(L=8)

X_0

X_1

X_2

X_3

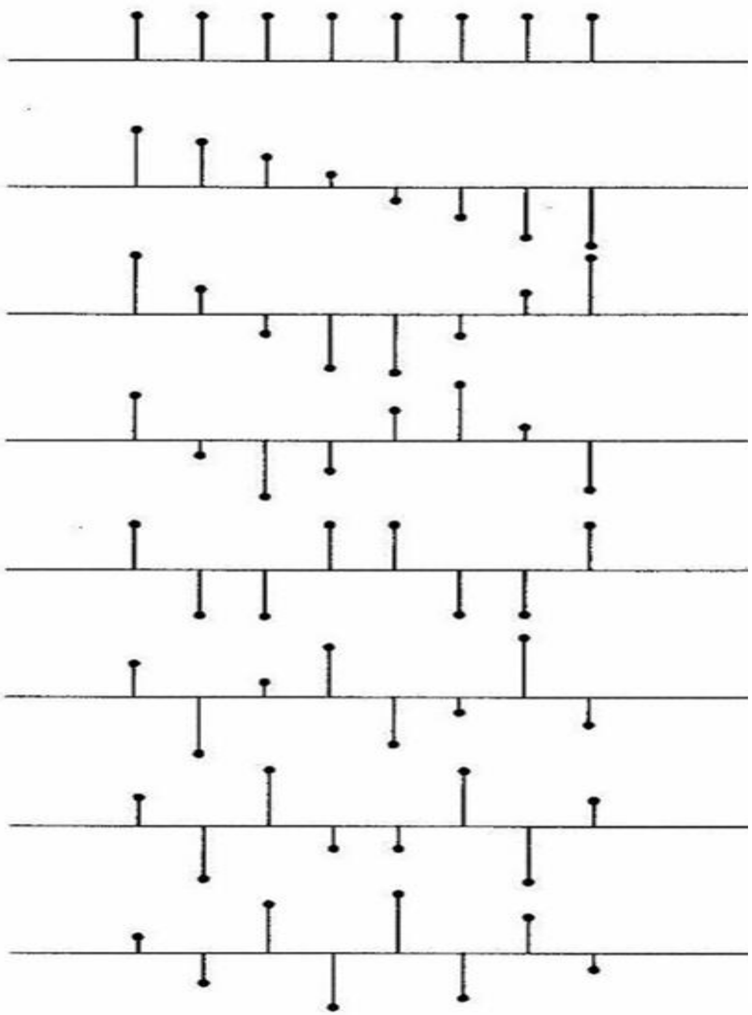
X_4

X_5

X_6

X_7

=



X_0

X_1

X_2

X_3

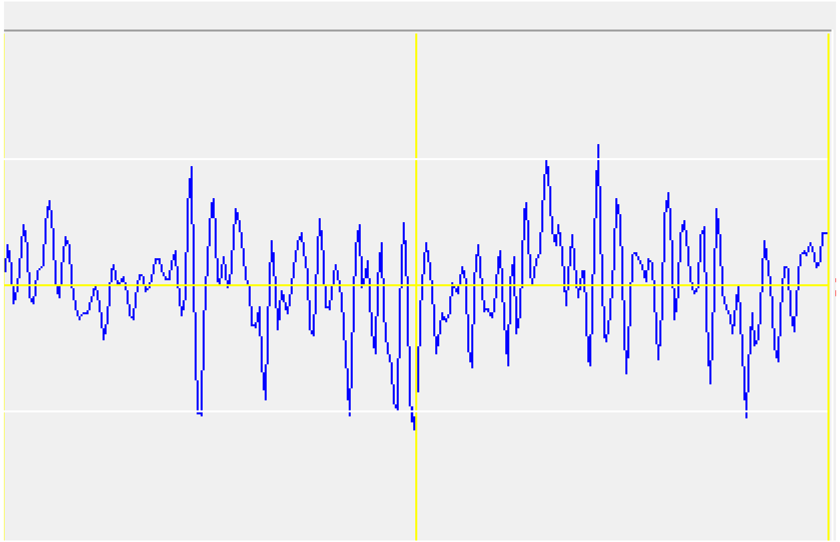
X_4

X_5

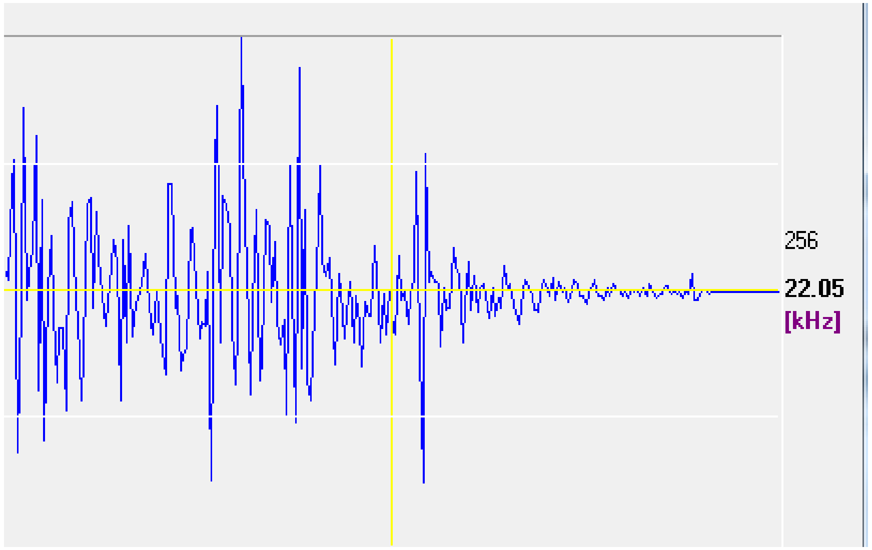
X_6

X_7

DCT of audio signal – an example



256 samples of audio signal,   fs=1/T = 44100 Hz



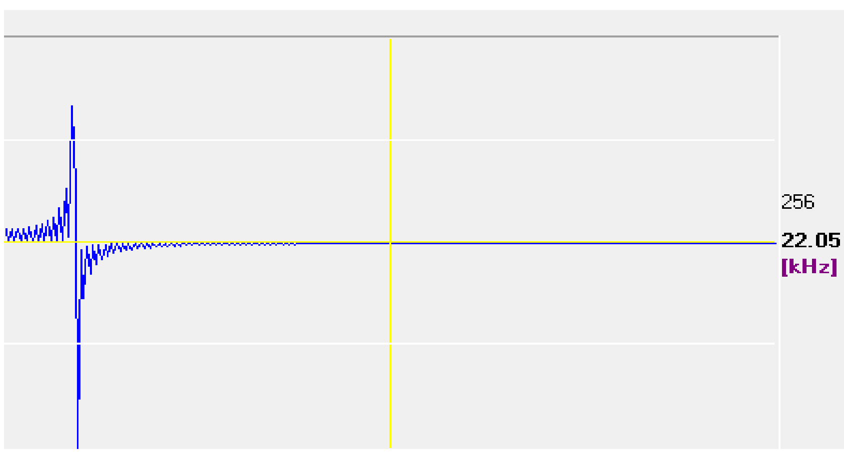
DCT   of audio signal, fs=1/T = 44100 Hz,
L=256 samples

DFT and DCT - comparison

DFT:	**DCT:**
complex	real
Frequency range from f=0 to sampling frequency	from f=0 to half of sampling freq.
Amplitude and power spectrum independent on signal phase	dependent on signal phase

DFT and DCT - comparison

|DFT| of $\sin(2\pi f_0 nT)$ and $\cos(2\pi f_0 nT)$, f_0=2000 Hz, fs=1/T = 44100 Hz, L=256 samples

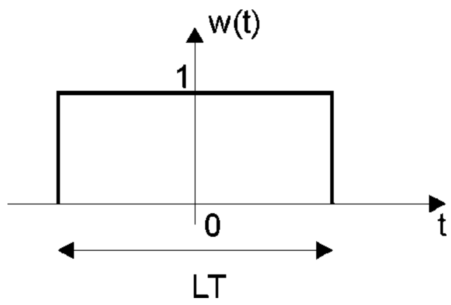


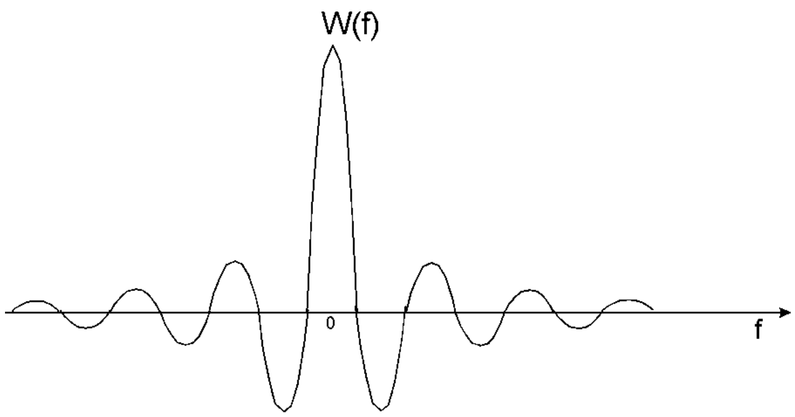
DCT of $\sin(2\pi f_0 nT)$, f_0=2000 Hz, fs=1/T = 44100 Hz, L=256 samples

DCT of $\cos(2\pi f_0 nT)$, f_0=2000 Hz, fs=1/T = 44100 Hz, L=256 samples

Frequency resolution

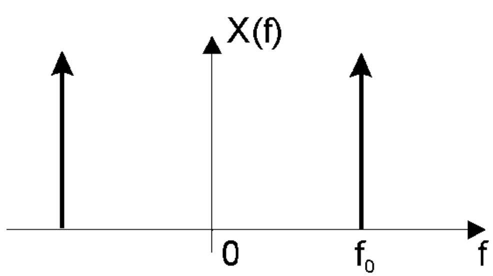
Using discrete transforms, we analyse signal of finite duration, i.e. we multiply the signal x(t) by the window w(t), thus obtaining x(t) w(t) (L samples, LT seconds).
Spectrum of x(t) w(t) is a covolution of both spectra: X(f)*W(f)

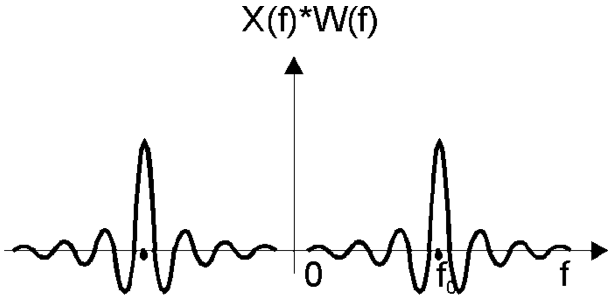




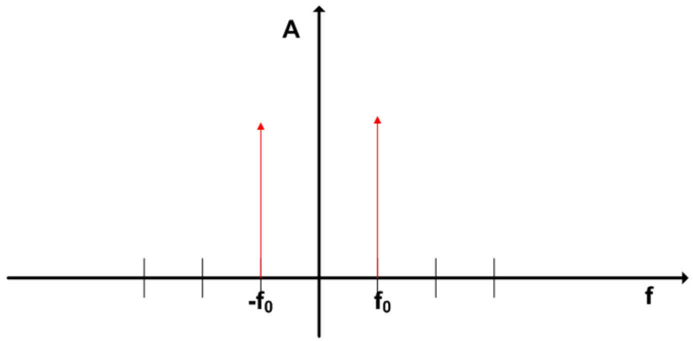
zero crossings at f=1/(LT), -1/(LT), 2/(LT), -2/(LT),...

E.g. for $x(t)=\cos(2\pi f_0 t)$

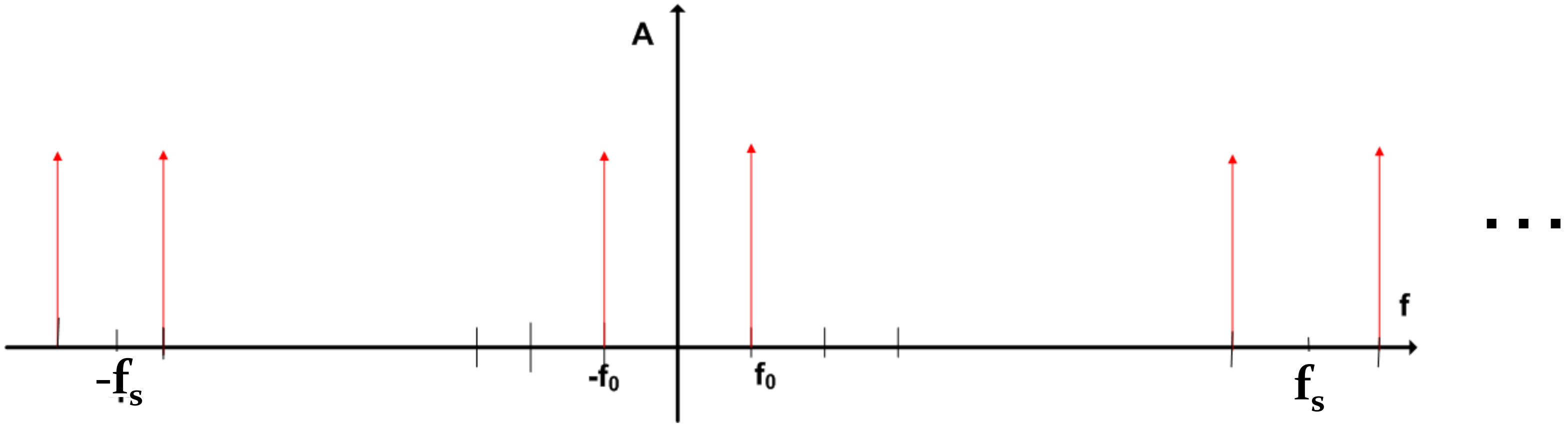




Spectrum of $\cos(2\pi f_0 t)$

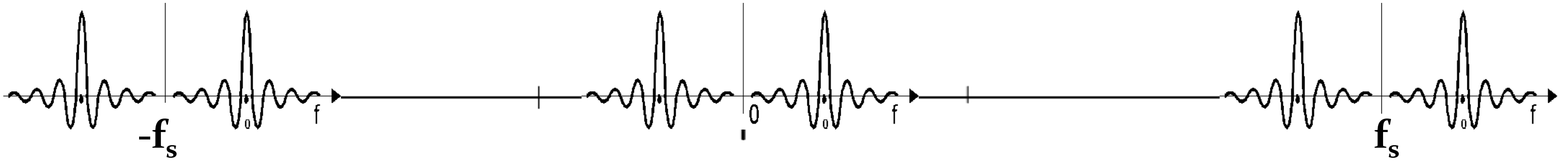


Fourier transform



DTFT : Fourier transform of the sampled signal (f_s – sampling frequency) calculated for infinite number of signal samples

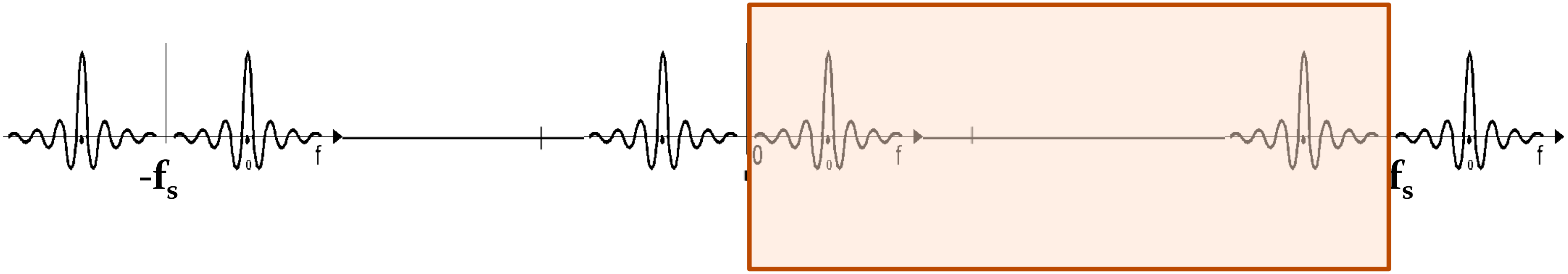
Spectrum of $\cos(2\pi f_0 t)$



DTFT of *L* samples of $\cos(2\pi f_0 t)$

Discrete Fourier Transform (DFT)

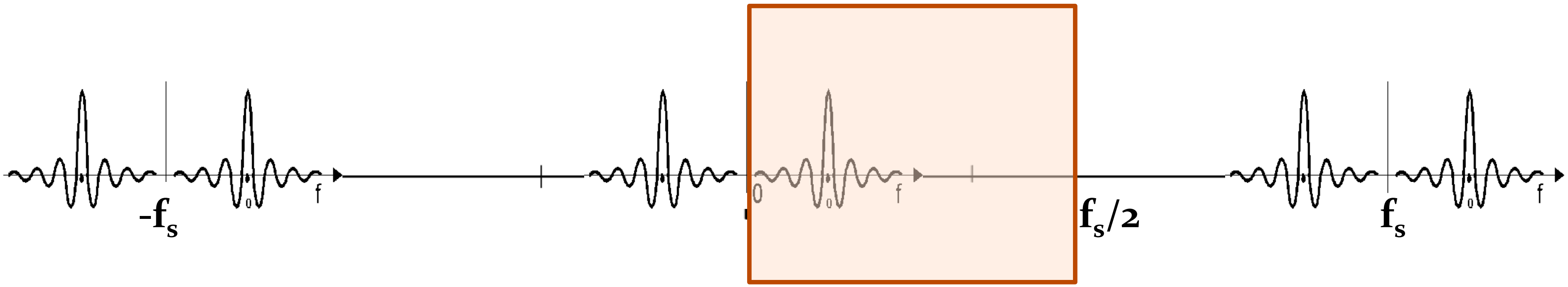
frequency range from 0 to f_s



L spectrum samples from frequency 0 to the sampling frequency f_s

Discrete Cosine Transform (DCT)

frequency range from 0 to $f_s/2$



L spectrum samples from frequency 0 to $f_s/2$

Example: spectral analysis of cosine signal: DFT and DCT

Signal cos(2πft) of frequency f=1 kHz is sampled with sampling frequency f_s=1/T= 20 kHz.
L =100 samples are used for spectral analysis using DFT.
DFT coefficients are obtained: X_0, X_1, …, X_{99}.
Which coefficients have maximum absolute values?

DFT coefficients are samples of DTFT at frequencies $f_k = \dfrac{k}{TL}, \qquad k = 0, 1, \ldots, L-1$

Frequency f=1 kHz points to k = f L T = 1 kHz x 100 / 20 kHz , that is to k=5.
Maximum value has the DFT coefficient X_5.
Note that the same value will have the mirror coefficient of mirror frequency 20-1=19 kHz
Its numer is 19 x 100 /20 = 95.

And what about spectrum analysis using DCT?
Here we have L coefficients from frequency 0 to half of sampling frequency: f_k = k / (2TL).
Maximum value is expected for DCT coefficient number k=10.
There is no mirror image coefficient.

2-dimensional discrete transforms

1-dimensional transform: straight $\bar{y} = \bar{W}\,\bar{x}$, inverse $\bar{x} = \bar{W}^{-1}\,\bar{y}$

DFT: $\bar{W}^{-1} = \frac{1}{N}\bar{W}^{*}$, DCT $\bar{W}^{-1} = \bar{W}^{t}$

t – transposition, * - complex conjugate

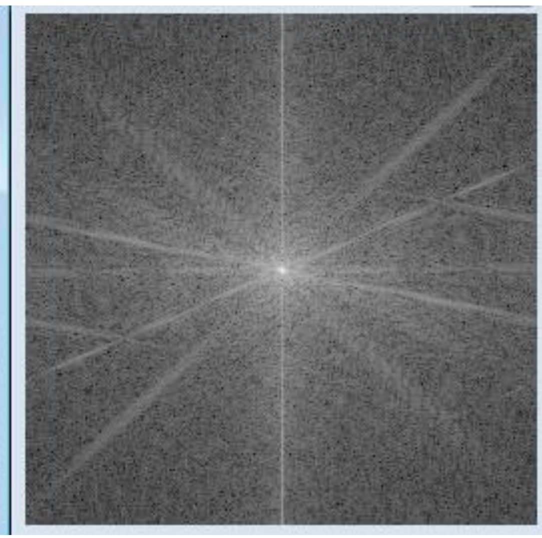
2-dimensional transform: straight $\bar{Y} = \bar{W}\,\bar{X}\,\bar{W}^{-1}$, inverse $\bar{X} = \bar{W}^{-1}\,\bar{Y}\,\bar{W}$

DFT: $\bar{Y} = \frac{1}{N}\bar{W}\,\bar{X}\,\bar{W}^{*}$, $\bar{X} = \frac{1}{N}\bar{W}^{*}\,\bar{Y}\,\bar{W}$

DCT: $\bar{Y} = \bar{W}\,\bar{X}\,\bar{W}^{t}$, $\bar{X} = \bar{W}^{t}\,\bar{Y}\,\bar{W}$

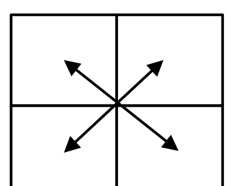
$\bar{y}, \bar{x}$ – N-dimensional vectors (columns), $\bar{Y}, \bar{W}, \bar{X}$ - NxN matrices

2-dimensional DFT and DCT



DFT
(quarters of Y interchanged)

DCT



2-dimensional DCT – basis images

Straight DCT: $Y = W X W^t$

$$Y(u,v) = \alpha(u)\alpha(v)\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} X(i,j)\cdot\cos\left[\frac{(2i+1)u\pi}{2N}\right]\cdot\cos\left[\frac{(2j+1)v\pi}{2N}\right]$$

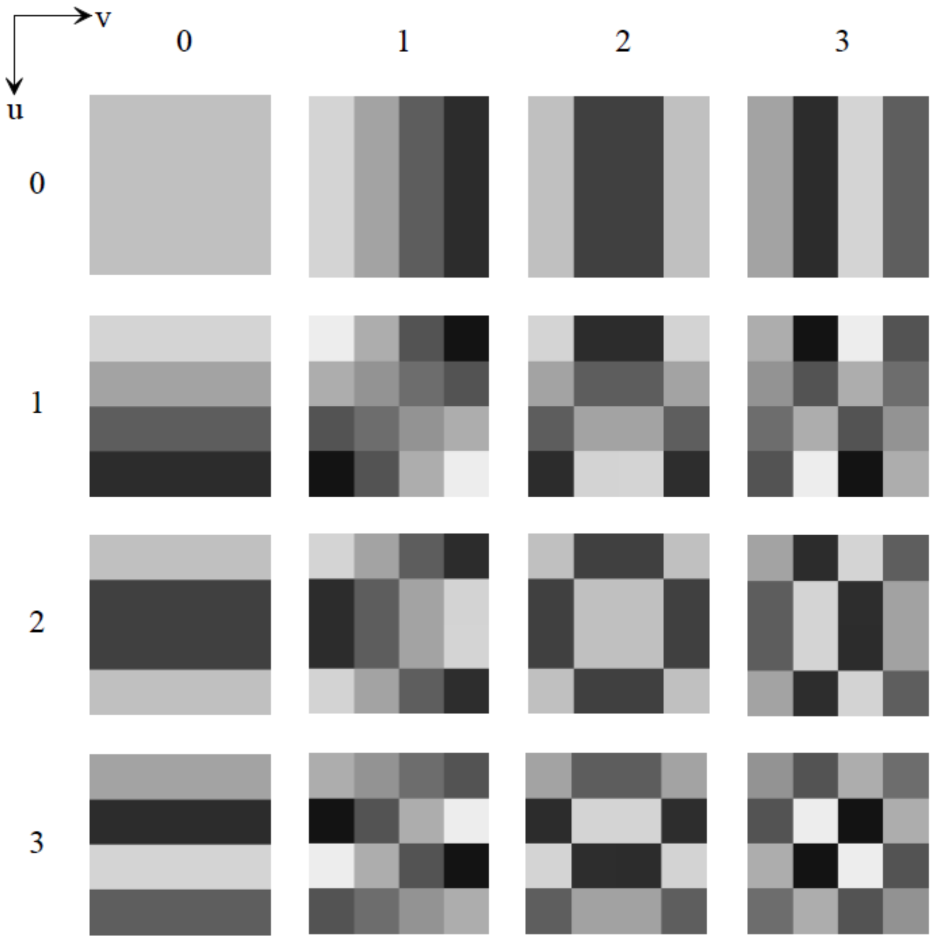
Inverse DCT (IDCT): $X = W^t Y W$

$$X(i,j) = \sum_{u=0}^{N-1}\sum_{v=0}^{N-1} \alpha(u)\alpha(v) Y(u,v)\cdot\cos\left[\frac{(2i+1)u\pi}{2N}\right]\cdot\cos\left[\frac{(2j+1)v\pi}{2N}\right]$$

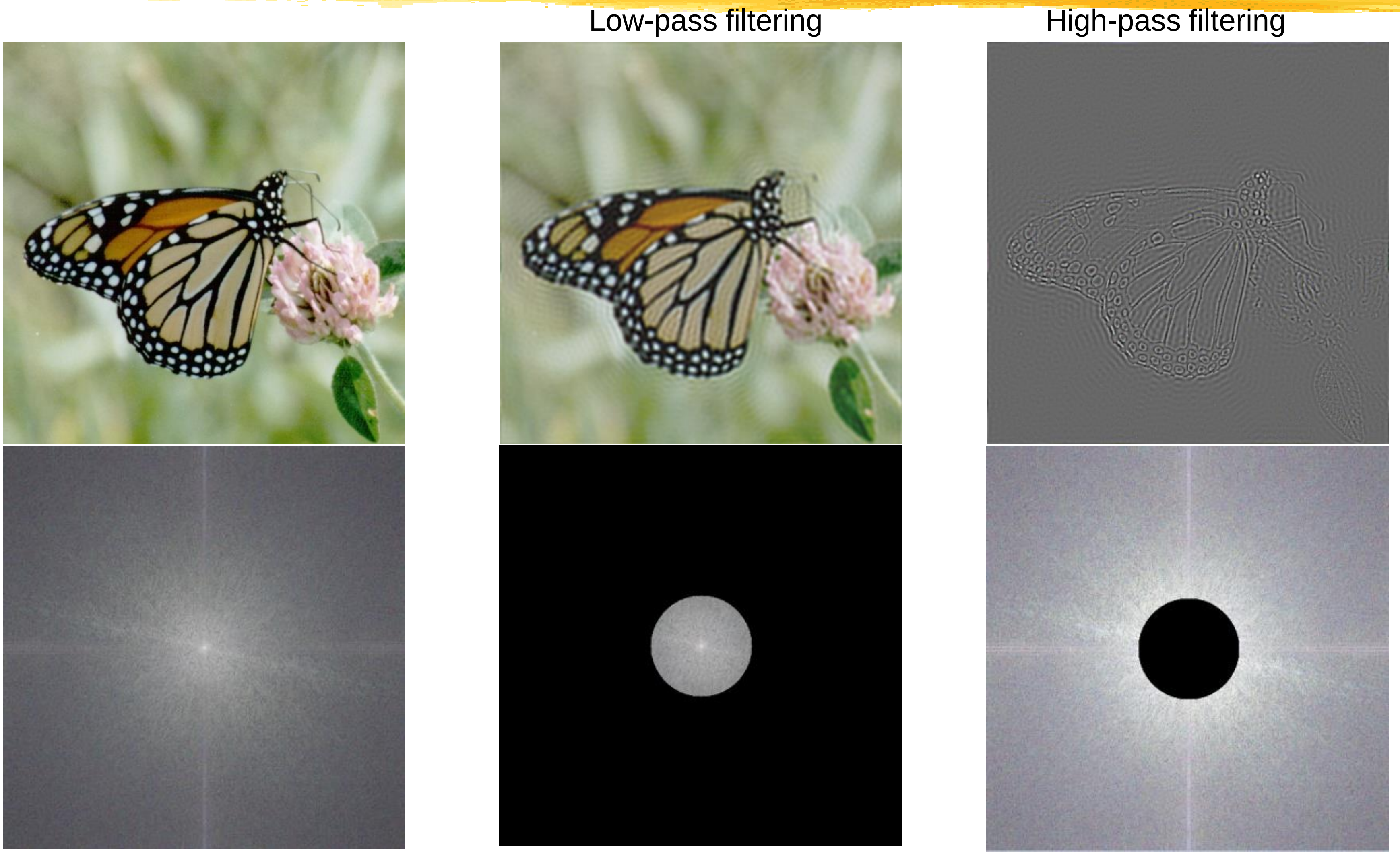
X is a sum of N*N basis images.

2-dimensional DCT – basis images

Basis images for 2-dimensional DCT for *N*=4



Filtering in transform domain



Denoising in transform domain

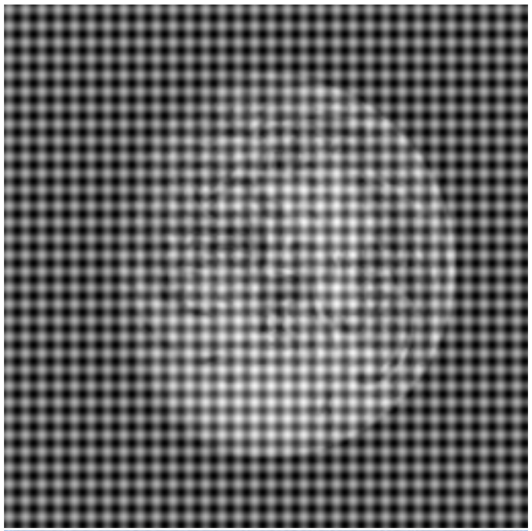
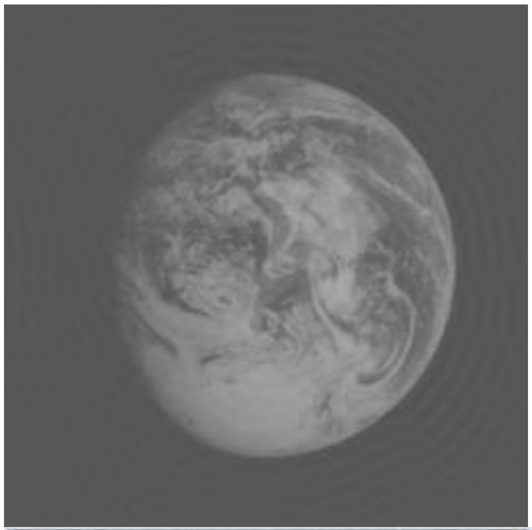


Image with
narrowband
distorsion

Denoised
image



DFT

Filtering
in transform
domain