MTE544 Assignment 1

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Q1

To prove $G^{-1} = \begin{bmatrix} R^T & -R^T q \\ 0 & 1 \end{bmatrix}$, we know that to represent a point, p^B, in the reference frame {B} in the reference frame {A}, we need the transformation matrix $G_B^A = \begin{bmatrix} R_B^A & q^A \\ 0 & 1 \end{bmatrix}$ which consist of the rotation matrix R_B^A and the vector q pointing from {A} to {B}.

To do the opposite, represent a point in the reference frame {A} in the reference frame {B}, then we would need the transformation matrix $G_A^B = \begin{bmatrix} R_A^B & q^B \\ 0 & 1 \end{bmatrix}$.

Form this, we know from class that

$$R_A^B = \left(R_B^A\right)^{-1} = \left(R_B^A\right)^T$$

Additionally, since we know q^A goes from {A} to {B} then simply multiplying it by -1 and applying the rotation matrix R_A^B to it would result in q^B which is a vector from {B} to {A}:

$$q^B = -R_A^B q^A = -\left(R_B^A\right)^T q^A$$

Therefore

$$G_A^B = \begin{bmatrix} R_A^B & q^B \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \left(R_B^A\right)^T & -\left(R_B^A\right)^T q^A \\ 0 & 1 \end{bmatrix}$$

If we multiply G_A^B by G_A^B we get the identity matrix:

$$G_A^B G_A^B = \begin{bmatrix} R_B^A & q^A \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_A^B & q^B \\ 0 & 1 \end{bmatrix} = I$$

This means

$$G_A^B = (G_A^B)^{-1} = G_A^B$$

And so, in general

$$G^{-1} = \begin{bmatrix} R^T & -R^T q \\ 0 & 1 \end{bmatrix}$$

Q2

Q2a

The transformation matrix for each reference frame is as follows:

$$G_e^3 = \begin{bmatrix} R_e^3 & q^e \\ 0 & 1 \end{bmatrix}, R_e^3 = \begin{bmatrix} \sin(\psi) & -sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{bmatrix}, q^e = \begin{bmatrix} l_3 \\ 0 \end{bmatrix}$$

$$G_3^2 = \begin{bmatrix} R_3^2 & q^3 \\ 0 & 1 \end{bmatrix}, R_3^2 = \begin{bmatrix} \sin(\theta) & -sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, q^3 = \begin{bmatrix} l_2 \\ 0 \end{bmatrix}$$

$$G_2^1 = \begin{bmatrix} R_2^1 & q^2 \\ 0 & 1 \end{bmatrix}, R_2^1 = \begin{bmatrix} \sin(\gamma) & -sin(\gamma) \\ \sin(\gamma) & \cos(\gamma) \end{bmatrix}, q^2 = \begin{bmatrix} l_1 \\ 0 \end{bmatrix}$$

$$G_1^s = \begin{bmatrix} R_1^s & q^1 \\ 0 & 1 \end{bmatrix}, R_1^s = \begin{bmatrix} \sin(\alpha) & -sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}, q^1 = \begin{bmatrix} x \\ y \end{bmatrix}$$

Then, transformation matrix from {e} to {s} is as follows:

$$G_e^s = G_1^s G_2^1 G_3^2 G_e^3$$
$$= \begin{bmatrix} \cos(\psi+\theta+\gamma+\alpha) & -\sin(\psi+\theta+\gamma+\alpha) & y\cos(\psi+\theta+\gamma) - x\sin(\psi+\theta+\gamma) + l_3 + l_2\cos(\psi) + l_1\cos(\psi+\theta) \\ \sin(\psi+\theta+\gamma+\alpha) & \cos(\psi+\theta+\gamma+\alpha) & y\sin(\psi+\theta+\gamma) + x\cos(\psi+\theta+\gamma) + l_2\sin(\psi) + l_1\sin(\psi+\theta) \\ 0 & 0 & 1 \end{bmatrix}$$

Q2b

At the configuration of the robot specified in the question, the coordinates of the points p_1^e and p_2^e in spatial frame {s} would be:

$$p_1^s = p_1^e G_e^s = \begin{bmatrix} 0.8411 \\ -2.7247 \\ 1 \end{bmatrix}$$

$$p_2^s = p_2^e G_e^s = \begin{bmatrix} 3.0731 \\ -2.5908 \\ 1 \end{bmatrix}$$

Q2c

If the robot is located at the base at (0,0), the following equation applies:

$$\dot{\bar{p}}^s = \dot{G}_2^s \bar{p}^2 = \begin{bmatrix} \widehat{\omega} R_2^s & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p^2 \\ 1 \end{bmatrix}$$

To calculate R_2^s:

$$R_2^s = R_1^s R_2^1$$
$$R_1^s = \begin{bmatrix} \cos(\alpha) & -sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$
$$R_2^1 = \begin{bmatrix} \cos(\gamma) & -sin(\gamma) \\ \sin(\gamma) & \cos(\gamma) \end{bmatrix}$$

$$\rightarrow R_2^s = \begin{bmatrix} \cos(\alpha+\gamma) & -sin(\alpha+\gamma) \\ \sin(\alpha+\gamma) & \cos(\alpha+\gamma) \end{bmatrix}$$

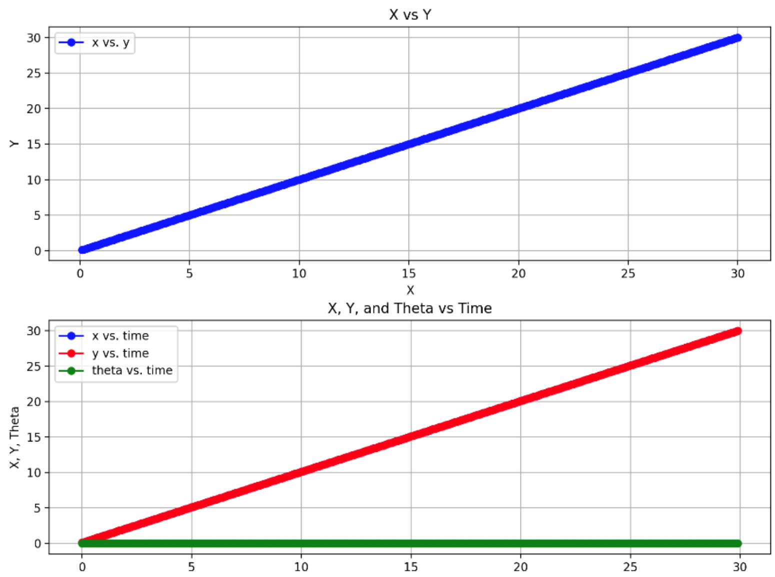
$$\rightarrow \dot{\bar{p}}^s = \begin{bmatrix} \omega\cos(\alpha+\gamma) & -\omega\sin(\alpha+\gamma) & v_x \\ \omega\sin(\alpha+\gamma) & \omega\cos(\alpha+\gamma) & v_y \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} x'\omega\cos(\alpha+\gamma) - y'\omega\sin(\alpha+\gamma) + v_x \\ x'\omega\sin(\alpha+\gamma) + y'\omega\cos(\alpha+\gamma) + v_y \\ 0 \end{bmatrix}$$

Q3

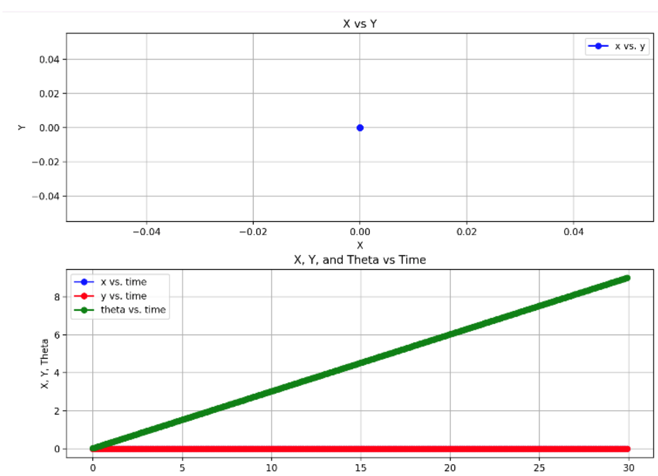
Q3a

The simulation was programmed in Python.

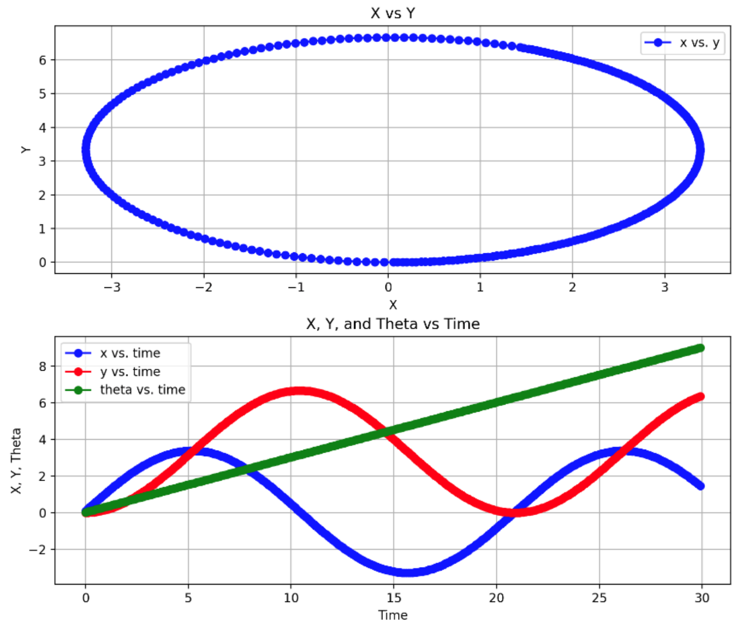
For $\begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$:



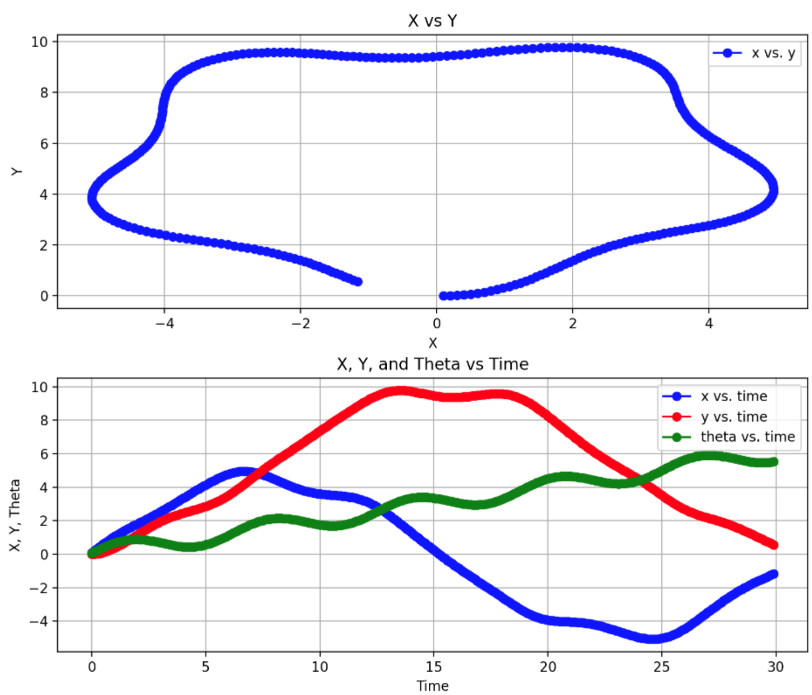
For $\begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} 0 \\ 0.3 \end{bmatrix}$:



For $\begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} 1 \\ 0.3 \end{bmatrix}$:



For $\begin{bmatrix} v(t) \\ \omega(t) \end{bmatrix} = \begin{bmatrix} 1 + 0.1\sin(t) \\ 0.2 + 0.5\cos(t) \end{bmatrix}$:



Q3b

If $T = 0.2m$ and $r = 0.1m$, then

$$u_r = \frac{1}{r}\left(v + \frac{T\omega}{2}\right)$$
$$u_l = \frac{1}{r}\left(v - \frac{T\omega}{2}\right)$$

For $v = 1$ m/s and $\omega = 0$ m/s:

$$u_r = \frac{1}{0.1}\left(1 + \frac{0.2(0)}{2}\right) = 10\, rad/s$$
$$u_l = \frac{1}{0.1}\left(1 - \frac{0.2(0)}{2}\right) = 10\, rad/s$$

For $v = 0$ m/s and $\omega = 0.3$ m/s:

$$u_r = \frac{1}{0.1}\left(0 + \frac{0.2(0.3)}{2}\right) = 0.03\, rad/s$$
$$u_l = \frac{1}{0.1}\left(0 - \frac{0.2(0.3)}{2}\right) = -0.03\, rad/s$$

For $v = 1$ m/s and $\omega = 0.3$ m/s:

$$u_r = \frac{1}{0.1}\left(1 + \frac{0.2(0.3)}{2}\right) = 10.3\, rad/s$$
$$u_l = \frac{1}{0.1}\left(1 - \frac{0.2(0.3)}{2}\right) = 9.7\, rad/s$$

If the input control parameters were wheel speeds, the simulation could simply be modified to calculate linear velocity v, and angular velocity ω first by using equations for u_r and u_l.

Q4

Q4a

To derive the equation of motion for the omnidirectional wheeled robot with three wheels, we can use the general equations derived in class:

$$u_i = \frac{1}{r_i}\begin{bmatrix}1 & \tan(\gamma_i)\end{bmatrix} g(\theta)\dot{q}$$

Where $g(\theta)$ is defined by:

$$g(\theta) = \begin{bmatrix} \cos(\theta+\beta_i) & \sin(\theta+\beta_i) & x_i\sin(\beta_i) - y_i\cos(\beta_i) \\ -\sin(\theta+\beta_i) & \cos(\theta+\beta_i) & x_i\cos(\beta_i) + y_i\sin(\beta_i) \end{bmatrix}$$

We know $\gamma_i = 0$, so $\tan(\gamma_i) = 0$:

$$u_i = \frac{1}{r_i}\begin{bmatrix}1 & 0\end{bmatrix} g(\theta)\dot{q}$$

The following constants can be derived from the figure provided:

$$\beta_1 = \frac{\pi}{2}, \beta_2 = \frac{7\pi}{6}, \beta_3 = \frac{11}{6}\pi$$

$$x_1 = l,\ x_2 = -l\sin\left(\frac{\pi}{6}\right), x_3 = -l\sin\left(\frac{\pi}{6}\right)$$

$$y_1 = 0,\ y_2 = l\cos\left(\frac{\pi}{6}\right),\ y_3 = -l\cos\left(\frac{\pi}{6}\right)$$

The following results from the derived constants:

$$u_i = \frac{1}{r}\begin{bmatrix} \cos(\theta+\beta_1) & \sin(\theta+\beta_1) & x_1\sin(\beta_1) - y_1\cos(\beta_1) \\ \cos(\theta+\beta_2) & \sin(\theta+\beta_2) & x_2\sin(\beta_2) - y_2\cos(\beta_2) \\ \cos(\theta+\beta_3) & \sin(\theta+\beta_3) & x_3\sin(\beta_3) - y_3\cos(\beta_3) \end{bmatrix}\begin{bmatrix}\dot{x} \\ \dot{y} \\ \dot{\theta}\end{bmatrix}$$

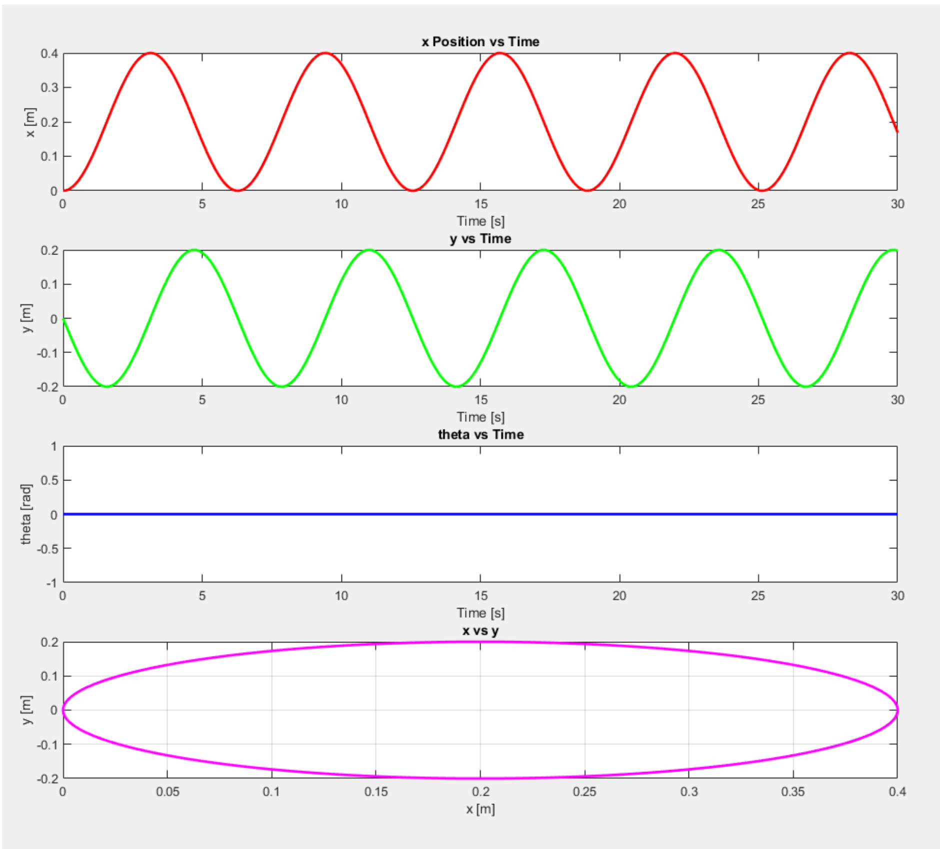
$$u_i = G(\theta)\dot{q}$$

Q4bi

For wheel speeds of $u_1 = -2$, $u_2 = 1.0$, and $u_3 = 1.0$, we can determine $\dot{x}, \dot{y}, \dot{\theta}$ by the following equation:

$$\dot{q} = G^{-1}(\theta)u_i$$

The plot below shows the variables against time and the x and y trajectory.



Q4bii1

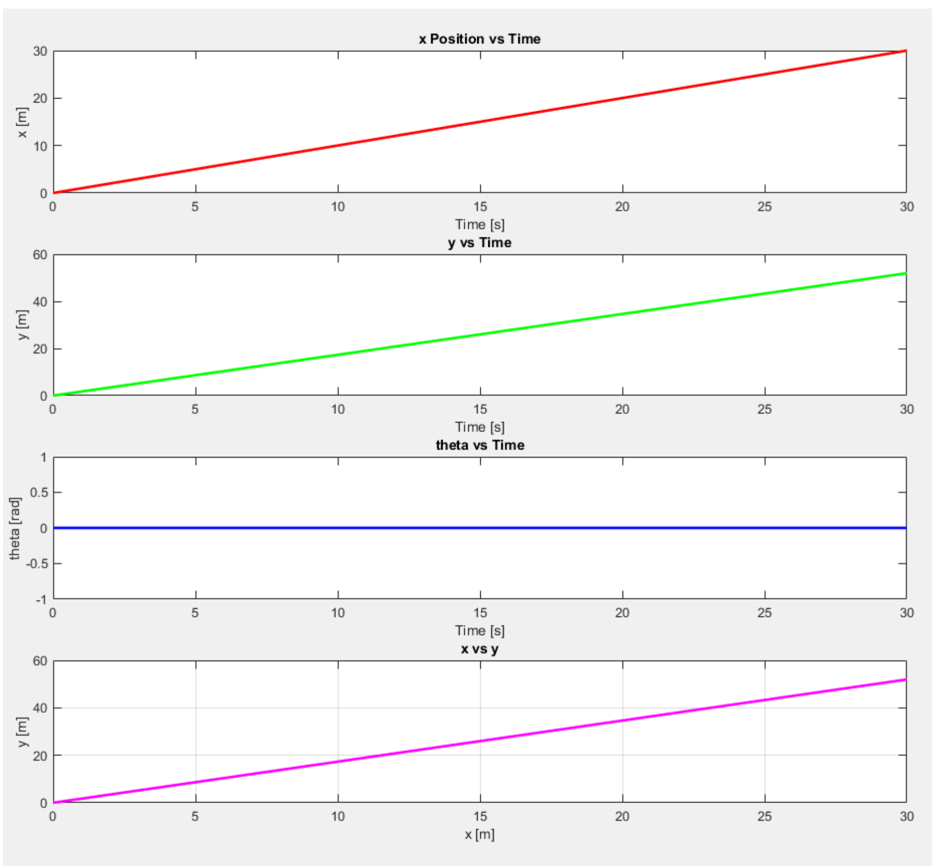
To design u_1, u_2, u_3 for the robot to move in a straight line with a slope of 60 degrees, we need to determine $\dot{x}, \dot{y}, \dot{\theta}$ accordingly.

Based on the slope of $60° = \frac{\pi}{3}$, we know that the relationship between $\dot{x}$ and $\dot{y}$:

$$\dot{y} = \dot{x}\tan\left(\frac{\pi}{3}\right)$$

If we choose $\dot{x} = 1$, then $\dot{y} = \tan\left(\frac{\pi}{3}\right)$. We can set $\dot{\theta} = 0$. If we apply the equation for u_i, we can determine u_1, u_2, u_3 and use these as inputs to the simulation in part a.

The plot below shows the variables against time and the x and y trajectory.



Q4bii2

To design u_1, u_2, u_3 for the robot to move in a 2-meter diameter circle, we need to determine $\dot{x}, \dot{y}, \dot{\theta}$ accordingly.

Since this robot does not need to rotate to travel in a circle, we will set $\dot{\theta} = 0$. Then, based on the equation of a circle of $\sin^2(t) + \cos^2(t) = 1$, then we can set $\dot{x} = \sin(t)$ and $\dot{y} = \cos(t)$ for a circle of diameter of 2m.

The plot below shows the variables against time and the x and y trajectory.