EPRST: Probability and Statistics
Problem set 10

1. The joint density of a bivariate normal vector (X_1, X_2) is

$$f(x,y) = \frac{1}{8\pi} \exp\left\{-\frac{1}{32}\left[5(x+1)^2 - 4(x+1)(y+1) + 4(y+1)^2\right]\right\}.$$

   Find $\mathbb{E}X_i$, $\operatorname{Var} X_i$, $i = 1, 2$ and the correlation coefficient of X_1 and X_2.

2. The distribution of X is defined by $\mathbb{P}(X = -1) = \mathbb{P}(X = 1) = 1/4$, $\mathbb{P}(X = 0) = 1/2$, and the distribution of Y is given by $\mathbb{P}(Y = -1) = \mathbb{P}(Y = 1) = 1/2$. Find the convolution of these distributions.

3. Suppose that X_1, X_2 are independent and both are uniformly distributed on the set $\{0, 1, \ldots, n\}$. Find the distribution of $X_1 + X_2$.

4. Random variables X and Y are independent and have the same geometric distribution with parameter p. Compute the convolution of X and Y. Is it a geometric distribution?

5. Random variables X and Y are independent. They both have the standard normal distribution. What is the convolution of distributions of X^2 and Y^2?

6. Find the convolution of the uniform distributions $\mathrm{U}(0, 1)$ and $\mathrm{U}(-1, 0)$.

7. Let $Z = 2X + Y$, $T = 2X - Y$ and

$$(X, Y) \sim \mathcal{N}\left(\begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ -1 & 4 \end{pmatrix}\right).$$

   Compute $\mathbb{E}[X(X + 3Y)]$ and $\operatorname{Var}(2Z + T - 3)$.

8. Random vector (X, Y) has the normal distribution with the density

$$f(x,y) = \frac{1}{4\sqrt{2}\pi} \exp\left\{-\frac{1}{8}\left[3(x-1)^2 - 4(x-1)(y+3) + 2(y+3)^2\right]\right\}.$$

   (a) Compute $\mathbb{E}(XY)$.

   (b) Let $Z = 2X + Y$ and $T = 3Y - X$. Find the correlation coefficient $\rho(2T, Z + 2)$.

9. Let X and Y be independent random variables with distributions: $X \sim \mathcal{N}(-1, 2)$, $Y \sim \mathcal{N}(1, 4)$. Find the pdf of $Z = X + Y$.

10. Let the random vector (X, Y) have bivariate normal distribution with $\mathbb{E}X = \mathbb{E}Y = 0$, $\operatorname{Var}(X) = \operatorname{Var}(Y) = 1$ and correlation coefficient $\rho_{X,Y} = -0.5$. What are the values of a for which the random variables $V = aX + Y$ and $W = X + aY$ are independent?