Problem 1.

For a projectile motion the speed v is described by

$$v^2 = v_0^2 - 2gy$$

In this equation $y_0 = 0$.

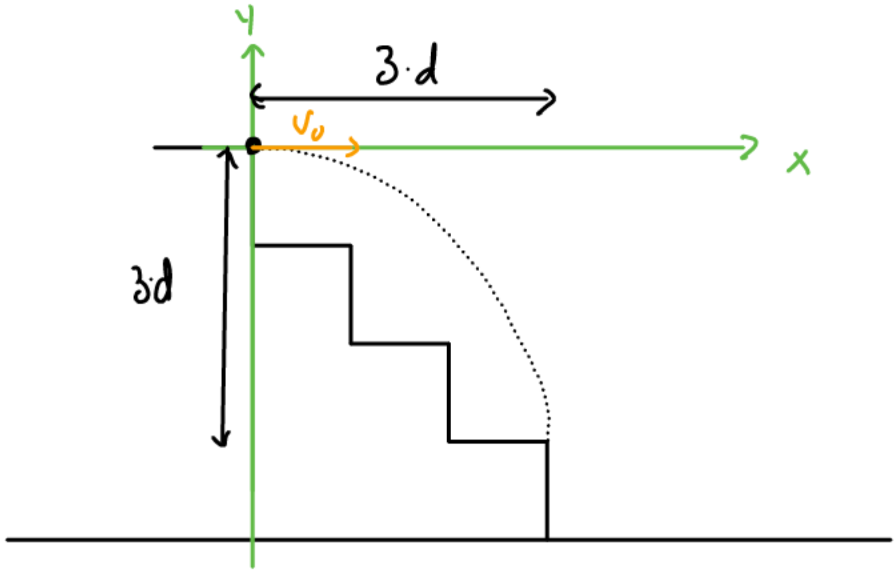
As y increases the speed v decreases. Therefore, the relationship must be

$$v_2 < v_3 < v_1$$

The answer is option E.

Problem 2.

A sketch of the physical situation and coordinate system can be seen below.



We write up the equations of motion in both the x- and y-directions

$$x(t) = v_0 t$$
$$y(t) = -\frac{1}{2}gt^2$$

We introduce the time variable T, where the ball passes the top of the lowest step. At this point in time the positions are $x(T) = 3d$ og $y(T) = -3d$:

$$v_0 T = 3d \qquad (1)$$
$$-\frac{1}{2}gT^2 = -3d \qquad (2)$$

Isolating T in equation (2)

$$3d = \frac{1}{2}gT^2 \Leftrightarrow T^2 = \frac{6d}{g} \Leftrightarrow T = \sqrt{\frac{6d}{g}}$$

By inserting the expression for T in equation (1) and solving for v_0

$$v_0 = \frac{3d}{T} = \frac{3d}{\sqrt{\frac{6d}{g}}} = \sqrt{\frac{3}{2}dg}$$

The answer is option E.

Problem 3.

a) The radial acceleration of a person moving in circular motion can be calculated by noting that the speed is constant

$$a = \frac{v^2}{R}$$

The speed is given by

$$v = \frac{\text{circumference}}{\text{period}} = \frac{2\pi R}{T}$$

Where R is the radius of the Earth. The revolution period is introduced into the formula

$$a = \frac{\left(\frac{2\pi R}{T}\right)^2}{R} = \frac{4\pi^2 R}{T^2} \qquad (1)$$

Inserting $R = 6370 \cdot 10^3$ m and $T = 24\text{ h} = 86400$ s:

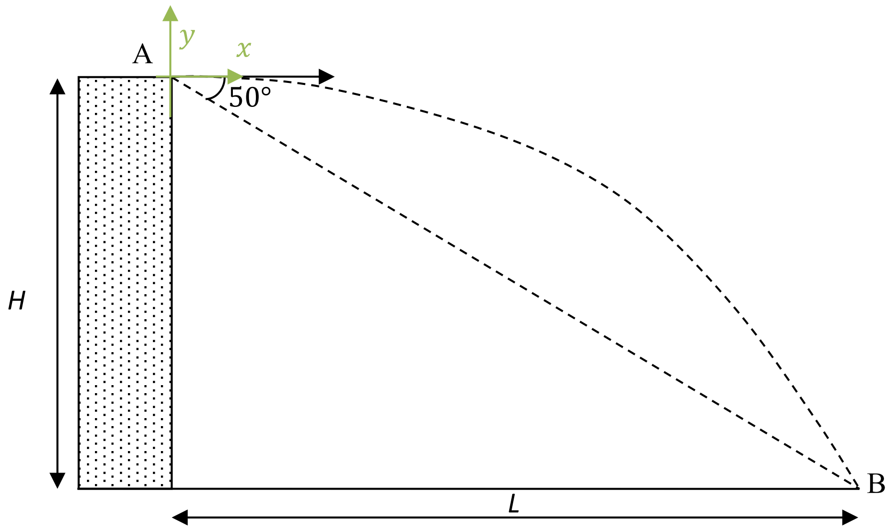
$$a = 0.0337\,\frac{\text{m}}{\text{s}^2}$$

b) By inserting $a = g$ into equation (1) and isolating T

$$g = \frac{4\pi^2 R}{T^2} \Leftrightarrow T^2 = \frac{4\pi^2 R}{g} \Leftrightarrow T = 2\pi\sqrt{\frac{R}{g}} = 5060\text{ s} = 1.41\text{ h}$$

Problem 4.

A sketch of the problem can be seen below with a coordinate system and variables H and L.



The rock is in the air for $T = 3.5$ s and the angle between point A and B is $\theta = 50°$. The height of the tower is H and the horizontal distance between A and B is denoted L.

The relationship between H and T is given by

$$y(t = T) = -H \Leftrightarrow -\frac{1}{2}gT^2 = -H \Leftrightarrow$$
$$H = \frac{1}{2}gT^2 \qquad (1)$$

Particle motion in the x-direction gives the relationship between L and T

$$x(t = T) = L \Leftrightarrow$$
$$v_0 T = L \qquad (2)$$

The angle θ can be found by the ratio between the opposite and adjacent sides of the triangle

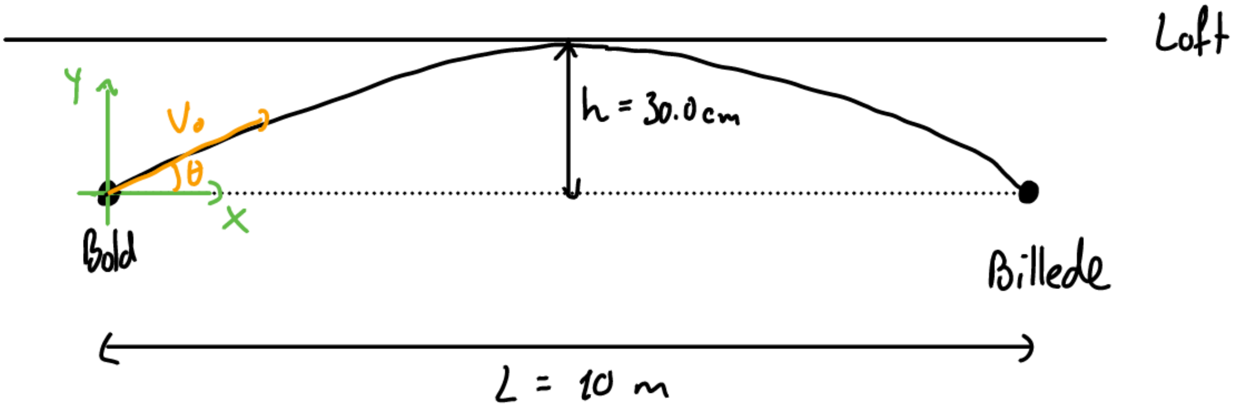
$$\tan\theta = \frac{H}{L} \qquad (3)$$

By inserting (1-2) in (3) and solving for v_0:

$$\tan\theta = \frac{\frac{1}{2}gT^2}{v_0 T} = \frac{gT}{2v_0} \Leftrightarrow$$
$$v_0 = \frac{gT}{2\tan\theta} = 14.4\,\frac{\mathrm{m}}{\mathrm{s}}$$

Problem 5.

A sketch of the problem can be seen below.



The ball starts and ends in the same distance from the ceiling at $y = 0$. This is a special case of the general projectile motion, where the simplified equations for the height h and length L can be used.

$$h = \frac{v_0^2 \sin^2 \theta}{2g}$$
$$L = \frac{v_0^2 \sin(2\theta)}{g}$$

By dividing the expression for h by the expression for L and utilizing the trigonometric relation $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$$\frac{h}{L} = \frac{\frac{v_0^2 \sin^2 \theta}{2g}}{\frac{v_0^2 \sin(2\theta)}{g}} = \frac{\sin^2 \theta}{4 \sin\theta \cos\theta} = \frac{1}{4}\frac{\sin\theta}{\cos\theta} = \frac{1}{4}\tan\theta \Leftrightarrow$$
$$\theta = \tan^{-1}\left(\frac{4h}{L}\right) = 6.84°$$

The initial velocity can now be determined from the expression for L

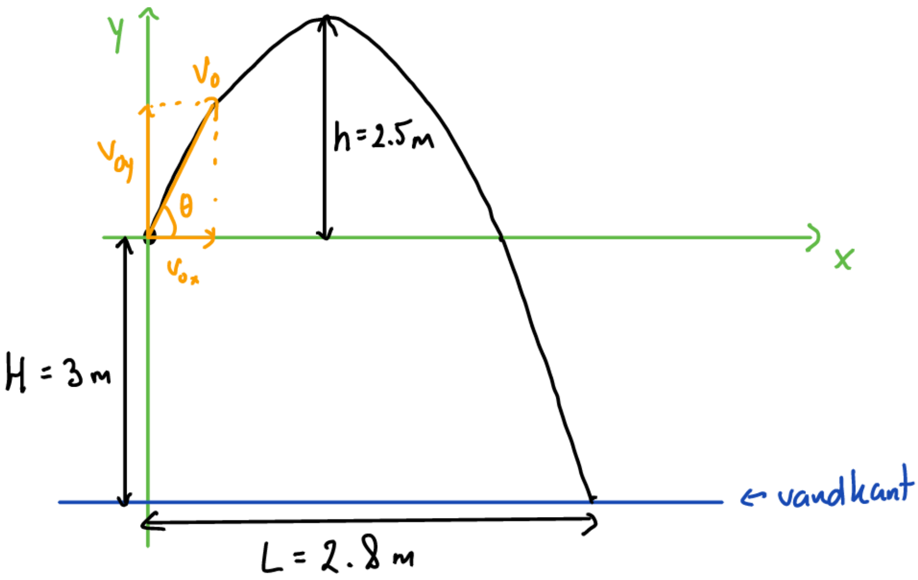
$$L = \frac{v_0^2 \sin(2\theta)}{g} \Leftrightarrow$$
$$v_0 = \sqrt{\frac{Lg}{\sin(2\theta)}} = 20.4\,\frac{\mathrm{m}}{\mathrm{s}}$$

The duration of flight T can now be calculated

$$T = \frac{2v_0 \sin\theta}{g} = 0.495\ \mathrm{s}$$

Problem 6.

The figure is a sketch of the jump



We must determine v_{0x}, v_{0y}, θ as well as the duration of the flight T. The equations of motion in the x- and y-directions are

$$x(t) = v_{0x}t$$
$$y(t) = v_{0y}t - \frac{1}{2}gt^2$$
$$v_{0x} = v_0 \cos\theta$$
$$v_{0y} = v_0 \sin\theta$$

The diver hits the water at $x(T) = L$ og $y(T) = -H$

$$L = v_{0x}T \qquad (1)$$
$$-H = v_{0y}T - \frac{1}{2}gT^2 \qquad (2)$$

At time t^* when the diver is at the top of his jump the vertical velocity is $v_y = 0$

$$v_y = v_{0y} - gt \Leftrightarrow$$
$$0 = v_{0y} - gt^* \qquad (3)$$

At this point the vertical distance is $y(t^*) = h$:

$$h = v_{0y}t^* - \frac{1}{2}g{t^*}^2 \qquad (4)$$

In addition, we also have the relation

$$\frac{v_{0y}}{v_{0x}} = \frac{v_0 \sin\theta}{v_0 \cos\theta} \Leftrightarrow$$
$$\tan\theta = \frac{v_{0y}}{v_{0x}} \qquad (5)$$

Now we have 5 equations with 5 unknowns, and these equations (1-5) are solved in Maple with respect to v_{0x}, v_{0y}, θ, T and t^*:

$$t^* = 0.71\ \mathrm{s}$$
$$v_{0x} = 1.6\,\frac{\mathrm{m}}{\mathrm{s}}$$
$$v_{0y} = 7\,\frac{\mathrm{m}}{\mathrm{s}}$$
$$T = 1.8\ \mathrm{s}$$
$$\theta = 77°$$

Problem 7.

The projectile is shot vertically with speed v, but as the shopping cart is moving horizontally with speed v, the projectile will have both horizontal and vertical velocity components

$$v_{0x} = v$$
$$v_{0y} = v$$

The equations of motion in x- and y-directions are

$$x(t) = vt$$
$$y(t) = vt - \frac{1}{2}gt^2$$

The projectile lands in the toy cannon at time T. At this time $x(T) = L$ and $y(T) = 0$

$$x(T) = L \Leftrightarrow L = vt \Leftrightarrow T = \frac{L}{v} \qquad (1)$$
$$y(T) = 0 \Leftrightarrow 0 = vt - \frac{1}{2}gT^2 \qquad (2)$$

By inserting the expression for T from equation (1) into equation (2), v can be isolated

$$0 = v\left(\frac{L}{v}\right) - \frac{1}{2}g\left(\frac{L}{v}\right)^2 \Leftrightarrow L = \frac{\frac{1}{2}gL^2}{v^2}$$
$$v = \sqrt{\frac{gL}{2}}$$

The answer is option D.

Problem 8.

The ball is thrown and caught at the same height. The motion is a special case of projectile motion, where we can use the simplified equations to calculate the distance the ball has travelled L.

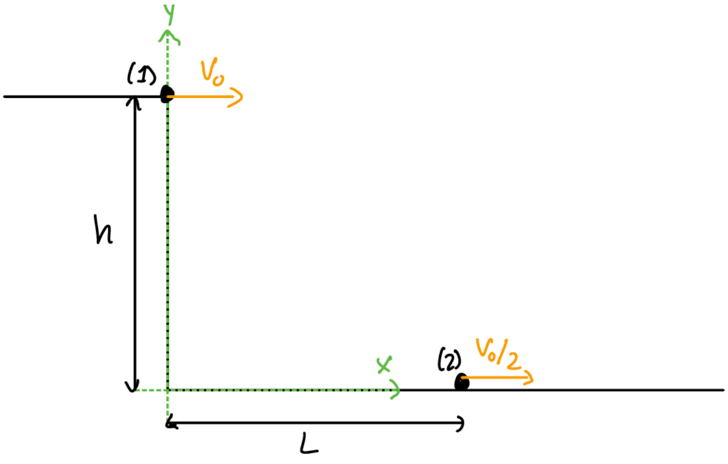
$$L = \frac{v_0^2 \sin(2\theta)}{g} \Leftrightarrow$$
$$v_0 = \sqrt{\frac{Lg}{\sin(2\theta)}}$$

The initial speed is equal to the final speed when the ball is caught. Here it can be seen that the lowest initial speed occurs if $\sin(2\theta)$ attains its largest value

$$\sin(2\theta) = 1 \Leftrightarrow \theta = 45°$$
$$v_0 = \sqrt{gL}$$

Problem 9.

A sketch of the problem can be seen with the chosen coordinate system



a) The equations of motion along the x-axis for both particles are given by

$$x_1(t) = v_0 t$$
$$x_2(t) = L + \frac{v_0}{2} t$$

We would like to determine the time T, where the particles collide. At this point the particles will have the same x position

$$x_1(t = T) = x_2(t = T) \Leftrightarrow$$
$$v_0 T = L + \frac{v_0}{2} T \Leftrightarrow v_0 T - \frac{v_0}{2} T = L \Leftrightarrow \frac{v_0}{2} T = L \Leftrightarrow$$
$$T = \frac{2L}{v_0}$$

b) The movement of the first particle in the y-direction is given by

$$y_1(t) = h - \frac{1}{2} g t^2$$

And hits the ground at T, where $y_1(T) = 0$

$$y_1(t = T) = 0 \Leftrightarrow 0 = h - \frac{1}{2} g T^2 \Leftrightarrow$$
$$h = \frac{1}{2} g T^2 \qquad (1)$$

Inserting the expression for T from question a), we express h as a function of L and v_0

$$h = \frac{1}{2} g \left(\frac{2L}{v_0}\right)^2 = \frac{2gL^2}{v_0^2}$$

c) The value of h is now known. The velocity in the x- and y- directions of the first particle is

$$v_x^{(1)} = v_0$$
$$v_y^{(2)} = -gt$$

The collision occurs at time T, which can be calculated from equation (1) in question b):

$$h = \frac{1}{2}gT^2 \Leftrightarrow T = \sqrt{\frac{2h}{g}}$$

The velocity of particle 1 before the collision is

$$v_x^{(1)} = v_0$$
$$v_y^{(1)} = -gT = -\sqrt{2gh}$$

The velocity vector $\overrightarrow{v_1}$ is therefore

$$\overrightarrow{v_1} = \begin{pmatrix} v_0 \\ -\sqrt{2gh} \end{pmatrix}$$

The second particle has only a velocity in the x-direction. The velocity vector $\overrightarrow{v_2}$ is then

$$\overrightarrow{v_2} = \begin{pmatrix} v_0/2 \\ 0 \end{pmatrix}$$

The relative velocity between the two particles can now be calculated

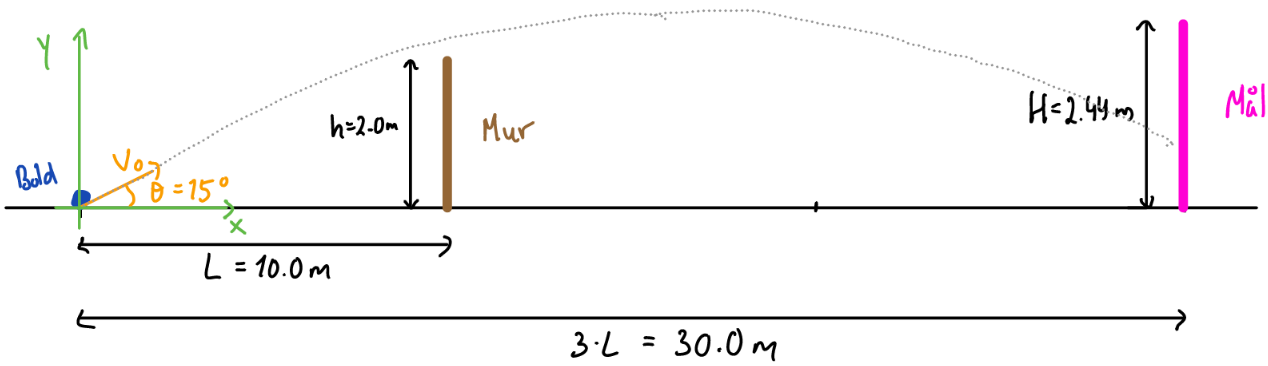
$$\overrightarrow{v_{\text{rel}}} = \overrightarrow{v_1} - \overrightarrow{v_2} = \begin{pmatrix} v_0 \\ -\sqrt{2gh} \end{pmatrix} - \begin{pmatrix} v_0/2 \\ 0 \end{pmatrix} = \begin{pmatrix} v_0/2 \\ -\sqrt{2gh} \end{pmatrix}$$

The magnitude of the relative velocity vector is

$$|\overrightarrow{v_{\text{rel}}}| = \sqrt{\left(\frac{v_0}{2}\right)^2 + \left(-\sqrt{2gh}\right)^2} = \sqrt{\frac{v_0^2}{4} + 2gh}$$

Problem 10.

a) We draw a sketch of the problem



b) The parabola $y(x)$ for the projectile motion is

$$y(x) = \tan(\theta)\,x - \frac{g}{2v_0^2\cos^2(\theta)}x^2$$

Now isolating v_0

$$y - \tan(\theta)\,x = -\frac{g}{2v_0^2\cos^2(\theta)}x^2 \Leftrightarrow$$
$$2v_0^2\cos^2(\theta)\,(\tan(\theta)\,x - y) = gx^2 \Leftrightarrow$$
$$v_0 = \frac{1}{\cos(\theta)}\sqrt{\frac{gx^2}{2(\tan(\theta)\,x - y)}} \qquad (1)$$

A path of motion that travels above the defenders and into the goal is found by inserting $x = L = 10.0\text{ m}$ and $y = h = 2.0\text{ m}$ into equation (1):

$$v_1 = 27.8\frac{\text{m}}{\text{s}}$$

c) We calculate the initial velocity for the two extreme cases where the ball lands on the goal line $(x, y) = (3L, 0)$, and where it hits the goal post $(x, y) = (3L, H)$.

When the ball hits the goal line $(x, y) = (3L, 0)$

$$v_0 = \frac{1}{\cos(\theta)}\sqrt{\frac{3gL}{2\tan\theta}} = 24.3\frac{\text{m}}{\text{s}}$$

Notice that the ball will collide with the wall of defenders, see question b).

When the ball hits the goal post $(x, y) = (3L, H)$, we have

$$v_0 = \frac{1}{\cos(\theta)}\sqrt{\frac{9gL^2}{2(\tan(\theta)\,3L - H)}} = 29.1\,\frac{\mathrm{m}}{\mathrm{s}}$$

This means that the velocity of the ball should be in the interval

$$27.8\,\frac{\mathrm{m}}{\mathrm{s}} < v_0 < 29.1\,\frac{\mathrm{m}}{\mathrm{s}}$$

This is a small interval, so we can conclude that it is not an easy goal to make.

Problem 11.

a) The tangential acceleration is

$$a_{\text{tan}} = \frac{dv}{dt} = a_0$$

We integrate with respect to time

$$\int_0^{v_0} \mathrm{d}v = \int_0^t a_0 \, \mathrm{d}t \Leftrightarrow$$
$$v(t) = a_0 t$$

The radial acceleration is given by the centripetal acceleration.

$$a_{\text{rad}} = \frac{v^2(t)}{R} = \frac{a_0^2 t^2}{R}$$

b) The tangential acceleration is constant during the circular motion, but the radial acceleration is increasing with t. We must determine the time T it takes to perform one revolution.

The motion is along the line which is a 1-dimensional movement with length $2\pi R$ and constant acceleration a_0.

$$x(t) = \frac{1}{2} a_0 t^2$$

Note that here x is the tangential length!

After one revolution $x(T) = 2\pi R$:

$$2\pi R = \frac{1}{2} a_0 T^2 \Leftrightarrow$$
$$T = 2\sqrt{\frac{\pi R}{a_0}}$$

The radial acceleration after one revolution (after time T has passed) can be determined

$$a_{\text{rad}}(t = T) = \frac{a_0^2 T^2}{R} = \frac{a_0^2}{R} \frac{4\pi R}{a_0} = 4\pi a_0$$

We now have the two components of the acceleration to find the magnitude.

$$a = \sqrt{a_{\text{tan}}^2 + a_{\text{rad}}^2} = \sqrt{a_0^2 + 16\pi^2 a_0^2} = a_0\sqrt{1 + 16\pi^2}.$$