Linear differential equations with constant coefficients

1. Homogeneous equations

Consider the following linear homogeneous differential equation with constant coefficients

$$y^{(n)} + a_{n-1}y^{n-1} + \cdots + a_1 y' + a_0 y = 0. \tag{1}$$

The polynomial

$$w(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 \tag{2}$$

is called a characteristic polynomial of equation (1).

The general solution of (1) is given by

$$y(x) = C_1 y_1(x) + \cdots + C_n y_n(x),$$

where $y_1, \ldots, y_n$ are linear independent solutions of (1).

How to find functions $y_1, \ldots, y_n$?

To find n linear independent solutions of (1) we need to look on the roots of characteristic polynomial (2). Let us assume that $\lambda_1, \ldots, \lambda_n$ are roots of characteristic polynomial. There are several cases now.

(a) All of roots $\lambda_1, \ldots, \lambda_n$ are real and have multiciplity 1. Then functions

$$y_1(x) = \mathrm{e}^{\lambda_1 x},\ y_2(x) = \mathrm{e}^{\lambda_2 x}, \ldots, y_n(x) = \mathrm{e}^{\lambda_n x}$$

are linear independent solutions of (1). Therefore, the general solution is

$$y(x) = C_1\mathrm{e}^{\lambda_1 x} + C_2\mathrm{e}^{\lambda_2 x} + \cdots + C_n\mathrm{e}^{\lambda_n x}$$

where $C_1, \ldots, C_n \in \mathbb{R}$ are constants.

(b) All of roots $\lambda_1, \ldots, \lambda_n$ are real but one of them has multiciplity greater than 1, i. e. there exists $1 \leq i \leq n$ such that λ_i has multiciplity $k > 1$. Then, we are not able to find n linear independent solutions using the same method as in (a), because some k of functions from (a) will be equal. In that case, functions

$$y_1(x) = \mathrm{e}^{\lambda_i x},\ y_2(x) = x\mathrm{e}^{\lambda_i x},\ y_3(x) = x^2\mathrm{e}^{\lambda i x}, \ldots, y_k(x) = x^{k-1}\mathrm{e}^{\lambda_i x}$$

will be linear independent solutions connected with λ_i.

(c) There exists a 1-multiple complex root $\lambda_i = a + bi$ of (2). Then, the fuctions

$$y_1(x) = \mathrm{Re}\ \mathrm{e}^{\lambda_i x} = \mathrm{e}^{ax}\cos(bx),\ ,y_2(x) = \mathrm{Im}\ \mathrm{e}^{\lambda i x} = \mathrm{e}^{ax}\sin(bx)$$

are two linear independent solutions of (1) corresponding to λ_i.

(d) There exists a k-multiple complex root $\lambda_i = a+bi$ of (2) where $k > 1$. Then, the functions

$$y_1(x) = \mathrm{e}^{ax}\cos(bx),\ , y_2(x) = x\mathrm{e}^{ax}\cos(bx), \dots, y_k(x) = x^{k-1}\mathrm{e}^{ax}\cos(bx),$$
$$y_{k+1}(x) = \mathrm{e}^{ax}\sin(bx),\ y_{k+2}(x) = x\mathrm{e}^{ax}\sin(bx), \dots, y_{2k}(x) = x^{k-1}\mathrm{e}^{ax}\sin(bx)$$

are $2k$ linear independent solutions of (1) connected with λ_i.

Example: Solve the differential equation $y^{(3)} - 3y' + 2y = 0$.

Solution: The characteristic polynomial of considered equation is equal

$$w(\lambda) = \lambda^3 - 3\lambda + 2$$

It is easy to see that

$$w(\lambda) = (\lambda - 1)^2(\lambda + 2).$$

Therefore, $\lambda_1 = 1$ is 2-multiple root of considered equation and $\lambda_2 = -2$ is 1-multiple root. In that case, the general solution is

$$y(x) = C_1 x\mathrm{e}^x + C_2\mathrm{e}^x + C_3\mathrm{e}^{-2x},$$

where $C_1, C_2, C_3 \in \mathbb{R}$.

2. Non-homogeneous equations

Consider the following linear non-homogeneous differential equation with constant coefficients

$$y^{(n)} + a_{n-1}y^{n-1} + \cdots + a_1 y' + a_0 y = f(x). \tag{3}$$

To find general solution of (3) we need to find GSHE and PSNE. Using methods from the previous part we may find GSHE. To find PSNE we can use the prediction method.

If $f(x) = \mathrm{e}^{ax}(P_1(x)\cos(bx) + P_2(x)\sin(bx))$, where P_1, P_2 are some polynomials, we can predict PSNE in following form:

(a) If $a + bi$ is not a root of characteristic polynomial, then we may look for PSNE in form

$$y(x) = \mathrm{e}^{ax}(Q_1(x)\cos(bx) + Q_2(x)\sin(bx)),$$

where Q_1, Q_2 are unknown polynomials such that $\deg Q_1 = \deg Q_2 = \max\{\deg P_1, \deg P_2\}$.

(b) If $a + bi$ is a p-multiple root of characteristic polynomial, then we may look for PSNE in form

$$y(x) = x^p\mathrm{e}^{ax}(Q_1(x)\cos(bx) + Q_2(x)\sin(bx)).$$