CS 70 Spring 2023 — Discrete Mathematics and Probability Theory — Satish Rao and Babak Ayazifar — HW 00

1 Administrivia

(a) Make sure you are on the course Ed (for Q&A) and Gradescope (for submitting homeworks, including this one). Find and familiarize yourself with the course website. What is its homepage's URL?

(b) Read the policies page on the course website.

(i) What is the breakdown of how your grade is calculated?

(ii) What is the attendance policy for discussions?

(iii) When are homeworks released and when are they due?

(iv) How many "drops" do you get for vitamins? For homework?

(v) When is the midterm? When is the final?

Solution:

(a) The course website is located at `https://www.eecs70.org/`.

(b) (i) With HW: Discussion Attendance: 5%, Vitamins: 5%, Homework: 20%, Midterm: 25%, Final: 45%.

Without HW: Discussion Attendance: 6.25%, Vitamins: 6.25%, Midterm: 31.25%, Final: 56.25%.

(ii) You will receive 1 attendance point for every discussion, and will need at least 13 points in order to receive full credit for discussion attendance. You are welcome to attend other discussion sections, but your attendance will only be counted for the section you are actually assigned.

(iii) The homework for the current week is released on Gradescope on Sunday. The homework is due on Gradescope the following Saturday at 4:00 pm (grace period until 6:00pm); the solutions for that homework will be released on Sunday.

(iv) You can drop the lowest 2 homework scores in the discrete mathematics portion of the course, and the lowest 2 homework scores in the probability portion of the course, and the same policy applies for vitamins. However, please save these drops for emergencies. We do not have the bandwidth to make personalized exceptions to this rule.

(v) The midterm is on 3/8/23 from 7-9pm, and the final is on 5/11/23 from 3-6pm.

2 Course Policies

Go to the course website and read the course policies carefully. Leave a followup on Ed if you have any questions. Are the following situations violations of course policy? Write "Yes" or "No", and a short explanation for each.

(a) Alice and Bob work on a problem in a study group. They write up a solution together and submit it, noting on their submissions that they wrote up their homework answers together.

(b) Carol goes to a homework party and listens to Dan describe his approach to a problem on the board, taking notes in the process. She writes up her homework submission from her notes, crediting Dan.

(c) Erin comes across a proof that is part of a homework problem while studying course material. She reads it and then, after she has understood it, writes her own solution using the same approach. She submits the homework with a citation to the website.

(d) Frank is having trouble with his homework and asks Grace for help. Grace lets Frank look at her written solution. Frank copies it onto his notebook and uses the copy to write and submit his homework, crediting Grace.

(e) Heidi has completed her homework using LaTeX. Her friend Irene has been working on a homework problem for hours, and asks Heidi for help. Heidi sends Irene her PDF solution, and Irene uses it to write her own solution with a citation to Heidi.

(f) Joe found homework solutions before they were officially released, and every time he got stuck, he looked at the solutions for a hint. He then cited the solutions as part of his submission.

Solution:

(a) Yes, this is a violation of course policy. All solutions must be written entirely by the student submitting the homework. Even if students collaborate, each student must write a unique, individual solution. In this case, both Alice and Bob would be culpable.

(b) No, this is not a violation of course policy. While sharing *written solutions* is not allowed, sharing *approaches* to problems is allowed and encouraged. Because Carol only copied down *notes*, not *Dan's solution*, and properly cited Dan's contribution, this is an actively encouraged form of collaboration.

(c) No, this is not a violation of course policy. Using external sources to help with homework problems, while less encouraged than peer collaboration, is fine as long as (i) the student makes sure to understand the solution; (ii) the student uses understanding to write a new solution, and does not copy from the external source; and (iii) the student credits the external source. However, looking up a homework problem online is a violation of course policies; the correct course of action upon finding homework solutions online is to close the tab.

(d) Yes, this is a violation of course policy, and both Frank and Grace would be culpable. Even though Frank credits Grace, written solutions should never be shared in the first place, and certainly not copied down. This is to ensure that each student learns how to write and present clear and convincing arguments. To be safe, try not to let anybody see your written solutions at any point in the course—restrict your collaboration to *approaches* and *verbal communication*.

(e) Yes, this is a violation of course policy. Once again, a citation does not make up for the fact that written solutions should never be shared, in written or typed form. In this case, both Heidi and Irene would be culpable.

(f) Yes, this is a violation of course policy. Joe should not be reading solutions before they are officially released. Instead, Joe should ask for help when he is stuck through Ed or Office Hours.

3 Use of Ed

Ed is incredibly useful for Q&A in such a large-scale class. We will use Ed for all important announcements. You should check it frequently. We also highly encourage you to use Ed to ask questions and answer questions from your fellow students.

(a) Read the Ed Etiquette section of the course policies and explain what is wrong with the following hypothetical student question: "Can someone explain the proof of Theorem XYZ to me?" (Assume Theorem XYZ is a complicated concept.)

(b) When are the weekly posts released? Are they required reading?

(c) If you have a question or concern not directly related to the course content, where should you direct it?

Solution:

(a) There are two things wrong with this question. First, this question does not pass the 5-minute test. This concept takes longer than 5 minutes to explain, and therefore is better suited to Office Hours. Second, this question does not hone in on a particular concept with which the student is struggling. Questions on Ed should be narrow, and should include every step of reasoning that led up to the question. A better question in this case might be: “I understood every step of the proof of Theorem XYZ in Note 2, except for the very last step. I tried to reason it like this, but I didn’t see how it yielded the result. Can someone explain where I went wrong?”

(b) The weekly posts are released every Sunday. They’re required reading.

(c) Please send an email to `sp23@eecs70.org`.

4 Academic Integrity

Please write or type out the following pledge in print, and sign it.

> I pledge to uphold the university's honor code: to act with honesty, integrity, and respect for others, including their work. By signing, I ensure that all written homework I submit will be in my own words, that I will acknowledge any collaboration or help received, and that I will neither give nor receive help on any examinations.

5 Propositional Practice

Note 1

In parts (a)-(c), convert the English sentences into propositional logic. In parts (d)-(f), convert the propositions into English. In part (f), let $P(a)$ represent the proposition that a is prime.

(a) There is one and only one real solution to the equation $x^2 = 0$.

(b) Between any two distinct rational numbers, there is another rational number.

(c) If the square of an integer is greater than 4, that integer is greater than 2 or it is less than -2.

(d) $(\forall x \in \mathbb{R})\ (x \in \mathbb{C})$

(e) $(\forall x, y \in \mathbb{Z})(x^2 - y^2 \neq 10)$

(f) $(\forall x \in \mathbb{N})\ [\ (x > 1) \implies (\exists a, b \in \mathbb{N})\ ((a + b = 2x) \wedge P(a) \wedge P(b))\ ]$

Solution:

(a) Let $p(x) = x^2$. The sentence can be read: "There is a solution x to the equation $p(x) = 0$, and any other solution y is equal to x". Or,

$$(\exists x \in \mathbb{R})\left((p(x) = 0) \wedge ((\forall y \in \mathbb{R})(p(y) = 0) \implies (x = y))\right).$$

(b) The sentence can be read "If x and y are distinct rational numbers, then there is a rational number z between x and y." Or,

$$(\forall x, y \in \mathbb{Q})((x \neq y) \implies ((\exists z \in \mathbb{Q})(x < z < y \vee y < z < x))).$$

Equivalently,

$$(\forall x, y \in \mathbb{Q})((x = y) \vee (\exists z \in \mathbb{Q})(x < z < y \vee y < z < x)).$$

Note that $x < z < y$ is mathematical shorthand for $(x < z) \wedge (z < y)$, so the above statement is equivalent to

$$(\forall x, y \in \mathbb{Q})(x = y) \vee ((\exists z \in \mathbb{Q})\left((x < z) \wedge (z < y)\right) \vee ((y < z) \wedge (z < x))).$$

(c) $(\forall x \in \mathbb{Z})\ ((x^2 > 4) \implies ((x > 2) \vee (x < -2)))$

(d) All real numbers are complex numbers.

(e) There are no integer solutions to the equation $x^2 - y^2 = 10$.

(f) For any natural number greater than 1, there are some prime numbers a and b such that $2x = a + b$.

In other words: Any even integer larger than 2 can be written as the sum of two primes.

Aside: This statement is known as Goldbach's Conjecture, and it is a famous unsolved problem in number theory (`https://xkcd.com/1310/`).

6 Implication

Note 1

Which of the following assertions are true no matter what proposition Q represents? For any false assertion, state a counterexample (i.e. come up with a statement $Q(x,y)$ that would make the implication false). For any true assertion, give a brief explanation for why it is true.

(a) $\exists x \exists y Q(x,y) \implies \exists y \exists x Q(x,y)$.

(b) $\forall x \exists y Q(x,y) \implies \exists y \forall x Q(x,y)$.

(c) $\exists x \forall y Q(x,y) \implies \forall y \exists x Q(x,y)$.

(d) $\exists x \exists y Q(x,y) \implies \forall y \exists x Q(x,y)$.

Solution:

(a) True. There exists can be switched if they are adjacent; $\exists x, \exists y$ and $\exists y, \exists x$ means there exists x and y in our universe.

(b) False. Let $Q(x,y)$ be $x < y$, and the universe for x and y be the integers. Or let $Q(x,y)$ be $x = y$ and the universe be any set with at least two elements. In both cases, the antecedent is true and the consequence is false, thus the entire implication statement is false.

(c) True. The first statement says that there is an x, say x' where for every y, $Q(x,y)$ is true. Thus, one can choose $x = x'$ for the second statement and that statement will be true again for every y. Note: 4c and 4d are not logically equivalent. In fact, the converse of 4d is 4c, which we saw is false.

(d) False. Suppose Q is the statement "y is 5, and x is any integer". The antecedent is true when $y = 5$, but for $y \neq 5$, there is no x that will make it true.

7 Equivalences with Quantifiers

Note 1

Evaluate whether the expressions on the left and right sides are equivalent in each part, and briefly justify your answers.

(a) $\forall x \exists y\,(P(x) \implies Q(x,y)) \stackrel{?}{\equiv} \forall x\,(P(x) \implies \exists y\, Q(x,y))$

(b) $\forall x\,((\exists y\, Q(x,y)) \implies P(x)) \stackrel{?}{\equiv} \forall x \exists y\,(Q(x,y) \implies P(x))$

(c) $\neg \exists x \forall y\,(P(x,y) \implies \neg Q(x,y)) \stackrel{?}{\equiv} \forall x\,((\exists y\, P(x,y)) \land (\exists y\, Q(x,y)))$

Solution:

(a) Equivalent.

Justification: We can rewrite the left side as

$$\forall x \exists y\,(P(x) \implies Q(x,y)) \equiv \forall x \exists y(\neg P(x) \lor Q(x,y)).$$

We can also rewrite the right side as

$$\forall x\,(P(x) \implies \exists y\, Q(x,y)) \equiv \forall x(\neg P(x) \lor \exists y\, Q(x,y)).$$

Clearly, the two sides are the same if $\neg P(x)$ is true. If $\neg P(x)$ is false, then the two sides are still the same, because

$$\forall x \exists y\,(\text{False} \lor Q(x,y)) \equiv \forall x \exists y Q(x,y) \equiv \forall x\,(\text{False} \lor (\exists y\, Q(x,y)))\,.$$

(b) Not equivalent.

Justification: We can rewrite the left side as

$$\begin{aligned}\forall x\,((\exists y\, Q(x,y)) \implies P(x)) &\equiv \forall x\,((\neg(\exists y\, Q(x,y))) \lor P(x))\\ &\equiv \forall x\,((\forall y\, \neg Q(x,y)) \lor P(x))\\ &\equiv \forall x \forall y\,(\neg Q(x,y) \lor P(x))\,,\end{aligned}$$

noting that we can extract the $\forall y$ out of the inner $\lor$ expression, since $P(x)$ does not depend on y. (This can be shown in a similar fashion as the previous part.)

We can also rewrite the right side as

$$\forall x \exists y\,(Q(x,y) \implies P(x)) \equiv \forall x \exists y\,(\neg Q(x,y) \lor P(x))\,.$$

This gives us

$$\forall x \forall y\,(\neg Q(x,y) \lor P(x)) \not\equiv \forall x \exists y\,(\neg Q(x,y) \lor P(x))\,,$$

so the two sides are not equivalent.

Another approach to the problem is to consider a linguistic example. Let x and y span the universe of all people, and let $Q(x,y)$ mean "Person x is Person y's offspring", and let $P(x)$ mean "Person x likes tofu".

The right side claims that, for all Persons x, there exists some Person y such that either Person x is not Person y's offspring or that Person x likes tofu.

The left side claims that, for all Persons x, if there exists a parent of Person x, then Person x likes tofu.

It should be clear that these are not the same.

(c) Not equivalent.

Justification: Using De Morgan's Laws to distribute the negation on the left side yields

$$\begin{aligned}\neg\exists x\forall y(P(x,y) \implies \neg Q(x,y)) &\equiv \forall x\neg\forall y(P(x,y) \implies \neg Q(x,y))\\ &\equiv \forall x\exists y\neg(P(x,y) \implies \neg Q(x,y))\\ &\equiv \forall x\exists y\neg(\neg P(x,y)\vee\neg Q(x,y))\\ &\equiv \forall x\exists y(P(x,y)\wedge Q(x,y))\end{aligned}$$

But $\exists$ does not distribute over $\wedge$. There could exist different values of y such that $P(x,y)$ and $Q(x,y)$ for a given x, but not necessarily the same value. This means that the two sides are not equivalent.

8 Preserving Set Operations

Note 0
Note 2

For a function f, define the image of a set X to be the set $f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}$. Define the inverse image or preimage of a set Y to be the set $f^{-1}(Y) = \{x \mid f(x) \in Y\}$. Prove the following statements, in which A and B are sets. By doing so, you will show that inverse images preserve set operations, but images typically do not.

Recall: For sets X and Y, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$. To prove that $X \subseteq Y$, it is sufficient to show that $(\forall x)\ ((x \in X) \implies (x \in Y))$.

(a) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(b) $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.

(c) $f(A \cap B) \subseteq f(A) \cap f(B)$, and give an example where equality does not hold.

(d) $f(A \setminus B) \supseteq f(A) \setminus f(B)$, and give an example where equality does not hold.

Solution:

In order to prove equality $A = B$, we need to prove that A is a subset of B, $A \subseteq B$ and that B is a subset of A, $B \subseteq A$. To prove that LHS is a subset of RHS we need to prove that if an element is a member of LHS then it is also an element of the RHS.

(a) Suppose x is such that $f(x) \in A \cap B$. Then $f(x)$ lies in both A and B, so x lies in both $f^{-1}(A)$ and $f^{-1}(B)$, so $x \in f^{-1}(A) \cap f^{-1}(B)$. So $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$.

Now, suppose that $x \in f^{-1}(A) \cap f^{-1}(B)$. Then, x is in both $f^{-1}(A)$ and $f^{-1}(B)$, so $f(x) \in A$ and $f(x) \in B$, so $f(x) \in A \cap B$, so $x \in f^{-1}(A \cap B)$. So $f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$.

(b) Suppose x is such that $f(x) \in A \setminus B$. Then, $f(x) \in A$ and $f(x) \notin B$, which means that $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$, which means that $x \in f^{-1}(A) \setminus f^{-1}(B)$. So $f^{-1}(A \setminus B) \subseteq f^{-1}(A) \setminus f^{-1}(B)$.

Now, suppose that $x \in f^{-1}(A) \setminus f^{-1}(B)$. Then, $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$, so $f(x) \in A$ and $f(x) \notin B$, so $f(x) \in A \setminus B$, so $x \in f^{-1}(A \setminus B)$. So $f^{-1}(A) \setminus f^{-1}(B) \subseteq f^{-1}(A \setminus B)$.

(c) Suppose $x \in A \cap B$. Then, x lies in both A and B, so $f(x)$ lies in both $f(A)$ and $f(B)$, so $f(x) \in f(A) \cap f(B)$. Hence, $f(A \cap B) \subseteq f(A) \cap f(B)$.

Consider when there are elements $a \in A$ and $b \in B$ with $f(a) = f(b)$, but A and B are disjoint. Here, $f(a) = f(b) \in f(A) \cap f(B)$, but $f(A \cap B)$ is empty (since $A \cap B$ is empty).

(d) Suppose $y \in f(A) \setminus f(B)$. Since y is not in $f(B)$, there are no elements in B which map to y. Let x be any element of A that maps to y; by the previous sentence, x cannot lie in B. Hence, $x \in A \setminus B$, so $y \in f(A \setminus B)$. Hence, $f(A) \setminus f(B) \subseteq f(A \setminus B)$.

Consider when $B = \{0\}$ and $A = \{0, 1\}$, with $f(0) = f(1) = 0$. One has $A \setminus B = \{1\}$, so $f(A \setminus B) = \{0\}$. However, $f(A) = f(B) = \{0\}$, so $f(A) \setminus f(B) = \varnothing$.