CMSC691 Optimization, VCU
Fall 2016, Assignment 3: Convex Optimization Problems

Due: Friday, November 4, at 11:59 PM.
Submit via Blackboard under Assignments tab.

Total marks: 45 marks + 15 marks bonus.

1 Exercises

1. (5 marks + 5 marks bonus) Consider the optimization problem

$$\begin{array}{ll} \min & f_0(x_1, x_2) \\ \text{s.t.} & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \qquad . \end{array}$$

For each of the following objective functions, give the set of optimal solutions and the corresponding optimal value. (Hint: Sketch the feasible region first, and then consider level sets of the objective function. A full proof of each solution is not needed, simply argue based on your sketch. You don't need to submit your sketch.)

   (a) (5 marks) $f_0(x_1, x_2) = x_1 + x_2$.

   (b) (5 marks bonus) $f_0(x_1, x_2) = x_1^2 + 9x_2^2$.

2. (10 marks) Formulate the following problems as LPs. Below, you are given $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. You do not need to put your LP into standard form. (Hint: Consider the epigraph form of optimization problems.)

   (a) (5 marks) Minimize $\| Ax - b \|_\infty$ (l_∞-norm approximation).

   (b) (5 marks) Minimize $\| Ax - b \|_1$ (l_1-norm approximation).

3. (10 marks) Suppose you are given two disjoint sets of points in $\mathbb{R}^n$, $V = \left\{v^1, \dots, v^K\right\}$ and $W = \left\{w^1, \dots, w^L\right\}$. Formulate the following two problems as LP feasibility problems. (Note: The question asks for a *feasibility* problem, i.e. the objective function should be just the constant (say) 0.)

   (a) (5 marks) Determine a hyperplane that separates the two sets, i.e. find $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ with $a \neq 0$ such that

$$a^T v^i \leq b, \quad i = 1, \dots, K, \qquad\qquad a^T w^i \geq b, \quad i = 1, \dots, L.$$

   Note that we require $a \neq 0$, so you have to make sure your formulation excludes the trivial solution $a = 0, b = 0$. (Hint: The tricky part is ensuring $a \neq 0$. To do so, start by observing that since V and W are disjoint finite sets, they can be strictly separated, i.e. there exists (a, b) and $\epsilon > 0$ such that $a^T v^i \leq b - \epsilon$ and $a^T w^i \geq b + \epsilon$. Why can we assume without loss of generality that (say) $\epsilon = 1$?)

   (b) (5 marks) Determine a sphere separating the two sets of points, i.e. find $x_c \in \mathbb{R}^n$ and $R \geq 0$ such that

$$\left\| v^i - x_c \right\|_2 \leq R, \quad i = 1, \dots, K, \qquad\qquad \left\| w^i - x_c \right\|_2 \geq R, \quad i = 1, \dots, L.$$

   (Here, x_c is the center of the sphere, and R is its radius.) (Hint: Your LP does not need to explicitly compute R.)

4. (5 marks) Give an explicit solution of the following QCQP, where $A \in S^n_{++}$ and $c \neq 0$:

$$\begin{array}{ll} \text{min} & c^T x \\ \text{s.t.} & x^T A x \leq 1. \end{array}$$

(Hints: Observe that this is just optimizing a linear function over an ellipsoid centered at the origin. Thus, the optimal value is attained on the boundary of the ellipsoid (i.e. $x^T Ax = 1$). Here are some first steps to get you started: (1) Convert to an equivalent maximization problem. (2) Observe that since $A \succ 0$, there exists operator $A^{1/2}$ such that $A = A^{1/2}A^{1/2}$. Moreover, $A^{1/2}$ is symmetric. (3) Rewrite the constraint in terms of a new variable y you will introduce, so that the constraint becomes $\| y \|_2 = 1$. What now happens to the objective function?)

5. (5 marks + 5 marks bonus) Suppose $A : \mathbb{R}^n \mapsto S^m$ is affine, i.e. $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$ for $A_i \in S^m$. Let $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_m(x)$ denote the eigenvalues of $A(x)$. Phrase the following problems as SDPs.

   (a) (5 marks) Minimize the maximum eigenvalue $\lambda_1(x)$.

   (b) (5 marks bonus) Minimize the condition number of $A(x)$, subject to $A(x) \succ 0$. The condition number is defined as $\kappa(A(x)) = \lambda_1(x)/\lambda_m(x)$, with domain $\{x \mid A(x) \succ 0\}$. You may assume that $A(x) \succ 0$ for at least one x. (Hint: You need to minimize λ/γ, subject to $0 \prec \gamma I \preceq A(x) \preceq \lambda I$. Change variables to $y = x/\gamma, t = \lambda/\gamma, s = 1/\gamma$.)

6. (10 marks + 5 marks bonus) [Nes00, Par00] *Sum-of-squares representation via LMIs*. Consider polyonimal $p : \mathbb{R}^n \mapsto \mathbb{R}$ of degree $2k$. The polynomial is called positive semidefinite (PSD) if $p(x) \geq 0$ for all $x \in \mathbb{R}^n$. A famous *sufficient* condition for p to be PSD is that it have form

$$p(x) = \sum_{i=1}^{r} q_i(x)^2,$$

for some polynomials q_i, with degree no more than k. A polynomial p with this "sum-of-squares" form is called SOS.

   (a) (5 marks) Let $f_1, \ldots, f_s$ be all monomials of degree k or less. (Here, a monomial is defined as[1] $x_1^{m_1} \ldots x_n^{m_n}$ for $m_i \in \mathbb{Z}_+$.) Show that p can be expressed as a positive semidefinite quadratic form $p = f^T V f$ with $V \in S^s_+$ if and only if p is SOS. Here, f is the vector whose ith entry is monomial f_i. (Hint: Use the Cholesky decomposition of V.)

   (b) (5 marks) Show that the condition $p = f^T V f$ is a set of linear equality constraints relating the coefficients of p and the matrix V. Why does this imply that determining whether p is SOS can be phrased as an SDP?

   (c) (5 marks bonus) Work out the linear matrix inequality conditions for SOS explicitly for the case where p is polynomial of degree four in two variables.

[1]This is the standard definition of a monomial, and differs from the one we used when discussing geometric programs.