Assignment 3

Saturday, February 29, 2020 6:13 PM

Q1)

5.12

if : $y = b_i - a_i^T x > 0$

minimize $-\sum_{i=1}^{m} \log y_i = f_0(y)$

subject to $y_i = -a_i^T x + b_i$

$\Rightarrow y_i + a_i^T x - b_i = 0$

$L(x,y,\lambda) = f_0(y) + \sum_{i=1}^{m} \lambda_i (y_i + a_i^T x - b_i)$

$g(\lambda) = \min_{x,y} \ -\sum_{i=1}^{m} \log y_i + \sum_{i=1}^{m} \lambda_i (y_i + a_i^T x - b_i)$

$= \min_{x,y} -\sum_{i=1}^{m} \log y_i + \sum_{i=1}^{m} \lambda_i y_i + \sum_{i=1}^{m} \lambda_i a_i^T x - \sum_{i=1}^{m} \lambda_i b_i$

if $\sum_{i=1}^{m} \lambda_i a_i^T x = 0$

$g(\lambda) = \min_{y} \ -\sum_{i=1}^{m} \log y_i + \sum_{i=1}^{m} \lambda_i y_i - \sum_{i=1}^{m} \lambda_i b_i$

else $\min_{x} \sum_{i=1}^{m} \lambda_i a_i^T x = -\infty \Rightarrow g(\lambda) \to -\infty$

$\therefore$ max $g(\lambda) = \min_{y} \ -\sum_{i=1}^{m} \log y_i + \sum_{i=1}^{m} \lambda_i y_i - \sum_{i=1}^{m} \lambda_i b_i$

is the dual problem of the primal problem

Q2)

5.27 minimize$_x$ $\|Ax - b\|_2^2$

subject to $Gx = h$

$L(x,\lambda) = (Ax-b)^T (Ax-b) + \lambda^T (Gx - h)$

$= (x^T A^T - b^T)(Ax - b) + \lambda^T G x - \lambda^T h$

$= x^T A^T A x - x^T A^T b - b^T A x + b^T b + \lambda^T G x - \lambda^T h$

$= x^T A^T A x + \lambda^T G x - 2x^T A^T b - \lambda^T h + \|b\|_2^2$

$\therefore g(\lambda) = \min_{x} x^T A^T A x + \lambda^T G x - 2x^T A^T b - \lambda^T h + \|b\|_2^2$

$= \min_{x} \underbrace{x^T A^T A x + (\lambda^T G - 2b^T A) x - \lambda^T h}_{h(x)}$

($A^T A$ is PSD, $h(x)$ is convex)

$\nabla_x h(x) = 2A^T A x + (\lambda^T G - 2b^T A)^T = 0$ ($\underset{n\times m}{A^T} \underset{m\times n}{A} \in R^{n\times n}$, since Rank$(A) = n$, its inverse $(A^T A)^{-1}$ exists)

$\Rightarrow x^* = -\frac{1}{2} (A^T A)^{-1} (\lambda^T G - 2b^T A)^T$

$\nabla_x h(x) = 2A^TAx + (\lambda^T G - 2b^TA)^T = 0$ $\quad$ $(A^TA \in R^{n\times n}$, since $\mathrm{Rank}(A) = n$,

$\Rightarrow x^* = -\frac{1}{2}(A^TA)^{-1}(\lambda^T G - 2b^TA)^T$ $\quad$ its inverse $(A^TA)^{-1}$ exists)

$= -\frac{1}{2}(A^TA)^{-1}(G^T\lambda^* - 2A^Tb)$ $\quad$ (by KKT condition)

$\therefore g(\lambda) = \min_x h(x) = \frac{1}{4}\left[(A^TA)^{-1}(G^T\lambda - 2A^Tb)\right]^T A^TA\left[(A^TA)^{-1}(G^T\lambda - 2A^Tb)\right]$

$-\frac{1}{2}(\lambda^T G - 2b^TA)\left[(A^TA)^{-1}(G^T\lambda - 2A^Tb)\right] - \lambda^T h$

For λ^*: $\quad Gx^* = h$

$-\frac{1}{2}G(A^TA)^{-1}(G^T\lambda^* - 2A^Tb) = h$

$\Rightarrow \quad G(A^TA)^{-1}G^T\lambda^* - 2G(A^TA)^{-1}A^Tb = -2h$

$\Rightarrow \quad \underbrace{\overset{p\times n}{G}\;\overset{n\times n}{(A^TA)^{-1}}\;\overset{n\times p}{G^T}}_{p\times p}\lambda^* = 2G(A^TA)^{-1}A^Tb - 2h$

$\lambda^* = \left[G(A^TA)^{-1}G^T\right]^{-1}\left[2G(A^TA)^{-1}A^Tb - 2h\right]$

Rank $G = p$, $\therefore \underbrace{\left[G(A^TA)^{-1}G^T\right]^{-1}}_{\in R^{p\times p}}$ exist

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Q3) . 5.35

unperterbed:

$L(y,\lambda,\mu) = \log f_0(y) + \sum_{i=1}^{m}\lambda_i \log f_i(y) + \sum_{i=1}^{p}\mu_i \log h_i(y)$

$d^* = \log p^*(0,0) = g(\lambda^*,\mu^*) \le L(y,\lambda,\mu)$

$\le \log f_0(y) + \lambda^T u + \mu^T v$

(since: $\log f_i(y) \le u_i$, $\log h_i(y) = v_i$)

when: $f_0(y) = p^*(u,v)$

$\log p^*(0,0) \le \log p^*(u,v) + \lambda^{*T}u + \mu^{*T}v$

$\Rightarrow \quad \log p^*(u,v) \ge \log p^*(0,0) - \lambda^{*T}u - \mu^{*T}v$

let: $v = 0$, for very small value u

$\log p^*(u,0) \ge \log p^*(0,0) - \lambda^{*T}u$

1) : if $u > 0$

$\underbrace{\frac{\log p^*(u_i,0) - \log p^*(0,0)}{u_i}} \ge -\lambda_i^*$

$$\frac{\partial \log p^*(0,0)}{\partial u}$$

2) : if $u<0$

$$\frac{\partial \log p^*(0,0)}{\partial u_i} \le -\lambda_i^*$$

$$\therefore \quad \frac{\partial \log p^*(0,0)}{\partial u_i} = -\lambda_i^*$$

let: $\partial u_i = \alpha \longrightarrow \underbrace{\partial \log p^*(0,0)}_{\text{change of unperterbed objective}} = -\alpha\lambda_i^* \quad , i=1,\dots m$

Q4) 5.42

$$\begin{aligned} &\underset{x}{\text{minimize}} && c^T x \\ &\text{subject to} && Ax \preceq_K b \end{aligned}$$

$$\begin{aligned} L(x,\lambda) &= c^T x + \lambda^T(Ax-b) \\ &= (c^T - \lambda^T A)x - \lambda^T b \end{aligned}$$

$$g(\lambda) = \min_x L = \begin{cases} -\lambda^T b & c^T - \lambda^T A = 0 \\ -\infty & \text{else} \end{cases}$$

$\therefore$ dual problem

$$\begin{aligned} &\underset{\lambda}{\max} && -\lambda^T b \\ &\text{subject to} && \lambda \succeq_{K^*} 0 \quad \text{dual cone} \\ &&& c^T - \lambda^T A = 0 \end{aligned}$$

Q5)

$$\begin{aligned} &\text{minimize} && c^T x = f_0(x) \\ &\text{subject to} && Ax \ge b \rightarrow b - Ax \le 0 \\ &&& x \ge 0 \rightarrow -x \le 0 \end{aligned}$$

$$L(x,\lambda_1,\lambda_2) = f_0(x) + \lambda_1^T(b-Ax) + \lambda_2^T(-x)$$

$$\begin{aligned} g(\lambda_1,\lambda_2) &= \min_x \underline{c^T x} + \lambda_1^T b \underline{- \lambda_1^T A x} \underline{- \lambda_2^T x} \\ &= \min_x (c^T - \lambda_1^T A - \lambda_2^T)x + \lambda_1^T b \end{aligned}$$

if $c^T - \lambda_1^T A - \lambda_2^T = 0$: $g(\lambda_1,\lambda_2) = \lambda_1^T b$

$\wedge$

if $c^T - \lambda_1^T A - \lambda_2^T = 0$: $g(\lambda_1, \lambda_2) = \lambda_1^T b$

else : $g(\lambda_1, \lambda_2) = -\infty$

$\therefore \max_{\lambda_1, \lambda_2} g(\lambda_1, \lambda_2) = \underset{\lambda_1}{\text{maximize}} \ \lambda_1^T b$

subject to $c^T - \lambda_1^T A - \lambda_2^T = 0$

$\lambda_1 \geqslant 0$

$\lambda_2 \geqslant 0$

$\Rightarrow$ maximize $\lambda_1^T b$

subject to $(c^T - \lambda_1^T A)^T = c - A^T \lambda_1 = \lambda_2 \geqslant 0 \quad (*)$

$\lambda_1 \geqslant 0$

a). if primal $f_0(x)$ is unbounded, it means $c^T < 0$, then for $(*)$:

$c - A^T \lambda_1 \geqslant 0$

it is not feasible as $c - A^T \lambda_1 \to -\infty$

b) if primal is infeasible, it means either $Ax < b$ or $x < 0$, or both;

• If: $\begin{cases} x < 0 \\ Ax \geqslant b \end{cases}$, the constraint of the dual will be

$\lambda_2 = A^T \lambda_1 - c \geqslant 0 \Rightarrow A^T \lambda_1 \geqslant c > 0$ since primal should be bounded

$\lambda_1 \geqslant 0$

therefore: $A > 0$, $b \leq Ax < 0$

then for its dual: $g(\lambda_1) = \min \lambda_1^T b$ is unbounded

• If: $\begin{cases} x < 0 \\ Ax < b \end{cases}$, $g(\lambda_1, \lambda_2) = \min (c^T + \lambda_1^T A + \lambda_2^T) x - \lambda_1^T b$

$= \begin{cases} \lambda_1^T b & c^T + \lambda_1^T A + \lambda_2^T = 0 \\ -\infty & \text{else} \end{cases}$

$\max g(\lambda_1, \lambda_2) = \max \ -\lambda_1^T b$

s.t. $\lambda_1 \geqslant 0$

$-A^T \lambda_1 - c \geqslant 0 \Rightarrow A^T \lambda_1 < 0$ ($c > 0$ so that primary is bounded)

therefore: $A^T < 0$, plus $\begin{cases} Ax < b \\ x < 0 \end{cases}$, we know

$b > 0$. Since $\lambda_1 \geqslant 0$, $\max -\lambda_1^T b = 0$ when $\lambda_1 = 0$,

which contradict to $A^T \lambda_1 < 0$, so it is infeasible;

If we don't violate the constraint of the dual, the problem is unbounded

Q b) <a>

$$\max_{y} \; x^T P y \qquad \Rightarrow \qquad \underset{y}{\text{minimize}} \; -x^T P y$$

subject to $Cy \le d$ $\qquad$ subject to $Cy - d \le 0$

$$L(y, \lambda) = -x^T P y + \lambda^T (Cy - d)$$

$$= (-x^T P + \lambda^T C) y - \lambda^T d$$

$$g(\lambda) = \min_{y} L = \begin{cases} -\lambda^T d & \text{if } -x^T P + \lambda^T C = 0 \\ -\infty & \text{else} \end{cases}$$

$\therefore \max g(\lambda) = \underset{x, \lambda}{\text{minimize}} \; \lambda^T d = d^T \lambda$

subject to $-x^T P + \lambda^T C = 0 \rightarrow C^T \lambda = P^T x$

$\lambda \ge 0$

$Ax \le b$

<b> $\underset{x}{\text{minimize}} \; x^T P y$

subject to $Ax \le b$

$$L(x, \nu) = x^T P y + \nu^T (Ax - b)$$

$$= (y^T P^T + \nu^T A) x - \nu^T b$$

$$g(\nu) = \min_{x} L = \begin{cases} -\infty & \text{else} \\ \underset{-b^T \nu}{\overset{\shortparallel}{-\nu^T b}} & \text{if } y^T P^T + \nu^T A = 0 \rightarrow A^T \nu + P y = 0 \end{cases}$$

$\therefore \max_{y, \lambda} g(\nu) = \text{maximize} \; -b^T \nu$

subject to $A^T \nu + P y = 0$

$Cy \le d$

$\nu \ge 0$

<c> For <a>

$$L(x, \lambda, \mu_1, \mu_2, \mu_3) = d^T \lambda + \mu_1^T (C^T \lambda - P^T x) + \mu_2^T (Ax - b) + \mu_3^T (-\lambda)$$

$$= (d^T + \mu_1^T C^T - \mu_3^T) \lambda + (\mu_2^T A - \mu_1^T P^T) x - \mu_2^T b$$

$\therefore\ g(\mu_1,\mu_2,\mu_3) = \min_{x,\lambda} L = \begin{cases} -\infty & \text{else} \\ -\mu_2^T b = -b^T\mu_2 & \text{if} \begin{cases} d^T+\mu_1^T c^T = \mu_3^T \rightarrow c\mu_1 + d = \mu_3 \\ \mu_2^T A - \mu_1^T p^T = 0 \rightarrow A^T\mu_2 = p\,\mu_1 \end{cases} \end{cases}$

$\therefore$ maximize$_{\mu_1,\mu_2,\mu_3}$ g = maximize $-b^T\mu_2$

subject to

$$\left.\begin{aligned} &\mu_2 \geq 0 \\ &\mu_3 \geq 0 \\ &c\mu_1 + d = \mu_3 \geq 0 \\ &A^T\mu_2 = p\,\mu_1 \end{aligned}\right\} \Rightarrow \begin{aligned} &A^T\mu_2 + p(-\mu_1) = 0 \\ &c(-\mu_1) \leq d \qquad (*) \\ &\mu_2 \geq 0 \end{aligned}$$

compare (*) to <b>, we can see that by substitution, $y=-\mu_1$, $v=\mu_2$, we can recover <b> from (*), therefore, <a> is the dual of <b>, therefore here, min-max = max-min

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7. <a>. let: $t = \begin{bmatrix} |p_1| \\ \vdots \\ |p_i| \end{bmatrix}$, $f = \begin{bmatrix} p_1 \\ \vdots \\ p_i \end{bmatrix}$ and $i=10$ (my f is p in the question. I don't want to change it because I want to be consistant with my Matlab Code)

rewrite the optimization using epigraph approach

minimize$_{t,f}$ $0^T f + 1^T t = [0^T \ \ 1^T]\begin{bmatrix} f \\ t \end{bmatrix}$

subject to

$$\left.\begin{aligned} &t \geq f \rightarrow f - t \leq 0 \\ &t \geq -f \rightarrow -f - t \leq 0 \end{aligned}\right\} \Rightarrow \begin{bmatrix} I & -I \\ -I & -I \end{bmatrix}\begin{bmatrix} f \\ t \end{bmatrix} \leq \begin{bmatrix} \underset{n\times 1}{0} \\ \underset{n\times 1}{0} \end{bmatrix}_{2n\times 1}$$

$n=10$ here

$1^T f = 0$ (velocity constraint)

$A^T f = 1$ where $A = [9.5\ 8.5\ 7.5\ \ldots\ 0.5]$

(position constraint. For calculation of A, see attach Matlab pages)

<b>

i)

$$|w^T v| = \left|\sum_{i=1}^n w_i v_i\right| \leq \sum_{i=1}^n |w_i||v_i| \leq \max_i v_i \sum_{i=1}^n |w_i| = \|v\|_\infty \|w\|_1$$

ii)

$$\left.\begin{aligned} &\text{minimize} \quad \|z\|_1 \\ &\text{subject to} \quad Az = y \end{aligned}\right\} p^*$$

$$\Rightarrow \quad \text{minimize} \quad 1^T t$$
$$\text{subject to} \quad t \geq z$$
$$t \geq -z$$
$$Az = b$$

$$g(z, t, \lambda, \mu, \nu) = (1^T - \mu^T \nu^T) t + (\lambda^T A + \mu^T - \nu^T) z - \lambda^T y$$
$$= \begin{cases} -\lambda^T y & \text{if } \begin{cases} \nu + \mu = 1 \\ \nu - \mu = A^T \lambda \end{cases} \\ -\infty & \text{else} \end{cases}$$

$\therefore$ dual problem:
$$\underset{\lambda, \mu, \nu}{\text{maximize}} \quad -\lambda^T y$$
$$\text{subject to} \quad \left.\begin{array}{l} 1 = \nu + \mu \\ A^T \lambda = \nu - \mu \\ \mu \geq 0 \\ \nu \geq 0 \end{array}\right\} \|A^T \lambda\|_\infty \leq 1$$

$$\Rightarrow \quad \left.\begin{array}{l} \underset{\lambda}{\text{maximize}} \quad -\lambda^T y \\ \text{subject to} \quad \|A^T \lambda\|_\infty \leq 1 \end{array}\right\} d^*$$

claim: for: $\underset{\lambda}{\text{maximize}} \ \dfrac{-\lambda^T y}{\|A^T \lambda\|_\infty} = d^{**}$, $\quad d^* = d^{**}$

proof: 1). $d^* \geq 0$. since, say, for a optimum λ' to make the optima d', say, $d' < 0$, then $-\lambda'$ is also feasible since $\|A^T - \lambda'\|_\infty = \|A^T \lambda'\|_\infty \leq 1$. the obtained $d'' > 0$, and $d'' > d$, which is contradictory to the hypothesis that d' is optima (maximum value). Therefore, $d^* \geq 0$

therefore $-\lambda^T y = |\lambda^T y|$

d' is optima (maximum value). therefore, $d' \geq 0$

therefore, $-\lambda^T y = |\lambda^T y|$

2) when $\|A^T \lambda'\|_\infty = 1$, d has the optima d'.

say if not, then assume $\|A^T \lambda\|_\infty = \alpha < 1$ and $\alpha > 0$

therefore $\frac{\lambda'}{\alpha}$ is also feasible since $\frac{\|A^T \lambda\|_\infty}{\alpha} = \|A^T \frac{\lambda}{\alpha}\|_\infty = 1$

and $\frac{d'}{\alpha} > d' \geq 0$, which again is contradictory to

the assumption that d' is optima.

3) since $\|A^T \lambda\|_\infty$ is non-negative

$\frac{-v^T b}{\|A^T \lambda\|_\infty}$ is just simply a scaled version of $\overset{-v^T b}{\cancel{\|A^T \lambda\|_\infty}}$.

by $\frac{1}{\|A^T \lambda\|_\infty}$. When λ is found to have $\|A^T \lambda\|_\infty = 1$, $d^* = d^{**}$

Therefore: $b^* = b^{**}$.

With the general weak duality property:

$$p^* \geq b^* = b^{**}$$

$$\therefore \quad \|z\|_1 \geq \frac{|\lambda^T y|}{\|A^T \lambda\|_\infty}$$

iii)

write the gestion in 2a) in Lagrange multiplier form:

$$\begin{aligned}
\underset{t, f}{\text{minimize}} \quad & 1^T t \\
\text{subject to} \quad & t \geq f \\
& t \geq -f \\
& 1^T f = 0 \\
& A^T f = 1
\end{aligned}$$

We have two equality constraints, let λ_1 with $1^T f = 0$,

and λ_2 with $A^T f = 1$

therefore from ii), we have

$$\|f\|_1 \geqslant \frac{|\lambda_2^T|}{\|A\lambda_2 + 1\lambda_1\|_\infty}$$

well here lambda_1 and lambda_2 are scaler...

when $\|f\|_1 = \dfrac{|\lambda_2^T|}{\|A\lambda_2 + 1\lambda_1\|_\infty}$, this gives optimal values

This is because, analogy to ii, $d^* = d^* = \dfrac{|\lambda_2^T|}{\|A\lambda_2 + 1\lambda_1\|_\infty}$.

it is the dual optima of question (a).

when strong duality holds, $p^* = d^*$

2 c).

$$\underset{f,t}{\text{minimize}} \quad 0^T f + t = [0^T \; 1]\begin{bmatrix} f \\ t \end{bmatrix}$$

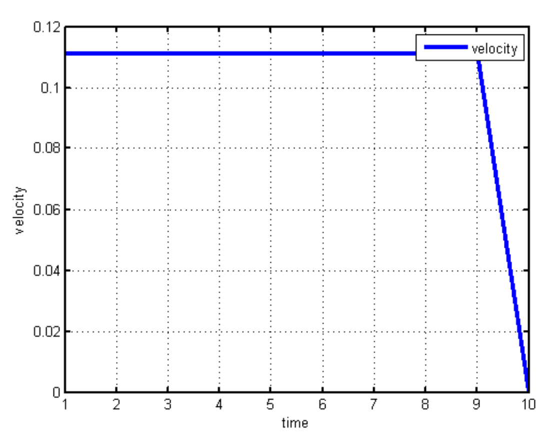
subject to

$$\left.\begin{aligned} f \le t1 &\rightarrow f - t1 \le 0 \\ -f \le t1 &\rightarrow -f - t1 \le 0 \end{aligned}\right\} \Rightarrow \underbrace{\begin{bmatrix} I & -1 \\ -I & -1 \end{bmatrix}}_{20 \times 11} \begin{bmatrix} f \\ t \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{n \times 1}$$

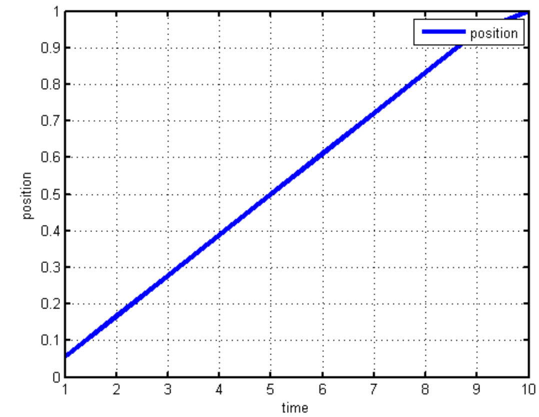
$$1^T f = 0$$

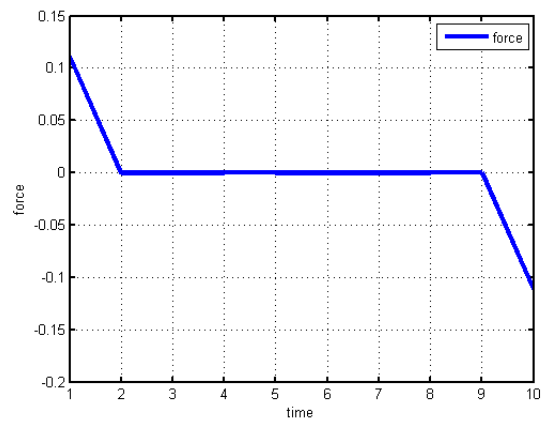
$$A^T f = 1$$

```
%%  Q7 (a)

close all
clear all

S = [9.5 8.5 7.5 6.5 5.5 4.5 3.5 2.5 1.5 0.5];
t = 10; % 10 time instances
y = zeros(t,1);

obj = [zeros(1,t),ones(1,t)];
A = [eye(t,t),-eye(t,t); -eye(t,t), -eye(t,t)];
b = [y;y];
Aeq = [ones(1,t),zeros(1,t); S, zeros(1,t)];
beq = [0; 1];

% formulate the LP problem in MATLAB
[opt_sol, fval] = linprog(obj,A,b,Aeq,beq );
optima = opt_sol(1:10);
x = optima;

v = zeros(10,1);        % velocity
for t = 1:1:10
    v(t,:) = sum(x(1:t));
end
```

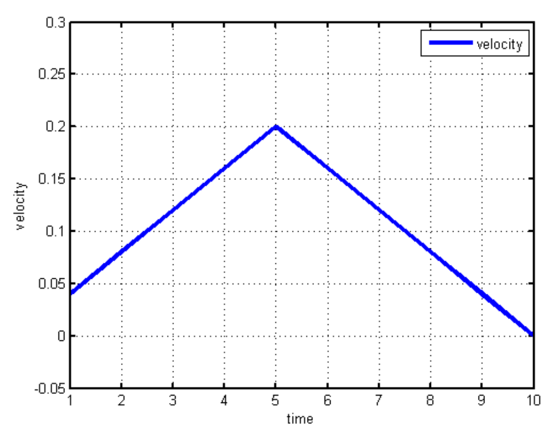


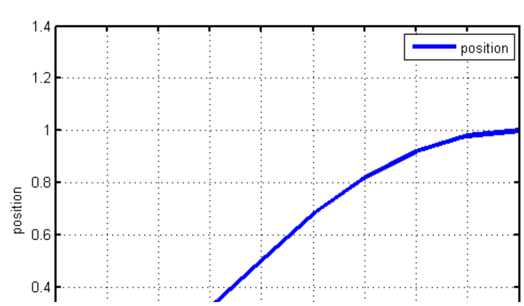
```
end

a = [0;x];          % add a zero to the front, acceleration
s = zeros(10,1);     % displacement
for t = 1:1:10
   s(t,:) = sum(a(1:t)) + 0.5 * x(t,:);
end

p = zeros(10,1);      % position
for t = 1:1:10
   p(t,:) = sum(s(1:t));
end

figure(1)
p1 = plot(x);
set(gca,'linewidth',2)
set(p1, 'linewidth',3)
xlabel('time')
ylabel('force')
legend('force')
grid on

figure(2)
p1 = plot(v);
set(gca,'linewidth',2)
set(p1, 'linewidth',3)
xlabel('time')
ylabel('velocity')
legend('velocity')
grid on

figure(3)
p1 = plot(p);
set(gca,'linewidth',2)
set(p1, 'linewidth',3)
xlabel('time')
ylabel('position')
legend('position')
grid on
```






```
%% Q7(b)

close all
clear all

S = [9.5 8.5 7.5 6.5 5.5 4.5 3.5 2.5 1.5 0.5];
t = 10; % 10 time instances
y = zeros(t,1);

obj = [zeros(1,t),1];
A = [eye(t,t),-ones(t,1); -eye(t,t), -ones(t,1)];
b = [y;y];
Aeq = [ones(1,t),0; S, 0];
beq = [0; 1];

% formulate the LP problem in MATLAB
[opt_sol, fval] = linprog(obj,A,b,Aeq,beq );
optima = opt_sol(1:10);
x = optima;

v = zeros(10,1);      % velocity
for t = 1:1:10
   v(t,:) = sum(x(1:t));
end

a = [0;x];          % add a zero to the front, acceleration
s = zeros(10,1);     % displacement
for t = 1:1:10
   s(t,:) = sum(a(1:t)) + 0.5 * x(t,:);
end

p = zeros(10,1);      % position
for t = 1:1:10
   p(t,:) = sum(s(1:t));
end

figure(1)
p1 = plot(x);
set(gca,'linewidth',2)
set(p1, 'linewidth',3)
xlabel('time')
ylabel('force')
legend('force')
```





```
for t = 1:1:10
    s(t,:) = sum(a(1:t)) + 0.5 * x(t,:);
end

p = zeros(10,1);       % position
for t = 1:1:10
    p(t,:) = sum(s(1:t));
end

figure(1)
p1 = plot(x);
set(gca,'linewidth',2)
set(p1, 'linewidth',3)
xlabel('time')
ylabel('force')
legend('force')
grid on

figure(2)
p1 = plot(v);
set(gca,'linewidth',2)
set(p1, 'linewidth',3)
xlabel('time')
ylabel('velocity')
legend('velocity')
grid on

figure(3)
p1 = plot(p);
set(gca,'linewidth',2)
set(p1, 'linewidth',3)
xlabel('time')
ylabel('position')
legend('position')
grid on
```