Higher order linear differential equations

The n-th order linear differential equation is an equation of the form

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = p(x), \tag{1}$$

where $a_{n-1}, \ldots, a_1, a_0, p$ are given continuous functions defined on some interval $I \subset \mathbb{R}$. If $p(x) \equiv 0$, then equation (1) is called **homogeneous**. Otherwise, we call that equation **non-homogeneous**.

Let us consider firstly homogeneous equation

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0. \tag{2}$$

Let V denote the set of all solutions of equation (2). It can be proven that V is n-dimensional vector subspace of $C(I)$ (here $C(I)$ denotes space of all continuous functions on interval I). Therefore, to find all solutions of (2) we need to find some basis $\{y_1, \ldots, y_n\}$ of V. If we find such basis, then every solution y of (2) may be written as a linear combination of $y_1, \ldots, y_n$. So

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x),$$

for some real constants $C_1, \ldots, C_n \in \mathbb{R}$.

To find all solutions of non-homogeneous equation (1) we need to solve firstly corresponding homogeneous equation (2) (i. e. we find GSHE - general solution of homogeneous equation). Then, we need one particular solution of non-homogeneous equation (1) (i. e. PSNE). Then, general solution of non-homogeneous equation (GSNE) is given by

$$\text{GSNE} = \text{GSHE} + \text{PSNE}.$$

Task 1. Find all solutions of differential equation

$$y'' + \frac{y'}{x} - \frac{y}{x^2} = 0$$

knowing that one solution is $y_1(x) = x$.

Sketch of solution. The space of solutions of considered equation is 2-dimensional. Therefore, if we find other solution y_2 such that functions y_1, y_2 are linear independent, we get basis of this space. The general solution will be given by

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

where $C_1, C_2 \in \mathbb{R}$. The idea is to look for function y_2 in form

$$y_2(x) = y_1(x) \cdot u(x) = xu(x)$$

for some unknown function u. After substituting function y_2 to original equation we get

$$xu'' + 3u' = 0.$$

After substituting $w = u'$ we get $xw' + 3w = 0$. Therefore $w = \frac{C}{x^3}$. We need only one particular solution of original equation, so we may fix $C = 1$ (of course, any other non-zero constant C will be okay). Then $u' = \frac{1}{x^3}$ which provides that $u = -\frac{1}{2x^2} + A$. We may fix now $A = 0$ since we are looking for one particular non-zero solution. Now we have

$$y_2(x) = xu(x) = -\frac{1}{2x}.$$

It is easy to see that functions y_1, y_2 are linear independent. Therefore, the general solution is

$$y(x) = C_1 x + C_2 \cdot \frac{-1}{2x}, \text{ where } C_1, C_2 \in \mathbb{R}.$$

Remark. Of course, we can write general solution on infinitely many ways. For example, if we take $C = 3$ and $A = 1$, we get another basis of solution space. The general solution will be

$$y(x) = C_1 x + C_2 \left(\frac{-3}{2x} + x \right).$$

Similarly, the vector space $\mathbb{R}^2$ also many different bases. For example, both $\{(1,0), (0,1)\}$ and $\{(1,1), (0,3)\}$ are bases of $\mathbb{R}^2$.

Task 2. Using the same approach solve the differential equation

$$y'' + \frac{y'}{x} - \frac{y}{x^2} = \cos x.$$

Sketch of solution. Since GSNE $=$ GSHE $+$ PSNE and we already know GSHE from task 1., we only need to find PSNE. Using the same substitution $y = xu$ as in previous task, we get

$$xu'' + 3u' = \cos x.$$

Then, after substituting $w = u'$ we get first order linear non-homogeneous equation

$$xw' + 3w = \cos x. \tag{3}$$

By using variation of constants method to solve (3) we get

$$w(x) = \frac{x^2 \sin x + 2x \cos x - 2 \sin x}{x^3}.$$

Then, we can calculate that

$$u(x) = \int w(x) \mathrm{d}x = \frac{\sin x - x \cos x}{x^2} + C.$$

However, we look for one particular solution, so we can take $C = 0$ (or any other constant). Then, the PSNE is

$$y(x) = xu(x) = \frac{\sin x - x\cos x}{x}.$$

Since GSNE = GSHE + PSNE we get that GSNE is

$$y(x) = C_1 x + C_2 \cdot \left(-\frac{1}{2x}\right) + \frac{\sin x - x\cos x}{x}, \text{ for } C_1, C_2 \in \mathbb{R}.$$

Task 3. (Homework) Find all solutions of differential equation

$$(x-1)y'' - xy' + y = 0$$

knowing that one solution is $y_1(x) = \mathrm{e}^x$. Then, find all solutions of non-homogeneous equation

$$(x-1)y'' - xy' + y = 1.$$

Answer. GSHE: $y(x) = C_1\mathrm{e}^x + C_2 x$, where $C_1, C_2 \in \mathbb{R}$, GSNE: $y(x) = C_1\mathrm{e}^x + C_2 x + 1$, where $C_1, C_2 \in \mathbb{R}$.

Task 4. (Homework) Find all solutions of differential equation

$$x^2y'' + 3xy' - 3y = 0$$

knowing that one solution is $y_1(x) = x$. Then, find all solutions of non-homogeneous equation

$$x^2y'' + 3xy' - 3y = 12.$$

Answer. GSHE: $y(x) = C_1 x + C_2 \cdot \frac{1}{x^3}$, where $C_1, C_2 \in \mathbb{R}$, GSNE: $y(x) = C_1 x + C_2 \cdot \frac{1}{x^3} - 4$, where $C_1, C_2 \in \mathbb{R}$.

Remark. In tasks 3. and 4. it is easy to guess PSNE since the right side of equation is constant. We don't have to do any calculations - it is enough to notice that certain constant function are particular solutions.