Circuits and Signals

Powers in AC circuits

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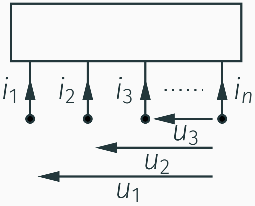
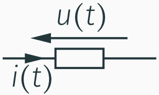
Instantaneous power — recap



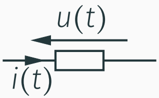
$$p(t) = i_1(t)u_1(t) + i_2(t)u_2(t) + \cdots + i_{n-1}(t)u_{n-1}(t).$$

For a one-port:



$$p(t) = u(t)i(t).$$

Instantaneous power in an AC circuit



$$p(t) = u(t)i(t).$$

$$i(t) = I_m \cos(\omega t + \phi), \qquad u(t) = U_m \cos(\omega t + \psi).$$

$$p(t) = I_m U_m \cos(\omega t + \phi) \cos(\omega t + \psi) \overset{\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha+\beta)+\cos(\alpha-\beta))}{=}$$

$$= \frac{1}{2} U_m I_m \cos(\phi - \psi) + \frac{1}{2} U_m I_m \cos(2\omega t + \phi + \psi).$$

Mean power and real power

Mean power:

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} p(t)\mathrm{d}t, \qquad T = \frac{2\pi}{\omega}.$$

In the AC circuit context, the mean power is called also the real power.

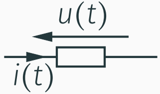
If

$$p(t) = \frac{1}{2}U_m I_m \cos(\phi - \psi) + \frac{1}{2}U_m I_m \cos(2\omega t + \phi + \psi),$$

then

$$P = \frac{1}{2}U_m I_m \cos(\phi - \psi).$$

Real power and the phasors



$$p(t) = u(t)i(t).$$

$$i(t) = I_m \cos(\omega t + \phi), \qquad u(t) = U_m \cos(\omega t + \psi).$$

$$P = \frac{1}{2} U_m I_m \cos(\phi - \psi).$$

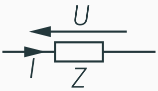
It occurs that

$$\boxed{P = \frac{1}{2} \mathrm{Re}(U\bar{I})}, \qquad U = U_m e^{\jmath\psi},\ I = I_m e^{\jmath\phi}.$$

Indeed

$$U\bar{I} = U_m e^{\jmath\psi} I_m e^{-\jmath\phi} = U_m I_m e^{\jmath(\psi-\phi)} = U_m I_m (\cos(\psi - \phi) + \jmath \sin(\psi - \phi)).$$

Real power and the impedance



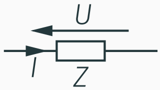
$$P = \frac{1}{2}\operatorname{Re}(U\bar{I})$$

$$P \overset{U=ZI}{=} \frac{1}{2}\operatorname{Re}(Z\underbrace{I\bar{I}}_{|I|^2}) = \frac{1}{2}|I|^2 \operatorname{Re} Z.$$

And also

$$P \overset{I=U/Z}{=} \frac{1}{2}\operatorname{Re}(\underbrace{U\bar{U}}_{|U|^2}/\bar{Z}) = \frac{1}{2}|U|^2 \operatorname{Re}\frac{1}{\bar{Z}} \overset{\operatorname{Re} z = \operatorname{Re}\bar{z}}{=} \frac{1}{2}|U|^2 \operatorname{Re}\frac{1}{Z}.$$

Real power — examples



$$P = \frac{1}{2}|I|^2 \operatorname{Re} Z = \frac{1}{2}|U|^2 \operatorname{Re} \frac{1}{Z}.$$

Inductor ($Z = \jmath\omega L$): $\boxed{P = 0}$.

Capacitor ($Z = \frac{1}{\jmath\omega C} = -\jmath\frac{1}{\omega C}$): $\boxed{P = 0}$.

Resistor ($Z = R$): $\boxed{P = \frac{1}{2}|I|^2 R = \frac{1}{2}\frac{|U|^2}{R}}$.

RMS value of a signal

RMS (root mean square) value X_{RMS} of a T-periodic signal x:

$$X_{\mathrm{RMS}} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} x^2(t)\,\mathrm{d}t}, \qquad T = \frac{2\pi}{\omega}.$$

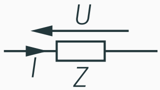
For a harmonic signal $x(t) = X_m \cos(\omega t + \phi)$:

$$X_{\mathrm{RMS}} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} X_m^2 \cos^2(\omega t + \phi)\,\mathrm{d}t} \overset{\cos^2\alpha = \frac{1}{2}(\cos 2\alpha + 1)}{=}$$

$$= \sqrt{\frac{1}{T}\underbrace{\int_{t_0}^{t_0+T} \frac{1}{2}X_m^2\,\mathrm{d}t}_{T\frac{1}{2}X_m^2} + \frac{1}{T}\underbrace{\int_{t_0}^{t_0+T} \frac{1}{2}X_m^2 \cos(2\omega t + 2\phi)\,\mathrm{d}t}_{0}} =$$

$$= \sqrt{\frac{1}{2}X_m^2} = \frac{X_m}{\sqrt{2}}.$$

Real power and RMS values



$$P = \frac{1}{2}|I|^2 \operatorname{Re} Z = \frac{1}{2}|U|^2 \operatorname{Re} \frac{1}{Z}.$$

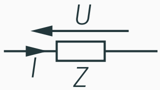
Therefore

$$P = I_{\mathrm{RMS}}^2 \operatorname{Re} Z = U_{\mathrm{RMS}}^2 \operatorname{Re} \frac{1}{Z}.$$

For a resistor $Z = R$ we thus obtain „the same” formulas as those for DC case

$$P = I_{\mathrm{RMS}}^2 R = \frac{U_{\mathrm{RMS}}^2}{R}.$$

Powers in AC circuits



Real power: $$P = \frac{1}{2}\,\mathrm{Re}(U\bar{I}) \quad [\mathrm{W}].$$

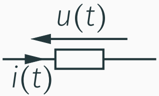
Reactive power: $$Q = \frac{1}{2}\,\mathrm{Im}(U\bar{I}) \quad [\mathrm{VAr}].$$

Complex power: $$S = \frac{1}{2}U\bar{I} \quad [\mathrm{VA}].$$

Apparent power: $$|S| = \frac{1}{2}U_m I_m \quad [\mathrm{VA}].$$

$$S = P + \jmath Q$$

Power factor



$$p(t) = u(t)i(t).$$

$$i(t) = I_m \cos(\omega t + \phi), \qquad u(t) = U_m \cos(\omega t + \psi).$$

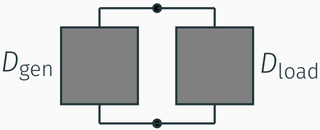
$$p(t) = \underbrace{\frac{1}{2}U_m I_m \cos(\phi - \psi)}_{P} + \underbrace{\frac{1}{2}U_m I_m}_{|S|} \cos(2\omega t + \phi + \psi).$$

Power factor P.F.

$$\text{P.F.} = \frac{P}{|S|} \qquad -1 \le \text{P.F.} \le 1.$$

$$P = |S| \times \text{P.F.}, \qquad \text{P.F.} = \cos \angle(U, I) = \cos\arg\frac{U}{I} = \cos\arg Z.$$

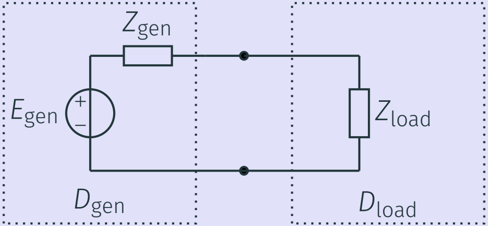
Maximum power transfer theorem — introduction



Assumption: D_{gen} is fixed (we cannot change it).

For what one-port D_{load} the real power delivered to D_{load} is maximal?

Maximum Power Transfer Theorem (AC case)



If $E_{\text{gen}} \neq 0$ and Z_{gen} are fixed parameters and if $\operatorname{Re} Z_{\text{gen}} > 0$, then the maximal real power that can be delivered (transferred) to Z_{load} equals

$$P_{\max} = \frac{|E_{\text{gen}}|^2}{8 \operatorname{Re} Z_{\text{gen}}}.$$

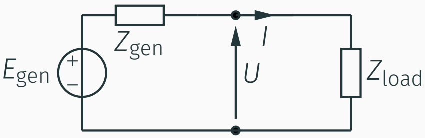
Such power is delivered to Z_{load} if and only if

$$Z_{\text{load}} = \overline{Z_{\text{gen}}}.$$

MPT theorems in DC and AC cases — comparison

	„DC case"	„AC case"
assumptions	$E_{gen} \neq 0$, $R_{gen} > 0$	$E_{gen} \neq 0$, $\mathrm{Re}\, Z_{gen} > 0$
MPT condition	$R_{load} = R_{gen}$	$Z_{load} = \overline{Z_{gen}}$
maximal power	$\frac{E_{gen}^2}{4R_{gen}}$	$\frac{\lvert E_{gen}\rvert^2}{8\,\mathrm{Re}\, Z_{gen}}$
max. power („Norton counterpart")	$\frac{J_{gen}^2}{4} R_{gen}$	$\frac{\lvert J_{gen}\rvert^2}{8\,\mathrm{Re}\,\frac{1}{Z_{gen}}}$

MPT theorem — proof



$$P = \frac{1}{2}\left|\frac{E_{gen}}{Z_{gen}+Z_{load}}\right|^2 \operatorname{Re} Z_{load} = \frac{|E_{gen}|^2}{8\operatorname{Re} Z_{gen}} \frac{2\operatorname{Re} Z_{load}}{|Z_{gen}+Z_{load}|} \frac{2\operatorname{Re} Z_{gen}}{|Z_{gen}+Z_{load}|}$$

$$\frac{2\operatorname{Re} Z_{load}}{|Z_{gen}+Z_{load}|} \frac{2\operatorname{Re} Z_{gen}}{|Z_{gen}+Z_{load}|} \overset{4ab\leq(a+b)^2}{\leq} \left(\frac{\operatorname{Re} Z_{load} + \operatorname{Re} Z_{gen}}{|Z_{gen}+Z_{load}|}\right)^2 =$$

$$\frac{\operatorname{Re}^2(Z_{load}+Z_{gen})}{\operatorname{Re}^2(Z_{load}+Z_{gen}) + \operatorname{Im}^2(Z_{load}+Z_{gen})} \overset{|x|/(|x|+|y|)\leq 1}{\leq} 1.$$

Thus $P \leq \frac{|E_{gen}|^2}{8\operatorname{Re} Z_{gen}}$ and „=” $\overset{a=b,\ |y|=0}{\iff}$ $\operatorname{Re} Z_{load} = \operatorname{Re} Z_{gen}$ and $\operatorname{Im} Z_{load} = -\operatorname{Im} Z_{gen}$.