Matrix

> ✎ **Definition**
>
> Matrix is a table of rows and columns containing data. Matrix is commonly used to describe linear transformations. We often build matrix with Vector as it's column.

Notation

Dimensions of matrix are often described with letters **m** and **n**. We often note matrix of size m, n.

- m stands for number of rows
- n stands for number of columns

Example of 3×2 matrix:

$$M = \begin{bmatrix} 3 & 2 \\ 2 & 4 \\ 4 & 5 \end{bmatrix}$$

If we want to access specific element we use subscript notation and say the index of row and column. It's common to use letter i to determine a row and j as column.
In general: $M_{i,j} = x$
Example:
For matrix M above $M_{1,1} = 3$ and $M_{3,2} = 5$. Notice it is 3,2 not 2,3. order of arguments matters.

Identity transformation

Identity transformations a transformation that does nothing - result is the same as if we didn't apply it at all.

Related ideas

Matrix multiplication
Determinant of a matrix
Inverse matrix

Rank of matrix

Rank of a matrix is a number of dimensions in the output of the transformation.

Full rank

When the matrix has maximum rank as it can have (n^{th} rank for $n \times n$ matrix)

Null space (kernel)

A set of vectors that fall into origin point after linear transformation is applied. Also known as kernel

Column space

A span of basis vectors

Vector

Ways of interpreting vector

Common way of interpreting the vector is to associate it with a force or movement in certain direction. Also it's common to associate it with a point in space.

1. In Physics we often visualize a vector with an arrow in space
2. In Computer Science we can represent vectors with an ordered list of numbers

Mathematics tries to generalise the idea of a vector and use it whenever it makes sense to use functions like addition or multiplication of vectors

Later on, in maths it is common to treat vectors as a matrix of size $m \times 1$ where m is the number of dimensions a vector has and m is a number of columns a matrix would have.

Notation

In 2d space: {x, y} or $\begin{bmatrix} a \\ b \end{bmatrix}$

In 3d space: {x, y, z} or $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Related Ideas

Vector scaling
Vector addition
Unit Vector
Linear transformation
Linear dependence
Linear combination
Dot product
Cross product
Magnitude
Argument

Eigenvalue

Definition

Factor by which Eigenvector are scaled during Linear transformation.

Warning

Not every linear transformation has eigenvalue!

Calculation

First it may be beneficial to rewrite equation defining eigenvector.
This way, after some transformations we obtain:

$$(A - \lambda I)\vec{v} = \vec{0}$$

To solve this we could say that $\vec{v} = 0$ but that does not get us closer to finding true solution.
Rather, we will focus on $A - \lambda I$ part. In practice it's just subtracting λ from diagonal axis.
Now, we can exploit that Eigenvector is on a null space (kernel) of the new matrix.
This way if we have a transformation that "squishes" space to smaller dimension.
This of course only happens when we have determinant equal to 0.
Equation to make it more visual:

$$det(A - \lambda I) = 0$$

Now, if we don't remember what determinant is we may be tempted to just try all possible lambdas and see what happens.
We can also just expand Determinant of a matrix equation and solve for λ.

Trick for calculating eigenvalues for 2x2 matrix

Motivation

This method make it much faster and more direct to get eigenvalues of this small matrix

Let M be our 2×2 matrix:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

3 facts we need to know:

1. trace of matrix (sum od diagonal components) is equal to sum of eigenvalues
   - $a + d = \lambda_1 + \lambda_2$
2. determinant of matrix is equal to product of eigenvalues
   - $ad - bc = \lambda_1 \cdot \lambda_2$
3. now let us define m as mean of $a + d$. Then our final solution will be
   - $\lambda_1, \lambda_2 = m \pm \sqrt{m^2 - p}$

Eigenvector

Definition

A nonzero vector that changes at most by a scalar factor when that linear transformation is applied to it. The corresponding Eigenvalue, often denoted by λ, is the factor by which the eigenvector is scaled.

Warning

Not every linear transformation has eigenvectors or enough vectors to describe original system of coordinates fully!

Notation

Every eigenvector must satisfy the equation:

$$A \cdot \vec{v} = \lambda \cdot \vec{v}$$

Where A is a matrix describing transformation, $\vec{v}$ is an eigenvector and λ is it's eigenvalue.

Calculation

We start with the calculation of eigenvalue.
Then, we solve the System of linear equations for given data.

Cramer's Rule

We use Cramer's rule to calculate a System of linear equations.
For a square matrix $M_{n \times n}$ of any size defining the Linear transformation, and a vector $\vec{v}$ of size n result vector can be calculated using following algorithm:

1. calculate a $det(M)$
2. for each column in the matrix M create new matrix with that column substituted with vector $\vec{v}$
3. calculate a determinant of this new matrix
4. save result to a corresponding component in result vector
5. multiply vector by $\frac{1}{det(M)}$

Result vector for 3d matrix:

$$\begin{bmatrix} det(Msub_1\vec{v})/det(M) \\ det(Msub_2\vec{v})/det(M) \\ det(Msub_3\vec{v})/det(M) \end{bmatrix}$$

Ex:

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Where:

$$Msub_1\vec{v} = \begin{bmatrix} x & b & c \\ y & e & f \\ z & h & i \end{bmatrix}, Msub_2\vec{v} = \begin{bmatrix} a & x & c \\ d & y & f \\ g & z & i \end{bmatrix}, Msub_3\vec{v} = \begin{bmatrix} a & b & x \\ y & e & y \\ g & h & z \end{bmatrix}$$

Determinant of a matrix

Notation

We denote a determinant of a matrix M as:

- det(M)
- $|M|$

Intuition

Determinant is a factor by which a space is scaled by the matrix after a Linear transformation.

Special cases:

- If the determinant of a matrix is 0 then a transformation reduces a dimensionality of space. In other words makes the basis vectors linearly dependent. For example turns a 2d space into a 1d line.
- If the determinant of a matrix is negative then a transformation "flips" the space as in flipping a sheet of paper.

Computation

> ⚠ **Warning**
>
> Only possible if square matrixes!!

For 2d

$Let\ M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $det(M) = ad - bc$

For 3d

$$Let\ M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$det(M) = a \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \begin{bmatrix} d & e \\ g & h \end{bmatrix} = aei + bfg + cdh - ath - bdi - ceg$$

Notice: first 3 components are products of components of diagonals from one side. Other three are products of components of diagonals from the other side.

For general n $\times$ n matrix

In general if is possible to calculate a determinant of any matrix recursively similarly to the way shown in 3d example. It is important to remember that the signs of components switches between + and -.
However, if we want to start from different column you can! Remember that + and - sign follow chess board pattern like the one below:

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

> ✎ **Sidenote**
>
> For each index sign of the index will be determined by equation: -1^{i+j}.

For example for 4d matrix:

$$m = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 0 \end{bmatrix}$$

$$det(M) = +1 \begin{bmatrix} 0 & 2 & 0 \\ 1 & 2 & 3 \\ 3 & 0 & 0 \end{bmatrix} - 2 \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 2 & 0 & 0 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 2 & 3 & 0 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 3 & 0 \end{bmatrix} = \ldots$$

OR

$$det(M) = -1 \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 3 & 0 & 0 \end{bmatrix} + 0 \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 3 \\ 2 & 0 & 0 \end{bmatrix} - 2 \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 2 & 3 & 0 \end{bmatrix} + 0 \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 3 & 0 \end{bmatrix} = \ldots$$

Notice that in the second approach we can skip the calculations of 2nd or 4th determinants because of multiplication by zero.
Also notice: when creating a determinant of lower dimensions we skipped the rows and columns of given index.