

# MTE544 Assignment 1

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## Q1

To prove  $G^{-1} = \begin{bmatrix} R^T & -R^T q \\ 0 & 1 \end{bmatrix}$ , we know that to represent a point,  $p^B$ , in the reference frame {B} in the reference frame {A}, we need the transformation matrix  $G_A^B = \begin{bmatrix} R_B^A & q^A \\ 0 & 1 \end{bmatrix}$  which consist of the rotation matrix  $R_B^A$  and the vector  $q$  pointing from {A} to {B}.

To do the opposite, represent a point in the reference frame {A} in the reference frame {B}, then we would need the transformation matrix  $G_A^B = \begin{bmatrix} R_A^B & q^B \\ 0 & 1 \end{bmatrix}$ .

Form this, we know from class that

$$R_A^B = (R_B^A)^{-1} = (R_B^A)^T$$

Additionally, since we know  $q^A$  goes from {A} to {B} then simply multiplying it by -1 and applying the rotation matrix  $R_A^B$  to it would result in  $q^B$  which is a vector from {B} to {A}:

$$q^B = -R_A^B q^A = -(R_B^A)^T q^A$$

Therefore

$$G_A^B = \begin{bmatrix} R_A^B & q^B \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (R_B^A)^T & -(R_B^A)^T q^A \\ 0 & 1 \end{bmatrix}$$

If we multiply  $G_A^B$  by  $G_A^B$  we get the identity matrix:

$$G_A^B G_A^B = \begin{bmatrix} R_B^A & q^A \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_A^B & q^B \\ 0 & 1 \end{bmatrix} = I$$

This means

$$G_A^B = (G_A^B)^{-1} = G_A^B$$

And so, in general

$$G^{-1} = \begin{bmatrix} R^T & -R^T q \\ 0 & 1 \end{bmatrix}$$

## Q2

### Q2a

The transformation matrix for each reference frame is as follows:

$$G_e^3 = \begin{bmatrix} R_e^3 & q^e \\ 0 & 1 \end{bmatrix}, R_e^3 = \begin{bmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{bmatrix}, q^e = \begin{bmatrix} l_3 \\ 0 \end{bmatrix}$$

$$G_3^2 = \begin{bmatrix} R_3^2 & q^3 \\ 0 & 1 \end{bmatrix}, R_3^2 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, q^3 = \begin{bmatrix} l_2 \\ 0 \end{bmatrix}$$

$$G_2^1 = \begin{bmatrix} R_2^1 & q^2 \\ 0 & 1 \end{bmatrix}, R_2^1 = \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) \\ \sin(\gamma) & \cos(\gamma) \end{bmatrix}, q^2 = \begin{bmatrix} l_1 \\ 0 \end{bmatrix}$$

$$G_1^s = \begin{bmatrix} R_1^s & q^1 \\ 0 & 1 \end{bmatrix}, R_1^s = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}, q^1 = \begin{bmatrix} x \\ y \end{bmatrix}$$

Then, transformation matrix from {e} to {s} is as follows:

$$G_e^s = G_1^s G_2^1 G_3^2 G_e^3$$

$$= \begin{bmatrix} \cos(\psi + \theta + \gamma + \alpha) & -\sin(\psi + \theta + \gamma + \alpha) & y \cos(\psi + \theta + \gamma) - x \sin(\psi + \theta + \gamma) + l_3 + l_2 \cos(\psi) + l_1 \cos(\psi + \theta) \\ \sin(\psi + \theta + \gamma + \alpha) & \cos(\psi + \theta + \gamma + \alpha) & y \sin(\psi + \theta + \gamma) + x \cos(\psi + \theta + \gamma) + l_2 \sin(\psi) + l_1 \sin(\psi + \theta) \\ 0 & 0 & 1 \end{bmatrix}$$

### Q2b

At the configuration of the robot specified in the question, the coordinates of the points  $p_1^e$  and  $p_2^e$  in spatial frame {s} would be:

$$p_1^s = G_e^s p_1^e = \begin{bmatrix} 2.5731 \\ 3.2071 \\ 1 \end{bmatrix}$$

$$p_2^s = G_e^s p_2^e = \begin{bmatrix} 4.8052 \\ 3.3411 \\ 1 \end{bmatrix}$$

### Q2c

If the robot is located at the base at (0,0), the following equation applies:

$$\dot{p}^s = \dot{G}_2^s \bar{p}^2 = \begin{bmatrix} \hat{\omega} R_2^s & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p^2 \\ 1 \end{bmatrix}$$

To calculate  $R_2^s$ :

$$R_2^s = R_1^s R_2^1$$

$$R_1^s = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

$$R_2^1 = \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) \\ \sin(\gamma) & \cos(\gamma) \end{bmatrix}$$

$$\rightarrow R_2^s = \begin{bmatrix} \cos(\alpha + \gamma) & -\sin(\alpha + \gamma) \\ \sin(\alpha + \gamma) & \cos(\alpha + \gamma) \end{bmatrix}$$

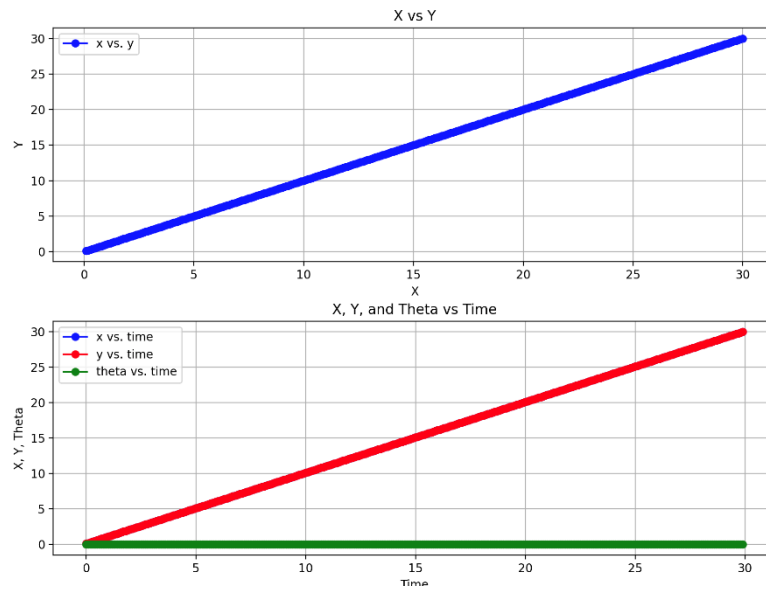
$$\rightarrow \dot{\vec{p}}^S = \begin{bmatrix} \omega \cos(\alpha + \gamma) & -\omega \sin(\alpha + \gamma) & v_x \\ \omega \sin(\alpha + \gamma) & \omega \cos(\alpha + \gamma) & v_y \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} x' \omega \cos(\alpha + \gamma) - y' \omega \sin(\alpha + \gamma) + v_x \\ x' \omega \sin(\alpha + \gamma) + y' \omega \cos(\alpha + \gamma) + v_y \\ 0 \end{bmatrix}$$

Q3

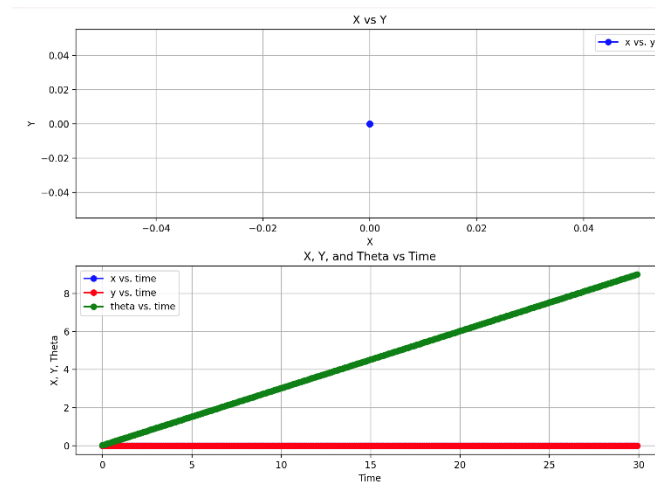
Q3a

The simulation was programmed in Python.

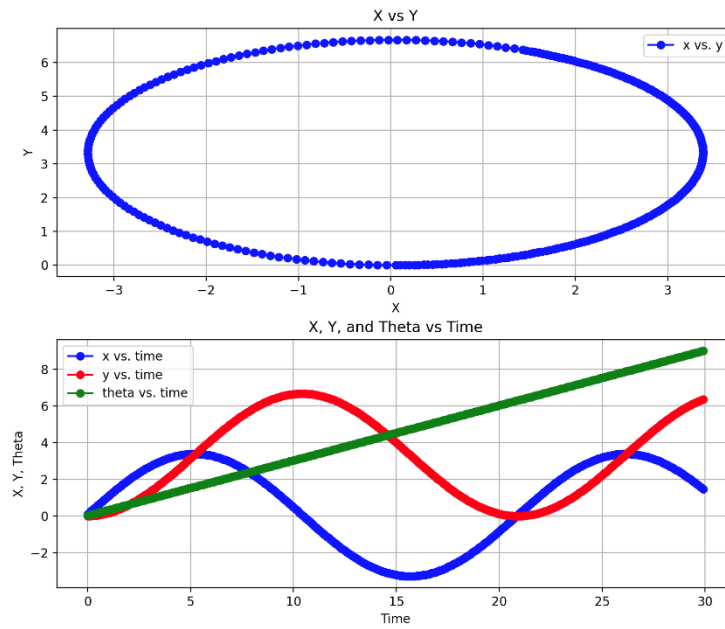
For  $\begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ :



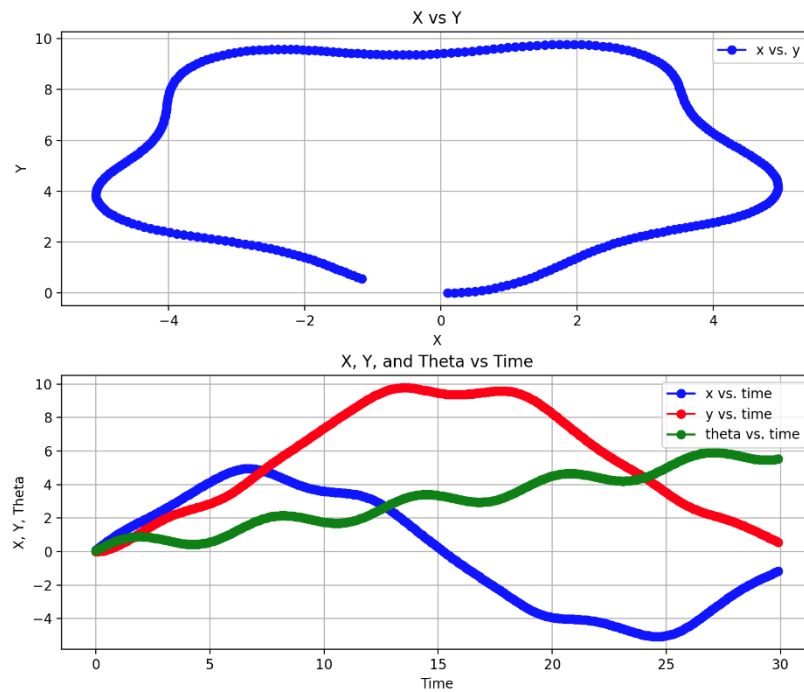
For  $\begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} 0 \\ 0.3 \end{bmatrix}$ :



For  $\begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} 1 \\ 0.3 \end{bmatrix}$ :



For  $\begin{bmatrix} v(t) \\ \omega(t) \end{bmatrix} = \begin{bmatrix} 1 + 0.1 \sin(t) \\ 0.2 + 0.5 \cos(t) \end{bmatrix}$ :



Q3b

If  $T = 0.2m$  and  $r = 0.1m$ , then

$$u_r = \frac{1}{r} \left( v + \frac{T\omega}{2} \right)$$

$$u_l = \frac{1}{r} \left( v - \frac{T\omega}{2} \right)$$

For  $v = 1$  m/s and  $\omega = 0$  m/s:

$$u_r = \frac{1}{0.1} \left( 1 + \frac{0.2(0)}{2} \right) = 10 \text{ rad/s}$$

$$u_l = \frac{1}{0.1} \left( 1 - \frac{0.2(0)}{2} \right) = 10 \text{ rad/s}$$

For  $v = 0$  m/s and  $\omega = 0.3$  m/s:

$$u_r = \frac{1}{0.1} \left( 0 + \frac{0.2(0.3)}{2} \right) = 0.03 \text{ rad/s}$$

$$u_l = \frac{1}{0.1} \left( 0 - \frac{0.2(0.3)}{2} \right) = -0.03 \text{ rad/s}$$

For  $v = 1$  m/s and  $\omega = 0.3$  m/s:

$$u_r = \frac{1}{0.1} \left( 1 + \frac{0.2(0.3)}{2} \right) = 10.3 \text{ rad/s}$$

$$u_l = \frac{1}{0.1} \left( 1 - \frac{0.2(0.3)}{2} \right) = 9.7 \text{ rad/s}$$

If the input control parameters were wheel speeds, the simulation could simply be modified to calculate linear velocity  $v$ , and angular velocity  $\omega$  first by using equations for  $u_r$  and  $u_l$ .

## Q4

### Q4a

To derive the equation of motion for the omnidirectional wheeled robot with three wheels, we can use the general equations derived in class:

$$u_i = \frac{1}{r_i} [1 \quad \tan(\gamma_i)] g(\theta) \dot{q}$$

Where  $g(\theta)$  is defined by:

$$g(\theta) = \begin{bmatrix} \cos(\theta + \beta_i) & \sin(\theta + \beta_i) & x_i \sin(\beta_i) - y_i \cos(\beta_i) \\ -\sin(\theta + \beta_i) & \cos(\theta + \beta_i) & x_i \cos(\beta_i) + y_i \sin(\beta_i) \end{bmatrix}$$

We know  $\gamma_i = 0$ , so  $\tan(\gamma_i) = 0$ :

$$u_i = \frac{1}{r_i} [1 \quad 0] g(\theta) \dot{q}$$

The following constants can be derived from the figure provided:

$$\beta_1 = \frac{\pi}{2}, \beta_2 = \frac{7\pi}{6}, \beta_3 = \frac{11\pi}{6}$$

$$x_1 = l, x_2 = -l \sin\left(\frac{\pi}{6}\right), x_3 = -l \sin\left(\frac{\pi}{6}\right)$$

$$y_1 = 0, y_2 = l \cos\left(\frac{\pi}{6}\right), y_3 = -l \cos\left(\frac{\pi}{6}\right)$$

The following results from the derived constants:

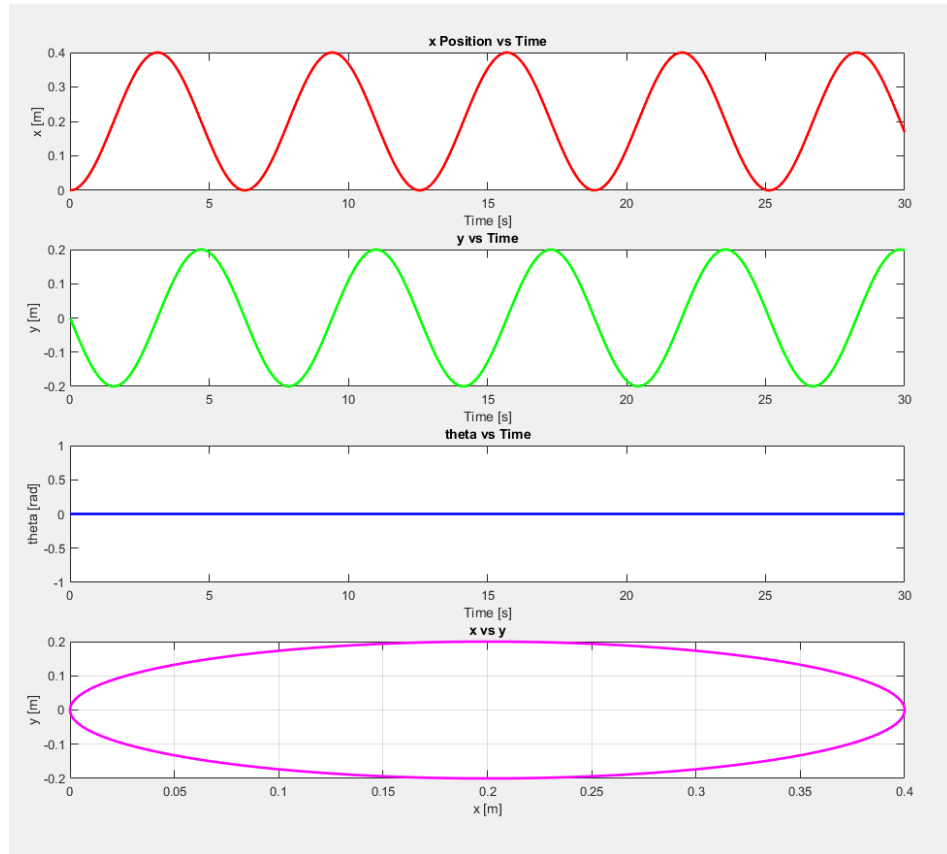
$$u_i = \frac{1}{r} \begin{bmatrix} \cos(\theta + \beta_1) & \sin(\theta + \beta_1) & x_1 \sin(\beta_1) - y_1 \cos(\beta_1) \\ \cos(\theta + \beta_2) & \sin(\theta + \beta_2) & x_2 \sin(\beta_2) - y_2 \cos(\beta_2) \\ \cos(\theta + \beta_3) & \sin(\theta + \beta_3) & x_3 \sin(\beta_3) - y_3 \cos(\beta_3) \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}$$
$$u_i = G(\theta) \dot{q}$$

#### Q4bi

For wheel speeds of  $u_1 = -2$ ,  $u_2 = 1.0$ , and  $u_3 = 1.0$ , we can determine  $\dot{x}, \dot{y}, \dot{\theta}$  by the following equation:

$$\dot{q} = G^{-1}(\theta)u_i$$

The plot below shows the variables against time and the x and y trajectory.





#### Q4bii1

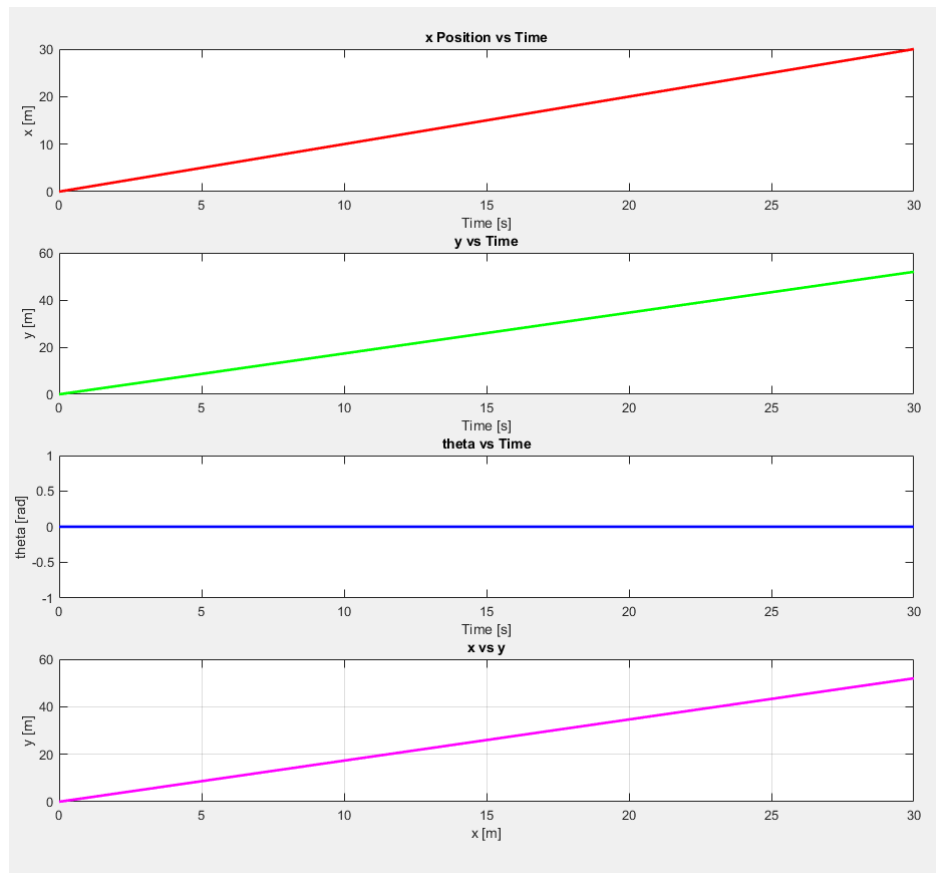
To design  $u_1, u_2, u_3$  for the robot to move in a straight line with a slope of 60 degrees, we need to determine  $\dot{x}, \dot{y}, \dot{\theta}$  accordingly.

Based on the slope of  $60^\circ = \frac{\pi}{3}$ , we know that the relationship between  $\dot{x}$  and  $\dot{y}$ :

$$\dot{y} = \dot{x} \tan\left(\frac{\pi}{3}\right)$$

If we choose  $\dot{x} = 1$ , then  $\dot{y} = \tan\left(\frac{\pi}{3}\right)$ . We can set  $\dot{\theta} = 0$ . If we apply the equation for  $u_i$ , we can determine  $u_1, u_2, u_3$  and use these as inputs to the simulation in part a.

The plot below shows the variables against time and the x and y trajectory.



#### Q4bii2

To design  $u_1, u_2, u_3$  for the robot to move in a 2-meter diameter circle, we need to determine  $\dot{x}, \dot{y}, \dot{\theta}$  accordingly.

Since this robot does not need to rotate to travel in a circle, we will set  $\dot{\theta} = 0$ . Then, based on the equation of a circle of  $\sin^2(t) + \cos^2(t) = 1$ , then we can set  $\dot{x} = \sin(t)$  and  $\dot{y} = \cos(t)$  for a circle of diameter of 2m.

The plot below shows the variables against time and the x and y trajectory.

