1 Truth Tables

Note 1

Determine whether the following equivalences hold, by writing out truth tables. Clearly state whether or not each pair is equivalent.

(a) $P \wedge (Q \vee P) \equiv P \wedge Q$

(b) $(P \vee Q) \wedge R \equiv (P \wedge R) \vee (Q \wedge R)$

(c) $(P \wedge Q) \vee R \equiv (P \vee R) \wedge (Q \vee R)$

Solution:

(a) Not equivalent.

P	Q	$P \wedge (Q \vee P)$	$P \wedge Q$
T	T	T	T
T	F	T	F
F	T	F	F
F	F	F	F

(b) Equivalent.

P	Q	R	$(P \vee Q) \wedge R$	$(P \wedge R) \vee (Q \wedge R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

(c) Equivalent.

P	Q	R	$(P \wedge Q) \vee R$	$(P \vee R) \wedge (Q \vee R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	F	F
F	F	T	T	T
F	F	F	F	F

2 Propositional Practice

Note 1

Convert the following English sentences into propositional logic and the following propositions into English. State whether or not each statement is true with brief justification.

(a) There is a real number which is not rational.

(b) All integers are natural numbers or are negative, but not both.

(c) If a natural number is divisible by 6, it is divisible by 2 or it is divisible by 3.

(d) $(\forall x \in \mathbb{Z})\ (x \in \mathbb{Q})$

(e) $(\forall x \in \mathbb{Z})\ (((2 \mid x) \vee (3 \mid x)) \implies (6 \mid x))$

(f) $(\forall x \in \mathbb{N})\ ((x > 7) \implies ((\exists a, b \in \mathbb{N})\ (a + b = x)))$

Solution:

(a) $(\exists x \in \mathbb{R})\ (x \notin \mathbb{Q})$, or equivalently $(\exists x \in \mathbb{R})\ \neg(x \in \mathbb{Q})$. This is true, and we can use π as an example to prove it.

(b) $(\forall x \in \mathbb{Z})\ (((x \in \mathbb{N}) \vee (x < 0)) \wedge \neg((x \in \mathbb{N}) \wedge (x < 0)))$. This is true, since we define the naturals to contain all integers which are not negative.

(c) $(\forall x \in \mathbb{N})\ ((6 \mid x) \implies ((2 \mid x) \vee (3 \mid x)))$. This is true, since any number divisible by 6 can be written as $6k = (2 \cdot 3)k = 2(3k)$, meaning it must also be divisible by 2.

(d) All integers are rational numbers. This is true, since any integer number n can be written as $n/1$.

(e) Any integer that is divisible by 2 or 3 is also divisible by 6. This is false–2 provides the easiest counterexample. Note that this statement is false even though its converse (part c) is true.

(f) If a natural number is larger than 7, it can be written as the sum of two other natural numbers. This is trivially true, since we can take $a = x$ and $b = 0$.

(Aside: this is a refererence to the very weak Goldback Conjecture (https://xkcd.com/1310/).)

3 Converse and Contrapositive

Note 1
Note 2

Consider the statement "if a natural number is divisible by 4, it is divisible by 2".

(a) Write the statement in propositional logic. Prove that it is true or give a counterexample.

(b) Write the inverse of the implication in English and in propositional logic. Prove that it is true or give a counterexample. (The inverse of an implication $P \implies Q$ is $\neg P \implies \neg Q$.)

(c) Write the converse of the implication in English and in propositional logic. Prove that it is true or give a counterexample.

(d) Write the contrapositive of the implication in English and in propositional logic. Prove that it is true or give a counterexample. Consider using part (a).

Solution:

The notation $a \mid b$ ("a divides b") denotes that b is divisible by a.

(a) $(\forall x \in \mathbb{N})\ (4 \mid x \implies 2 \mid x)$. This statement is true. We know that if x is divisible by 4, we can write x as $4k$ for some integer k. But $4k = (2 \cdot 2)k = 2(2k)$, where $2k$ is also an integer. Thus, x must also be divisible by 2, since it can be written as 2 times an integer.

(b) The inverse is that if a natural number is not divisible by 4, it is not divisible by 2: $(\forall x \in \mathbb{N})\ (4 \nmid x \implies 2 \nmid x)$. This is false, since 2 is not divisible by 4, but is divisible by 2.

(c) The converse is that any natural number that is divisible by 2 is also divisible by 4: $(\forall x \in \mathbb{N})\ (2 \mid x \implies 4 \mid x)$. Again, this is false, since 2 is divisible by 2 but not by 4.

(d) The contrapositive is that any natural number that is not divisible by 2 is not divisible by 4: $(\forall x \in \mathbb{N})\ (2 \nmid x \implies 4 \nmid x)$. The simplest way is to use a proof by contraposition through part (a); since this is the contrapositive of (a), and we've shown that (a) is true, this statement must also be true.

    To show that this is true through a direct proof, first consider that saying that x is not divisible by 2 is equivalent to saying that $x/2$ is not an integer. And if we divide a non-integer by an integer, we get back another non-integer–so $(x/2)/2 = x/4$ must also not be an integer. But that is exactly the same as saying that x is not divisible by 4.

Note that the inverse and the converse will always be contrapositives of each other, and so will always be logically equivalent.

4 Logical Equivalence?

Note 1

Decide whether each of the following logical equivalences is correct and justify your answer.

(a) $\forall x\ \big(P(x) \wedge Q(x)\big) \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$

(b) $\forall x\ \big(P(x) \vee Q(x)\big) \equiv \forall x\, P(x) \vee \forall x\, Q(x)$

(c) $\exists x\ \big(P(x) \vee Q(x)\big) \equiv \exists x\, P(x) \vee \exists x\, Q(x)$

(d) $\exists x\ \big(P(x) \wedge Q(x)\big) \equiv \exists x\, P(x) \wedge \exists x\, Q(x)$

Solution:

(a) **Correct**.

    Assume that the left hand side is true. Then we know for an arbitrary x $P(x) \wedge Q(x)$ is true. This means that both $\forall x P(x)$ and $\forall x Q(x)$. Therefore the right hand side is true. Now for the other direction assume that the right hand side is true. Since for any x $P(x)$ and for any y $Q(y)$ holds, then for an arbitrary x both $P(x)$ and $Q(x)$ must be true. Thus the left hand side is true.

(b) **Incorrect**.

    Note, there are many possible counterexamples - here we present only one. Suppose that the universe (i.e. the values that x can take on) is $\{1,2\}$ and that P and Q are truth functions defined on this universe. If we set $P(1)$ to be true, $Q(1)$ to be false, $P(2)$ to be false and $Q(2)$ to be true, the left-hand side will be true, but the right-hand side will be false. Hence, we can find a universe and truth functions P and Q for which these two expressions have different values, so they must be different.

(c) **Correct**

Assuming that the left hand side is true, we know there exists some x such that one of $P(x)$ and $Q(x)$ is true. Thus $\exists x P(x)$ or $\exists x Q(x)$ and the right hand side is true. To prove the other direction, assume the left hand side is false. Then there does not exists an x for which $P(x) \vee Q(x)$ is true, which means there is no x for which $P(x)$ or $Q(x)$ is true. Therefore the right hand side is false.

(d) **Incorrect**.

Note, there are many possible counterexamples - here we present only one. Suppose that the universe (i.e. the values that x can take on) is the natural numbers $\mathbb{N}$, and that P and Q are truth functions defined on this universe. If we set $P(1)$ to be true and $P(x)$ to be false for all other x, and $Q(2)$ to be true and $Q(x)$ to be false for all other x, then the right hand side would be true. However, there would be no value of x at which both $P(x)$ and $Q(x)$ would be simultaneously true, so the left hand side would be false. Hence, we can find a universe and truth functions P and Q for which these two expressions have different values, so they must be different.