Chapter 8: Formula Cheat Sheet

Cross product: $\mathbf{a}\times\mathbf{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$

Trigonometric Identities:

$$\sin(\alpha)^2 + \cos(\alpha)^2 = 1$$
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Determinant:

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$
$$= aei + bfg + cdh - ceg - bdi - afh$$

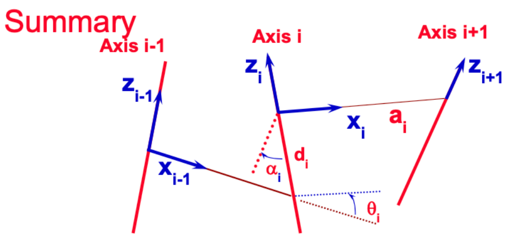
A manipulator may have special configurations, called "isotropic points", that are characterized by the Jacobi matrix having orthogonal columns of equal length, thus $J^TJ = \delta I$ for some $\delta \in \mathbb{R}$.

8.1 Denavit-Hartenberg Parameters

DH-table (link-index i):

i	a_{i-1}	α_{i-1}	d_i	θ_i
1	...	...	...	...
...	...	...	...	...

1. Shift Z_{i-1} by a_{i-1} along X_{i-1}.
2. Rot. Z_{i-1} by α_{i-1} about X_{i-1} & shift by d_i along Z_i.
3. Rot. X_{i-1} by θ_i about Z_i & move it to Z_i.



$a_i =$ *the distance from* $\hat{Z}_i$ *to* $\hat{Z}_{i+1}$ *measured along* $\hat{X}_i$;

$\alpha_i =$ *the angle from* $\hat{Z}_i$ *to* $\hat{Z}_{i+1}$ *measured about* $\hat{X}_i$;

$d_i =$ *the distance from* $\hat{X}_{i-1}$ *to* $\hat{X}_i$ *measured along* $\hat{Z}_i$; and

$\theta_i =$ *the angle from* $\hat{X}_{i-1}$ *to* $\hat{X}_i$ *measured about* $\hat{Z}_i$.

Homogeneous transformation from link i to $i-1$: $${}^{i-1}_{\;\;i}T = \left[\begin{array}{ccc|c} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i\ c\alpha_{i-1} & c\theta_i\ c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}\ d_i \\ s\theta_i\ s\alpha_{i-1} & c\theta_i\ s\alpha_{i-1} & c\alpha_{i-1} & d_i\ c\alpha_{i-1} \\ \hline 0 & 0 & 0 & 1 \end{array}\right].$$

Inverse of the homogeneous transform: $${}^A_BT^{-1} = {}^B_AT = \left[\begin{array}{ccc|c} & {}^A_BR^T & & -{}^A_BR^T\,{}^AP_{Borg} \\ \hline 0 & 0 & 0 & 1 \end{array}\right].$$

8.2 Jacobian

Singularity: the end-effector locally looses at least 1 DOF. This happens, when Z-axes are aligned $\Leftrightarrow$ the Jacobian does not have full rank $\Leftrightarrow$ $det(J) = 0$. Small end-effector motions require large joint motions near singularities.

8.2.1 Velocity Propagation

Linear and angular velocities at joint $i+1$ (with scalar $\dot{d}_{i+1}$ or $\dot{\Theta}_{i+1}$ for prismatic/ revolute joints):

$${}^{i+1}\omega_{i+1} = {}^{i+1}_{\;\;\;i}R \cdot {}^i\omega_i + \dot{\Theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}$$
$${}^{i+1}v_{i+1} = {}^{i+1}_{\;\;\;i}R({}^iv_i + {}^i\omega_i \times {}^iP_{i+1}) + \dot{d}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}$$

Then read off the Jacobian from: $$\begin{bmatrix} \dot{x}_P \\ \dot{x}_R \end{bmatrix}_{6\times1} = \begin{bmatrix} J_{xP} \\ J_{xR} \end{bmatrix}_{6\times n} \dot{\boldsymbol{\Theta}} = J_{6\times n} \begin{bmatrix} \dot{\Theta}_1 \\ \dots \\ \dot{\Theta}_n \end{bmatrix}_{n\times1}.$$

8.2.2 Force/ Torque Propagation

Force f and torque n at joint i:

$$
\begin{aligned}
{}^{i}f_i &= {}^{i}_{i+1}R \cdot {}^{i+1}f_{i+1} \\
{}^{i}n_i &= {}^{i}_{i+1}R \cdot {}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_i
\end{aligned}
$$

The force/ torque exerted on the ith joint:

$$\tau_i = {}^{i}f_i^{\mathrm{T}\,i}Z_i = {}^{i}f_i^{\mathrm{T}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \tau_i = {}^{i}n_i^{\mathrm{T}\,i}Z_i = {}^{i}n_i^{\mathrm{T}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Then read off the Jacobian from:

$$\begin{pmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_n \end{pmatrix} = \tau = {}^{A}J^{T\,A}\mathcal{F} = {}^{A}J^{T} \begin{pmatrix} {}^{A}f \\ {}^{A}n \end{pmatrix},$$

where $\mathcal{F}$ is a 6×1 force-torque vector in frame $\{A\}$.

8.2.3 Explicit Form

Jacobian in frame $\{0\}$:

$${}^{0}J = \left[\begin{matrix} \frac{\partial}{\partial q_1}\left({}^{0}x_P\right) & \frac{\partial}{\partial q_2}\left({}^{0}x_P\right) & \cdots & \frac{\partial}{\partial q_n}\left({}^{0}x_P\right) \\ \bar{\epsilon}_1 \cdot ({}^{0}_{1}R \cdot Z) & \bar{\epsilon}_2 \cdot ({}^{0}_{2}R \cdot Z) & \cdots & \bar{\epsilon}_n \cdot ({}^{0}_{n}R \cdot Z] \end{matrix} \right)$$

with ${}^{0}Z_i = {}^{0}_{i}R^{i}Z_i$; $\quad {}^{i}Z_i = Z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. The indicator variable $\bar{\epsilon}_i$ is 1 if joint i is revolute, otherwise it is 0. Then, rotate ${}^{0}J$ into the required reference frame:

$${}^{A}J(\Theta) = \begin{pmatrix} {}^{A}_{B}R & \mathbf{0} \\ \mathbf{0} & {}^{A}_{B}R \end{pmatrix} {}^{B}J(\Theta)$$

8.3 Newton-Euler Method

Propagate lin. and ang. velocities and accelerations foward (account for gravity by setting ${}^{i}\dot{v}_i = -G$):

Rotational joint $i+1$	${}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R \cdot {}^{i}\omega_i + \dot{\Theta}_{i+1} \cdot {}^{i+1}Z_{i+1}$ ${}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R \cdot {}^{i}\dot{\omega}_i + {}^{i+1}_{i}R \cdot {}^{i}\omega_i \times \dot{\Theta}_{i+1} \cdot {}^{i+1}Z_{i+1} + \ddot{\Theta}_{i+1}{}^{i+1}Z_{i+1}$ ${}^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R \left({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{i+1}\right) + {}^{i}\dot{v}_i\right)$
Prismatic joint $i+1$	${}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R \cdot {}^{i}\omega_i$ ${}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R \cdot {}^{i}\dot{\omega}_i$ ${}^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R \left({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{i+1}\right) + {}^{i}\dot{v}_i\right)$ $+ 2 \cdot {}^{i+1}\omega_{i+1} \times \dot{d}_{i+1}{}^{i+1}Z_{i+1} + \ddot{d}_{i+1}{}^{i+1}Z_{i+1}$

At the center of mass we have (revolute/ prismatic joints): ${}^{i}\dot{v}_{C_i} = {}^{i}\dot{\omega}_i \times {}^{i}P_{C_i} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{C_i}) + {}^{i}\dot{v}_i$. Next, compute the forces and torques at the center of mass of each link (applied by the motion):

$$
\begin{aligned}
{}^{i}F_i &= m_i \cdot {}^{i}\dot{v}_{C_i} \\
{}^{i}N_i &= {}^{C_i}I_i \cdot {}^{i}\dot{\omega}_i + {}^{i}\omega_i \times {}^{C_i}I_i \cdot {}^{i}\omega_i
\end{aligned}
$$

Propagate forces and moments f_i and n_i backward:

$$
\begin{aligned}
{}^{i}f_i &= {}^{i}_{i+1}R \cdot {}^{i+1}f_{i+1} + {}^{i}F_i \\
{}^{i}n_i &= {}^{i}N_i + {}^{i}_{i+1}R \cdot {}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times {}^{i}_{i+1}R^{i+1}f_{i+1}
\end{aligned}
$$

The values of τ_i, compute as $\tau_i = {}^{i}n_i^{T} \cdot {}^{i}Z_i$ for a revolute joint and $\tau_i = {}^{i}f_i^{T} \cdot {}^{i}Z_i$ for a prismatic joint.

8.3.1 Parallel-Axes Theorem

How the inertia tensor changes under *translations* of the reference coordinate system (s.t. the axes remain parallel). It relates the inertia tensor w.r.t. the center of mass ${}^{C}I$ to the inertia tensor w.r.t. another reference frame ${}^{A}I$:

$$ {}^{A}I = {}^{C}I + m[P_c^T P_c I_3 - P_c P_c^T] $$

where $P_c = [x_c, y_c, z_c]^T$ locates the center of mass relative to A. I_3 is the identity matrix.

8.4 Lagrange Method

Kinetic energy of link i (requires ${}^{0}v_{C_i}$ (from ${}^{0}P_{C_i}$) and ω_i):

$$ k_i = \underbrace{\frac{1}{2} m_i v_{C_i}^{\mathrm{T}} \cdot v_{C_i}}_{\text{transl. energy of } C_i} + \underbrace{\frac{1}{2} {}^{i}\omega_i^{\mathrm{T}} \cdot {}^{C_i}I_i \cdot {}^{i}\omega_i}_{\text{rot. energy about } C_i} \qquad \text{or alternatively} \qquad k(\Theta, \dot{\Theta}) = \frac{1}{2} \dot{\Theta}^{\mathrm{T}} M(\Theta) \dot{\Theta} $$

where ${}^{C_i}I_i$ is the inertia of link i in $\{C_i\}$. Then compute the total kinetic energy, $k = \sum_{i=1}^{n} k_i$. Potential energy of link i:

$$ u_i = -m_i \cdot {}^{0}g^{\mathrm{T}} \cdot {}^{0}P_{C_i} + u_{\mathrm{ref}_i} $$

Finally, use the Lagrangian $L = k - u$ to compute the joint torques τ:

$$ \tau = \frac{d}{dt} \frac{\partial L}{\partial \dot{\Theta}} - \frac{\partial L}{\partial \Theta} = \frac{d}{dt} \frac{\partial k}{\partial \dot{\Theta}} - \frac{\partial k}{\partial \Theta} + \frac{\partial u}{\partial \Theta} \qquad \text{or per-joint:} \qquad \tau_i = \frac{d}{dt} \frac{\partial k}{\partial \dot{\Theta}_i} - \frac{\partial k}{\partial \Theta_i} + \frac{\partial u}{\partial \Theta_i} $$

8.4.1 M-V-G-form (State-Space-form)

$$ \tau = \underbrace{M(\Theta)}_{\substack{n \times n \text{ matrix} \\ \text{All coefficients of } \ddot{\Theta}}} \ddot{\Theta} + \underbrace{V(\Theta, \dot{\Theta})}_{\substack{n \times 1 \text{ vector} \\ \text{All summands with } \dot{\Theta}}} + \underbrace{G(\Theta)}_{\substack{n \times 1 \text{ vector} \\ \text{All summands with } g}} $$

V can be decomposed into B and C, yielding the M-B-C-G-form (configuration-space equation):

$$ \tau = M(\Theta)\ddot{\Theta} + \underbrace{B(\Theta)}_{n \times \frac{n(n-1)}{2} \text{ matrix}} \overbrace{[\dot{\Theta}\dot{\Theta}]}^{=(\dot{\Theta}_1\dot{\Theta}_2, \dot{\Theta}_1\dot{\Theta}_3, \ldots, \dot{\Theta}_{n-1}\dot{\Theta}_n)^T} + \underbrace{C(\Theta)}_{n \times n \text{ matrix}} \overbrace{[\dot{\Theta}^2]}^{=(\dot{\Theta}_1^2, \dot{\Theta}_2^2, \ldots, \dot{\Theta}_n^2)^{\mathrm{T}}} + G(\Theta) $$

The matrices B and C can be determined by finding the coefficients of $\Theta_i \Theta_j$ and Θ_i^2, respectively.

8.5 Control

8.5.1 Mass-Spring-System

Open-loop equation: $m\ddot{x} + b\dot{x} + kx = f \overbrace{= -k_p x - k_v \dot{x}}^{\text{PD-control force for critical damping}}$

Closed-loop equation: $m\ddot{x} + (b + k_v)\,\dot{x} + (k + k_p)\,x = 0$

Solve the characteristic equation to determine $x(t)$: $ms^2 + (b + k_v)\,s + (k + k_p) = 0$

$$ s_{1,2} = \frac{-b' \pm \sqrt{b'^2 - 4mk'}}{2m} \quad \Rightarrow \quad x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} \underbrace{=}_{\text{if } s_{1,2} = \lambda \pm \mu i} e^{\lambda t} \left(c_1 \cos(\mu t) + c_2 \sin(\mu t) \right) $$

If $s_1 = s_2 \in \mathbb{R}$ the system is critically damped ($\Rightarrow b'^2 - 4mk' = 0$ or $b' = 2\sqrt{mk'}$, where $b', k' > 0$). For an oscillating system, the equations can also be stated as:

$$ s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \qquad \text{with } \textit{damping ratio } \zeta = \frac{b'}{2\sqrt{k'm}} \text{ and } \textit{natural frequency } \omega_n = \sqrt{\frac{k'}{m}} $$

8.5.2 PD Control

Control law partitioning: separate model dependant parameters like mass, friction, gravitation from the ideal unit mass system: $\tau = \alpha\tau' + \beta$. The system appears as a unit mass system to the controller.

	Mass-Spring-System	**Multi-Body-System**
Control law partitioning	Into system dependant & servo part: $m\ddot{x} + b\dot{x} + kx = f = \alpha f' + \beta.$ with $\alpha = m$ and $\beta = b\dot{x} + kx$. Unit mass system: $\ddot{x} = \tau'$.	$M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta) = \tau = \alpha\tau' + \beta$ with $\alpha = M(\Theta)$ and $\beta = V(\Theta, \dot{\Theta}) + G(\Theta)$. Unit mass system if M^{-1} exists: $\ddot{\Theta} = \tau'$.
Position control	Control law & equation of motion: $f' = -k_v\dot{x} - k_p x$ $\ddot{x} + k_p x + k_v\dot{x} = 0$	$\tau' = -K_v\dot{\Theta} - K_p\Theta$ $\ddot{\Theta} + K_v\dot{\Theta} + K_p\Theta = 0$
Trajectory following	Control law & equation of motion: $f' = \ddot{x}_d + k_v\dot{e} + k_p e$ $\ddot{x} = \ddot{x}_d + k_v\dot{e} + k_p e$ $\Leftrightarrow 0 = \ddot{e} + k_v\dot{e} + k_p e$	$\tau' = \ddot{\Theta}_d + K_v\left(\dot{\Theta}_d - \dot{\Theta}\right) + K_p\left(\Theta_d - \Theta\right) = \ddot{\Theta}_d + K_v\dot{E} + K_p E$ where Θ_d is the vector of desired joint positions. Insert τ' for the error equation: $\tau = M(\Theta)\left(\ddot{\Theta}_d + K_v\dot{E} + K_p E\right) + V(\Theta, \dot{\Theta}) + G(\Theta)$ $\Leftrightarrow 0 = M(\Theta)\left(\ddot{\Theta}_d - \ddot{\Theta} + K_v\dot{E} + K_p E\right)$ $\Leftrightarrow 0 = \ddot{E} + K_v\dot{E} + K_p E$ with diagonal K_v and K_p.
Natural frequency & critical damping	Max. freq.: $\omega_n = \sqrt{k_p} \leq 0.5\omega_{\text{res}}$ Critical damping for: $k_v = 2\sqrt{k_p}$	$\omega_{ni} = \sqrt{k_{pi}}$ $k_{vi} = 2\sqrt{k_{pi}}$

8.5.3 Controller Block Diagram

General form (use the expressions for α and β). Complete the diagram according to the equation for τ'.