1.1 Sets

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The concept of set is basic to all of mathematics and mathematical applications. A **set** is simply a collection of objects. The objects are sometimes referred to as elements or members. If a set is finite and not too large, we can describe it by listing the elements in it. For example, the equation

$$A = \{1, 2, 3, 4\} \tag{1.1.1}$$

describes a set A made up of the four elements 1, 2, 3, and 4. A set is determined by its elements and not by any particular order in which the elements might be listed. Thus the set A might just as well be specified as $A = \{1, 3, 4, 2\}$. The elements making up a set are assumed to be distinct, and although for some reason we may have duplicates in our list, only one occurrence of each element is in the set. For this reason we may also describe the set A defined in (1.1.1) as $A = \{1, 2, 2, 3, 4\}$.

If a set is a large finite set or an infinite set, we can describe it by listing a property necessary for membership. For example, the equation

$$B = \{x \mid x \text{ is a positive, even integer}\} \tag{1.1.2}$$

describes the set B made up of all positive, even integers; that is, B consists of the integers 2, 4, 6, and so on. The vertical bar "|" is read "such that." Equation (1.1.2) would be read "B equals the set of all x such that x is a positive, even integer." Here the property necessary for membership is "is a positive, even integer." Note that the property appears after the vertical bar. The notation in (1.1.2) is called **set-builder notation**.

A set may contain *any* kind of elements whatsoever, and they need *not* be of the same "type." For example,

$$\{4.5, \text{Lady Gaga}, \pi, 14\}$$

is a perfectly fine set. It consists of four elements: the number 4.5, the person Lady Gaga, the number $\pi (= 3.1415\ldots)$, and the number 14.

A set may contain elements that are themselves sets. For example, the set

$$\{3, \{5, 1\}, 12, \{\pi, 4.5, 40, 16\}, \text{Henry Cavill}\}$$

consists of five elements: the number 3, the set $\{5, 1\}$, the number 12, the set $\{\pi, 4.5, 40, 16\}$, and the person Henry Cavill.

Some sets of numbers that occur frequently in mathematics generally, and in discrete mathematics in particular, are shown in Figure 1.1.1. The symbol **Z** comes from the German word, *Zahlen*, for *integer*. Rational numbers are quotients of integers, thus **Q** for *quotient*. The set of real numbers **R** can be depicted as consisting of all points on a straight line extending indefinitely in either direction (see Figure 1.1.2).†

Symbol	*Set*	*Example of Members*
Z	Integers	$-3, 0, 2, 145$
Q	Rational numbers	$-1/3, 0, 24/15$
R	Real numbers	$-3, -1.766, 0, 4/15, \sqrt{2}, 2.666\ldots, \pi$

Figure 1.1.1 Sets of numbers.

†The real numbers can be constructed by starting with a more primitive notion such as "set" or "integer," or they can be obtained by stating properties (axioms) they are assumed to obey. For our purposes, it suffices to think of the real numbers as points on a straight line. The construction of the real numbers and the axioms for the real numbers are beyond the scope of this book.

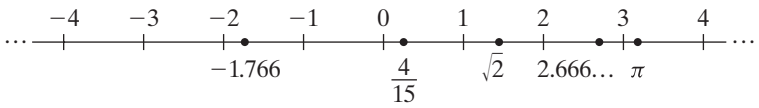

Figure 1.1.2 The real number line.

To denote the negative numbers that belong to one of **Z**, **Q**, or **R**, we use the superscript minus. For example, $\mathbf{Z}^-$ denotes the set of negative integers, namely $-1, -2, -3, \ldots$. Similarly, to denote the positive numbers that belong to one of the three sets, we use the superscript plus. For example, $\mathbf{Q}^+$ denotes the set of positive rational numbers. To denote the nonnegative numbers that belong to one of the three sets, we use the superscript *nonneg*. For example, $\mathbf{Z}^{nonneg}$ denotes the set of nonnegative integers, namely $0, 1, 2, 3, \ldots$.

If X is a finite set, we let $|X| =$ number of elements in X. We call $|X|$ the **cardinality** of X. There is also a notion of cardinality of infinite sets, although we will not discuss it in this book. For example, the cardinality of the integers, Z, is denoted $\aleph_0$, read "aleph null." Aleph is the first letter of the Hebrew alphabet.

Example 1.1.1 For the set A in (1.1.1), we have $|A| = 4$, and the cardinality of A is 4. The cardinality of the set $\{\mathbf{R}, \mathbf{Z}\}$ is 2 since it contains two elements, namely the two sets **R** and **Z**. ◀

Given a description of a set X such as (1.1.1) or (1.1.2) and an element x, we can determine whether or not x belongs to X. If the members of X are listed as in (1.1.1), we simply look to see whether or not x appears in the listing. In a description such as (1.1.2), we check to see whether the element x has the property listed. If x is in the set X, we write $x \in X$, and if x is not in X, we write $x \notin X$. For example, $3 \in \{1, 2, 3, 4\}$, but $3 \notin \{x \mid x \text{ is a positive, even integer}\}$.

The set with no elements is called the **empty** (or **null** or **void**) **set** and is denoted $\varnothing$. Thus $\varnothing = \{\ \}$.

Two sets X and Y are **equal** and we write $X = Y$ if X and Y have the same elements. To put it another way, $X = Y$ if the following two conditions hold:

- For every x, if $x \in X$, then $x \in Y$,

and

- For every x, if $x \in Y$, then $x \in X$.

The first condition ensures that every element of X is an element of Y, and the second condition ensures that every element of Y is an element of X.

Example 1.1.2 If $A = \{1, 3, 2\}$ and $B = \{2, 3, 2, 1\}$, by inspection, A and B have the same elements. Therefore $A = B$. ◀

Example 1.1.3 Show that if $A = \{x \mid x^2 + x - 6 = 0\}$ and $B = \{2, -3\}$, then $A = B$.

SOLUTION According to the criteria in the paragraph immediately preceding Example 1.1.2, we must show that for every x,

$$\text{if } x \in A, \text{ then } x \in B, \tag{1.1.3}$$

and for every x,

$$\text{if } x \in B, \text{ then } x \in A. \tag{1.1.4}$$

To verify condition (1.1.3), suppose that $x \in A$. Then

$$x^2 + x - 6 = 0.$$

Solving for x, we find that $x = 2$ or $x = -3$. In either case, $x \in B$. Therefore, condition (1.1.3) holds.

To verify condition (1.1.4), suppose that $x \in B$. Then $x = 2$ or $x = -3$. If $x = 2$, then

$$x^2 + x - 6 = 2^2 + 2 - 6 = 0.$$

Therefore, $x \in A$. If $x = -3$, then

$$x^2 + x - 6 = (-3)^2 + (-3) - 6 = 0.$$

Again, $x \in A$. Therefore, condition (1.1.4) holds. We conclude that $A = B$. ◀

For a set X to *not* be equal to a set Y (written $X \neq Y$), X and Y must *not* have the same elements: There must be at least one element in X that is not in Y or at least one element in Y that is not in X (or both).

Example 1.1.4 Let $A = \{1, 2, 3\}$ and $B = \{2, 4\}$. Then $A \neq B$ since there is at least one element in A (1 for example) that is not in B. [Another way to see that $A \neq B$ is to note that there is at least one element in B (namely 4) that is not in A.] ◀

Suppose that X and Y are sets. If every element of X is an element of Y, we say that X is a **subset** of Y and write $X \subseteq Y$. In other words, X is a subset of Y if for every x, if $x \in X$, then $x \in Y$.

Example 1.1.5 If $C = \{1, 3\}$ and $A = \{1, 2, 3, 4\}$, by inspection, every element of C is an element of A. Therefore, C is a subset of A and we write $C \subseteq A$. ◀

Example 1.1.6 Let $X = \{x \mid x^2 + x - 2 = 0\}$. Show that $X \subseteq \mathbf{Z}$.

SOLUTION We must show that for every x, if $x \in X$, then $x \in \mathbf{Z}$. If $x \in X$, then $x^2 + x - 2 = 0$. Solving for x, we obtain $x = 1$ or $x = -2$. In either case, $x \in \mathbf{Z}$. Therefore, for every x, if $x \in X$, then $x \in \mathbf{Z}$. We conclude that X is a subset of $\mathbf{Z}$ and we write $X \subseteq \mathbf{Z}$. ◀

Example 1.1.7 The set of integers $\mathbf{Z}$ is a subset of the set of rational numbers $\mathbf{Q}$. If $n \in \mathbf{Z}$, n can be expressed as a quotient of integers, for example, $n = n/1$. Therefore $n \in \mathbf{Q}$ and $\mathbf{Z} \subseteq \mathbf{Q}$. ◀

Example 1.1.8 The set of rational numbers $\mathbf{Q}$ is a subset of the set of real numbers $\mathbf{R}$. If $x \in \mathbf{Q}$, x corresponds to a point on the number line (see Figure 1.1.2) so $x \in \mathbf{R}$. ◀

For X to *not* be a subset of Y, there must be at least one member of X that is not in Y.

Example 1.1.9 Let $X = \{x \mid 3x^2 - x - 2 = 0\}$. Show that X is not a subset of $\mathbf{Z}$.

SOLUTION If $x \in X$, then $3x^2 - x - 2 = 0$. Solving for x, we obtain $x = 1$ or $x = -2/3$. Taking $x = -2/3$, we have $x \in X$ but $x \notin \mathbf{Z}$. Therefore, X is not a subset of $\mathbf{Z}$. ◀

Any set X is a subset of itself, since any element in X is in X. Also, the empty set is a subset of every set. If $\varnothing$ is *not* a subset of some set Y, according to the discussion preceding Example 1.1.9, there would have to be at least one member of $\varnothing$ that is not in Y. But this cannot happen because the empty set, by definition, has no members.

Notice the difference between the terms "subset" and "element of." The set X is a *subset* of the set $Y (X \subseteq Y)$, if every element of X is an element of Y; x is an *element of* $X (x \in X)$, if x is a member of the set X.

Example 1.1.10 Let $X = \{1, 3, 5, 7\}$ and $Y = \{1, 2, 3, 4, 5, 6, 7\}$. Then $X \subseteq Y$ since every element of X is an element of Y. But $X \notin Y$, since the *set* X is not a member of Y. Also, $1 \in X$, but 1 is not a subset of X. Notice the difference between the number 1 and the *set* $\{1\}$. The set $\{1\}$ *is* a subset of X. ◀

If X is a subset of Y and X does not equal Y, we say that X is a **proper subset** of Y and write $X \subset Y$.

Example 1.1.11 Let $C = \{1, 3\}$ and $A = \{1, 2, 3, 4\}$. Then C is a proper subset of A since C is a subset of A but C does not equal A. We write $C \subset A$. ◀

Example 1.1.12 Example 1.1.7 showed that $\mathbf{Z}$ is a subset of $\mathbf{Q}$. In fact, $\mathbf{Z}$ is a proper subset of $\mathbf{Q}$ because, for example, $1/2 \in \mathbf{Q}$, but $1/2 \notin \mathbf{Z}$. ◀

Example 1.1.13 Example 1.1.8 showed that $\mathbf{Q}$ is a subset of $\mathbf{R}$. In fact, $\mathbf{Q}$ is a proper subset of $\mathbf{R}$ because, for example, $\sqrt{2} \in \mathbf{R}$, but $\sqrt{2} \notin \mathbf{Q}$. (In Example 2.2.3, we will show that $\sqrt{2}$ is not the quotient of integers). ◀

The set of all subsets (proper or not) of a set X, denoted $\mathcal{P}(X)$, is called the **power set** of X.

Example 1.1.14 If $A = \{a, b, c\}$, the members of $\mathcal{P}(A)$ are

$$\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}.$$

All but $\{a, b, c\}$ are proper subsets of A. ◀

In Example 1.1.14, $|A| = 3$ and $|\mathcal{P}(A)| = 2^3 = 8$. In Section 2.4 (Theorem 2.4.6), we will give a formal proof that this result holds in general; that is, the power set of a set with n elements has 2^n elements.

Given two sets X and Y, there are various set operations involving X and Y that can produce a new set. The set

$$X \cup Y = \{x \mid x \in X \text{ or } x \in Y\}$$

is called the **union** of X and Y. The union consists of all elements belonging to either X or Y (or both).

The set

$$X \cap Y = \{x \mid x \in X \text{ and } x \in Y\}$$

is called the **intersection** of X and Y. The intersection consists of all elements belonging to both X and Y.

The set

$$X - Y = \{x \mid x \in X \text{ and } x \notin Y\}$$

is called the **difference** (or **relative complement**). The difference $X - Y$ consists of all elements in X that are not in Y.

Example 1.1.15 If $A = \{1, 3, 5\}$ and $B = \{4, 5, 6\}$, then

$$A \cup B = \{1, 3, 4, 5, 6\}$$
$$A \cap B = \{5\}$$
$$A - B = \{1, 3\}$$
$$B - A = \{4, 6\}.$$

Notice that, in general, $A - B \neq B - A$. ◀

Example 1.1.16 Since $\mathbf{Q} \subseteq \mathbf{R}$,

$$\mathbf{R} \cup \mathbf{Q} = \mathbf{R}$$
$$\mathbf{R} \cap \mathbf{Q} = \mathbf{Q}$$
$$\mathbf{Q} - \mathbf{R} = \varnothing.$$

The set $\mathbf{R} - \mathbf{Q}$, called the set of **irrational numbers,** consists of all real numbers that are not rational. ◀

We call a set $\mathcal{S}$, whose elements are sets, a **collection of sets** or a **family of sets**. For example, if

$$\mathcal{S} = \{\{1, 2\}, \{1, 3\}, \{1, 7, 10\}\},$$

then $\mathcal{S}$ is a collection or family of sets. The set $\mathcal{S}$ consists of the sets

$$\{1, 2\}, \{1, 3\}, \{1, 7, 10\}.$$

Sets X and Y are **disjoint** if $X \cap Y = \varnothing$. A collection of sets $\mathcal{S}$ is said to be **pairwise disjoint** if, whenever X and Y are distinct sets in $\mathcal{S}$, X and Y are disjoint.

Example 1.1.17 The sets $\{1, 4, 5\}$ and $\{2, 6\}$ are disjoint. The collection of sets $\mathcal{S} = \{\{1, 4, 5\}, \{2, 6\}, \{3\}, \{7, 8\}\}$ is pairwise disjoint. ◀

Sometimes we are dealing with sets, all of which are subsets of a set U. This set U is called a **universal set** or a **universe.** The set U must be explicitly given or inferred from the context. Given a universal set U and a subset X of U, the set $U - X$ is called the **complement** of X and is written $\overline{X}$.

Example 1.1.18 Let $A = \{1, 3, 5\}$. If U, a universal set, is specified as $U = \{1, 2, 3, 4, 5\}$, then $\overline{A} = \{2, 4\}$. If, on the other hand, a universal set is specified as $U = \{1, 3, 5, 7, 9\}$, then $\overline{A} = \{7, 9\}$. The complement obviously depends on the universe in which we are working. ◀

Example 1.1.19 Let the universal set be $\mathbf{Z}$. Then $\overline{\mathbf{Z}^-}$, the complement of the set of negative integers, is $\mathbf{Z}^{nonneg}$, the set of nonnegative integers. ◀

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Venn diagrams provide pictorial views of sets. In a Venn diagram, a rectangle depicts a universal set (see Figure 1.1.3). Subsets of the universal set are drawn as circles. The inside of a circle represents the members of that set. In Figure 1.1.3 we see two sets A and B within the universal set U. Region 1 represents $\overline{(A \cup B)}$, the elements in neither A nor B. Region 2 represents $A - B$, the elements in A but not in B. Region 3 represents $A \cap B$, the elements in both A and B. Region 4 represents $B - A$, the elements in B but not in A.

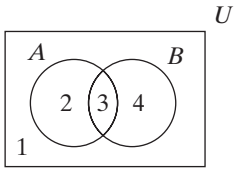

Figure 1.1.3 A Venn diagram.

Example 1.1.20 Particular regions in Venn diagrams are depicted by shading. The set $A \cup B$ is shown in Figure 1.1.4, and Figure 1.1.5 represents the set $A - B$. ◀

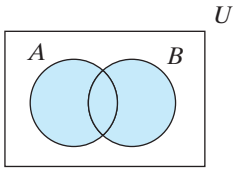

Figure 1.1.4 A Venn diagram of $A \cup B$.

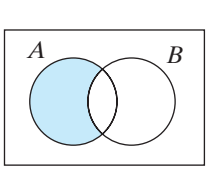

Figure 1.1.5 A Venn diagram of $A - B$.

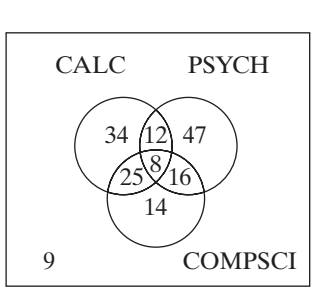

Figure 1.1.6 A Venn diagram of three sets CALC, PSYCH, and COMPSCI. The numbers show how many students belong to the particular region depicted.

To represent three sets, we use three overlapping circles (see Figure 1.1.6).

Example 1.1.21 Among a group of 165 students, 8 are taking calculus, psychology, and computer science; 33 are taking calculus and computer science; 20 are taking calculus and psychology; 24 are taking psychology and computer science; 79 are taking calculus; 83 are taking psychology; and 63 are taking computer science. How many are taking none of the three subjects?

SOLUTION Let CALC, PSYCH, and COMPSCI denote the sets of students taking calculus, psychology, and computer science, respectively. Let U denote the set of all 165 students (see Figure 1.1.6). Since 8 students are taking calculus, psychology, and computer science, we write 8 in the region representing CALC $\cap$ PSYCH $\cap$ COMPSCI. Of the 33 students taking calculus and computer science, 8 are also taking psychology; thus 25 are taking calculus and computer science but not psychology. We write 25 in the region representing CALC $\cap$ $\overline{\text{PSYCH}}$ $\cap$ COMPSCI. Similarly, we write 12 in the region representing CALC $\cap$ PSYCH $\cap$ $\overline{\text{COMPSCI}}$ and 16 in the region representing $\overline{\text{CALC}}$ $\cap$ PSYCH $\cap$ COMPSCI. Of the 79 students taking calculus, 45 have now been accounted for. This leaves 34 students taking only calculus. We write 34 in the region representing CALC $\cap$ $\overline{\text{PSYCH}}$ $\cap$ $\overline{\text{COMPSCI}}$. Similarly, we write 47 in the region representing $\overline{\text{CALC}}$ $\cap$ PSYCH $\cap$ $\overline{\text{COMPSCI}}$ and 14 in the region representing

$\overline{\text{CALC}} \cap \overline{\text{PSYCH}} \cap \text{COMPSCI}$. At this point, 156 students have been accounted for. This leaves 9 students taking none of the three subjects. ◀

Venn diagrams can also be used to visualize certain properties of sets. For example, by sketching both $\overline{(A \cup B)}$ and $\overline{A} \cap \overline{B}$ (see Figure 1.1.7), we see that these sets are equal. A formal proof would show that for every x, if $x \in \overline{(A \cup B)}$, then $x \in \overline{A} \cap \overline{B}$, and if $x \in \overline{A} \cap \overline{B}$, then $x \in \overline{(A \cup B)}$. We state many useful properties of sets as Theorem 1.1.22.

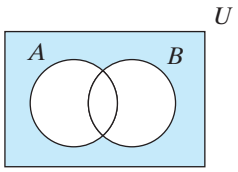

Figure 1.1.7 The shaded region depicts both $\overline{(A \cup B)}$ and $\overline{A} \cap \overline{B}$; thus these sets are equal.

Theorem 1.1.22

Let U be a universal set and let A, B, and C be subsets of U. The following properties hold.

(a) Associative laws:

$$(A \cup B) \cup C = A \cup (B \cup C), \quad (A \cap B) \cap C = A \cap (B \cap C)$$

(b) Commutative laws:

$$A \cup B = B \cup A, \quad A \cap B = B \cap A$$

(c) Distributive laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(d) Identity laws:

$$A \cup \varnothing = A, \quad A \cap U = A$$

(e) Complement laws:

$$A \cup \overline{A} = U, \quad A \cap \overline{A} = \varnothing$$

(f) Idempotent laws:

$$A \cup A = A, \quad A \cap A = A$$

(g) Bound laws:

$$A \cup U = U, \quad A \cap \varnothing = \varnothing$$

(h) Absorption laws:

$$A \cup (A \cap B) = A, \quad A \cap (A \cup B) = A$$

(i) Involution law:

$$\overline{\overline{A}} = A^{\dagger}$$

(j) 0/1 laws:

$$\overline{\varnothing} = U, \quad \overline{U} = \varnothing$$

(k) De Morgan's laws for sets:

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}, \quad \overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

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Proof The proofs are left as exercises (Exercises 46–56, Section 2.1) to be done after more discussion of logic and proof techniques. ◀

We define the union of a collection of sets $\mathcal{S}$ to be those elements x belonging to at least one set X in $\mathcal{S}$. Formally,

$$\cup \mathcal{S} = \{x \mid x \in X \text{ for some } X \in \mathcal{S}\}.$$

[†] $\overline{\overline{A}}$ denotes the complement of the complement of A, that is, $\overline{\overline{A}} = \overline{(\overline{A})}$.

Similarly, we define the intersection of a collection of sets $\mathcal{S}$ to be those elements x belonging to every set X in $\mathcal{S}$. Formally,

$$\cap\mathcal{S} = \{x \mid x \in X \text{ for all } X \in \mathcal{S}\}.$$

Example 1.1.23 Let $\mathcal{S} = \{\{1, 2\}, \{1, 3\}, \{1, 7, 10\}\}$. Then $\cup S = \{1, 2, 3, 7, 10\}$ since each of the elements 1, 2, 3, 7, 10 belongs to at least one set in $\mathcal{S}$, and no other element belongs to any of the sets in $\mathcal{S}$. Also $\cap\mathcal{S} = \{1\}$ since only the element 1 belong to every set in $\mathcal{S}$. ◀

If

$$\mathcal{S} = \{A_1, A_2, \ldots, A_n\},$$

we write

$$\bigcup\mathcal{S} = \bigcup_{i=1}^{n} A_i, \qquad \bigcap\mathcal{S} = \bigcap_{i=1}^{n} A_i,$$

and if

$$\mathcal{S} = \{A_1, A_2, \ldots\},$$

we write

$$\bigcup\mathcal{S} = \bigcup_{i=1}^{\infty} A_i, \qquad \bigcap\mathcal{S} = \bigcap_{i=1}^{\infty} A_i.$$

Example 1.1.24 For $i \geq 1$, define $A_i = \{i, i + 1, \ldots\}$ and $\mathcal{S} = \{A_1, A_2, \ldots\}$. As examples, $A_1 = \{1, 2, 3, \ldots\}$ and $A_2 = \{2, 3, 4, \ldots\}$. Then

$$\bigcup\mathcal{S} = \bigcup_{i=1}^{\infty} A_i = \{1, 2, \ldots\}, \qquad \bigcap\mathcal{S} = \bigcap_{i=1}^{\infty} A_i = \varnothing.$$ ◀

A partition of a set X divides X into nonoverlapping subsets. More formally, a collection $\mathcal{S}$ of nonempty subsets of X is said to be a **partition** of the set X if every element in X belongs to exactly one member of $\mathcal{S}$. Notice that if $\mathcal{S}$ is a partition of X, $\mathcal{S}$ is pairwise disjoint and $\cup\mathcal{S} = X$.

Example 1.1.25 Since each element of $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ is in exactly one member of $\mathcal{S} = \{\{1, 4, 5\}, \{2, 6\}, \{3\}, \{7, 8\}\}$, $\mathcal{S}$ is a partition of X. ◀

At the beginning of this section, we pointed out that a set is an unordered collection of elements; that is, a set is determined by its elements and not by any particular order in which the elements are listed. Sometimes, however, we do want to take order into account. An **ordered pair** of elements, written (a, b), is considered distinct from the ordered pair (b, a), unless, of course, $a = b$. To put it another way, $(a, b) = (c, d)$ precisely when $a = c$ and $b = d$. If X and Y are sets, we let $X \times Y$ denote the set of all ordered pairs (x, y) where $x \in X$ and $y \in Y$. We call $X \times Y$ the **Cartesian product** of X and Y.

Example 1.1.26 If $X = \{1, 2, 3\}$ and $Y = \{a, b\}$, then

$$X \times Y = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

$$Y \times X = \{(a, 1), (b, 1), (a, 2), (b, 2), (a, 3), (b, 3)\}$$

$$X \times X = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

$$Y \times Y = \{(a, a), (a, b), (b, a), (b, b)\}.$$ ◀

Example 1.1.26 shows that, in general, $X \times Y \neq Y \times X$.

Notice that in Example 1.1.26, $|X \times Y| = |X| \cdot |Y|$ (both are equal to 6). The reason is that there are 3 ways to choose an element of X for the first member of the ordered pair, there are 2 ways to choose an element of Y for the second member of the ordered pair, and $3 \cdot 2 = 6$ (see Figure 1.1.8). The preceding argument holds for arbitrary finite sets X and Y; it is always true that $|X \times Y| = |X| \cdot |Y|$.

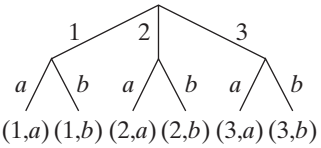


Figure 1.1.8 $|X \times Y| = |X| \cdot |Y|$, where $X = \{1, 2, 3\}$ and $Y = \{a, b\}$. There are 3 ways to choose an element of X for the first member of the ordered pair (shown at the top of the diagram) and, for each of these choices, there are 2 ways to choose an element of Y for the second member of the ordered pair (shown at the bottom of the diagram). Since there are 3 groups of 2, there are $3 \cdot 2 = 6$ elements in $X \times Y$ (labeled at the bottom of the figure).

Example 1.1.27 A restaurant serves four appetizers,

$$r = \text{ribs}, \quad n = \text{nachos}, \quad s = \text{shrimp}, \quad f = \text{fried cheese},$$

and three entrees,

$$c = \text{chicken}, \quad b = \text{beef}, \quad t = \text{trout}.$$

If we let $A = \{r, n, s, f\}$ and $E = \{c, b, t\}$, the Cartesian product $A \times E$ lists the 12 possible dinners consisting of one appetizer and one entree. ◀

Ordered lists need not be restricted to two elements. An ***n*-tuple,** written $(a_1, a_2, \ldots, a_n)$, takes order into account; that is,

$$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$$

precisely when

$$a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n.$$

The Cartesian product of sets $X_1, X_2, \ldots, X_n$ is defined to be the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ where $x_i \in X_i$ for $i = 1, \ldots, n$; it is denoted $X_1 \times X_2 \times \cdots \times X_n$.

Example 1.1.28 If $X = \{1, 2\}$, $Y = \{a, b\}$, and $Z = \{\alpha, \beta\}$, then

$$X \times Y \times Z = \{(1, a, \alpha), (1, a, \beta), (1, b, \alpha), (1, b, \beta), (2, a, \alpha), (2, a, \beta), (2, b, \alpha), (2, b, \beta)\}.$$ ◀

Notice that in Example 1.1.28, $|X \times Y \times Z| = |X| \cdot |Y| \cdot |Z|$. In general,

$$|X_1 \times X_2 \times \cdots \times X_n| = |X_1| \cdot |X_2| \cdots |X_n|.$$

We leave the proof of this last statement as an exercise (see Exercise 27, Section 2.4).

Example 1.1.29 If A is a set of appetizers, E is a set of entrees, and D is a set of desserts, the Cartesian product $A \times E \times D$ lists all possible dinners consisting of one appetizer, one entree, and one dessert. ◀

1.1 Problem-Solving Tips

- To verify that two sets A and B are equal, written $A = B$, show that for every x, if $x \in A$, then $x \in B$, and if $x \in B$, then $x \in A$.
- To verify that two sets A and B are *not* equal, written $A \neq B$, find at least one element that is in A but not in B, or find at least one element that is in B but not in A. One or the other conditions suffices; you need not (and may not be able to) show both conditions.
- To verify that A is a subset of B, written $A \subseteq B$, show that for every x, if $x \in A$, then $x \in B$. Notice that if A is a subset of B, it is possible that $A = B$.
- To verify that A is *not* a subset of B, find at least one element that is in A but not in B.
- To verify that A is a proper subset of B, written $A \subset B$, verify that A is a subset of B as described previously, and that $A \neq B$, that is, that there is at least one element that is in B but not in A.
- To visualize relationships among sets, use a Venn diagram. A Venn diagram can suggest whether a statement about sets is true or false.
- A set of elements is determined by its members; order is irrelevant. On the other hand, ordered pairs and n-tuples take order into account.

1.1 Review Exercises

†**1.** What is a set?

2. What is set notation?

3. Describe the sets $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{Z}^+$, $\mathbf{Q}^+$, $\mathbf{R}^+$, $\mathbf{Z}^-$, $\mathbf{Q}^-$, $\mathbf{R}^-$, $\mathbf{Z}^{nonneg}$, $\mathbf{Q}^{nonneg}$, and $\mathbf{R}^{nonneg}$, and give two examples of members of each set.

4. If X is a finite set, what is $|X|$?

5. How do we denote *x is an element of the set X*?

6. How do we denote *x is not an element of the set X*?

7. How do we denote the empty set?

8. Define *set X is equal to set Y*. How do we denote *X is equal to Y*?

9. Explain a method of verifying that sets X and Y are equal.

10. Explain a method of verifying that sets X and Y are *not* equal.

11. Define *X is a subset of Y*. How do we denote *X is a subset of Y*?

12. Explain a method of verifying that X is a subset of Y.

13. Explain a method of verifying that X is *not* a subset of Y.

14. Define *X is a proper subset of Y*. How do we denote *X is a proper subset of Y*?

15. Explain a method of verifying that X is a proper subset of Y.

16. What is the power set of X? How is it denoted?

17. Define *X union Y*. How is the union of X and Y denoted?

18. If $\mathcal{S}$ is a family of sets, how do we define the union of $\mathcal{S}$? How is the union denoted?

19. Define *X intersect Y*. How is the intersection of X and Y denoted?

20. If $\mathcal{S}$ is a family of sets, how do we define the intersection of $\mathcal{S}$? How is the intersection denoted?

21. Define *X and Y are disjoint sets*.

22. What is a pairwise disjoint family of sets?

23. Define the *difference* of sets X and Y. How is the difference denoted?

24. What is a universal set?

25. What is the complement of the set X? How is it denoted?

26. What is a Venn diagram?

27. Draw a Venn diagram of three sets and identify the set represented by each region.

†Exercise numbers in color indicate that a hint or solution appears at the back of the book in the section following the References.

28. State the associative laws for sets.

29. State the commutative laws for sets.

30. State the distributive laws for sets.

31. State the identity laws for sets.

32. State the complement laws for sets.

33. State the idempotent laws for sets.

34. State the bound laws for sets.

35. State the absorption laws for sets.

36. State the involution law for sets.

37. State the 0/1 laws for sets.

38. State De Morgan's laws for sets.

39. What is a partition of a set X?

40. Define the *Cartesian product* of sets X and Y. How is this Cartesian product denoted?

41. Define the *Cartesian product* of the sets $X_1, X_2, \ldots, X_n$. How is this Cartesian product denoted?

1.1 Exercises

In Exercises 1–16, let the universe be the set $U = \{1, 2, 3, \ldots, 10\}$. Let $A = \{1, 4, 7, 10\}$, $B = \{1, 2, 3, 4, 5\}$, and $C = \{2, 4, 6, 8\}$. List the elements of each set.

1. $A \cup B$

2. $B \cap C$

3. $A - B$

4. $B - A$

5. $\overline{A}$

6. $U - C$

7. $\overline{U}$

8. $A \cup \varnothing$

9. $B \cap \varnothing$

10. $A \cup U$

11. $B \cap U$

12. $A \cap (B \cup C)$

13. $\overline{B} \cap (C - A)$

14. $(A \cap B) - C$

15. $\overline{A \cap B} \cup C$

16. $(A \cup B) - (C - B)$

In Exercises 17–27, let the universe be the set $\mathbf{Z}^+$. Let $X = \{1, 2, 3, 4, 5\}$ and let Y be the set of positive, even integers. In set-builder notation, $Y = \{2n \mid n \in \mathbf{Z}^+\}$. In Exercises 18–27, give a mathematical notation for the set by listing the elements if the set is finite, by using set-builder notation if the set is infinite, or by using a predefined set such as $\varnothing$.

17. Describe $\overline{Y}$ in words.

18. $\overline{X}$

19. $\overline{Y}$

20. $X \cap Y$

21. $X \cup Y$

22. $\overline{X} \cap Y$

23. $\overline{X} \cup Y$

24. $X \cap \overline{Y}$

25. $X \cup \overline{Y}$

26. $\overline{X} \cap \overline{Y}$

27. $\overline{X} \cup \overline{Y}$

28. What is the cardinality of $\varnothing$?

29. What is the cardinality of $\{\varnothing\}$?

30. What is the cardinality of $\{a, b, a, c\}$?

31. What is the cardinality of $\{\{a\}, \{a, b\}, \{a, c\}, a, b\}$?

In Exercises 32–35, show, as in Examples 1.1.2 and 1.1.3, that $A = B$.

32. $A = \{3, 2, 1\},\ B = \{1, 2, 3\}$

33. $C = \{1, 2, 3\},\ D = \{2, 3, 4\},\ A = \{2, 3\},\ B = C \cap D$

34. $A = \{1, 2, 3\},\ B = \{n \mid n \in \mathbf{Z}^+ \text{ and } n^2 < 10\}$

35. $A = \{x \mid x^2 - 4x + 4 = 1\},\ B = \{1, 3\}$

In Exercises 36–39, show, as in Example 1.1.4, that $A \neq B$.

36. $A = \{1, 2, 3\},\ B = \varnothing$

37. $A = \{1, 2\},\ B = \{x \mid x^3 - 2x^2 - x + 2 = 0\}$

38. $A = \{1, 3, 5\},\ B = \{n \mid n \in \mathbf{Z}^+ \text{ and } n^2 - 1 \leq n\}$

39. $B = \{1, 2, 3, 4\},\ C = \{2, 4, 6, 8\},\ A = B \cap C$

In Exercises 40–43, determine whether each pair of sets is equal.

40. $\{1, 2, 2, 3\}, \{1, 2, 3\}$

41. $\{1, 1, 3\}, \{3, 3, 1\}$

42. $\{x \mid x^2 + x = 2\}, \{1, -1\}$

43. $\{x \mid x \in \mathbf{R} \text{ and } 0 < x \leq 2\}, \{1, 2\}$

In Exercises 44–47, show, as in Examples 1.1.5 and 1.1.6, that $A \subseteq B$.

44. $A = \{1, 2\},\ B = \{3, 2, 1\}$

45. $A = \{1, 2\},\ B = \{x \mid x^3 - 6x^2 + 11x = 6\}$

46. $A = \{1\} \times \{1, 2\},\ B = \{1\} \times \{1, 2, 3\}$

47. $A = \{2n \mid n \in \mathbf{Z}^+\},\ B = \{n \mid n \in \mathbf{Z}^+\}$

In Exercises 48–51, show, as in Example 1.1.9, that A is not a subset of B.

48. $A = \{1, 2, 3\},\ B = \{1, 2\}$

49. $A = \{x \mid x^3 - 2x^2 - x + 2 = 0\},\ B = \{1, 2\}$

50. $A = \{1, 2, 3, 4\},\ C = \{5, 6, 7, 8\},\ B = \{n \mid n \in A \text{ and } n + m = 8 \text{ for some } m \in C\}$

51. $A = \{1, 2, 3\},\ B = \varnothing$

In Exercises 52–59, draw a Venn diagram and shade the given set.

52. $A \cap \overline{B}$

53. $\overline{A} - B$

54. $B \cup (B - A)$

55. $(A \cup B) - B$

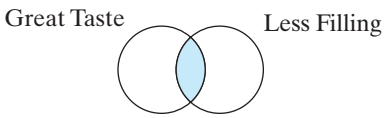
56. $B \cap \overline{(C \cup A)}$

57. $(\overline{A} \cup B) \cap (\overline{C} - A)$

58. $((C \cap A) - \overline{(B - A)}) \cap C$

59. $(B - \overline{C}) \cup ((B - \overline{A}) \cap (C \cup B))$

60. A television commercial for a popular beverage showed the following Venn diagram



What does the shaded area represent?

Exercises 61–65 refer to a group of 191 students, of which 10 are taking French, business, and music; 36 are taking French and business; 20 are taking French and music; 18 are taking business and music; 65 are taking French; 76 are taking business; and 63 are taking music.

61. How many are taking French and music but not business?

62. How many are taking business and neither French nor music?

63. How many are taking French or business (or both)?

64. How many are taking music or French (or both) but not business?

65. How many are taking none of the three subjects?

66. A television poll of 151 persons found that 68 watched "Law and Disorder"; 61 watched "25"; 52 watched "The Tenors"; 16 watched both "Law and Disorder" and "25"; 25 watched both "Law and Disorder" and "The Tenors"; 19 watched both "25" and "The Tenors"; and 26 watched none of these shows. How many persons watched all three shows?

67. In a group of students, each student is taking a mathematics course or a computer science course or both. One-fifth of those taking a mathematics course are also taking a computer science course, and one-eighth of those taking a computer science course are also taking a mathematics course. Are more than one-third of the students taking a mathematics course?

In Exercises 68–71, let $X = \{1, 2\}$ *and* $Y = \{a, b, c\}$. *List the elements in each set.*

68. $X \times Y$

69. $Y \times X$

70. $X \times X$

71. $Y \times Y$

In Exercises 72–75, let $X = \{1, 2\}$, $Y = \{a\}$, *and* $Z = \{\alpha, \beta\}$. *List the elements of each set.*

72. $X \times Y \times Z$

73. $X \times Y \times Y$

74. $X \times X \times X$

75. $Y \times X \times Y \times Z$

In Exercises 76–82, give a geometric description of each set in words. Consider the elements of the sets to be coordinates. For example, $\mathbf{R} \times \mathbf{Z}$ *is the set* $\{(x, n) \mid x \in \mathbf{R}$ *and* $n \in \mathbf{Z}\}$. *Interpreting the ordered pairs* (x, n) *as coordinates in the plane, the graph of all such ordered pairs is the set of all parallel horizontal lines spaced one unit apart, one of which passes through* (0,0).

76. $\mathbf{R} \times \mathbf{R}$

77. $\mathbf{Z} \times \mathbf{R}$

78. $\mathbf{R} \times \mathbf{Z}^{nonneg}$

79. $\mathbf{Z} \times \mathbf{Z}$

80. $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$

81. $\mathbf{R} \times \mathbf{R} \times \mathbf{Z}$

82. $\mathbf{R} \times \mathbf{Z} \times \mathbf{Z}$

In Exercises 83–86, list all partitions of the set.

83. $\{1\}$

84. $\{1, 2\}$

85. $\{a, b, c\}$

86. $\{a, b, c, d\}$

In Exercises 87–92, answer true or false.

87. $\{x\} \subseteq \{x\}$

88. $\{x\} \in \{x\}$

89. $\{x\} \in \{x, \{x\}\}$

90. $\{x\} \subseteq \{x, \{x\}\}$

91. $\{2\} \subseteq \mathcal{P}(\{1, 2\})$

92. $\{2\} \in \mathcal{P}(\{1, 2\})$

93. List the members of $\mathcal{P}(\{a, b\})$. Which are proper subsets of $\{a, b\}$?

94. List the members of $\mathcal{P}(\{a, b, c, d\})$. Which are proper subsets of $\{a, b, c, d\}$?

95. If X has 10 members, how many members does $\mathcal{P}(X)$ have? How many proper subsets does X have?

96. If X has n members, how many proper subsets does X have?

In Exercises 97–100, what relation must hold between sets A and B in order for the given condition to be true?

97. $A \cap B = A$

98. $A \cup B = A$

99. $\overline{A} \cap U = \varnothing$

100. $\overline{A \cap B} = \overline{B}$

The symmetric difference *of two sets A and B is the set*

$$A \triangle B = (A \cup B) - (A \cap B).$$

101. If $A = \{1, 2, 3\}$ and $B = \{2, 3, 4, 5\}$, find $A \triangle B$.

102. Describe the symmetric difference of sets A and B in words.

103. Given a universe U, describe $A \triangle A$, $A \triangle \overline{A}$, $U \triangle A$, and $\varnothing \triangle A$.

104. Let C be a circle and let $\mathcal{D}$ be the set of all diameters of C. What is $\cap \mathcal{D}$? (Here, by "diameter" we mean a line segment through the center of the circle with its endpoints on the circumference of the circle.)

†★**105.** Let P denote the set of integers greater than 1. For $i \geq 2$, define

$$X_i = \{ik \mid k \in P\}.$$

Describe $P - \bigcup_{i=2}^{\infty} X_i$.

†A starred exercise indicates a problem of above-average difficulty.

Applying (2.3.1) to expressions 4 and 7, we derive

8. d.

Now 5 and 8 combine to give a contradiction, and the proof is complete. ◀

It can be shown that resolution is *correct* and *refutation complete*. "Resolution is correct" means that if resolution derives a contradiction from a set of clauses, the clauses are inconsistent (i.e., the clauses are not all true). "Resolution is refutation complete" means that resolution will be able to derive a contradiction from a set of inconsistent clauses. Thus, if a conclusion follows from a set of hypotheses, resolution will be able to derive a contradiction from the hypotheses and the negation of the conclusion. Unfortunately, resolution does not tell us which clauses to combine in order to deduce the contradiction. A key challenge in automating a reasoning system is to help guide the search for clauses to combine. References on resolution and automated reasoning are [Gallier; Genesereth; and Wos].

2.3 Problem-Solving Tips

To construct a resolution proof, first replace any of the hypotheses or conclusion that is not a clause with one or more clauses. Then replace pairs of hypotheses of the form $p \vee q$ and $\neg p \vee r$ with $q \vee r$ until deriving the conclusion. Remember that resolution can be combined with proof by contradiction.

2.3 Review Exercises

1. What rule of logic does proof by resolution use?
2. What is a clause?
3. Explain how a proof by resolution proceeds.

2.3 Exercises

1. Write a truth table that proves (2.3.1).

Use resolution to derive each conclusion in Exercises 2–6. Hint: In Exercises 5 and 6, replace $\rightarrow$ and $\leftrightarrow$ with logically equivalent expressions that use or and and.

2. $\neg p \vee q \vee r$
   $\neg q$
   $\neg r$
   $\therefore \neg p$

3. $\neg p \vee r$
   $\neg r \vee q$
   p
   $\therefore q$

4. $\neg p \vee t$
   $\neg q \vee s$
   $\neg r \vee st$
   $p \vee q \vee r \vee u$
   $\therefore s \vee t \vee u$

5. $p \rightarrow q$
   $p \vee q$
   $\therefore q$

6. $p \leftrightarrow r$
   r
   $\therefore p$

7. Use resolution and proof by contradiction to re-prove Exercises 2–6.
8. Use resolution and proof by contradiction to re-prove Example 2.3.6.

2.4 Mathematical Induction

Go Online
For more on mathematical induction, see goo.gl/gHgyey

Suppose that a sequence of blocks numbered 1, 2, . . . sits on an (infinitely) long table (see Figure 2.4.1) and that some blocks are marked with an "X." (All of the blocks visible in Figure 2.4.1 are marked.) Suppose that

The first block is marked. **(2.4.1)**

For all n, if block n is marked, then block $n + 1$ is also marked. **(2.4.2)**

We claim that (2.4.1) and (2.4.2) imply that every block is marked.

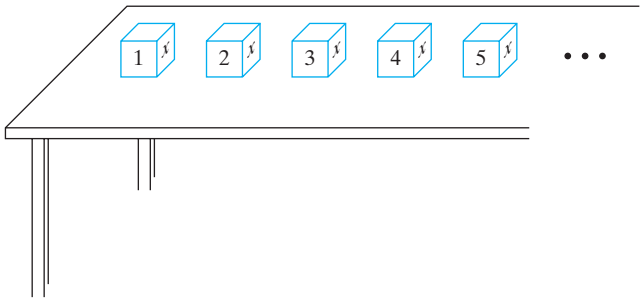


Figure 2.4.1 Numbered blocks on a table.

We examine the blocks one by one. Statement (2.4.1) explicitly states that block 1 is marked. Consider block 2. Since block 1 is marked, by (2.4.2) (taking $n = 1$), block 2 is also marked. Consider block 3. Since block 2 is marked, by (2.4.2) (taking $n = 2$), block 3 is also marked. Continuing in this way, we can show that every block is marked. For example, suppose that we have verified that blocks 1–5 are marked, as shown in Figure 2.4.1. To show that block 6, which is not shown in Figure 2.4.1, is marked, we note that since block 5 is marked, by (2.4.2) (taking $n = 5$), block 6 is also marked.

The preceding example illustrates the **Principle of Mathematical Induction.** To show how mathematical induction can be used in a more profound way, let S_n denote the sum of the first n positive integers:

$$S_n = 1 + 2 + \cdots + n. \tag{2.4.3}$$

Suppose that someone claims that

$$S_n = \frac{n(n+1)}{2} \qquad \text{for all } n \geq 1. \tag{2.4.4}$$

A sequence of statements is really being made, namely,

$$S_1 = \frac{1(2)}{2} = 1, \quad S_2 = \frac{2(3)}{2} = 3, \quad S_3 = \frac{3(4)}{2} = 6, \ldots.$$

$$\begin{aligned}
S_1 &= \frac{1(2)}{2} && \times \\
S_2 &= \frac{2(3)}{2} && \times \\
&\vdots \\
S_{n-1} &= \frac{(n-1)n}{2} && \times \\
S_n &= \frac{n(n+1)}{2} && \times \\
S_{n+1} &= \frac{(n+1)(n+2)}{2} && ? \\
&\vdots
\end{aligned}$$

Figure 2.4.2 A sequence of statements. True statements are marked with $\times$.

Suppose that each true equation has an "$\times$" placed beside it (see Figure 2.4.2). Since the first equation is true, it is marked. Now suppose we can show that for all n, if equation n is marked, then equation $n + 1$ is also marked. Then, as in the example involving the blocks, all of the equations are marked; that is, all the equations are true and the formula (2.4.4) is verified.

We must show that for all n, if equation n is true, then equation $n + 1$ is also true. Equation n is

$$S_n = \frac{n(n+1)}{2}. \tag{2.4.5}$$

Assuming that this equation is true, we must show that equation $n + 1$

$$S_{n+1} = \frac{(n+1)(n+2)}{2}$$

is true. According to definition (2.4.3),

$$S_{n+1} = 1 + 2 + \cdots + n + (n+1).$$

We note that S_n is contained within S_{n+1}, in the sense that

$$S_{n+1} = 1 + 2 + \cdots + n + (n+1) = S_n + (n+1). \tag{2.4.6}$$

Because of (2.4.5) and (2.4.6), we have

$$S_{n+1} = S_n + (n+1) = \frac{n(n+1)}{2} + (n+1).$$

Since

$$\begin{aligned}\frac{n(n+1)}{2} + (n+1) &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2},\end{aligned}$$

we have

$$S_{n+1} = \frac{(n+1)(n+2)}{2}.$$

Therefore, assuming that equation n is true, we have proved that equation $n+1$ is true. We conclude that all of the equations are true.

Our proof using mathematical induction consisted of two steps. First, we verified that the statement corresponding to $n = 1$ was true. Second, we *assumed* that statement n was true and then *proved* that statement $n+1$ was also true. In proving statement $n+1$, we were permitted to make use of statement n; indeed, the trick in constructing a proof using mathematical induction is to relate statement n to statement $n+1$.

We next formally state the Principle of Mathematical Induction.

Principle of Mathematical Induction

Suppose that we have a propositional function $S(n)$ whose domain of discourse is the set of positive integers. Suppose that

$S(1)$ is true; **(2.4.7)**

for all $n \geq 1$, if $S(n)$ is true, then $S(n+1)$ is true. **(2.4.8)**

Then $S(n)$ is true for every positive integer n.

Condition (2.4.7) is sometimes called the **Basis Step** and condition (2.4.8) is sometimes called the **Inductive Step.** Hereafter, "induction" will mean "mathematical induction."

After defining n factorial, we illustrate the Principle of Mathematical Induction with another example.

Definition 2.4.1 ▶ *n factorial* is defined as

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n(n-1)(n-2)\cdots 2 \cdot 1 & \text{if } n \geq 1. \end{cases}$$

That is, if $n \geq 1$, $n!$ is equal to the product of all the integers between 1 and n inclusive. As a special case, 0! is defined to be 1. ◀

Example 2.4.2

$$0! = 1! = 1, \quad 3! = 3 \cdot 2 \cdot 1 = 6, \quad 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$ ◀

Example 2.4.3 Use induction to show that

$$n! \geq 2^{n-1} \qquad \text{for all } n \geq 1. \qquad \textbf{(2.4.9)}$$

SOLUTION

Basis Step ($n = 1$)

[Condition (2.4.7)] We must show that (2.4.9) is true if $n = 1$. This is easily accomplished, since $1! = 1 \geq 1 = 2^{1-1}$.

Inductive Step

[Condition (2.4.8)] We assume that the inequality is true for $n \geq 1$; that is, we assume that

$$n! \geq 2^{n-1} \tag{2.4.10}$$

is true. We must then prove that the inequality is true for $n + 1$; that is, we must prove that

$$(n+1)! \geq 2^n \tag{2.4.11}$$

is true. We can relate (2.4.10) and (2.4.11) by observing that $(n + 1)! = (n + 1)(n!)$. Now

$$\begin{aligned}(n+1)! &= (n+1)(n!) \\ &\geq (n+1)2^{n-1} && \text{by (2.4.10)} \\ &\geq 2 \cdot 2^{n-1} && \text{since } n+1 \geq 2 \\ &= 2^n.\end{aligned}$$

Therefore, (2.4.11) is true. We have completed the Inductive Step.

Since the Basis Step and the Inductive Step have been verified, the Principle of Mathematical Induction tells us that (2.4.9) is true for every positive integer n. ◀

If we want to verify that the statements $S(n_0), S(n_0 + 1), \ldots$, where $n_0 \neq 1$, are true, we must change the Basis Step to $S(n_0)$ *is true.* In words, the Basis Step is to prove that the propositional function $S(n)$ is true for the smallest value n_0 in the domain of discourse.

The Inductive Step then becomes

for all $n \geq n_0$, if $S(n)$ is true, then $S(n + 1)$ is true.

Example 2.4.4 **Geometric Sum** Use induction to show that if $r \neq 1$,

$$a + ar^1 + ar^2 + \cdots + ar^n = \frac{a(r^{n+1} - 1)}{r - 1} \tag{2.4.12}$$

for all $n \geq 0$.

The sum on the left is called the **geometric sum.** In the geometric sum in which $a \neq 0$ and $r \neq 0$, the ratio of adjacent terms $[(ar^{i+1})/(ar^i) = r]$ is constant.

SOLUTION

Basis Step ($n = 0$)

Since the smallest value in the domain of discourse $\{n \mid n \geq 0\}$ is $n = 0$, the Basis Step is to prove that (2.4.12) is true for $n = 0$. For $n = 0$, (2.4.12) becomes

$$a = \frac{a(r^1 - 1)}{r - 1},$$

which is true.

Inductive Step

Assume that statement (2.4.12) is true for n. Now

$$\begin{aligned} a + ar^1 + ar^2 + \cdots + ar^n + ar^{n+1} &= \frac{a(r^{n+1} - 1)}{r - 1} + ar^{n+1} \\ &= \frac{a(r^{n+1} - 1)}{r - 1} + \frac{ar^{n+1}(r - 1)}{r - 1} \\ &= \frac{a(r^{n+2} - 1)}{r - 1}. \end{aligned}$$

Since the Basis Step and the Inductive Step have been verified, the Principle of Mathematical Induction tells us that (2.4.12) is true for all $n \geq 0$. ◀

As an example of the use of the geometric sum, if we take $a = 1$ and $r = 2$ in (2.4.12), we obtain the formula

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^n = \frac{2^{n+1} - 1}{2 - 1} = 2^{n+1} - 1.$$

The reader has surely noticed that in order to prove the previous formulas, one has to be given the correct formulas in advance. A reasonable question is: How does one come up with the formulas? There are many answers to this question. One technique to derive a formula is to experiment with small values and try to discover a pattern. (Another technique is discussed in Exercises 70–73). For example, consider the sum $1+3+\cdots+(2n-1)$. The following table gives the values of this sum for $n = 1, 2, 3, 4$.

n	$1 + 3 + \cdots + (2n - 1)$
1	1
2	4
3	9
4	16

Since the second column consists of squares, we conjecture that

$$1 + 3 + \cdots + (2n - 1) = n^2 \qquad \text{for every positive integer } n.$$

The conjecture is correct and the formula can be proved by mathematical induction (see Exercise 1).

At this point, the reader may want to read the Problem-Solving Corner that follows this section. This Problem-Solving Corner gives an extended, detailed exposition of how to do proofs by mathematical induction.

Our final examples show that induction is not limited to proving formulas for sums and verifying inequalities.

Example 2.4.5 Use induction to show that $5^n - 1$ is divisible by 4 for all $n \geq 1$.

SOLUTION

Basis Step ($n = 1$)

If $n = 1$, $5^n - 1 = 5^1 - 1 = 4$, which is divisible by 4.

Inductive Step

We assume that $5^n - 1$ is divisible by 4. We must then show that $5^{n+1} - 1$ is divisible by 4. We use the fact that if p and q are each divisible by k, then $p + q$ is also divisible by k. In our case, $k = 4$. We leave the proof of this fact to the exercises (see Exercise 74).

We relate the $(n+1)$st case to the nth case by writing

$$5^{n+1} - 1 = 5^n - 1 + \text{to be determined.}$$

Now, by the inductive assumption, $5^n - 1$ is divisible by 4. If "to be determined" is also divisible by 4, then the preceding sum, which is equal to $5^{n+1} - 1$, will also be divisible by 4, and the Inductive Step will be complete. We must find the value of "to be determined."

Now

$$5^{n+1} - 1 = 5 \cdot 5^n - 1 = 4 \cdot 5^n + 1 \cdot 5^n - 1.$$

Thus, "to be determined" is $4 \cdot 5^n$, which is divisible by 4. Formally, we could write the Inductive Step as follows.

By the inductive assumption, $5^n - 1$ is divisible by 4 and, since $4 \cdot 5^n$ is divisible by 4, the sum

$$(5^n - 1) + 4 \cdot 5^n = 5^{n+1} - 1$$

is divisible by 4.

Since the Basis Step and the Inductive Step have been verified, the Principle of Mathematical Induction tells us that $5^n - 1$ is divisible by 4 for all $n \geq 1$. ◀

We next give the proof promised in Section 1.1 that if a set X has n elements, the power set of X, $\mathcal{P}(X)$, has 2^n elements.

Theorem 2.4.6

If $|X| = n$, then

$$|\mathcal{P}(X)| = 2^n \tag{2.4.13}$$

for all $n \geq 0$.

Proof The proof is by induction on n.

Basis Step ($n = 0$)

If $n = 0$, X is the empty set. The only subset of the empty set is the empty set itself; thus,

$$|\mathcal{P}(X)| = 1 = 2^0 = 2^n.$$

Thus, (2.4.13) is true for $n = 0$.

Inductive Step

Assume that (2.4.13) holds for n. Let X be a set with $n+1$ elements. Choose $x \in X$. We claim that exactly half of the subsets of X contain x, and exactly half of the subsets of X do not contain x. To see this, notice that each subset S of X that contains x can be paired uniquely with the subset obtained by removing x from S (see Figure 2.4.3). Thus exactly half of the subsets of X contain x, and exactly half of the subsets of X do not contain x.

If we let Y be the set obtained from X by removing x, Y has n elements. By the inductive assumption, $|\mathcal{P}(Y)| = 2^n$. But the subsets of Y are precisely the subsets of X that do not contain x. From the argument in the preceding paragraph, we conclude that

$$|\mathcal{P}(Y)| = \frac{|\mathcal{P}(X)|}{2}.$$

Therefore,

$$|\mathcal{P}(X)| = 2|\mathcal{P}(Y)| = 2 \cdot 2^n = 2^{n+1}.$$

Thus (2.4.13) holds for $n+1$ and the inductive step is complete. By the Principle of Mathematical Induction, (2.4.13) holds for all $n \geq 0$. ◀

Subsets of X that contain a	*Subsets of X that do not contain a*
$\{a\}$	$\emptyset$
$\{a, b\}$	$\{b\}$
$\{a, c\}$	$\{c\}$
$\{a, b, c\}$	$\{b, c\}$

Figure 2.4.3 Subsets of $X = \{a, b, c\}$ divided into two classes: those that contain a and those that do not contain a. Each subset in the right column is obtained from the corresponding subset in the left column by deleting the element a from it.

Example 2.4.7

A Tiling Problem A *right tromino,* hereafter called simply a *tromino,* is an object made up of three squares, as shown in Figure 2.4.4. A tromino is a type of polyomino. Since polyominoes were introduced by Solomon W. Golomb in 1954 (see [Golomb, 1954]), they have been a favorite topic in recreational mathematics. A *polyomino of order s* consists of s squares joined at the edges. A tromino is a polyomino of order 3. Three squares in a row form the only other type of polyomino of order 3. (No one has yet found a simple formula for the number of polyominoes of order s.) Numerous problems using polyominoes have been devised (see [Martin]).

Go Online
For more on trominoes, see goo.gl/gHgyey

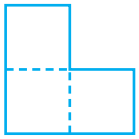


Figure 2.4.4 A tromino.

We give Golomb's inductive proof (see [Golomb, 1954]) that if we remove one square from an $n \times n$ board, where n is a power of 2, we can tile the remaining squares with right trominoes (see Figure 2.4.5). By a *tiling* of a figure by trominoes, we mean an exact covering of the figure by trominoes without having any of the trominoes overlap each other or extend outside the figure. We call a board with one square missing a *deficient board*.

We now use induction on k to prove that we can tile a $2^k \times 2^k$ deficient board with trominoes for all $k \geq 1$.

Basis Step ($k = 1$)

If $k = 1$, the 2×2 deficient board is itself a tromino and can therefore be tiled with one tromino.

Inductive Step

Assume that we can tile a $2^k \times 2^k$ deficient board. We show that we can tile a $2^{k+1} \times 2^{k+1}$ deficient board.

Consider a $2^{k+1} \times 2^{k+1}$ deficient board. Divide the board into four $2^k \times 2^k$ boards, as shown in Figure 2.4.6. Rotate the board so that the missing square is in the upper-left quadrant. By the inductive assumption, the upper-left $2^k \times 2^k$ board can be tiled. Place one tromino T in the center, as shown in Figure 2.4.6, so that each square of T is in each of the other quadrants. If we consider the squares covered by T as missing, each of these quadrants is a $2^k \times 2^k$ deficient board. Again, by the inductive assumption, these boards can be tiled. We now have a tiling of the $2^{k+1} \times 2^{k+1}$ board. By the Principle of Mathematical Induction, it follows that any $2^k \times 2^k$ deficient board can be tiled with trominoes, $k = 1, 2, \ldots$.

If we can tile an $n \times n$ deficient board, where n is not necessarily a power of 2, then the number of squares, $n^2 - 1$, must be divisible by 3. [Chu] showed that the converse is true, except when n is 5. More precisely, if $n \neq 5$, any $n \times n$ deficient board can be tiled

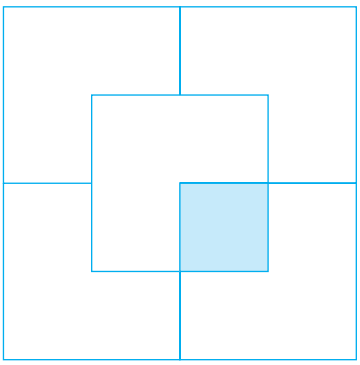


Figure 2.4.5 Tiling a 4×4 deficient board with trominoes.

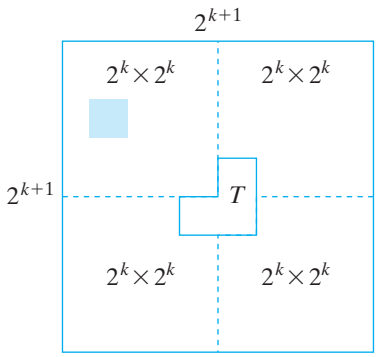


Figure 2.4.6 Using mathematical induction to tile a $2^{k+1} \times 2^{k+1}$ deficient board with trominoes.

with trominoes if and only if 3 divides $n^2 - 1$ (see Exercises 28 and 29, Section 2.5). [Some 5×5 deficient boards can be tiled and some cannot (see Exercises 33–35).]

Some real-world problems can be modeled as tiling problems. One example is the *VLSI layout problem*—the problem of packing many components on a computer chip (see [Wong]). (VLSI is short for very large scale integration.) The problem is to tile a rectangle of minimum area with the desired components. The components are sometimes modeled as rectangles and L-shaped figures similar to (right) trominoes. In practice, other constraints are imposed such as the proximity of various components that must be interconnected and restrictions on the ratios of width to height of the resulting rectangle. ◀

A **loop invariant** is a statement about program variables that is true just before a loop begins executing and is also true after each iteration of the loop. In particular, a loop invariant is true after the loop finishes, at which point the invariant tells us something about the state of the variables. Ideally, this statement tells us that the loop produces the expected result, that is, that the loop is correct. For example, a loop invariant for a while loop

```
while (condition)
    // loop body
```

is true just before *condition* is evaluated the first time, and it is also true each time the loop body is executed.

We can use mathematical induction to prove that an invariant has the desired behavior. The Basis Step proves that the invariant is true before the condition that controls looping is tested for the first time. The Inductive Step assumes that the invariant is true and then proves that if the condition that controls looping is true (so that the loop body is executed again), the invariant is true after the loop body executes. Since a loop iterates a finite number of times, the form of mathematical induction used here proves that a *finite* sequence of statements is true, rather than an infinite sequence of statements as in our previous examples. Whether the sequence of statements is finite or infinite, the steps needed for the proof by mathematical induction are the same. We illustrate a loop invariant with an example.

Example 2.4.8 Use a loop invariant to prove that when the pseudocode

```
i = 1
fact = 1
while (i < n) {
    i = i + 1
    fact = fact * i
}
```

terminates, *fact* is equal to $n!$.

SOLUTION We prove that $fact = i!$ is an invariant for the while loop. Just before the while loop begins executing, $i = 1$ and $fact = 1$, so $fact = 1!$. We have proved the Basis Step.

Assume that $fact = i!$. If $i < n$ is true (so that the loop body executes again), i becomes $i + 1$ and *fact* becomes

$$fact * (i + 1) = i! * (i + 1) = (i + 1)!.$$

We have proved the Inductive Step. Therefore, $fact = i!$ is an invariant for the while loop.

The while loop terminates when $i = n$. Because $fact = i!$ is an invariant, at this point, $fact = n!$. ◀

2.4 Problem-Solving Tips

To prove

$$a_1 + a_2 + \cdots + a_n = F(n) \qquad \text{for all } n \geq 1,$$

where $F(n)$ is the formula for the sum, first verify the equation for $n = 1$: $a_1 = F(1)$ (Basis Step). This is usually straightforward.

Now assume that the statement is true for n; that is, assume

$$a_1 + a_2 + \cdots + a_n = F(n).$$

Add a_{n+1} to both sides to get

$$a_1 + a_2 + \cdots + a_n + a_{n+1} = F(n) + a_{n+1}.$$

Finally, show that

$$F(n) + a_{n+1} = F(n+1).$$

To verify the preceding equation, use algebra to manipulate the left-hand side of the equation $[F(n) + a_{n+1}]$ until you get $F(n+1)$. *Look at $F(n+1)$ so you know where you're headed.* (It's somewhat like looking up the answer in the back of the book!) You've shown that

$$a_1 + a_2 + \cdots + a_{n+1} = F(n+1),$$

which is the Inductive Step. Now the proof is complete.

Proving an inequality is handled in a similar fashion. The difference is that instead of obtaining equality $[F(n) + a_{n+1} = F(n+1)$ in the preceding discussion], you obtain an *inequality*.

In general, the key to devising a proof by induction is to find case n "within" case $n+1$. Review the tiling problem (Example 2.4.7), which provides a striking example of case n "within" case $n+1$.

2.4 Review Exercises

1. State the Principle of Mathematical Induction.
2. Explain how a proof by mathematical induction proceeds.
3. Give a formula for the sum $1 + 2 + \cdots + n$.
4. What is the geometric sum? Give a formula for it.

2.4 Exercises

In Exercises 1–12, using induction, verify that each equation is true for every positive integer n.

1. $1 + 3 + 5 + \cdots + (2n-1) = n^2$
2. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$
3. $1(1!) + 2(2!) + \cdots + n(n!) = (n+1)! - 1$
4. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$
5. $1^2 - 2^2 + 3^2 - \cdots + (-1)^{n+1} n^2 = \dfrac{(-1)^{n+1} n(n+1)}{2}$
6. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$
7. $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \cdots + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$

8. $\dfrac{1}{2\cdot4}+\dfrac{1\cdot3}{2\cdot4\cdot6}+\dfrac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot8}+\cdots+\dfrac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n+2)}$

$$=\frac{1}{2}-\frac{1\cdot3\cdot5\cdots(2n+1)}{2\cdot4\cdot6\cdots(2n+2)}$$

9. $\dfrac{1}{2^2-1}+\dfrac{1}{3^2-1}+\cdots+\dfrac{1}{(n+1)^2-1}$

$$=\frac{3}{4}-\frac{1}{2(n+1)}-\frac{1}{2(n+2)}$$

10. $1\cdot2^2+2\cdot3^2+\cdots+n(n+1)^2=\dfrac{n(n+1)(n+2)(3n+5)}{12}$

★11. $\cos x+\cos 2x+\cdots+\cos nx=\dfrac{\cos[(x/2)(n+1)]\sin(nx/2)}{\sin(x/2)}$

provided that $\sin(x/2)\neq0$.

★12. $1\sin x+2\sin 2x+\cdots+n\sin nx$

$$=\frac{\sin[(n+1)x]}{4\sin^2(x/2)}-\frac{(n+1)\cos[(2n+1)x/2]}{2\sin(x/2)}$$

provided that $\sin(x/2)\neq0$.

In Exercises 13–18, using induction, verify the inequality.

13. $\dfrac{1}{2n}\le\dfrac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)},\ n=1,2,\ldots$

★14. $\dfrac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}\le\dfrac{1}{\sqrt{n+1}},\ n=1,2,\ldots$

15. $2n+1\le2^n,\ n=3,4,\ldots$

★16. $2^n\ge n^2,\ n=4,5,\ldots$

★17. $(a_1a_2\cdots a_{2^n})^{1/2^n}\le\dfrac{a_1+a_2+\cdots+a_{2^n}}{2^n},\ n=1,2,\ldots,$ and the a_i are positive numbers

18. $(1+x)^n\ge1+nx$, for $x\ge-1$ and $n\ge1$

19. Use the geometric sum to prove that

$$r^0+r^1+\cdots+r^n<\frac{1}{1-r}$$

for all $n\ge0$ and $0<r<1$.

★20. Prove that

$$1\cdot r^1+2\cdot r^2+\cdots+nr^n<\frac{r}{(1-r)^2}$$

for all $n\ge1$ and $0<r<1$. *Hint:* Using the result of the previous exercise, compare the sum of the terms in

$$\begin{array}{llllll}
r & r^2 & r^3 & r^4 & \cdots & r^n\\
r^2 & r^3 & r^4 & \cdots & r^n & \\
r^3 & r^4 & \cdots & r^n & & \\
r^4 & \cdots & & & & \\
\vdots & \vdots & & & & \\
r^{n-1} & r^n & & & & \\
r^n & & & & &
\end{array}$$

in the diagonal direction (↙) with the sum of the terms by columns.

21. Prove that

$$\frac{1}{2^1}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{n}{2^n}<2$$

for all $n\ge1$.

In Exercises 22–25, use induction to prove the statement.

22. 7^n-1 is divisible by 6, for all $n\ge1$.

23. 11^n-6 is divisible by 5, for all $n\ge1$.

24. $6\cdot7^n-2\cdot3^n$ is divisible by 4, for all $n\ge1$.

★25. 3^n+7^n-2 is divisible by 8, for all $n\ge1$.

26. Use induction to prove that if $X_1,\ldots,X_n$ and X are sets, then

(a) $X\cap(X_1\cup X_2\cup\cdots\cup X_n)=(X\cap X_1)\cup(X\cap X_2)\cup\cdots\cup(X\cap X_n)$.

(b) $\overline{X_1\cap X_2\cap\cdots\cap X_n}=\overline{X_1}\cup\overline{X_2}\cup\cdots\cup\overline{X_n}$.

27. Use induction to prove that if $X_1,\ldots,X_n$ are sets, then

$$|X_1\times X_2\times\cdots\times X_n|=|X_1|\cdot|X_2|\cdots|X_n|.$$

28. Prove that the number of subsets S of $\{1,2,\ldots,n\}$, with $|S|$ even, is 2^{n-1}, $n\ge1$.

29. By experimenting with small values of n, guess a formula for the given sum,

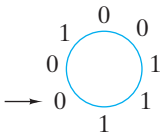
$$\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\cdots+\frac{1}{n(n+1)};$$

then use induction to verify your formula.

30. Use induction to show that n straight lines in the plane divide the plane into $(n^2+n+2)/2$ regions. Assume that no two lines are parallel and that no three lines have a common point.

31. Show that the regions of the preceding exercise can be colored red and green so that no two regions that share an edge are the same color.

32. Given n 0's and n 1's distributed in any manner whatsoever around a circle (see the following figure), show, using induction on n, that it is possible to start at some number and proceed clockwise around the circle to the original starting position so that, at any point during the cycle, we have seen at least as many 0's as 1's. In the following figure, a possible starting point is marked with an arrow.



33. Give a tiling of a 5×5 board with trominoes in which the upper-left square is missing.

34. Show a 5×5 deficient board that is impossible to tile with trominoes. Explain why your board cannot be tiled with trominoes.

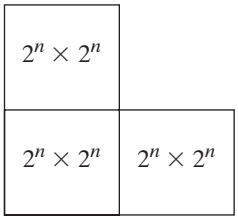
35. Which 5×5 deficient boards can be tiled?

36. Show that any $(2i)\times(3j)$ board, where i and j are positive integers, with no square missing, can be tiled with trominoes.

★37. Show that any 7×7 deficient board can be tiled with trominoes.

38. Show that any 11×11 deficient board can be tiled with trominoes. *Hint:* Subdivide the board into overlapping 7×7 and 5×5 boards and two 6×4 boards. Then, use Exercises 33, 36, and 37.

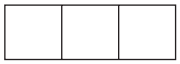
39. This exercise and the one that follows are due to Anthony Quas. A $2^n \times 2^n$ *L-shape*, $n \geq 0$, is a figure of the form



with no missing squares. Show that any $2^n \times 2^n$ L-shape can be tiled with trominoes.

40. Use the preceding exercise to give a different proof that any $2^n \times 2^n$ deficient board can be tiled with trominoes.

A straight tromino *is an object made up of three squares in a row:*

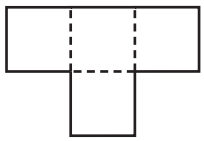


41. Which 4×4 deficient boards can be tiled with straight trominoes? *Hint:* Number the squares of the 4×4 board, left to right, top to bottom: 1, 2, 3, 1, 2, 3, and so on. Note that if there is a tiling, each straight tromino covers exactly one 2 and exactly one 3.

42. Which 5×5 deficient boards can be tiled with straight trominoes?

43. Which 8×8 deficient boards can be tiled with straight trominoes?

★44. A *T-tetromino* is an object made up of four squares



Prove that an $m \times n$ rectangle can be tiled with T-tetrominoes if and only if 4 divides m and 4 divides n.

45. Use a loop invariant to prove that when the pseudocode

```
i = 1
pow = 1
while (i ≤ n) {
    pow = pow * a
    i = i + 1
}
```

terminates, *pow* is equal to a^n.

46. Prove that, after the following pseudocode terminates, $a[h] = val$; for all p, $i \leq p < h$, $a[p] < val$; and for all p, $h < p \leq j$, $a[p] \geq val$. In particular, val is in the position in the array $a[i], \ldots, a[j]$ where it would be if the array were sorted.

```
val = a[i]
h = i
for k = i + 1 to j
    if (a[k] < val) {
        h = h + 1
        swap(a[h], a[k])
    }
swap(a[i], a[h])
```

Hint: Use the loop invariant: $h < k$; for all p, $i < p \leq h$, $a[p] < val$; and, for all p, $h < p < k$, $a[p] \geq val$. (A picture is helpful.)

This technique is called *partitioning*. This particular version is due to Nico Lomuto. Partitioning can be used to find the kth smallest element in an array and to construct a sorting algorithm called *quicksort*.

A 3D-septomino *is a three-dimensional* $2 \times 2 \times 2$ *cube with one* $1 \times 1 \times 1$ *corner cube removed. A* deficient cube *is a* $k \times k \times k$ *cube with one* $1 \times 1 \times 1$ *cube removed.*

47. Prove that a $2^n \times 2^n \times 2^n$ deficient cube can be tiled by 3D-septominoes.

48. Prove that if a $k \times k \times k$ deficient cube can be tiled by 3D-septominoes, then 7 divides one of $k-1, k-2, k-4$.

49. Suppose that $S_n = (n+2)(n-1)$ is (incorrectly) proposed as a formula for

$$2 + 4 + \cdots + 2n.$$

(a) Show that the Inductive Step is satisfied but that the Basis Step fails.

★(b) If S'_n is an arbitrary expression that satisfies the Inductive Step, what form must S'_n assume?

★50. What is wrong with the following argument, which allegedly shows that any two positive integers are equal?

We use induction on n to "prove" that if a and b are positive integers and $n = \max\{a, b\}$, then $a = b$.

Basis Step ($n = 1$)

If a and b are positive integers and $1 = \max\{a, b\}$, we must have $a = b = 1$.

Inductive Step

Assume that if a' and b' are positive integers and $n = \max\{a', b'\}$, then $a' = b'$. Suppose that a and b are positive integers and that $n + 1 = \max\{a, b\}$. Now $n = \max\{a-1, b-1\}$. By the inductive hypothesis, $a - 1 = b - 1$. Therefore, $a = b$.

Since we have verified the Basis Step and the Inductive Step, by the Principle of Mathematical Induction, any two positive integers are equal!

51. What is wrong with the following "proof" that

$$\frac{1}{2} + \frac{2}{3} + \cdots + \frac{n}{n+1} \neq \frac{n^2}{n+1}$$

for all $n \geq 2$?

Suppose by way of contradiction that

$$\frac{1}{2}+\frac{2}{3}+\cdots+\frac{n}{n+1}=\frac{n^2}{n+1}. \tag{2.4.14}$$

Then also

$$\frac{1}{2}+\frac{2}{3}+\cdots+\frac{n}{n+1}+\frac{n+1}{n+2}=\frac{(n+1)^2}{n+2}.$$

We could prove statement (2.4.14) by induction. In particular, the Inductive Step would give

$$\left(\frac{1}{2}+\frac{2}{3}+\cdots+\frac{n}{n+1}\right)+\frac{n+1}{n+2}=\frac{n^2}{n+1}+\frac{n+1}{n+2}.$$

Therefore,

$$\frac{n^2}{n+1}+\frac{n+1}{n+2}=\frac{(n+1)^2}{n+2}.$$

Multiplying each side of this last equation by $(n+1)(n+2)$ gives

$$n^2(n+2)+(n+1)^2=(n+1)^3.$$

This last equation can be rewritten as

$$n^3+2n^2+n^2+2n+1=n^3+3n^2+3n+1$$

or

$$n^3+3n^2+2n+1=n^3+3n^2+3n+1,$$

which is a contradiction. Therefore,

$$\frac{1}{2}+\frac{2}{3}+\cdots+\frac{n}{n+1}\neq\frac{n^2}{n+1},$$

as claimed.

52. Use mathematical induction to prove that

$$\frac{1}{2}+\frac{2}{3}+\cdots+\frac{n}{n+1}<\frac{n^2}{n+1}$$

for all $n \geq 2$. This inequality gives a correct proof of the statement of the preceding exercise.

In Exercises 53–57, suppose that $n > 1$ people are positioned in a field (Euclidean plane) so that each has a unique nearest neighbor. Suppose further that each person has a pie that is hurled at the nearest neighbor. A survivor *is a person that is not hit by a pie.*

53. Give an example to show that if n is even, there might be no survivor.

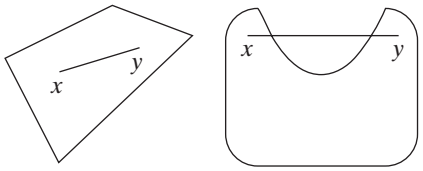
54. Give an example to show that there might be more than one survivor.

⋆55. [Carmony] Use induction on n to show that if n is odd, there is always at least one survivor.

56. Prove or disprove: If n is odd, one of two persons farthest apart is a survivor.

57. Prove or disprove: If n is odd, a person who throws a pie the greatest distance is a survivor.

Exercises 58–61 deal with plane convex sets. A plane convex set, *subsequently abbreviated to "convex set," is a nonempty set X in the plane having the property that if x and y are any two points in X, the straight-line segment from x to y is also in X. The following figures illustrate.*



convex set nonconvex set

58. Prove that if X and Y are convex sets and $X \cap Y$ is nonempty, $X \cap Y$ is a convex set.

⋆59. Suppose that X_1, X_2, X_3, X_4 are convex sets, each three of which have a common point. Prove that all four sets have a common point.

⋆60. Prove *Helly's Theorem:* Suppose that $X_1, X_2, \ldots, X_n$, $n \geq 4$, are convex sets, each three of which have a common point. Prove that all n sets have a common point.

61. Suppose that $n \geq 3$ points in the plane have the property that each three of them are contained in a circle of radius 1. Prove that there is a circle of radius 1 that contains all of the points.

62. If a and b are real numbers with $a < b$, an *open interval* (a, b) is the set of all real numbers x such that $a < x < b$. Prove that if $I_1, \ldots, I_n$ is a set of $n \geq 2$ open intervals such that each pair has a nonempty intersection, then

$$I_1 \cap I_2 \cap \cdots \cap I_n$$

is nonempty.

Flavius Josephus was a Jewish soldier and historian who lived in the first century (see [Graham, 1994; Schumer]). He was one of the leaders of a Jewish revolt against Rome in the year 66. The following year, he was among a group of trapped soldiers who decided to commit suicide rather than be captured. One version of the story is that, rather than being captured, they formed a circle and proceeded around the circle killing every third person. Josephus, being proficient in discrete math, figured out where he and a buddy should stand so they could avoid being killed.

Exercises 63–69 concern a variant of the Josephus Problem *in which every* second *person is eliminated. We assume that n people are arranged in a circle and numbered* $1, 2, \ldots, n$ *clockwise. Then, proceeding clockwise, 2 is eliminated, 4 is eliminated, and so on, until there is one survivor, denoted* $J(n)$.

63. Compute $J(4)$.

64. Compute $J(6)$.

65. Compute $J(10)$.

66. Use induction to show that $J(2^i) = 1$ for all $i \geq 1$.

67. Given a value of $n \geq 2$, let 2^i be the greatest power of 2 with $2^i \leq n$. (Examples: If $n = 10$, $i = 3$. If $n = 16$, $i = 4$.) Let $j = n - 2^i$. (After subtracting 2^i, the greatest power of 2 less than or equal to n, from n, j is what is left over.) By using the result of Exercise 66 or otherwise, prove that

$$J(n) = 2j + 1.$$

68. Use the result of Exercise 67 to compute $J(1000)$.

69. Use the result of Exercise 67 to compute $J(100{,}000)$.

If $a_1, a_2, \ldots$ is a sequence, we define the difference operator Δ *to be*

$$\Delta a_n = a_{n+1} - a_n.$$

The formula of Exercise 70 can sometimes be used to find *a formula for a sum as opposed to using induction to* prove *a formula for a sum (see Exercises 71–73).*

70. Suppose that $\Delta a_n = b_n$. Show that

$$b_1 + b_2 + \cdots + b_n = a_{n+1} - a_1.$$

This formula is analogous to the calculus formula $\int_c^d f(x)\,dx = g(d) - g(c)$, where $Dg = f$ (D is the derivative operator). In the calculus formula, sum is replaced by integral, and Δ is replaced by derivative.

71. Let $a_n = n^2$, and compute Δa_n. Use Exercise 70 to find a formula for

$$1 + 2 + 3 + \cdots + n.$$

72. Use Exercise 70 to find a formula for

$$1(1!) + 2(2!) + \cdots + n(n!).$$

(Compare with Exercise 3.)

73. Use Exercise 70 to find a formula for

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}.$$

(Compare with Exercise 29.)

74. Prove that if p and q are divisible by k, then $p + q$ is divisible by k.

Problem-Solving Corner

Mathematical Induction

Problem

Define

$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \qquad (1)$$

for all $k \geq 1$. The numbers $H_1, H_2, \ldots$ are called the *harmonic numbers*. Prove that

$$H_{2^n} \geq 1 + \frac{n}{2} \qquad (2)$$

for all $n \geq 0$.

Attacking the Problem

It's often a good idea to begin attacking a problem by looking at some concrete examples of the expressions under consideration. Let's look at H_k for some small values of k. The smallest value of k for which H_k is defined is $k = 1$. In this case, the last term $1/k$ in the definition of H_k equals $1/1 = 1$. Since the first and last terms coincide, $H_1 = 1$. For $k = 2$, the last term $1/k$ in the definition of H_k equals $1/2$, so

$$H_2 = 1 + \frac{1}{2}.$$

Similarly, we find that

$$H_3 = 1 + \frac{1}{2} + \frac{1}{3},$$

$$H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}.$$

We observe that H_1 appears as the first term of H_2, H_3, and H_4, that H_2 appears as the first two terms of H_3 and H_4, and that H_3 appears as the first three terms of H_4. In general, H_m appears as the first m terms of H_k if $m \leq k$. This observation will help us later because the Inductive Step in a proof by induction must relate smaller instances of a problem to larger instances of the problem.

In general, it's a good strategy to delay combining terms and simplifying until as late as possible, which is why, for example, we left H_4 as the sum of four terms rather than writing $H_4 = 25/12$. Since we left H_4 as the sum of four terms, we were able to see that each of H_1, H_2, and H_3 appears in the expression for H_4.

Finding a Solution

The Basis Step is to prove the given statement for the smallest value of n, which here is $n = 0$. For $n = 0$, inequality (2) that we must prove becomes

$$H_{2^0} \geq 1 + \frac{0}{2} = 1.$$

We have already observed that $H_1 = 1$. Thus inequality (2) is true when $n = 0$; in fact, the inequality is an equality. (Recall that by definition, if $x = y$ is true, then $x \geq y$ is also true.)

Let's move to the Inductive Step. It's a good idea to write down what is assumed (here the case n),

$$H_{2^n} \geq 1 + \frac{n}{2}, \qquad (3)$$

and what needs to be proved (here the case $n + 1$),

$$H_{2^{n+1}} \geq 1 + \frac{n+1}{2}. \qquad (4)$$

It's also a good idea to write the formulas for any expressions that occur. Using equation (1), we may write

$$H_{2^n} = 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} \qquad (5)$$

and

$$H_{2^{n+1}} = 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n+1}}.$$

It's not so evident from the last equation that H_{2^n} appears as the first 2^n terms of $H_{2^{n+1}}$. Let's rewrite the last equation as

$$H_{2^{n+1}} = 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} + \frac{1}{2^n + 1} + \cdots + \frac{1}{2^{n+1}} \quad (6)$$

to make it clear that H_{2^n} appears as the first 2^n terms of $H_{2^{n+1}}$.

For clarity, we have written the term that follows $1/2^n$. Notice that the denominators increase by one, so the term that follows $1/2^n$ is $1/(2^n+1)$. Also notice that there is a big difference between $1/(2^n + 1)$, the term that follows $1/2^n$, and $1/2^{n+1}$, the last term in equation (6).

Using equations (5) and (6), we may relate H_{2^n} to $H_{2^{n+1}}$ explicitly by writing

$$H_{2^{n+1}} = H_{2^n} + \frac{1}{2^n + 1} + \cdots + \frac{1}{2^{n+1}}. \quad (7)$$

Combining (3) and (7), we obtain

$$H_{2^{n+1}} \geq 1 + \frac{n}{2} + \frac{1}{2^n + 1} + \cdots + \frac{1}{2^{n+1}}. \quad (8)$$

This inequality shows that $H_{2^{n+1}}$ is greater than or equal to

$$1 + \frac{n}{2} + \frac{1}{2^n + 1} + \cdots + \frac{1}{2^{n+1}},$$

but our goal (4) is to show that $H_{2^{n+1}}$ is greater than or equal to $1 + (n + 1)/2$. We will achieve our goal if we show that

$$1 + \frac{n}{2} + \frac{1}{2^n + 1} + \cdots + \frac{1}{2^{n+1}} \geq 1 + \frac{n+1}{2}.$$

In general, to prove an inequality, we replace terms in the larger expression with smaller terms so that the resulting expression equals the smaller expression; or we replace terms in the smaller expression with larger terms so that the resulting expression equals the larger expression. Here let's replace each of the terms in the sum

$$\frac{1}{2^n + 1} + \cdots + \frac{1}{2^{n+1}}$$

by the smallest term $1/2^{n+1}$ in the sum. We obtain

$$\frac{1}{2^n + 1} + \cdots + \frac{1}{2^{n+1}} \geq \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+1}}.$$

Since there are 2^n terms in the latter sum, each equal to $1/2^{n+1}$, we may rewrite the preceding inequality as

$$\begin{aligned} \frac{1}{2^n + 1} + \cdots + \frac{1}{2^{n+1}} &\geq \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+1}} \\ &= 2^n \frac{1}{2^{n+1}} = \frac{1}{2}. \end{aligned} \quad (9)$$

Combining (8) and (9),

$$H_{2^{n+1}} \geq 1 + \frac{n}{2} + \frac{1}{2} = 1 + \frac{n+1}{2}.$$

We have the desired result, and the Inductive Step is complete.

Formal Solution

The formal solution could be written as follows.

Basis Step ($n = 0$)

$$H_{2^0} = 1 \geq 1 = 1 + \frac{0}{2}$$

Inductive Step

We assume (2). Now

$$\begin{aligned} H_{2^{n+1}} &= 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} + \frac{1}{2^n + 1} + \cdots + \frac{1}{2^{n+1}} \\ &= H_{2^n} + \frac{1}{2^n + 1} + \cdots + \frac{1}{2^{n+1}} \\ &\geq 1 + \frac{n}{2} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+1}} \\ &= 1 + \frac{n}{2} + 2^n \frac{1}{2^{n+1}} \\ &= 1 + \frac{n}{2} + \frac{1}{2} = 1 + \frac{n+1}{2}. \end{aligned}$$

Summary of Problem-Solving Techniques

- Look at concrete examples of the expressions under consideration, typically for small values of the variables.
- Look for expressions for small values of n to appear within expressions for larger values of n. In particular, the Inductive Step depends on relating case n to case $n + 1$.
- Delay combining terms and simplifying until as late as possible to help discover relationships among the expressions.
- Write out in full the specific cases to prove, specifically, the smallest value of n for the Basis Step, the case n that is assumed in the Inductive Step, and the case $n + 1$ to prove in the Inductive Step. Write out the formulas for the various expressions that appear.

- To prove an inequality, replace terms in the larger expression with smaller terms so that the resulting expression equals the smaller expression, or replace terms in the smaller expression with larger terms so that the resulting expression equals the larger expression.

Comments

The series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots,$$

which surfaces in calculus, is called the *harmonic series*. Inequality (2) shows that the harmonic numbers increase without bound. In calculus terminology, the harmonic series *diverges*.

Exercises

1. Prove that $H_{2^n} \leq 1 + n$ for all $n \geq 0$.

2. Prove that

$$H_1 + H_2 + \cdots + H_n = (n+1)H_n - n$$

for all $n \geq 1$.

3. Prove that

$$H_n = H_{n+1} - \frac{1}{n+1}$$

for all $n \geq 1$.

4. Prove that

$$1 \cdot H_1 + 2 \cdot H_2 + \cdots + nH_n = \frac{n(n+1)}{2} H_{n+1} - \frac{n(n+1)}{4}$$

for all $n \geq 1$.

5. Prove that

$$\frac{H_1}{1} + \frac{H_2}{2} + \cdots + \frac{H_n}{n} = \frac{H_n^2}{2} + \frac{1}{2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\right]$$

for all $n \geq 1$.

2.5 Strong Form of Induction and the Well-Ordering Property

In the Inductive Step of mathematical induction presented in Section 2.4, we assume that statement n is true, and then prove that statement $n + 1$ is true. In other words, to prove that a statement is true (statement $n + 1$), we assume the truth of its immediate predecessor (statement n). In some cases in the Inductive Step, to prove a statement is true, it is helpful to assume the truth of *all* of the preceding statements (not just the immediate predecessor). The **Strong Form of Mathematical Induction** allows us to assume the truth of all of the preceding statements. Following the usual convention, the statement to prove is denoted n rather than $n+1$. We next formally state the Strong Form of Mathematical Induction.

Strong Form of Mathematical Induction

Suppose that we have a propositional function $S(n)$ whose domain of discourse is the set of integers greater than or equal to n_0. Suppose that

$S(n_0)$ is true;

for all $n > n_0$, if $S(k)$ is true for all k, $n_0 \leq k < n$, then $S(n)$ is true.

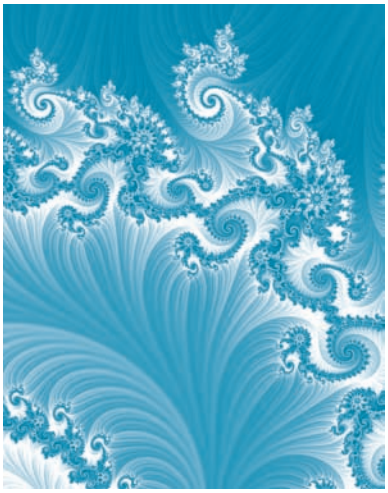
Then $S(n)$ is true for every integer $n \geq n_0$.

In the Inductive Step of the Strong Form of Mathematical Induction, we let n denote an arbitrary integer, $n > n_0$. Then, assuming that $S(k)$ is true for all k satisfying

$$n_0 \leq k < n, \tag{2.5.1}$$

we prove that $S(n)$ is true. In inequality (2.5.1), k indexes a statement $S(k)$ that is an *arbitrary* predecessor of the statement $S(n)$ (thus $k < n$), which we are to prove true. In inequality (2.5.1), $n_0 \leq k$ ensures that k is in the domain of discourse

$$\{n_0, n_0 + 1, n_0 + 2, \ldots\}.$$



Chapter 3

FUNCTIONS, SEQUENCES, AND RELATIONS

All of mathematics, as well as subjects that rely on mathematics, such as computer science and engineering, make use of functions, sequences, and relations.

A function assigns to each member of a set X exactly one member of a set Y. Functions are used extensively in discrete mathematics; for example, functions are used to analyze the time needed to execute algorithms.

A sequence is a special kind of function. A list of the letters as they appear in a word is an example of a sequence. Unlike a set, a sequence takes order into account. (Order is obviously important since, for example, *form* and *from* are different words.)

Relations generalize the notion of functions. A relation is a set of ordered pairs. The presence of the ordered pair (a, b) in a relation is interpreted as indicating a relationship from a to b. The relational database model that helps users access information in a database (a collection of records manipulated by a computer) is based on the concept of relation.

3.1 Functions

Credit card numbers typically consist of 13, 15, or 16 digits. For example,

$$4690\,3582\,1375\,4657 \qquad \textbf{(3.1.1)}$$

is a hypothetical credit card number. The first digit designates the system. In (3.1.1), the first digit, 4, shows that the card would be a Visa card. The following digits specify other information such as the account number and the bank number. (The precise meaning depends on the type of card.) The last digit is special; it is computed from the preceding digits and is called a *check digit*. In (3.1.1), the check digit is 7 and is computed from the preceding digits 4690 3582 1375 465. Credit card check digits are used to identify certain erroneous card numbers. It is not a security measure, but rather it is used to help detect errors such as giving a credit card number over the phone and having it transcribed improperly or detecting an error in entering a credit card number while ordering a product online.

†This section can be omitted without loss of continuity.

The check digit is computed as follows. Starting from the right and skipping the check digit, double every other number. If the result of doubling is a two-digit number, add the digits; otherwise, use the original digit. The other digits are not modified.

4	6	9	0	3	5	8	2	1	3	7	5	4	6	5	
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	Double every other digit.
8	6	18	0	6	5	16	2	2	3	14	5	8	6	10	
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	Add digits of two-digit numbers.
8	6	9	0	6	5	7	2	2	3	5	5	8	6	1	

Sum the resulting digits

$$8+6+9+0+6+5+7+2+2+3+5+5+8+6+1=73.$$

If the last digit of the sum is 0, the check digit is 0. Otherwise, subtract the last digit of the sum from 10 to get the check digit, $10-3=7$. Verify the check digit on your favorite Visa, MasterCard, American Express, or Diners Club card. This method of calculating a check digit is called the *Luhn algorithm*. It is named after Hans Peter Luhn (1896–1964), who invented it while at IBM. Although originally patented, it is now in the public domain and is widely used.

One common error in copying a number is to change one digit. Each undoubled digit contributes a unique value to the sum ($0 \to 0,\ 1 \to 1$, etc.). Each doubled digit also contributes a unique value to the sum ($0 \to 0,\ 1 \to 2, \ldots, 4 \to 8,\ 5 \to 1,\ 6 \to 3, \ldots, 9 \to 9$). Thus if a single digit is changed in a credit card number, the sum used in the Luhn algorithm will change by an absolute amount less than 10, and the check digit will change. In the preceding example if 1 is changed to 7, the Luhn algorithm calculation becomes

4	6	9	0	3	5	8	2	7	3	7	5	4	6	5	
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	Double every other digit
8	6	18	0	6	5	16	2	14	3	14	5	8	6	10	
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	Add digits of two-digit numbers.
8	6	9	0	6	5	7	2	5	3	5	5	8	6	1	

and the sum becomes

$$8+6+9+0+6+5+7+2+5+3+5+5+8+6+1=76.$$

Therefore the check digit changes to 4. Thus, if 1 is inadvertently transcribed as 7, the error will be detected.

Another common error is transposition of adjacent digits. For example, if 82 is inadvertently written as 28, the error will be detected by the Luhn algorithm because the check digit will change (check this). In fact, the Luhn algorithm will detect every transposition of adjacent digits except for 90 and 09 (see Computer Exercise 4).

Go Online
For more on functions, see goo.gl/V3y4pS

The Luhn algorithm gives an example of a **function**. A function assigns to each member of a set X exactly one member of a set Y. (The sets X and Y may or may not be the same.) The Luhn algorithm assigns to each integer 10 or greater (so there is a number available to compute a check digit) a single-digit integer, the check digit. In the preceding example, the integer 469035821375465 is assigned the value 7, and the integer 469035827375465 is assigned the value 4. We can represent these assignments as ordered pairs:

$$(469035821375465, 7) \quad \text{and} \quad (469035827375465, 4).$$

Formally, we *define* a function to be a particular kind of set of ordered pairs.

Definition 3.1.1 ▶ Let X and Y be sets. A *function* f from X to Y is a subset of the Cartesian product $X \times Y$ having the property that for each $x \in X$, there is exactly one $y \in Y$ with $(x, y) \in f$. We sometimes denote a function f from X to Y as $f: X \to Y$.

The set X is called the *domain of* f and the set Y is called the *codomain of* f. The set

$$\{y \mid (x, y) \in f\}$$

(which is a subset of the codomain Y) is called the *range of* f. ◀

Example 3.1.2 For the check digit function, the domain is the set of positive integers 10 or greater and the range is the set of single-digit integers. We can take the codomain to be any set containing the set of single-digit integers, for example, the set of nonnegative integers. ◀

Example 3.1.3 The set $f = \{(1, a), (2, b), (3, a)\}$ is a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$. Each element of X is assigned a unique value in Y: 1 is assigned the unique value a; 2 is assigned the unique value b; and 3 is assigned the unique value a. We can depict the situation as shown in Figure 3.1.1, where an arrow from j to x means that we assign the letter x to the integer j. We call a picture such as Figure 3.1.1 an **arrow diagram.** For an arrow diagram to be a function, Definition 3.1.1 requires that there is exactly one arrow from each element in the domain. Notice that Figure 3.1.1 has this property.

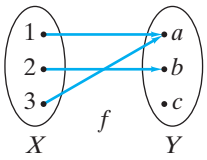


Figure 3.1.1 The arrow diagram of the function of Example 3.1.3. There is exactly one arrow from each element in X.

Definition 3.1.1 allows us to reuse elements in Y. For the function f, the element a in Y is used twice. Further, Definition 3.1.1 does *not* require us to use all the elements in Y. No element in X is assigned to the element c in Y. The domain of f is X, the codomain of f is Y, and the range of f is $\{a, b\}$. ◀

Example 3.1.4 The set

$$\{(1, a), (2, a), (3, b)\} \tag{3.1.2}$$

is not a function from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c\}$ because the element 4 in X is not assigned to an element in Y. It is also apparent from the arrow diagram (see Figure 3.1.2) that this set is not a function because there is no arrow from 4. The set (3.1.2) *is* a function from $X' = \{1, 2, 3\}$ to $Y = \{a, b, c\}$.

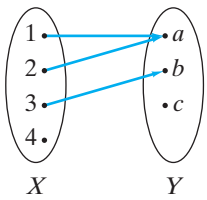


Figure 3.1.2 The arrow diagram of the set in Example 3.1.4, which is not a function because there is no arrow from 4.

◀

Example 3.1.5 The set $\{(1, a), (2, b), (3, c), (1, b)\}$ is not a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$ because 1 is not assigned a *unique* element in Y (1 is assigned the values a *and* b). It is also apparent from the arrow diagram (see Figure 3.1.3) that this set is not a function because there are two arrows from 1.

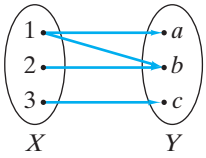


Figure 3.1.3 The arrow diagram of the set in Example 3.1.5, which is not a function because there are two arrows from 1. ◀

Given a function f from X to Y, according to Definition 3.1.1, for each element x in the domain X, there is exactly one y in the codomain Y with $(x, y) \in f$. This unique value y is denoted $f(x)$. In other words, $f(x) = y$ is another way to write $(x, y) \in f$.

Example 3.1.6 For the function f of Example 3.1.3, we may write $f(1) = a$, $f(2) = b$, and $f(3) = a$. ◀

Example 3.1.7 If we call the check digit function L, we may write

$$L(469035821375465) = 7 \quad \text{and} \quad L(469035827375465) = 4.$$ ◀

The next example shows how we sometimes use the $f(x)$ notation to define a function.

Example 3.1.8 Let f be the function defined by the rule $f(x) = x^2$. For example, $f(2) = 4$, $f(-3.5) = 12.25$, and $f(0) = 0$. Although we frequently find functions defined in this way, the definition is incomplete since the domain and codomain are not specified. If we are told that the domain is the set of all real numbers and the codomain is the set of all nonnegative real numbers, in ordered-pair notation, we would have

$$f = \{(x, x^2) \mid x \text{ is a real number}\}.$$

The range of f is the set of all nonnegative real numbers. ◀

Example 3.1.9 Most calculators have a $1/x$ key. If you enter a number and hit the $1/x$ key, the reciprocal of the number entered (or an approximation to it) is displayed. This function can be defined by the rule

$$R(x) = \frac{1}{x}.$$

The domain is the set of all numbers that can be entered into the calculator and whose reciprocals can be computed and displayed by the calculator. The range is the set of all the reciprocals that can be computed and displayed. We could define the codomain also to be the set of all the reciprocals that can be computed and displayed. Notice that by the nature of the calculator, the domain and range are finite sets. ◀

Another way to visualize a function is to draw its graph. The **graph of a function** f whose domain and codomain are subsets of the real numbers is obtained by plotting points in the plane that correspond to the elements in f. The domain is contained in the horizontal axis and the codomain is contained in the vertical axis.

Example 3.1.10 The graph of the function $f(x) = x^2$ is shown in Figure 3.1.4. ◀

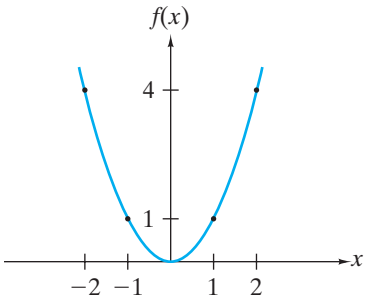


Figure 3.1.4 The graph of $f(x) = x^2$.

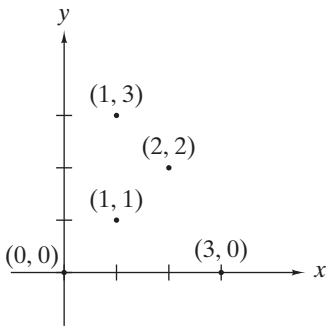


Figure 3.1.5 A set that is not a function. The vertical line $x = 1$ intersects two points in the set.

We note that a set S of points in the plane defines a function precisely when each vertical line intersects at most one point of S. If some vertical line contains two or more points of some set, the domain point does not assign a *unique* codomain point and the set does not define a function (see Figure 3.1.5).

Functions involving the **modulus operator** play an important role in mathematics and computer science.

Definition 3.1.11 ▶ If x is an integer and y is a positive integer, we define x mod y to be the remainder when x is divided by y. ◀

Example 3.1.12 We have

$$6 \bmod 2 = 0, \quad 5 \bmod 1 = 0, \quad 8 \bmod 12 = 8, \quad 199673 \bmod 2 = 1.$$ ◀

Example 3.1.13 The check digit calculated by the Luhn algorithm can be written

$$[10 - (S \bmod 10)] \bmod 10,$$

where S is the sum used in the intermediate step of the calculation. The last digit in S is given by S mod 10. It this digit is 1 through 9, inclusive, $10 - (S \bmod 10)$ gives the check digit and the last "mod 10" is unnecessary, but harmless. However, if the last digit in S is 0, $10 - (S \bmod 10) = 10$. In this case, adding the last "mod 10" gives the check digit as 0. ◀

Example 3.1.14 What day of the week will it be 365 days from Wednesday?

SOLUTION Seven days after Wednesday, it is Wednesday again; 14 days after Wednesday, it is Wednesday again; and in general, if n is a positive integer, $7n$ days after Wednesday, it is Wednesday again. Thus we need to subtract as many 7's as possible from 365 and see how many days are left, which is the same as computing 365 mod 7. Since $365 \bmod 7 = 1$, 365 days from Wednesday, it will be one day later, namely Thursday. This explains why, except for leap year, when an extra day is added to February, the identical month and date in consecutive years move forward one day of the week. ◀

Example 3.1.15

Hash Functions Suppose that we have cells in a computer memory indexed from 0 to 10 (see Figure 3.1.6). We wish to store and retrieve arbitrary nonnegative integers in these cells. One approach is to use a **hash function.** A hash function takes a data item to be stored or retrieved and computes the first choice for a location for the item. For example, for our problem, to store or retrieve the number n, we might take as the first choice for a location, n mod 11. Our hash function becomes $h(n) = n \bmod 11$. Figure 3.1.6 shows the result of storing 15, 558, 32, 132, 102, and 5, in this order, in initially empty cells.

Go Online
For more on hash functions, see goo.gl/V3y4pS

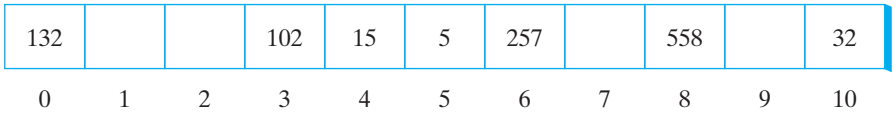


Figure 3.1.6 Cells in a computer memory.

Now suppose that we want to store 257. Since $h(257) = 4$, then 257 should be stored at location 4; however, this position is already occupied. In this case we say that a **collision** has occurred. More precisely, a collision occurs for a hash function H if $H(x) = H(y)$, but $x \neq y$. To handle collisions, a **collision resolution policy** is required. One simple collision resolution policy is to find the next highest (with 0 assumed to follow 10) unoccupied cell. If we use this collision resolution policy, we would store 257 at location 6 (see Figure 3.1.6).

If we want to locate a stored value n, we compute $m = h(n)$ and begin looking at location m. If n is not at this position, we look in the next-highest position (again, 0 is assumed to follow 10); if n is not in this position, we proceed to the next-highest position, and so on. If we reach an empty cell or return to our original position, we conclude that n is not present; otherwise, we obtain the position of n.

If collisions occur infrequently, and if when one does occur it is resolved quickly, then hashing provides a very fast method of storing and retrieving data. As an example, personnel data are frequently stored and retrieved by hashing on employee identification numbers. ◀

Example 3.1.16

Pseudorandom Numbers Computers are often used to simulate random behavior. A game program might simulate rolling dice, and a client service program might simulate the arrival of customers at a bank. Such programs generate numbers that appear random and are called **pseudorandom numbers.** For example, the dice-rolling program would need pairs of pseudorandom numbers, each between 1 and 6, to simulate the outcome of rolling dice. Pseudorandom numbers are not truly random; if one knows the program that generates the numbers, one could predict what numbers would occur.

The method usually used to generate pseudorandom numbers is called the **linear congruential method.** This method requires four integers: the modulus m, the multiplier a, the increment c, and a seed s satisfying $2 \leq a < m$, $0 \leq c < m$, and $0 \leq s < m$. We then set $x_0 = s$. The sequence of pseudorandom numbers generated, $x_1, x_2, \ldots$, is given by the formula

$$x_n = (ax_{n-1} + c) \bmod m.$$

The formula computes the next pseudorandom number using its immediate predecessor. For example, if $m = 11$, $a = 7$, $c = 5$, and $s = 3$, then

$$x_1 = (ax_0 + c) \bmod m = (7 \cdot 3 + 5) \bmod 11 = 4$$

and

$$x_2 = (ax_1 + c) \bmod m = (7 \cdot 4 + 5) \bmod 11 = 0.$$

Similar computations show that the sequence continues:

$$x_3 = 5,\ x_4 = 7,\ x_5 = 10,\ x_6 = 9,\ x_7 = 2,\ x_8 = 8,\ x_9 = 6,\ x_{10} = 3.$$

Since $x_{10} = 3$, which is the value of the seed, the sequence now repeats: 3, 4, 0, 5, 7, ...

Much effort has been invested in finding good values for a linear congruential method. Critical simulations such as those involving aircraft and nuclear research require "good" random numbers. In practice, large values are used for m and a. Commonly used values are $m = 2^{31} - 1 = 2{,}147{,}483{,}647$; $a = 7^5 = 16{,}807$; and $c = 0$, which generate a sequence of $2^{31} - 1$ integers before repeating a value. ◀

In the 1990s, Daniel Corriveau of Quebec won three straight games of a computer keno game in Montreal, each time choosing 19 of 20 numbers correctly. The odds against this feat are 6 billion to 1. Suspicious officials at first refused to pay him. Although Corriveau attributed his success to chaos theory, what in fact happened was that whenever power was cut, the random number generator started with the same seed, thus generating the same sequence of numbers. The embarrassed casino finally paid Corriveau the $600,000 due him.

We next define the **floor** and **ceiling** of a real number.

Definition 3.1.17 ▶ The *floor* of x, denoted $\lfloor x \rfloor$, is the greatest integer less than or equal to x. The *ceiling* of x, denoted $\lceil x \rceil$, is the least integer greater than or equal to x. ◀

Example 3.1.18 $\lfloor 8.3 \rfloor = 8,\ \lceil 9.1 \rceil = 10,\ \lfloor -8.7 \rfloor = -9,\ \lceil -11.3 \rceil = -11,\ \lceil 6 \rceil = 6,\ \lceil -8 \rceil = -8$ ◀

The floor of x "rounds x down" while the ceiling of x "rounds x up." We will use the floor and ceiling functions throughout the book.

Example 3.1.19 Figure 3.1.7 shows the graphs of the floor and ceiling functions. A bracket, [or], indicates that the point is to be included in the graph; a parenthesis, (or), indicates that the point is to be excluded from the graph.

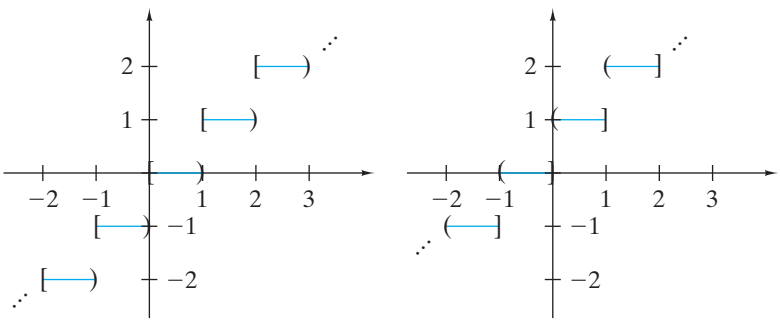


Figure 3.1.7 The graphs of the floor (left graph) and ceiling (right graph) functions. ◀

Example 3.1.20 The first-class postage rate for mail up to 13 ounces is 92 cents for the first ounce or fraction thereof and 20 cents for each additional ounce or fraction thereof. The postage $P(w)$ as a function of weight w is given by the equation

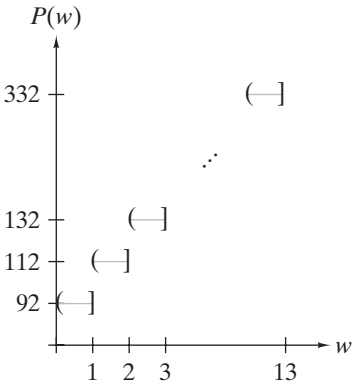


Figure 3.1.8 The graph of the postage function $P(w) = 92 + 20\lceil w - 1\rceil$.

$$P(w) = 92 + 20\lceil w - 1\rceil \qquad 13 \geq w > 0.$$

The expression $\lceil w-1\rceil$ counts the number of additional ounces beyond 1, with a fraction counting as one additional ounce. As examples,

$$P(3.7) = 92 + 20\lceil 3.7 - 1\rceil = 92 + 20\lceil 2.7\rceil = 92 + 20 \cdot 3 = 152,$$
$$P(2) = 92 + 20\lceil 2 - 1\rceil = 92 + 20\lceil 1\rceil = 92 + 20 \cdot 1 = 112.$$

The graph of the function P is shown in Figure 3.1.8. ◀

The Quotient-Remainder Theorem (Theorem 2.5.6) states that if d and n are integers, $d > 0$, there exist integers q (quotient) and r (remainder) satisfying

$$n = dq + r \qquad 0 \leq r < d.$$

Dividing by d, we obtain

$$\frac{n}{d} = q + \frac{r}{d}.$$

Since $0 \leq r/d < 1$,

$$\left\lfloor \frac{n}{d} \right\rfloor = \left\lfloor q + \frac{r}{d} \right\rfloor = q.$$

Thus, we may compute the quotient q as $\lfloor n/d \rfloor$. Having computed the quotient q, we may compute the remainder as $r = n - dq$. We previously introduced the notation n mod d for the remainder.

Example 3.1.21 We have $36844/2427 = 15.18088\ldots$; thus the quotient is $q = \lfloor 36844/2427 \rfloor = 15$. Therefore, the remainder 36844 mod 2427 is $r = 36844 - 2427 \cdot 15 = 439$. We have $n = dq + r$ or $36844 = 2427 \cdot 15 + 439$. ◀

Definition 3.1.22 ▶ A function f from X to Y is said to be *one-to-one* (or *injective*) if for all $x_1,\ x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$. ◀

An equivalent way to state Definition 3.1.22 is: If y is an element of the range of f, then there is *exactly one* x in the domain of f such that $f(x) = y$. If there were two distinct elements x_1 and x_2 of the domain of f with $f(x_1) = y = f(x_2)$, then we would have $f(x_1) = f(x_2)$ but $x_1 \neq x_2$—a counterexample to the claim that f is one-to-one.

Because the amount of potential data is usually so much larger than the available memory, hash functions are usually not one-to-one (see Example 3.1.15). In other words, most hash functions produce collisions.

Example 3.1.23 The function $f = \{(1, b), (3, a), (2, c)\}$ from $X = \{1, 2, 3\}$ to $Y = \{a, b, c, d\}$ is one-to-one. ◀

Example 3.1.24 The function $f = \{(1, a), (2, b), (3, a)\}$ is not one-to-one since $f(1) = a = f(3)$. ◀

Example 3.1.25 If X is the set of persons who have social security numbers and we assign each person $x \in X$ his or her social security number $\mathrm{SS}(x)$, we obtain a one-to-one function since distinct persons are always assigned distinct social security numbers. It is because this correspondence is one-to-one that the government uses social security numbers as identifiers. ◀

Example 3.1.26 If a function from X to Y is one-to-one, each element in Y in its arrow diagram will have at most one arrow pointing to it (see Figure 3.1.9). If a function is not one-to-one, some element in Y in its arrow diagram will have two or more arrows pointing to it (see Figure 3.1.10).

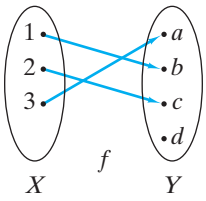


Figure 3.1.9 The function of Example 3.1.23. This function is one-to-one because each element in Y has at most one arrow pointing to it. This function is not onto Y because there is no arrow pointing to d.

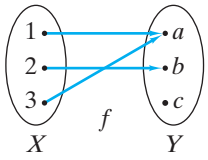


Figure 3.1.10 A function that is not one-to-one. This function is not one-to-one because a has two arrows pointing to it. This function is not onto Y because there is no arrow pointing to c. ◀

Example 3.1.27 Prove that the function $f(n) = 2n + 1$ from the set of positive integers to the set of positive integers is one-to-one.

SOLUTION We must show that for all positive integers n_1 and n_2, if $f(n_1) = f(n_2)$, then $n_1 = n_2$. So, suppose that $f(n_1) = f(n_2)$. Using the definition of f, this latter equation translates as $2n_1 + 1 = 2n_2 + 1$. Subtracting 1 from both sides of the equation and then dividing both sides of the equation by 2 yields $n_1 = n_2$. Therefore, f is one-to-one. ◀

Example 3.1.28 Prove that the function $f(n) = 2^n - n^2$ from the set of positive integers to the set of integers is *not* one-to-one.

SOLUTION We must find positive integers n_1 and n_2, $n_1 \neq n_2$, such that $f(n_1) = f(n_2)$. By checking the graph (see Figure 3.1.11) or otherwise, we find that $f(2) = f(4)$. Therefore, f is not one-to-one.

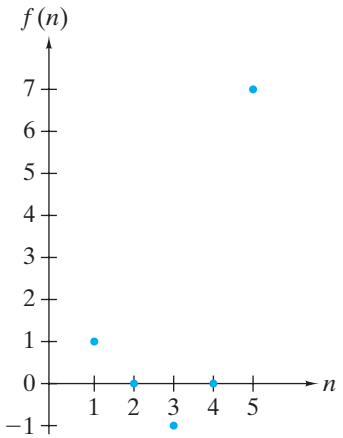


Figure 3.1.11 The graph of $f(n) = 2^n - n^2$. ◀

If the range of a function f is equal to its codomain Y, the function is said to be **onto** Y.

Definition 3.1.29 ▶ A function f from X to Y is said to be *onto* Y (or *surjective*) if for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$. ◀

Example 3.1.30 The function $f = \{(1, a), (2, c), (3, b)\}$ from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$ is one-to-one and onto Y. ◀

Example 3.1.31 The function $f = \{(1, b), (3, a), (2, c)\}$ from $X = \{1, 2, 3\}$ to $Y = \{a, b, c, d\}$ is *not* onto Y. ◀

Example 3.1.32 If a function from X to Y is onto, each element in Y in its arrow diagram will have at least one arrow pointing to it (see Figure 3.1.12). If a function from X to Y is not onto, some element in Y in its arrow diagram will fail to have an arrow pointing to it (see Figures 3.1.9 and 3.1.10).

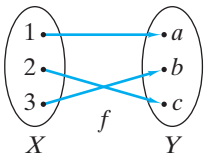


Figure 3.1.12 The function of Example 3.1.30. This function is one-to-one because each element in Y has at most one arrow. This function is onto because each element in Y has at least one arrow pointing to it. ◀

Example 3.1.33 Prove that the function

$$f(x) = \frac{1}{x^2}$$

from the set X of nonzero real numbers to the set Y of positive real numbers is onto Y.

SOLUTION We must show that for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$. Substituting the formula for $f(x)$, this last equation becomes

$$\frac{1}{x^2} = y.$$

Solving for x, we find

$$x = \pm\frac{1}{\sqrt{y}}.$$

Notice that $1/\sqrt{y}$ is defined because y is a positive real number. If we take x to be the positive square root

$$x = \frac{1}{\sqrt{y}},$$

then $x \in X$. (We could just as well have taken $x = -1/\sqrt{y}$.) Thus, for every $y \in Y$, there exists x, namely, $x = 1/\sqrt{y}$ such that

$$f(x) = f(1/\sqrt{y}) = \frac{1}{(1/\sqrt{y})^2} = y.$$

Therefore, f is onto Y. ◀

A function f from X to Y is *not* onto Y if for some $y \in Y$, for every $x \in X$, $f(x) \neq y$. In other words, y is a counterexample to the claim that for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

Example 3.1.34 Prove that the function $f(n) = 2n - 1$ from the set X of positive integers to the set Y of positive integers is *not* onto Y.

SOLUTION We must find an element $m \in Y$ such that for all $n \in X$, $f(n) \neq m$. Since $f(n)$ is an odd integer for all n, we may choose for y any positive, even integer, for example, $y = 2$. Then $y \in Y$ and $f(n) \neq y$ for all $n \in X$. Thus f is not onto Y. ◀

Definition 3.1.35 ▶ A function that is both one-to-one and onto is called a *bijection*. ◀

Example 3.1.36 The function f of Example 3.1.30 is a bijection. ◀

Example 3.1.37 If f is a bijection from a finite set X to a finite set Y, then $|X| = |Y|$, that is, the sets have the same cardinality and are the same size. For example, $f = \{(1, a), (2, b), (3, c), (4, d)\}$ is a bijection from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c, d)\}$. Both sets have four elements. In effect, f counts the elements in Y: $f(1) = a$ is the first element in Y; $f(2) = b$ is the second element in Y; and so on. ◀

Suppose that f is a one-to-one, onto function from X to Y. It can be shown (see Exercise 116) that $\{(y, x) \mid (x, y) \in f\}$ is a one-to-one, onto function from Y to X. This new function, denoted f^{-1}, is called f **inverse.**

Example 3.1.38 For the function $f = \{(1, a), (2, c), (3, b)\}$, we have $f^{-1} = \{(a, 1), (c, 2), (b, 3)\}$. ◀

Example 3.1.39 Given the arrow diagram for a one-to-one, onto function f from X to Y, we can obtain the arrow diagram for f^{-1} simply by reversing the direction of each arrow (see Figure 3.1.13, which is the arrow diagram for f^{-1}, where f is the function of Figure 3.1.12).

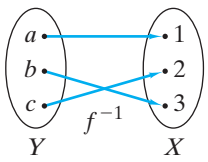


Figure 3.1.13 The inverse of the function in Figure 3.1.12. The inverse is obtained by reversing all of the arrows in Figure 3.1.12.

◀

Example 3.1.40 The function $f(x) = 2^x$ is a one-to-one function from the set $\mathbf{R}$ of all real numbers onto the set $\mathbf{R}^+$ of all positive real numbers. Derive a formula for $f^{-1}(y)$.

SOLUTION Suppose that (y, x) is in f^{-1}; that is,

$$f^{-1}(y) = x. \tag{3.1.3}$$

Then $(x, y) \in f$. Thus, $y = 2^x$. By the definition of logarithm,

$$\log_2 y = x. \tag{3.1.4}$$

Combining (3.1.3) and (3.1.4), we have $f^{-1}(y) = x = \log_2 y$. That is, for each $y \in \mathbf{R}^+$, $f^{-1}(y)$ is the logarithm to the base 2 of y. We can summarize the situation by saying that the inverse of the exponential function is the logarithm function. ◀

Let g be a function from X to Y and let f be a function from Y to Z. Given $x \in X$, we may apply g to determine a unique element $y = g(x) \in Y$. We may then apply f to determine a unique element $z = f(y) = f(g(x)) \in Z$. This compound action is called **composition.**

Definition 3.1.41 ▶ Let g be a function from X to Y and let f be a function from Y to Z. The *composition of f with g*, denoted $f \circ g$, is the function

$$(f \circ g)(x) = f(g(x))$$

from X to Z. ◀

Example 3.1.42 Given $g = \{(1, a), (2, a), (3, c)\}$, a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$, and $f = \{(a, y), (b, x), (c, z)\}$, a function from Y to $Z = \{x, y, z\}$, the composition function from X to Z is the function $f \circ g = \{(1, y), (2, y), (3, z)\}$. ◀

Example 3.1.43 Given the arrow diagram for a function g from X to Y and the arrow diagram for a function f from Y to Z, we can obtain the arrow diagram for the composition $f \circ g$ simply by "following the arrows" (see Figure 3.1.14).

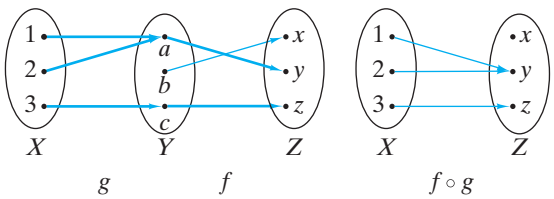


Figure 3.1.14 The composition of the functions of Example 3.1.42. The composition is obtained by drawing an arrow from x in X to z in Z provided that there are arrows from x to some y in Y and from y to z. ◀

Example 3.1.44 If $f(x) = \log_3 x$ and $g(x) = x^4$, then $f(g(x)) = \log_3(x^4)$, and $g(f(x)) = (\log_3 x)^4$. ◀

Example 3.1.45 A store offers 15% off the price of certain items. A coupon is also available that offers \$20 off the price of the same items. The store will honor both discounts. The function $D(p) = 0.85p$ gives the cost with 15% off the price p. The function $C(p) = p - 20$ gives the cost using the \$20 coupon. The composition

$$(D \circ C)(p) = 0.85(p - 20) = 0.85p - 17$$

gives the cost using first the coupon and then the 15% discount. The composition $(C \circ D)(p) = 0.85p - 20$ gives the cost using first the 15% discount and then the coupon.

We see that regardless of the price of an item, it is always cheapest to use the discount first. ◀

Example 3.1.46 Composition sometimes allows us to decompose complicated functions into simpler functions. For example, the function $f(x) = \sqrt{\sin 2x}$ can be decomposed into the functions

$$g(x) = \sqrt{x}, \qquad h(x) = \sin x, \qquad w(x) = 2x.$$

We can then write $f(x) = g(h(w(x)))$. This decomposition technique is important in differential calculus since there are rules for differentiating simple functions such as g, h, and w and also rules about how to differentiate the composition of functions. Combining these rules, we can differentiate more complicated functions. ◀

A **binary operator** on a set X associates with each ordered pair of elements in X one element in X.

Definition 3.1.47 ▶ A function from $X \times X$ to X is called a *binary operator* on X. ◀

Example 3.1.48 Let $X = \{1, 2, \ldots\}$. If we define $f(x, y) = x + y$, where $x, y \in X$, then f is a binary operator on X. ◀

Example 3.1.49 If X is a set of propositions, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$ are binary operators on X. ◀

A **unary operator** on a set X associates with each single element of X one element in X.

Definition 3.1.50 ▶ A function from X to X is called a *unary operator* on X. ◀

Example 3.1.51 Let U be a universal set. If we define $f(X) = \overline{X}$, where $X \in \mathcal{P}(U)$, then f is a unary operator on $\mathcal{P}(U)$. ◀

Example 3.1.52 If X is a set of propositions, $\neg$ is a unary operator on X. ◀

3.1 Problem-Solving Tips

The key to solving problems involving functions is clearly understanding the definition of function. A function f from X to Y can be thought of in many ways. Formally, f is a subset of $X \times Y$ having the property that for every $x \in X$, there is a unique $y \in Y$ such that $(x, y) \in X \times Y$. Informally, f can be thought of as a *mapping* of elements from X to Y. The arrow diagram emphasizes this view of a function. For an arrow diagram to be a function, there must be exactly one arrow from each element in X to some element in Y.

A function is a very general concept. *Any* subset of $X \times Y$ having the property that for every $x \in X$, there is a unique $y \in Y$ such that $(x, y) \in X \times Y$ is a function. A function may be defined by listing its members; for example, $\{(a, 1), (b, 3), (c, 2), (d, 1)\}$ is a function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$. Here, there is apparently no formula for membership; the definition just tells us which pairs make up the function.

On the other hand, a function may be defined by a formula. For example,

$$\{(n, n + 2) \mid n \text{ is a positive integer}\}$$

defines a function from the set of positive integers to the set of positive integers. The "formula" for the mapping is "add 2."

The $f(x)$ notation may be used to indicate which element in the codomain is associated with an element x in the domain or to define a function. For example, for the function $f = \{(a, 1), (b, 3), (c, 2), (d, 1)\}$, we could write $f(a) = 1, f(b) = 3$, and so on. Assuming that the domain of definition is the positive integers, the equation $g(n) = n+2$ defines the function $\{(n, n + 2) \mid n \text{ is a positive integer}\}$ from the set of positive integers to the set of positive integers.

To prove that a function f from X to Y is one-to-one, show that for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

To prove that a function f from X to Y is *not* one-to-one, find $x_1, x_2 \in X, x_1 \neq x_2$, such that $f(x_1) = f(x_2)$.

To prove that a function f from X to Y is onto, show that for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

To prove that a function f from X to Y is *not* onto, find $y \in Y$ such that $f(x) \neq y$ for all $x \in X$.

3.1 Review Exercises

1. What is a function from X to Y?
2. Explain how to use an arrow diagram to depict a function.
3. What is the graph of a function?
4. Given a set of points in the plane, how can we tell whether it is a function?
5. What is the value of x mod y?
6. What is a hash function?
7. What is a collision for a hash function?
8. What is a collision resolution policy?
9. What are pseudorandom numbers?
10. Explain how a linear congruential random number generator works, and give an example of a linear congruential random number generator.
11. What is the floor of x? How is the floor denoted?
12. What is the ceiling of x? How is the ceiling denoted?
13. Define *one-to-one function*. Give an example of a one-to-one function. Explain how to use an arrow diagram to determine whether a function is one-to-one.
14. Define *onto function*. Give an example of an onto function. Explain how to use an arrow diagram to determine whether a function is onto.
15. What is a bijection? Give an example of a bijection. Explain how to use an arrow diagram to determine whether a function is a bijection.
16. Define *inverse function*. Give an example of a function and its inverse. Given the arrow diagram of a function, how can we find the arrow diagram of the inverse function?
17. Define *composition of functions*. How is the composition of f and g denoted? Give an example of functions f and g and their composition. Given the arrow diagrams of two functions, how can we find the arrow diagram of the composition of the functions?
18. What is a binary operator? Give an example of a binary operator.
19. What is a unary operator? Give an example of a unary operator.

3.1 Exercises

In Exercises 1–6, determine which credit card numbers have correct check digits.

1. 5366-2806-9965-4138
2. 5194-1132-8860-3905
3. 4004-6067-3429-0019
4. 3419-6888-7169-444
5. 3016-4773-7532-21
6. 4629-9521-3698-0203
7. Show that when 82 in the valid credit card number 4690-3582-1375-4657 is transposed to 28, the check digit changes.

Determine whether each set in Exercises 8–12 is a function from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c, d\}$. If it is a function, find its do-

main and range, draw its arrow diagram, and determine if it is one-to-one, onto, or both. If it is both one-to-one and onto, give the description of the inverse function as a set of ordered pairs, draw its arrow diagram, and give the domain and range of the inverse function.

8. $\{(1, a), (2, a), (3, c), (4, b)\}$

9. $\{(1, c), (2, a), (3, b), (4, c), (2, d)\}$

10. $\{(1, c), (2, d), (3, a), (4, b)\}$

11. $\{(1, d), (2, d), (4, a)\}$

12. $\{(1, b), (2, b), (3, b), (4, b)\}$

Draw the graphs of the functions in Exercises 13–16. The domain of each function is the set of real numbers. The codomain of each function is also the set of real numbers.

13. $f(x) = \lceil x \rceil - \lfloor x \rfloor$

14. $f(x) = x - \lfloor x \rfloor$

15. $f(x) = \lceil x^2 \rceil$

16. $f(x) = \lfloor x^2 - x \rfloor$

Determine whether each function in Exercises 17–22 is one-to-one, onto, or both. Prove your answers. The domain of each function is the set of all integers. The codomain of each function is also the set of all integers.

17. $f(n) = n + 1$

18. $f(n) = n^2 - 1$

19. $f(n) = \lceil n/2 \rceil$

20. $f(n) = |n|$

21. $f(n) = 2n$

22. $f(n) = n^3$

Determine whether each function in Exercises 23–28 is one-to-one, onto, or both. Prove your answers. The domain of each function is $\mathbf{Z} \times \mathbf{Z}$. *The codomain of each function is* $\mathbf{Z}$.

23. $f(m, n) = m - n$

24. $f(m, n) = m$

25. $f(m, n) = mn$

26. $f(m, n) = m^2 + n^2$

27. $f(m, n) = n^2 + 1$

28. $f(m, n) = m + n + 2$

29. Prove that the function f from $\mathbf{Z}^+ \times \mathbf{Z}^+$ to $\mathbf{Z}^+$ defined by $f(m, n) = 2^m 3^n$ is one-to-one but not onto.

Determine whether each function in Exercises 30–35 is one-to-one, onto, or both. Prove your answers. The domain of each function is the set of all real numbers. The codomain of each function is also the set of all real numbers.

30. $f(x) = 6x - 9$

31. $f(x) = 3x^2 - 3x + 1$

32. $f(x) = \sin x$

33. $f(x) = 2x^3 - 4$

34. $f(x) = 3^x - 2$

35. $f(x) = \dfrac{x}{1 + x^2}$

36. Give an example of a function different from those presented in the text that is one-to-one but not onto, and prove that your function has the required properties.

37. Give an example of a function different from those presented in the text that is onto but not one-to-one, and prove that your function has the required properties.

38. Give an example of a function different from those presented in the text that is neither one-to-one nor onto, and prove that your function has the required properties.

39. Write the definition of "one-to-one" using logical notation (i.e., use $\forall$, $\exists$, etc.).

40. Use De Morgan's laws of logic to negate the definition of "one-to-one."

41. Write the definition of "onto" using logical notation (i.e., use $\forall$, $\exists$, etc.).

42. Use De Morgan's laws of logic to negate the definition of "onto."

Each function in Exercises 43–48 is one-to-one on the specified domain X. By letting $Y =$ *range of* f, *we obtain a bijection from X to Y. Find each inverse function.*

43. $f(x) = 4x + 2$, $X =$ set of real numbers

44. $f(x) = 3^x$, $X =$ set of real numbers

45. $f(x) = 3 \log_2 x$, $X =$ set of positive real numbers

46. $f(x) = 3 + \dfrac{1}{x}$, $X =$ set of nonzero real numbers

47. $f(x) = 4x^3 - 5$, $X =$ set of real numbers

48. $f(x) = 6 + 2^{7x-1}$, $X =$ set of real numbers

49. Given

$$g = \{(1, b), (2, c), (3, a)\},$$

a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c, d\}$, and

$$f = \{(a, x), (b, x), (c, z), (d, w)\},$$

a function from Y to $Z = \{w, x, y, z\}$, write $f \circ g$ as a set of ordered pairs and draw the arrow diagram of $f \circ g$.

50. Let f and g be functions from the positive integers to the positive integers defined by the equations

$$f(n) = 2n + 1, \qquad g(n) = 3n - 1.$$

Find the compositions $f \circ f$, $g \circ g$, $f \circ g$, and $g \circ f$.

51. Let f and g be functions from the positive integers to the positive integers defined by the equations

$$f(n) = n^2, \qquad g(n) = 2^n.$$

Find the compositions $f \circ f$, $g \circ g$, $f \circ g$, and $g \circ f$.

52. Let f and g be functions from the nonnegative real numbers to the nonnegative real numbers defined by the equations

$$f(x) = \lfloor 2x \rfloor, \qquad g(x) = x^2.$$

Find the compositions $f \circ f$, $g \circ g$, $f \circ g$, and $g \circ f$.

53. A store offers a (fixed, nonzero) percentage off the price of certain items. A coupon is also available that offers a (fixed, nonzero) amount off the price of the same items. The store will honor both discounts. Show that regardless of the price of an item, the percentage off the price, and amount off the price, it is always cheapest to use the coupon first.

In Exercises 54–59, decompose the function into simpler functions as in Example 3.1.46.

54. $f(x) = \log_2(x^2 + 2)$

55. $f(x) = \dfrac{1}{2x^2}$

56. $f(x) = \sin 2x$

57. $f(x) = 2 \sin x$

58. $f(x) = (3 + \sin x)^4$

59. $f(x) = \dfrac{1}{(\cos 6x)^3}$

60. Given

$$f = \{(x, x^2) \mid x \in X\},$$

a function from $X = \{-5, -4, \ldots, 4, 5\}$ to the set of integers, write f as a set of ordered pairs and draw the arrow diagram of f. Is f one-to-one or onto?

61. How many functions are there from $\{1, 2\}$ to $\{a, b\}$? Which are one-to-one? Which are onto?

62. Given

$$f = \{(a, b), (b, a), (c, b)\},$$

a function from $X = \{a, b, c\}$ to X:

(a) Write $f \circ f$ and $f \circ f \circ f$ as sets of ordered pairs.

(b) Define

$$f^n = f \circ f \circ \cdots \circ f$$

to be the n-fold composition of f with itself. Write f^9 and f^{623} as sets of ordered pairs.

63. Let f be the function from $X = \{0, 1, 2, 3, 4\}$ to X defined by

$$f(x) = 4x \bmod 5.$$

Write f as a set of ordered pairs and draw the arrow diagram of f. Is f one-to-one? Is f onto?

64. Let f be the function from $X = \{0, 1, 2, 3, 4, 5\}$ to X defined by

$$f(x) = 4x \bmod 6.$$

Write f as a set of ordered pairs and draw the arrow diagram of f. Is f one-to-one? Is f onto?

65. An International Standard Book Number (ISBN) is a code of 13 characters separated by dashes, such as 978-1-59448-950-1. An ISBN consists of five parts: a product code, a group code, a publisher code, a code that uniquely identifies the book among those published by the particular publisher, and a check digit. For 978-1-59448-950-1, the product code 978 identifies the product as a book (the same scheme is used for other products). The group code is 1, which identifies the book as one from an English-speaking country. The publisher code 59448 identifies the book as one published by Riverhead Books, Penguin Group. The code 950 uniquely identifies the book among those published by Riverhead Books, Penguin Group (Hosseini: *A Thousand Splendid Suns*, in this case). The check digit is 1.

Let S equal the sum of the first digit, plus three times the second digit, plus the third digit, plus three times the fourth digit, . . . , plus three times the twelfth digit. The check digit is equal to

$$[10 - (S \bmod 10)] \bmod 10.$$

Universal Product Codes (UPC), the bar codes that are scanned at the grocery store for example, use a similar method to compute the check digit.

Verify the check digit for this book.

In Exercises 66–71, determine which ISBNs (see Exercise 65) have correct check digits.

66. 978-1-61374-376-9

67. 978-0-8108-8139-2

68. 978-0-939460-91-5

69. 978-0-8174-3593-6

70. 978-1-4354-6028-7

71. 978-0-684-87018-0

72. Show that if a single digit of an ISBN is changed, the check digit will change. Thus, any single-digit error will be detected.

For each hash function in Exercises 73–76, show how the data would be inserted in the order given in initially empty cells. Use the collision resolution policy of Example 3.1.15.

73. $h(x) = x \bmod 11$; cells indexed 0 to 10; data: 53, 13, 281, 743, 377, 20, 10, 796

74. $h(x) = x \bmod 17$; cells indexed 0 to 16; data: 714, 631, 26, 373, 775, 906, 509, 2032, 42, 4, 136, 1028

75. $h(x) = x^2 \bmod 11$; cells and data as in Exercise 73

76. $h(x) = (x^2 + x) \bmod 17$; cells and data as in Exercise 74

77. Suppose that we store and retrieve data as described in Example 3.1.15. Will any problem arise if we delete data? Explain.

78. Suppose that we store data as described in Example 3.1.15 and that we never store more than 10 items. Will any problem arise when retrieving data if we stop searching when we encounter an empty cell? Explain.

79. Suppose that we store data as described in Example 3.1.15 and retrieve data as described in Exercise 78. Will any problem arise if we delete data? Explain.

Let g be a function from X to Y and let f be a function from Y to Z. For each statement in Exercises 80–87, if the statement is true, prove it; otherwise, give a counterexample.

80. If g is one-to-one, then $f \circ g$ is one-to-one.

81. If f is onto, then $f \circ g$ is onto.

82. If g is onto, then $f \circ g$ is onto.

83. If f and g are onto, then $f \circ g$ is onto.

84. If f and g are one-to-one and onto, then $f \circ g$ is one-to-one and onto.

85. If $f \circ g$ is one-to-one, then f is one-to-one.

86. If $f \circ g$ is one-to-one, then g is one-to-one.

87. If $f \circ g$ is onto, then f is onto.

If f is a function from X to Y and $A \subseteq X$ and $B \subseteq Y$, we define

$$f(A) = \{f(x) \mid x \in A\}, \qquad f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

We call $f^{-1}(B)$ the inverse image of B under f.

88. Let

$$g = \{(1, a), (2, c), (3, c)\}$$

be a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c, d\}$. Let $S = \{1\}$, $T = \{1, 3\}$, $U = \{a\}$, and $V = \{a, c\}$. Find $g(S)$, $g(T)$, $g^{-1}(U)$, and $g^{-1}(V)$.

⋆89. Let f be a function from X to Y. Prove that f is one-to-one if and only if

$$f(A \cap B) = f(A) \cap f(B)$$

for all subsets A and B of X. [When S is a set, we define $f(S) = \{f(x) \mid x \in S\}$.]

⋆90. Let f be a function from X to Y. Prove that f is one-to-one if and only if whenever g is a one-to-one function from any set A to X, $f \circ g$ is one-to-one.

⋆91. Let f be a function from X to Y. Prove that f is onto Y if and only if whenever g is a function from Y onto any set Z, $g \circ f$ is onto Z.

92. Let f be a function from X onto Y. Let

$$\mathcal{S} = \{f^{-1}(\{y\}) \mid y \in Y\}.$$

Show that $\mathcal{S}$ is a partition of X.

Let $\mathbf{R}^{\mathbf{R}}$ denote the set of functions from $\mathbf{R}$ to $\mathbf{R}$. We define the evaluation function E_a, *where $a \in \mathbf{R}$, from $\mathbf{R}^{\mathbf{R}}$ to $\mathbf{R}$ as*

$$E_a(f) = f(a).$$

93. Is E_1 one-to-one? Prove your answer.

94. Is E_1 onto? Prove your answer.

95. Let f be a function from $\mathbf{R}$ to $\mathbf{R}$ such that for some $r \in \mathbf{R}$, $f(rx) = rf(x)$ for all $x \in \mathbf{R}$. Prove that $f(r^n x) = r^n f(x)$ for all $x \in \mathbf{R}$, $n \in \mathbf{Z}^+$.

Exercises 96–100 use the following definitions. Let $X = \{a, b, c\}$. Define a function S from $\mathcal{P}(X)$ to the set of bit strings of length 3 as follows. Let $Y \subseteq X$. If $a \in Y$, set $s_1 = 1$; if $a \notin Y$, set $s_1 = 0$. If $b \in Y$, set $s_2 = 1$; if $b \notin Y$, set $s_2 = 0$. If $c \in Y$, set $s_3 = 1$; if $c \notin Y$, set $s_3 = 0$. Define $S(Y) = s_1 s_2 s_3$.

96. What is the value of $S(\{a, c\})$?

97. What is the value of $S(\varnothing)$?

98. What is the value of $S(X)$?

99. Prove that S is one-to-one.

100. Prove that S is onto.

Exercises 101–107 use the following definitions. Let U be a universal set and let $X \subseteq U$. Define

$$C_X(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \notin X. \end{cases}$$

We call C_X the characteristic function of X (in U). *(A look ahead at the next Problem-Solving Corner may help in understanding the following exercises.)*

101. Prove that $C_{X \cap Y}(x) = C_X(x) C_Y(x)$ for all $x \in U$.

102. Prove that $C_{X \cup Y}(x) = C_X(x) + C_Y(x) - C_X(x) C_Y(x)$ for all $x \in U$.

103. Prove that $C_{\overline{X}}(x) = 1 - C_X(x)$ for all $x \in U$.

104. Prove that $C_{X - Y}(x) = C_X(x)[1 - C_Y(x)]$ for all $x \in U$.

105. Prove that if $X \subseteq Y$, then $C_X(x) \leq C_Y(x)$ for all $x \in U$.

106. Find a formula for $C_{X \triangle Y}$. ($X \triangle Y$ is the symmetric difference of X and Y. The definition is given before Exercise 101, Section 1.1.)

107. Prove that the function f from $\mathcal{P}(U)$ to the set of characteristic functions in U defined by

$$f(X) = C_X$$

is one-to-one and onto.

108. Let X and Y be sets. Prove that there is a one-to-one function from X to Y if and only if there is a function from Y onto X.

A binary operator f on a set X is commutative *if $f(x, y) = f(y, x)$ for all $x, y \in X$. In Exercises 109–113, state whether the given function f is a binary operator on the set X. If f is not a binary operator, state why. State whether or not each binary operator is commutative.*

109. $f(x, y) = x + y, \quad X = \{1, 2, \ldots\}$

110. $f(x, y) = x - y, \quad X = \{1, 2, \ldots\}$

111. $f(x, y) = x \cup y, \quad X = \mathcal{P}(\{1, 2, 3, 4\})$

112. $f(x, y) = x/y, \quad X = \{0, 1, 2, \ldots\}$

113. $f(x, y) = x^2 + y^2 - xy, \quad X = \{1, 2, \ldots\}$

In Exercises 114 and 115, give an example of a unary operator [different from $f(x) = x$, for all x] on the given set.

114. $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

115. The set of all finite subsets of $\{1, 2, 3, \ldots\}$

116. Prove that if f is a one-to-one, onto function from X to Y, then

$$\{(y, x) \mid (x, y) \in f\}$$

is a one-to-one, onto function from Y to X.

In Exercises 117–119, if the statement is true for all real numbers, prove it; otherwise, give a counterexample.

117. $\lceil x + 3 \rceil = \lceil x \rceil + 3$

118. $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$

119. $\lfloor x + y \rfloor = \lfloor x \rfloor + \lceil y \rceil$

120. Prove that if n is an odd integer,

$$\left\lfloor \frac{n^2}{4} \right\rfloor = \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right).$$

121. Prove that if n is an odd integer,

$$\left\lceil \frac{n^2}{4} \right\rceil = \frac{n^2+3}{4}.$$

122. Find a value for x for which $\lceil 2x \rceil = 2\lceil x \rceil - 1$.

123. Prove that $2\lceil x \rceil - 1 \leq \lceil 2x \rceil \leq 2\lceil x \rceil$ for all real numbers x.

124. Prove that for all real numbers x and integers n, $\lceil x \rceil = n$ if and only if there exists ε, $0 \leq \varepsilon < 1$, such that $x + \varepsilon = n$.

125. State and prove a result analogous to Exercise 124 for $\lfloor x \rfloor$.

The months with Friday the 13th in year x are found in row

$$y = \left(x + \left\lfloor \frac{x-1}{4} \right\rfloor - \left\lfloor \frac{x-1}{100} \right\rfloor + \left\lfloor \frac{x-1}{400} \right\rfloor \right) \bmod 7$$

in the appropriate column:

y	Non-Leap Year	Leap Year
0	January, October	January, April, July
1	April, July	September, December
2	September, December	June
3	June	March, November
4	February, March, November	February, August
5	August	May
6	May	October

126. Find the months with Friday the 13th in 1945.

127. Find the months with Friday the 13th in the current year.

128. Find the months with Friday the 13th in 2040.

Problem-Solving Corner

Functions

Problem

Let U be a universal set and let $X \subseteq U$. Define

$$C_X(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \notin X. \end{cases}$$

[We call C_X the *characteristic function* of X (in U)]. Assume that X and Y are arbitrary subsets of the universal set U. Prove that $C_{X \cup Y}(x) = C_X(x) + C_Y(x)$ for all $x \in U$ if and only if $X \cap Y = \varnothing$.

Attacking the Problem

First, let's be clear what we must do. Since the statement is of the form p if and only if q, we have two tasks: (1) Prove if p then q. (2) Prove if q then p. It's a good idea to write out exactly what must be proved:

If $C_{X \cup Y}(x) = C_X(x) + C_Y(x)$ for all $x \in U$, then $X \cap Y = \varnothing$. **(1)**

If $X \cap Y = \varnothing$, then $C_{X \cup Y}(x) = C_X(x) + C_Y(x)$ for all $x \in U$. **(2)**

Consider the first statement in which we assume that $C_{X \cup Y}(x) = C_X(x) + C_Y(x)$ for all $x \in U$ and prove that $X \cap Y = \varnothing$. How do we prove that a set, $X \cap Y$ in this case, is the empty set? We have to show that $X \cap Y$ has no elements. How do we do that? There are several possibilities, but one thing that comes to mind is another question: What if $X \cap Y$ had an element? This suggests that we might prove the first statement by contradiction or by proving its contrapositive. If we let

p: $C_{X \cup Y}(x) = C_X(x) + C_Y(x)$ for all $x \in U$

q: $X \cap Y = \varnothing$,

the contrapositive is $\neg q \rightarrow \neg p$. Now the negation of q is

$$\neg q : X \cap Y \neq \varnothing,$$

and, using De Morgan's law (roughly, negating $\forall$ results in $\exists$), the negation of p is

$$\neg p : C_{X \cup Y}(x) \neq C_X(x) + C_Y(x) \text{ for at least one } x \in U.$$

Thus, the contrapositive is

If $X \cap Y \neq \varnothing$, then $C_{X \cup Y}(x) \neq C_X(x) + C_Y(x)$ for at least one $x \in U$. **(3)**

For the second statement, we assume that $X \cap Y = \varnothing$ and prove that $C_{X \cup Y}(x) = C_X(x) + C_Y(x)$ for all $x \in U$. Presumably, we can just use the definition of C_X to compute both sides of the equation for all $x \in U$ and verify that the two sides are equal. The definition of C_X suggests that we use proof by cases: $x \in X \cup Y$ (when $C_{X \cup Y}(x) = 1$) and $x \notin X \cup Y$ (when $C_{X \cup Y}(x) = 0$).

Finding a Solution

We first consider proving the contrapositive (3) of statement (1). Since we assume that $X \cap Y \neq \varnothing$, there exists an element $x \in X \cap Y$. Now let's compare the values of the expressions $C_{X\cup Y}(x)$ and $C_X(x) + C_Y(x)$. Since $x \in X \cup Y$, $C_{X\cup Y}(x) = 1$. Since $x \in X \cap Y$, $x \in X$ and $x \in Y$. Therefore

$$C_X(x) + C_Y(x) = 1 + 1 = 2.$$

We have proved that

$$C_{X\cup Y}(x) \neq C_X(x) + C_Y(x) \text{ for at least one } x \in U.$$

Now consider proving the statement (2). This time we assume that $X \cap Y = \varnothing$. Let's compute each side of the equation

$$C_{X\cup Y}(x) = C_X(x) + C_Y(x) \qquad \textbf{(4)}$$

for each $x \in U$. As suggested earlier, we consider the cases: $x \in X \cup Y$ and $x \notin X \cup Y$. If $x \in X \cup Y$, then $C_{X\cup Y}(x) = 1$. Since $X \cap Y = \varnothing$, either $x \in X$ or $x \in Y$ *but not both*. Therefore,

$$C_X(x) + C_Y(x) = 1 + 0 = 1 = C_{X\cup Y}(x)$$

or

$$C_X(x) + C_Y(x) = 0 + 1 = 1 = C_{X\cup Y}(x).$$

Equation (4) is true if $x \in X \cup Y$.

If $x \notin X \cup Y$, then $C_{X\cup Y}(x) = 0$. But if $x \notin X \cup Y$, then $x \notin X$ *and* $x \notin Y$. Therefore,

$$C_X(x) + C_Y(x) = 0 + 0 = 0 = C_{X\cup Y}(x).$$

Equation (4) is true if $x \notin X \cup Y$. Thus it is true for all $x \in U$.

Formal Solution

The formal proof could be written as follows.

CASE $\rightarrow$: If $C_{X\cup Y}(x) = C_X(x) + C_Y(x)$ for all $x \in U$, then $X \cap Y = \varnothing$.

We prove the equivalent contrapositive

If $X \cap Y \neq \varnothing$, then $C_{X\cup Y}(x) \neq C_X(x) + C_Y(x)$ for at least one $x \in U$.

Since $X \cap Y \neq \varnothing$, there exists $x \in X \cap Y$. Since $x \in X \cup Y$, $C_{X\cup Y}(x) = 1$. Since $x \in X \cap Y$, $x \in X$ and $x \in Y$. Therefore

$$C_X(x) + C_Y(x) = 1 + 1 = 2.$$

Thus,

$$C_{X\cup Y}(x) \neq C_X(x) + C_Y(x).$$

CASE $\leftarrow$: If $X \cap Y = \varnothing$, then $C_{X\cup Y}(x) = C_X(x) + C_Y(x)$ for all $x \in U$.

Suppose that $x \in X \cup Y$. Then $C_{X\cup Y}(x) = 1$. Since $X \cap Y = \varnothing$, either $x \in X$ or $x \in Y$ but not both. Therefore, $C_X(x) + C_Y(x) = 1$ and (4) holds.

If $x \notin X \cup Y$, then $C_{X\cup Y}(x) = 0$. If $x \notin X \cup Y$, then $x \notin X$ and $x \notin Y$. Therefore, $C_X(x) + C_Y(x) = 0$. Again, (4) holds. Therefore (4) holds for all $x \in U$.

Summary of Problem-Solving Techniques

- Write out exactly what must be proved.
- Instead of proving $p \rightarrow q$ directly, consider proving its contrapositive $\neg q \rightarrow \neg p$ or a proof by contradiction.
- For statements involving negation, De Morgan's laws can be very helpful.
- Look for definitions and theorems relevant to the expressions mentioned in the statements to be proved.
- A definition that involves cases suggests a proof by cases.

3.2 Sequences and Strings

Blue Taxi Inc. charges \$1 for the first mile and 50 cents for each additional mile. The following table shows the cost of traveling from 1 to 10 miles. In general, the cost C_n of traveling n miles is 1.00 (the cost of traveling the first mile) plus 0.50 times the number $(n - 1)$ of additional miles. That is, $C_n = 1 + 0.5(n - 1)$. As examples,

$$C_1 = 1 + 0.5(1 - 1) = 1 + 0.5 \cdot 0 = 1,$$
$$C_5 = 1 + 0.5(5 - 1) = 1 + 0.5 \cdot 4 = 1 + 2 = 3.$$