Assignment 2

Thursday, January 30, 2020 5:24 PM

4.11

<a> minimize $\|Ax-b\|_\infty$ is the Chebyshev approximation problem

LP: minimize t

subject to $-t\mathbf{1} \preceq Ax-b \preceq t\mathbf{1}$

with variables $x \in R^n$ and $t \in R$ (scalar), $\mathbf{1} \in R^n$

$t = \|Ax-b\|_\infty$, the maximum scalar value of $Ax-b$. x is the feasible set that will minimize the value of t

<c>. let: $R = Ax-b \in R^m$

$$\|Ax-b\|_1 = \|R\|_1 = \sum_{i=1}^{m} |r_i| = \mathbf{1}^T R$$

LP. minimize $\mathbf{1}^T R$

subject to $-R \preceq Ax-b \preceq R$

$-\mathbf{1} \preceq x \preceq \mathbf{1}$ $(\mathbf{1} \in R^n)$

<e> minimize $\overbrace{\mathbf{1}^T R}^{\|Ax-b\|_1} + \overbrace{t}^{\|x\|_\infty}$ $(\mathbf{1} \in R^m)$

subject to $-t\mathbf{1} \preceq x \preceq t\mathbf{1}$ $(\mathbf{1} \in R^n)$

$-R \preceq Ax-b \preceq R$

$R \in R^n$, $t \in R$

4.16.

$$F = \sum_{t=0}^{N-1} f(u(t)), \quad f(u(t)) = \begin{cases} |u(t)| & |u(t)| \le 1 \\ 2|u(t)| - 1 & |u(t)| > 1 \end{cases}$$

$x(1) = Ax(0) + bu(0)$

$x(2) = Ax(1) + bu(1)$

$x(1) = A x(0) + b u(0)$

$x(2) = A x(1) + b u(1)$

$= A(A x(0) + b u(0)) + b u(1)$

$= A^2 x(0) + A b u(0) + b u(1)$

$x(3) = A x(2) + b u(2)$

$= A[A^2 x(0) + A b u(0) + b u(1)] + b u(2)$

$= A^3 x(0) + A^2 b u(0) + A b u(1) + b u(2)$

$(x(0)=0)$

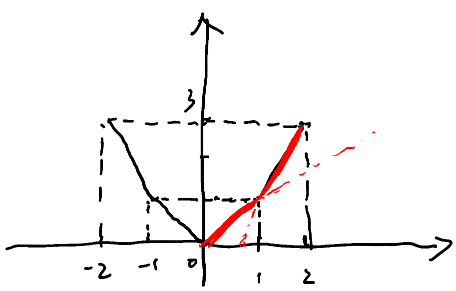
$\therefore\ x(N) = \cancel{A^N x(0)}^{\,0} + A^{N-1} b u(0) + A^{N-2} b u(1)$

$+ \dots + A b u(N-2) + b u(N-1)$

$$= \underbrace{[bA^{N-1}, bA^{N-2}, \dots bA^0]}_{V^T} \underbrace{\begin{bmatrix} \cancel{b}u(0) \\ \cancel{b}u(1) \\ \vdots \\ \cancel{b}u(N-1) \end{bmatrix}}_{W}$$

$= V^T W \quad V, W \in R^N$

$= X_{des}$



let: $[|U(0)|, \dots |U(N-1)|]^T = U$

$F \geq \|U\|_1 \longrightarrow$ if $|u(t)_i|_{all} < 1$, $\|U\|_1 > 2\|U\|_1 - N$

$F \geq 2\|U\|_1 - N \longrightarrow$ if$_{all}$ $|u(t)_i| \geq 1$, $2\|U\|_1 - N \geq \|U\|_1$

$F = 1^T U, \ 1 \in R^N$

LP: minimize $1^T \cancel{u}\, t$

subject to $V^T W = X_{des}$

$-U \leq W \leq U$

$t \geq U$

$t \geq 2U - N$

$\Big\} = \max\{U, 2U - N\}$

4.21 <a> Minimizing a linear function over an ellipsoid centered at the origin

$$\text{minimize} \quad c^T x$$
$$\text{subject to} \quad x^T A x \le 1$$

where $A \in S^n_{++}$ and $c \neq 0$. Consider only the convex case.

Let: $f_0(x^*) = x^{*T} A x^* = c^T x^*$

$$x^{*T} A x^* = (A^{\frac{1}{2}} x^*)^T (A^{\frac{1}{2}} x^*) = \|A^{\frac{1}{2}} x^*\|_2^2$$

let: $A^{\frac{1}{2}} x = y \rightarrow x = A^{-\frac{1}{2}} y$, since a linear function has its optima at the boundary of an ellipsoid centered at the origin, therefore the question is the same as:

$$\text{minimize} \quad c^T A^{-\frac{1}{2}} y = f$$
$$\text{s.t.} \quad \|y\|_2 = 1$$

$\therefore$ let: $c^T A^{-\frac{1}{2}} = \tilde{c}^T$, $\tilde{c} = (c^T A^{-\frac{1}{2}})^T = A^{-\frac{1}{2}} c$

$$\therefore -\frac{\tilde{c}}{\|\tilde{c}\|_2} = -\frac{A^{-\frac{1}{2}} c}{\|A^{-\frac{1}{2}} c\|_2} = y^*$$

$$\therefore x^* = A^{-\frac{1}{2}} y^* = -\frac{A^{-1} c}{\|A^{-\frac{1}{2}} c\|_2}$$

$$\therefore p^* = c^T x^* = -\frac{c^T A^{-1} c}{\|A^{-\frac{1}{2}} c\|_2}$$

4.25 *Linear separation of two sets of ellipsoids.* Suppose we are given $K + L$ ellipsoids

$$\mathcal{E}_i = \{P_i u + q_i \mid \|u\|_2 \le 1\}, \quad i = 1, \ldots, K + L,$$

where $P_i \in \mathbf{S}^n$. We are interested in finding a hyperplane that strictly separates $\mathcal{E}_1, \ldots, \mathcal{E}_K$ from $\mathcal{E}_{K+1}, \ldots, \mathcal{E}_{K+L}$, *i.e.*, we want to compute $a \in \mathbf{R}^n$, $b \in \mathbf{R}$ such that

$$a^T x + b > 0 \text{ for } x \in \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_K, \qquad a^T x + b < 0 \text{ for } x \in \mathcal{E}_{K+1} \cup \cdots \cup \mathcal{E}_{K+L},$$

or prove that no such hyperplane exists. Express this problem as an SOCP feasibility problem.

4.25

we have

$$\begin{cases} a^T(P_i u + q_i) + b > 0, & \forall i = 1, \dots K \quad ① \\ a^T(P_i u + q_i) + b < 0, & \forall i = K+1, \dots K+L \quad ② \end{cases}$$

For condition ① :

$$f_1 = a^T P_i u + a^T q_i + b = \langle a^T P_i, u \rangle + a^T q_i + b$$

$$= \|a^T P_i\|_2 \|u\|_2 \cos\langle a^T P_i, u \rangle + a^T q_i + b$$

For robustness, since $\|u\|_2 \leq 1$

$$\inf(f_1) = -\|a^T P_i\| + a^T q_i + b > 0$$

For condition ② :

$$f_2 = \langle a^T P_i, u \rangle + a^T q_i + b$$

For robustness

$$\sup(f_2) = \|a^T P_i\|_2 + a^T q_i + b < 0$$

$\therefore$ find a, b

subject to $\|a^T P_i\|_2 < a^T q_i + b$ ① $i = 1, \dots K$

$\|a^T P_i\|_2 < -a^T q_i - b$ ② $i = K+1, \dots K+L$

4.30 maximize Tr^2

subject to $T / T_{max} \leq 1$

$r / r_{max} \leq 1$

$w / w_{max} \leq 1$

$w / r \leq 0.1$

$$C_{max}^{-1}(\alpha_1 T r w^{-1} + \alpha_2 r + \alpha_3 r w + \alpha_4 T r^2) \leq 1$$

4.33 Express the following problems as convex opt. problems

<a> Maximize $\max\{p(x), q(x)\}$, where p and q are posynominals.

<b> Minimize $\exp(p(x)) + \exp(q(x))$, where p and q are posynomials.

<a>. let: $t = \max\{p(x), q(x)\}$

$\Rightarrow$ minimize t

subject to
$$\frac{p(y)}{t} \le 1$$
$$\frac{q(y)}{t} \le 1$$

where $y = \log x$, $x = e^y$ is convex

<b> Say: let: $y = \log x$, $b = \log c$

then
$$p(y) = \sum_{k=1}^{K} e^{a_{pk}^T y + b_{pk}}$$
$$q(y) = \sum_{n=1}^{N} e^{a_{qn}^T y + b_{qn}}$$

since $p(y)$, $q(y)$ are convex,

therefore $\exp(p(y)) + \exp(q(y))$ are also convex

$\Rightarrow$ minimize $\exp(p(y)) + \exp(q(y))$

$$\Rightarrow \quad \underset{y}{\text{minimize}} \quad \exp(p(y)) + \exp(q(y))$$

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7. $$\underset{x \in R^n, z \in R}{\text{minimize}} \sum_{m=1}^{M} \max(a_m^T x, z) + \tau \|x\|_2^2$$

let: $f_1 = \sum_{m=1}^{M} \max(a_m^T x, z)$

let: $t_m = \quad \|(a_m^T x, z)\|_\infty, \quad m = 1, \dots M$

$$t = \begin{bmatrix} t_1 \\ \vdots \\ t_m \end{bmatrix}$$

$\therefore f_1 = 1^T t$, and also:

subject to
$$\begin{cases} 1^T t \geqslant \sum_{m=1}^{M} a_m^T x \\ 1^T t \geqslant M z \end{cases}$$

let: $f_2 = \tau \|x\|_2^2 = \tau x^T x$

R.P.

$$f_2 = \tau \|x\|_2^2 = \tau x^T x$$

$\therefore$ combine f_1 & f_2:

$$\underset{x, t}{\text{minimize}} \quad \tau x^T x + 1^T t$$

subject to $\cancel{1^T t \geqslant \sum_{m=1}^{M} a_m^T x}$ $\quad t \geqslant Ax$

$\cancel{1^T t \geqslant M z}$ $\quad t \geqslant z$

8. Markowitz Portfolio Optimization

$$\begin{aligned} \text{minimize} \quad & x^T \Sigma x \\ \text{subject to} \quad & -\bar{p}^T x \le -r_{min} \\ & -x \le 0 \\ & 1^T x = 1 \end{aligned}$$

<a> <b> : see matlab code and figure