MATHEMATICS

distance from a point to a hyperplane

Asked 4 years, 10 months ago Active 1 year, 6 months ago Viewed 26k times

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I have an n-dimensional hyperplane: $w'x + b = 0$ and a point x_0. The shortest distance from this point to a hyperplane is $d = \frac{|w \cdot x_0 + b|}{||w||}$. I have no problem to prove this for 2 and 3 dimension space using algebraic manipulations, but fail to do this for an n-dimensional space. Can someone show a nice explanation for it?

linear-algebra geometry

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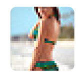
asked Mar 28 '15 at 20:28



Salvador Dali

1,954 2 16 35

2 Answers

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There are many ways to solve this problem. In principal one can use Lagrange multipliers and solve a large system of equations, but my attempt to do so met with a road block. However, since you are working in $\mathbb{R}^n$ we have the privilege of orthogonal projection via the dot product. To this end we need to construct a vector from the plane to x_0 to project onto a vector perpendicular to the plane. Then we compute the *length of the projection* to determine the distance from the plane to the point.

First, you have an affine hyperplane defined by $w \cdot x + b = 0$ and a point x_0. Suppose that $X \in \mathbb{R}^n$ is a point satisfying $w \cdot X + b = 0$, i.e. it is a point on the plane. You should construct the vector $x_0 - X$ which points from X to x_0 so that you can project it onto the unique vector *perpendicular* to the plane. Some quick reasoning should tell you that this vector is, in fact, w. So we want to compute $\|\text{proj}_w(x_0 - X)\|$. Some handy formulas give us

$$d = \|\text{proj}_w(x_0 - X)\| = \left\| \frac{(x_0 - X) \cdot w}{w \cdot w} w \right\| = |x_0 \cdot w - X \cdot w| \frac{\|w\|}{\|w\|^2} = \frac{|x_0 \cdot w - X \cdot w|}{\|w\|}$$

We chose X such that $w \cdot X = -b$ so we get

$$d = \|\text{proj}_w(x_0 - X)\| = \frac{|x_0 \cdot w + b|}{\|w\|}$$

as desired.

This almost seems like cheating and purely heuristic based on Euclidean geometry. Indeed, I would

To make this argument more concrete you should do each step in $\mathbb{R}^2$ for a line $y = mx + b$ and a point (x_0, y_0).

answered Mar 28 '15 at 22:55

Yeldarbskich
1,035 7 13

Can you please clarify how from $\left\|\frac{(x_0-X)\cdot w}{w\cdot w}w\right\|$ you ended up with $|x_0 \cdot w - X \cdot w|\frac{||w||}{||w||^2}$? In fact I am interested how w became $||w||$. – Salvador Dali Mar 29 '15 at 4:07

The dot product of two vectors is a real number. For any scalar $\lambda \in \mathbb{R}$ it is true that $||\lambda w|| = |\lambda|||w||$. You can verify this via the definition of the norm on $\mathbb{R}^n$. That is $||w|| = \sqrt{\sum_{i=1}^n w_i^2}$. – Yeldarbskich Mar 29 '15 at 4:18

oh, sorry, I have missed that the whole expression is in $||\ldots||$. Now everything is clear. Thank you. – Salvador Dali Mar 29 '15 at 4:27

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Here is a Lagrange multiplier based solution.

The goal is to minimize $(x_0 - x)'(x_0 - x)$ subject to $w'x + b = 0$

The Lagrangian is $(x_0 - x)'(x_0 - x) - L(w'x + b)$

The derivative of the Lagrangian is $2(x_0 - x) - Lw = 0$

Dot with w, we get $2w'(x_0 - x) - Lw'w = 0 \implies L = \frac{2w'(x_0-x)}{w'w}$

Dot with $(x_0 - x)$, we get

$$2(x_0 - x)'(x_0 - x) - L(x_0 - x)'w = 0 \implies 2(x_0 - x)'(x_0 - x) = \frac{2w'(x_0-x)}{w'w}(x_0 - x)'w \implies$$
$$(x_0 - x)'(x_0 - x) = \frac{(w'(x_0-x))^2}{w'w} \implies (x_0 - x)'(x_0 - x) = \frac{(w'x_0+b)^2}{w'w}$$

Taking square root gives the answer we wanted.

answered Jul 17 '18 at 3:36

Andrew Au
891 7 11

1 A solution based of Lagrange multiplier - Yeldarbskich – Andrew Au Jul 17 '18 at 3:37

Is there something that led you to dot with w and $(x_0 - x)$ other than that it happens to work out? – Alex Mar 11 '19 at 0:34

2 The problem with solving these vector equations is that you can't move the 'factor' to the other side if it is not a scalar. We can remedy that by computing the dot product with itself. After doing so, it turns scalar, and then we can move it. That's the idea anyway. – Andrew Au Mar 12 '19 at 1:24