

Assignment 3

Saturday, February 29, 2020 6:13 PM

Q1)

5.12

$$\text{if } y_i = b_i - a_i^T x > 0$$



$$\text{minimize} \quad -\sum_{i=1}^m \log y_i = f_0(y)$$

$$\text{subject to } y_i = -a_i^T x + b_i$$

$$\Rightarrow y_i + a_i^T x - b_i = 0$$

$$L(x, y, \lambda) = f_0(y) + \sum_{i=1}^m \lambda_i (y_i + a_i^T x - b_i)$$

$$g(\lambda) = \min_{x, y} -\sum_{i=1}^m \log y_i + \sum_{i=1}^m \lambda_i (y_i + a_i^T x - b_i)$$

$$= \min_{x, y} -\sum_{i=1}^m \log y_i + \sum_{i=1}^m \lambda_i y_i + \sum_{i=1}^m \lambda_i a_i^T x - \sum_{i=1}^m \lambda_i b_i$$

$$\text{if } \sum_{i=1}^m \lambda_i a_i^T x = 0$$

$$g(\lambda) = \min_y -\sum_{i=1}^m \log y_i + \sum_{i=1}^m \lambda_i y_i - \sum_{i=1}^m \lambda_i b_i$$

$$\text{else } \min_x \sum_{i=1}^m \lambda_i a_i^T x = -\infty \Rightarrow g(\lambda) \rightarrow -\infty$$

$$\therefore \max g(\lambda) = \min_y -\sum_{i=1}^m \log y_i + \sum_{i=1}^m \lambda_i y_i - \sum_{i=1}^m \lambda_i b_i$$

is the dual problem of the
primal problem

Q2)

$$5.27 \quad \underset{x}{\text{minimize}} \quad \|Ax - b\|_2^2$$

$$\text{subject to } Gx = h$$

$$\begin{aligned} L(x, \lambda) &= (Ax - b)^T (Ax - b) + \lambda^T (Gx - h) \\ &= (x^T A^T - b^T)(Ax - b) + \lambda^T Gx - \lambda^T h \\ &= x^T A^T Ax - x^T A^T b - b^T A x + b^T b + \lambda^T Gx - \lambda^T h \\ &= x^T A^T Ax + \lambda^T Gx - 2x^T A^T b - \lambda^T h + \|b\|_2^2 \end{aligned}$$

$$\therefore g(\lambda) = \min_x x^T A^T Ax + \lambda^T Gx - 2x^T A^T b - \lambda^T h + \|b\|_2^2$$

$$= \min_x \underbrace{x^T A^T Ax}_{\text{is PSD, hence is convex}} + \underbrace{(\lambda^T G - 2b^T A)x}_{\text{h(x)}} - \lambda^T h$$

$$\begin{aligned} \nabla_x h(x) &= 2A^T A x + (\lambda^T G - 2b^T A)^T = 0 & (A^T A \in R^{n \times n}, \text{ since } \text{Rank}(A) = n, \\ &\Rightarrow x^* = -\frac{1}{2} (A^T A)^{-1} (\lambda^T G - 2b^T A)^T & \text{if inverse } (A^T A)^{-1} \text{ exists} \end{aligned}$$

$$\begin{aligned}
 & \text{From } x^* = A^{-1}G - A^{-1}b \quad (\text{since } A^{-1} \text{ exists}) \\
 \Rightarrow x^* &= -\frac{1}{2}(A^T A)^{-1}(A^T G - 2b^T A)^T \quad (\text{as inverse } (A^T A)^{-1} \text{ exists}) \\
 &= -\frac{1}{2}(A^T A)^{-1}\underbrace{(G^T \lambda^* - 2A^T b)}_{\lambda^*} \quad (\text{by KKT condition})
 \end{aligned}$$

$$\therefore g(\lambda) = \min_x h(x) = \frac{1}{4} [(A^T A)^{-1}(G^T \lambda - 2A^T b)]^T A^T A [(A^T A)^{-1}(G^T \lambda - 2A^T b)]$$

$$-\frac{1}{2}(A^T G - 2b^T A)[(A^T A)^{-1}(G^T \lambda - 2A^T b)] - \lambda^T h$$

$$\text{For } \lambda^*: Gx^* = h$$

$$\begin{aligned}
 & -\frac{1}{2}G(A^T A)^{-1}(G^T \lambda^* - 2A^T b) = h \\
 \Rightarrow & G(A^T A)^{-1}G^T \lambda^* - 2G(A^T A)^{-1}A^T b = -2h \\
 \Rightarrow & \underbrace{G(A^T A)^{-1}G^T}_{P \times P} \lambda^* = 2G(A^T A)^{-1}A^T b - 2h \\
 & \lambda^* = [G(A^T A)^{-1}G^T]^{-1}[2G(A^T A)^{-1}A^T b - 2h]
 \end{aligned}$$

rank $G = p$, $\therefore [G(A^T A)^{-1}G^T]^{-1}$ exist
 $\in \mathbb{R}^{P \times P}$

Q37. 5.35

unperturbed:

$$L(y, \lambda, \mu) = \log f_0(y) + \sum_{i=1}^m \lambda_i \log f_i(y) + \sum_{i=1}^p \mu_i \log h_i(y)$$

$$\begin{aligned}
 d^* &= \log p^*(\omega, \sigma) = g(\lambda^*, \mu^*) \leq L(y, \lambda, \mu) \\
 &\leq \log f_0(y) + \lambda^T u + \mu^T v \\
 & \quad (\text{since: } \log f_i(y) \leq u_i, \log h_i(y) = v_i)
 \end{aligned}$$

$$\text{Then: } f_i(y) = p^*(u, v)$$

$$\log p^*(\omega, \sigma) \leq \log p^*(u, v) + \lambda^T u + \mu^T v$$

$$\Rightarrow \log p^*(u, v) \geq \log p^*(\omega, \sigma) - \lambda^T u - \mu^T v$$

Let: $v = 0$, for very small value u

$$\log p^*(u, 0) \geq \log p^*(\omega, 0) - \lambda^T u$$

Q: If $u > 0$

$$\underbrace{\frac{\log p^*(u, 0) - \log p^*(\omega, 0)}{u}}_{u_i} \geq -\lambda_i^*$$

$$\frac{\partial \log p^*(\omega, \alpha)}{\partial u}$$

27) if $u < 0$

$$\frac{\partial \log p^*(\omega, \alpha)}{\partial u_i} \leq -\lambda_i^*$$

$$\therefore \frac{\partial \log p^*(\omega, \alpha)}{\partial u_i} = -\lambda_i^*$$

$$\text{let: } \partial u_i = \alpha \rightarrow \underbrace{\partial \log p^*(\omega, \alpha)}_{\substack{\text{change of} \\ \text{unperturbed objective}}} = -\alpha \lambda_i^*, i=1, \dots, m$$

unperturbed objective

Q4) 5.42

$$\underset{x}{\text{minimize}} \quad c^T x$$

$$\text{subject to} \quad Ax \leq_k b$$

$$L(x, \lambda) = c^T x + \lambda^T (Ax - b)$$

$$= (c^T - \lambda^T A)x - \lambda^T b$$

$$g(\lambda) = \min_x L = \begin{cases} -\lambda^T b & c^T - \lambda^T A = 0 \\ -\infty & \text{else} \end{cases}$$

$$\therefore \text{dual problem} \quad \max_{\lambda} -\lambda^T b$$

dual cone

$$\text{subject to} \quad \lambda \geq_k 0$$

$$c^T - \lambda^T A = 0$$

Q5)

$$\underset{x}{\text{minimize}} \quad c^T x = f_0(x)$$

$$\text{subject to} \quad Ax \geq b \rightarrow b - Ax \leq 0$$

$$x \geq 0 \rightarrow -x \leq 0$$

$$L(x, \lambda_1, \lambda_2) = f_0(x) + \lambda_1^T (b - Ax) + \lambda_2^T (-x)$$

$$g(\lambda_1, \lambda_2) = \min_x \underbrace{c^T x}_{x} + \underbrace{\lambda_1^T b}_{-} - \underbrace{\lambda_1^T A x}_{-\lambda_2^T x} - \underbrace{\lambda_2^T x}_{0}$$

$$= \min_x (c^T - \lambda_1^T A - \lambda_2^T) x + \lambda_1^T b$$

$$\text{if } c^T - \lambda_1^T A - \lambda_2^T = 0 : g(\lambda_1, \lambda_2) = \lambda_1^T b$$

if $C^T - \lambda_1^T A - \lambda_2^T = 0 : g(\lambda_1, \lambda_2) = \lambda_1^T b$
else $: g(\lambda_1, \lambda_2) = -\infty$

$\therefore \max_{\lambda_1, \lambda_2} g(\lambda_1, \lambda_2) = \underset{\lambda_1}{\text{maximize}} \lambda_1^T b$
subject to $C^T - \lambda_1^T A - \lambda_2^T = 0$

$$\lambda_1 \geq 0$$

$$\lambda_2 \geq 0$$

$\Rightarrow \underset{\lambda_1}{\text{maximize}} \lambda_1^T b$
subject to $(C^T - \lambda_1^T A)^T = C - A^T \lambda_1 = \lambda_2 \geq 0 \quad (*)$
 $\lambda_1 \geq 0$

a). if primal $f(x)$ is unbounded \Rightarrow it

means $C^T < 0$, then for $(*)$:



$$C - A^T \lambda_1 \geq 0$$



it is not feasible as $C - A^T \lambda_1 \rightarrow -\infty$

b) if primal is infeasible, it means

either $Ax < b$ or $x < 0$, or both;

: $\begin{cases} x < 0, \text{ the constraint of the dual will be} \\ Ax \geq b \end{cases}$

$$\lambda_2 = A^T \lambda_1 - C \geq 0 \Rightarrow A^T \lambda_1 \geq C \geq 0 \text{ since primal should be bounded}$$

$$\lambda_1 \geq 0$$



therefore: $A \geq 0$, $b \leq Ax \leq 0$

then for its dual: $g(\lambda_1) = \min \lambda_1^T b$ is unbounded

$$\cdot \text{If: } \begin{cases} x < 0, g(\lambda_1, \lambda_2) = \min (C^T + \lambda_1^T A + \lambda_2^T) x - \lambda_1^T b \\ Ax \leq b \end{cases}$$

$$= \begin{cases} \lambda_1^T b & C^T + \lambda_1^T A + \lambda_2^T = 0 \\ -\infty & \text{else} \end{cases}$$

$$\max g(\lambda_1, \lambda_2) = \max -\lambda_1^T b$$

$$\text{s.t. } \lambda_1 \geq 0$$

$$-A^T \lambda_1 - C \geq 0 \Rightarrow A^T \lambda_1 \leq 0 \quad (\text{so } 0 \text{ so that primal is bounded})$$

therefore: $A^T \leq 0$, plus $\begin{cases} Ax \leq b \\ x = 0 \end{cases}$, we know



$b > 0$. Since $\lambda_1 \geq 0$, $\max -\lambda_1^T b = 0$ when $\lambda_1 = 0$,

which contradict to $A^T \lambda_1 \leq 0$, so it is infeasible;

If we don't violate the constraint of the dual, the problem is unbounded

\leftarrow

$$Q b) \max_y x^T p y \Rightarrow \min_y -x^T p y$$

subject to $Cy \leq d$ subject to $Cy - d \leq 0$

$L(y, \lambda) = -x^T p y + \lambda^T (Cy - d)$

 $= (-x^T p + \lambda^T C) y - \lambda^T d$

$$g(\lambda) = \min_y L = \begin{cases} -x^T d & \text{if } -x^T p + \lambda^T C = 0 \\ -\infty & \text{else} \end{cases}$$

$$\therefore \max_{x, \lambda} g(\lambda) = \min_{x, \lambda} \lambda^T d = d^T \lambda$$

subject to $-x^T p + \lambda^T C = 0 \rightarrow C^T \lambda = p^T x$

$$\lambda \geq 0$$

$$Ax \leq b$$

\leftarrow

 $\min_x x^T p y$

\leftarrow subject to $Ax \leq b$

$$L(x, v) = x^T p y + v^T (Ax - b)$$

$$= (y^T p^T + v^T A)x - v^T b$$

$$g(v) = \min_x L = \begin{cases} -v & \text{else} \\ -v^T b & \text{if } y^T p^T + v^T A = 0 \rightarrow A^T v + p y = 0 \\ -b^T v & \text{if } y^T p^T + v^T A = 0 \rightarrow A^T v + p y = 0 \end{cases}$$

$$\therefore \max_{y, \lambda} g(v) = \max_{y, \lambda} -b^T v$$

subject to $A^T v + p y = 0$

$$C y \leq d$$

$$v \geq 0$$

\leftrightarrow for \leftarrow

$$L(x, \lambda, \mu_1, \mu_2, \mu_3) = d^T \lambda + \mu_1^T (c^T \lambda - p^T x) + \mu_2^T (A x - b) + \mu_3^T (C y - d)$$

\leftarrow

$$= (d^T + \mu_1^T c^T - \mu_3^T) \lambda + (\mu_2^T A - \mu_1^T p^T) x - \mu_2^T b$$

$$\therefore g(\mu_1, \mu_2, \mu_3) = \min_{x, \lambda} L = \begin{cases} -\infty & \text{else} \\ -\mu_2^T b & \text{if } \begin{cases} d^T + \mu_1^T c^T = \mu_3^T \rightarrow c\mu_1 + d = \mu_3 \\ \mu_2^T A - \mu_1^T P^T = 0 \rightarrow A^T \mu_2 = P \mu_1 \end{cases} \\ -b^T \mu_2 & \end{cases}$$

$$\therefore \underset{\mu_1, \mu_2, \mu_3}{\text{maximize}} g = \underset{\mu_1, \mu_2, \mu_3}{\text{maximize}} -b^T \mu_2$$

subject to $\begin{array}{l} \mu_2 \geq 0 \\ \mu_3 \geq 0 \\ c\mu_1 + d = \mu_3 \geq 0 \\ A^T \mu_2 = P \mu_1 \end{array} \right\} \Rightarrow \begin{array}{l} A^T \mu_2 + P(-\mu_1) = 0 \\ c(-\mu_1) \leq d \\ \mu_2 \geq 0 \end{array} \quad (*)$

compare (*) to , we can see that by substitution, $y = -\mu_1$, $v = \mu_2$,

we can recover from (*), therefore, <a> is the dual of ,

therefore here, min-max = max-min

7. 2a7. let: $t = \begin{bmatrix} \|P\|_1 \\ i \\ \|P\|_1 \end{bmatrix}$, $f = \begin{bmatrix} P_1 \\ \vdots \\ P_i \end{bmatrix}$ and $i=10$ (my f is P in the question. I don't want to change it because I want to rewrite the optimization using epigraph approach be consistent with my Matlab Code)

$$\begin{array}{ll} \underset{t-f}{\text{minimize}} & 0^T f + 1^T t = [0^T \ 1^T] \begin{bmatrix} f \\ t \end{bmatrix} \\ \text{subject to} & \begin{array}{l} t \geq f \rightarrow f - t \leq 0 \\ t \geq -f \rightarrow -f - t \leq 0 \end{array} \Rightarrow \begin{bmatrix} I & -I \\ -I & -I \end{bmatrix} \begin{bmatrix} f \\ t \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array} \quad \begin{array}{l} n=10 \text{ here} \\ \rightarrow n \times 1 \\ 2n \times 1 \\ n \times 1 \end{array}$$

$$1^T f = 0 \quad (\text{velocity constraint})$$

$$A^T f = 1 \quad \text{where } A = [9.5 \ 8.5 \ 7.5 \ \dots \ 0.5]$$

(position constraint. For calculation of A , see attach Matlab pages)

i)

$$|w^T v| = \left| \sum_{i=1}^n w_i v_i \right| \leq \sum_{i=1}^n |w_i| |v_i| \leq \max_i |v_i| \sum_{i=1}^n |w_i| = \|v\|_\infty \|w\|_1$$

ii)

$$\begin{array}{ll} \underset{\text{subject to}}{\text{minimize}} & \|z\|_1 \\ A z = y & \end{array} \quad \left\{ \begin{array}{l} \\ P^* \end{array} \right.$$

$$\Rightarrow \text{minimize } l^T z$$

$$\text{subject to } z \geq 0$$

$$z \geq -x$$

$$Az = b$$

$$g(z, t, \lambda, \mu, v) = (l^T - \mu^T v^T) z + (\lambda^T A + \mu^T - v^T) z - x^T y$$

$$= \begin{cases} -v^T y & \text{if } \begin{cases} v + \mu = l \\ v - \mu = A^T \lambda \end{cases} \\ -\infty & \text{else} \end{cases}$$

$$\therefore \text{dual problem: } \underset{\lambda, \mu, v}{\text{maximize}} \quad -\lambda^T y$$

$$\text{subject to } \left. \begin{array}{l} l = v + \mu \\ A^T \lambda = v - \mu \\ \mu \geq 0 \\ v \geq 0 \end{array} \right\} \|A^T \lambda\|_\infty \leq 1$$

$$\Rightarrow \underset{\lambda}{\text{maximize}} \quad -\lambda^T y \quad \left. \right\} d^*$$

$$\text{subject to } \|A^T \lambda\|_\infty \leq 1$$

$$\text{claim: for: } \underset{\lambda}{\text{maximize}} \quad \frac{-\lambda^T y}{\|A^T \lambda\|_\infty} = d^{**}, \quad d^* = d^{**}$$

proof: 1) $d^* \geq 0$. since, say, for a optimum λ' to make the optima d' , say, $d' < 0$, then $-\lambda'$ is also feasible since $\|A^T - \lambda'\|_\infty = \|A^T \lambda'\|_\infty \leq 1$. the obtained $d'' > 0$, and $d'' > d$, which is contradictory to the hypothesis that d' is optima (maximum value). Therefore, $d^* \geq 0$
Moreover $-\lambda^T u = |\lambda^T u|$

d' is optima (maximum value). therefore, or \therefore
therefore, $-\lambda^T y = |\lambda^T y|$

2) when $\|A^T \lambda'\|_\infty = 1$, d has the optima d' .

Say if not, then assume $\|A^T \lambda'\|_\infty = \alpha < 1$ and $\alpha > 0$

therefore $\frac{\lambda'}{\alpha}$ is also feasible since $\frac{\|A^T \lambda'\|_\infty}{\alpha} = \|A^T \frac{\lambda}{\alpha}\|_\infty = 1$

and $\frac{d'}{\alpha} > d' \geq 0$, which again is contradictory to
the assumption that d' is optima.

3) since $\|A^T \lambda'\|_\infty$ is non-negative

$\frac{-b^T}{\|A^T \lambda'\|_\infty}$ is just simply a scaled version of $\|A^T \lambda'\|_\infty$.

by $\frac{1}{\|A^T \lambda'\|_\infty}$. When λ is found to have $\|A^T \lambda'\|_\infty = 1$, $d^* = d^{**}$

Therefore: $b^* = b^{**}$.

With the general weak duality property:

$$p^* \geq b^* = b^{**}$$

$$\therefore \|z\|_1 \geq \frac{|\lambda^T y|}{\|A^T \lambda'\|_\infty}$$

iii)

writ the question in 2a) in Lagrange multiplier form:

$$\underset{t-f}{\text{minimize}} \quad t^T t$$

$$\text{subject to} \quad t \geq f$$

$$t \geq -f$$

$$t^T f = 0$$

$$A^T f = 1$$

We have two equality constraints, let λ_1 with $1^T f = 0$, and λ_2 with $A^T f = 1$

therefore from ii), we have

$$\|f\|_1 \geq \frac{|\lambda_2^T|}{\|A\lambda_2 + \lambda_1\|_\infty}$$

when $\|f\|_1 = \frac{|\lambda_2^T|}{\|A\lambda_2 + \lambda_1\|_\infty}$, this gives optimal values

this is because, analogy to ii, $d^* = d^* = \frac{|\lambda_2^T|}{\|A\lambda_2 + \lambda_1\|_\infty}$

it is the dual optima of question (a).

when strong duality holds, $p^* = d^*$

2c7. minimize f, t $0^T f + t = [0^T \ 1] \begin{bmatrix} f \\ t \end{bmatrix}$

subject to $\begin{cases} f \leq t \\ -f \leq t \end{cases} \rightarrow \begin{cases} f - t \leq 0 \\ -f - t \leq 0 \end{cases} \Rightarrow \begin{bmatrix} I & -I \\ -I & -I \end{bmatrix} \begin{bmatrix} f \\ t \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$1^T f = 0$$

$$A^T f = 1$$

%% Q7 (a)

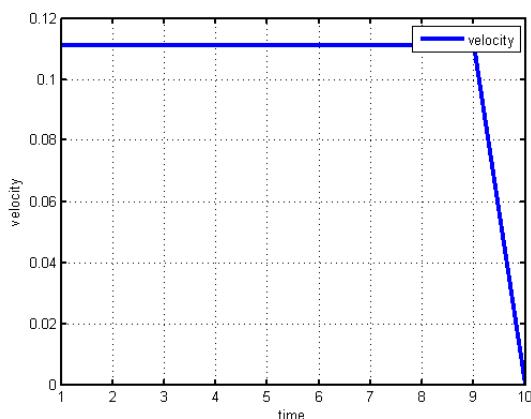
```
close all
clear all
```

```
S = [9.5 8.5 7.5 6.5 5.5 4.5 3.5 2.5 1.5 0.5];
t = 10; % 10 time instances
y = zeros(t,1);
```

```
obj = [zeros(1,t),ones(1,t)];
A = [eye(t,t),-eye(t,t),-eye(t,t),-eye(t,t)];
b = [y;y];
Aeq = [ones(1,t),zeros(1,t);S, zeros(1,t)];
beq = [0; 1];
```

```
% formulate the LP problem in MATLAB
[opt_sol, fval] = linprog(obj,A,b,Aeq,beq);
optima = opt_sol(1:10);
x = optima;
```

```
v = zeros(10,1); % velocity
for t = 1:1:10
    v(t,:) = sum(x(1:t));
end
```



```

end

a = [0;x];      % add a zero to the front, acceleration
s = zeros(10,1);    % displacement
for t = 1:1:10
    s(t,:) = sum(a(1:t)) + 0.5 * x(t,:);
end

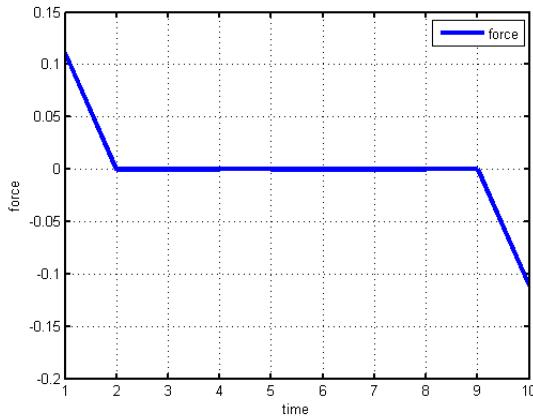
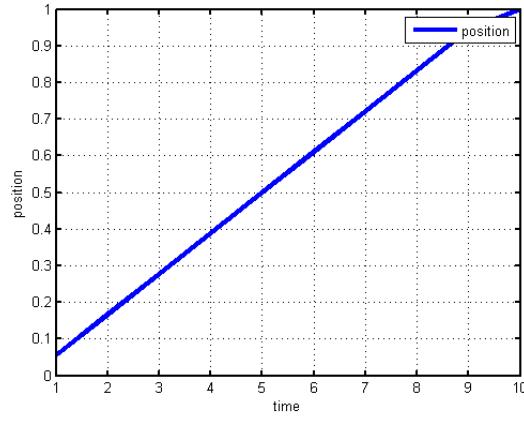
p = zeros(10,1);    % position
for t = 1:1:10
    p(t,:) = sum(s(1:t));
end

figure(1)
p1 = plot(x);
set(gca,'linewidth',2)
set(p1,'linewidth',3)
xlabel('time')
ylabel('force')
legend('force')
grid on

figure(2)
p1 = plot(v);
set(gca,'linewidth',2)
set(p1,'linewidth',3)
xlabel('time')
ylabel('velocity')
legend('velocity')
grid on

figure(3)
p1 = plot(p);
set(gca,'linewidth',2)
set(p1,'linewidth',3)
xlabel('time')
ylabel('position')
legend('position')
grid on

```



%% Q7(b)

```

close all
clear all

S = [9.5 8.5 7.5 6.5 5.5 4.5 3.5 2.5 1.5 0.5];
t = 10; % 10 time instances
y = zeros(t,1);

obj = [zeros(1,t),1];
A = [eye(t,t)-ones(t,1); -eye(t,t), -ones(t,1)];
b = [y;v];
Aeq = [ones(1,t),0; S, 0];
beq = [0; 1];

% formulate the LP problem in MATLAB
[opt_sol, fval] = linprog(obj,A,b,Aeq,beq);
optima = opt_sol(1:10);
x = optima;

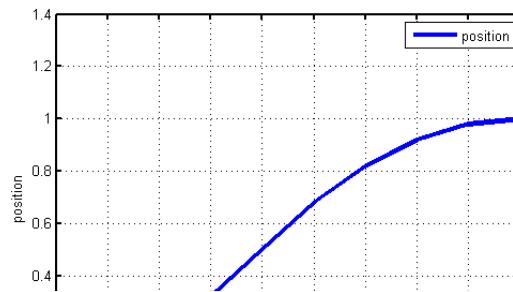
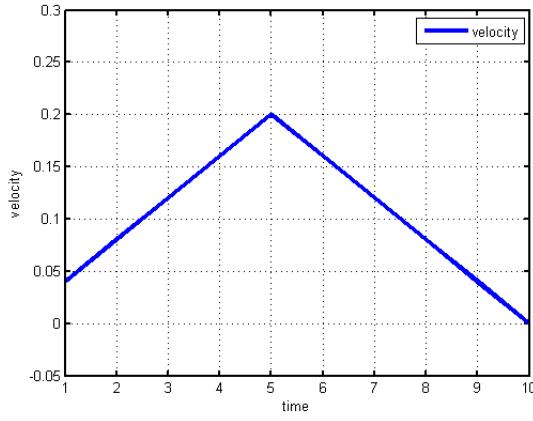
v = zeros(10,1);    % velocity
for t = 1:1:10
    v(t,:) = sum(x(1:t));
end

a = [0;x];      % add a zero to the front, acceleration
s = zeros(10,1);    % displacement
for t = 1:1:10
    s(t,:) = sum(a(1:t)) + 0.5 * x(t,:);
end

p = zeros(10,1);    % position
for t = 1:1:10
    p(t,:) = sum(s(1:t));
end

figure(1)
p1 = plot(x);
set(gca,'linewidth',2)
set(p1,'linewidth',3)
xlabel('time')
ylabel('force')
legend('force')

```



```

for t = 1:1:10
    s(:, :) = sum(a(1:t)) + 0.5 * x(t,:);
end

p = zeros(10,1); % position
for t = 1:1:10
    p(:, :) = sum(s(1:t));
end

figure(1)
p1 = plot(x);
set(gca,'linewidth',2)
set(p1,'linewidth',3)
xlabel('time')
ylabel('force')
legend('force')
grid on

figure(2)
p1 = plot(v);
set(gca,'linewidth',2)
set(p1,'linewidth',3)
xlabel('time')
ylabel('velocity')
legend('velocity')
grid on

figure(3)
p1 = plot(p);
set(gca,'linewidth',2)
set(p1,'linewidth',3)
xlabel('time')
ylabel('position')
legend('position')
grid on

```

