EPRST: Probability and Statistics
Problem set 3

1. Roll two 6 - sided dice (36 - possible outcomes are assumed to be equally likely). Given that at least one die faces up 6 find the probability that the other roll turns up 6 as well.

2. Toss a fair coin n times. Compute the probability of heads on the first toss given that r heads were obtained in the n tosses.

3. Flip a coin and roll a die. Let A be the event that the coin lands heads and B the event that the die lands six. Given that at least one of the two events has occurred, compute:

   (a) the probability that both events have occurred,

   (b) the probabllity that event A has occurred.

4. There are two urns: the one with 2 white and 3 black balls and the other one with 3 white and 3 black balls. We pick one urn at random and draw two balls out of it. What is the probability that the two balls have different colors?

5. We have 10 coins and one of them is non-symmetric (with probability of head equal 1/3). We toss a randomly selected coin 6 times, and obtain 3 tails. What is the probability that we tossed a symmetric coin?

6. There are three urns: i-th one contains i white and $3 - i$ black balls ($i = 1, 2, 3$). We pick an urn at random and then we pick two balls from the chosen urn (without replacement). It turned out that we picked two white balls. What is the probability that the balls were taken from the third urn?

7. In a binary transmition channel , a "1" is transmitted with probability 0.8 and "0" with probability 0.2. The conditional probability of receiving a "1" given that a "1" was sent is 0.95. The conditional probability of receiving a "0" when "0" was sent is 0.99. What is the probability that "1" had been sent when "1" was received?

8. Let A and B be independent events. Verify that events A and B' are independent. Conclude that A' and B' are also independent.

9. You are diagnosed with an uncommon disease. You know that there is only 1% of chance of getting it. Use the letter D for the event that "you have the disease" and T that "the test says so". It is known that the test is imperfect: $\mathbb{P}(T|D) = 0.98$ and $\mathbb{P}(T^c|D^c) = 0.95$. Given that you test positive, find the probability that you really have the disease.

10. An insect lays k eggs with probability $p_k = e^{-\lambda}\lambda^k/k!$, $k = 0, 1, 2, \ldots$. An insects hatches the egg with probability p, $0 < p < 1$ (independently of the other eggs).

    (a) Determine the probability that exactly j of the insect's descendants will hatch, $j = 0, 1, 2, \ldots$.

    (b) Given that j descendants of the insect have hatched, determine the probability that the insect has laid k eggs $k = 0, 1, 2, \ldots$.

11. Consider two independent fair coin tosses and the following events:

    - $H_1 = \{\text{1st toss is head}\}$,
    - $H_2 = \{\text{2nd toss is head}\}$,
    - $D = \{\text{the two tosses have different results}\}$.

    Are H_1, H_2 and D independent?

12. Let $(\Omega, \mathbb{P})$ be a probability space, $\Omega = [0,1]^2$, and $\mathbb{P}$ - a geometrical probability on Ω . Let's define events:

$$A = \left\{ (x,y) : x \le \frac{1}{2} \right\},$$

$$B = \left\{ (x,y) : y \le \frac{1}{2} \right\},$$

$$C = \{(x,y) : y \le x\}.$$

Are A, B and C independent?