

# Tarea 07: Unidad 03-B | splines cúbicos

## Métodos Numéricos

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## Tabla de Contenidos

<b>1 CONJUNTO DE EJERCICIOS</b>	<b>1</b>
1.1 1. Dados los puntos $(0, 1), (1, 5), (2, 3)$ , determine el <i>spline</i> cúbico. . . . .	1
1.2 2. Dados los puntos $(-1, 1), (1, 3)$ , determine el <i>spline</i> cúbico sabiendo que $f'(x_0) = 1, f'(x_n) = 2$ . . . . .	3
1.3 3. Diríjase al pseudocódigo del <i>spline</i> cúbico con frontera natural provisto en clase, en base a ese pseudocódigo complete la siguiente función: . . . . .	5
1.4 4. Usando la función anterior, encuentre el spline cúbico para: . . . . .	6
1.5 5. Usando la función anterior, encuentre el spline cúbico para: . . . . .	8
1.6 6. Use la función <code>cubic_spline_clamped</code> , provista en el enlace de Github, para graficar los datos de la siguiente tabla. . . . .	10
1.6.1 <b>Primera Curva</b> . . . . .	11
1.6.2 <b>Segunda Curva</b> . . . . .	13
1.6.3 <b>Tercera Curva</b> . . . . .	14

## 1 CONJUNTO DE EJERCICIOS

### 1.1 1. Dados los puntos $(0, 1), (1, 5), (2, 3)$ , determine el *spline* cúbico.

```
import numpy as np

xs = [0, 1, 2]
ys = [1, 5, 3]

splines = cubic_spline(xs=xs, ys=ys)
_ = [display(s) for s in splines]
```

```

print("-----")
_ = [display(s.expand()) for s in splines]

import matplotlib.pyplot as plt

x_vals = np.linspace(min(xs), max(xs), 200)
y_vals = np.zeros_like(x_vals)

for i, S in enumerate(splines):
    # Each spline is valid in [xs[i], xs[i+1]]
    mask = (x_vals >= xs[i]) & (x_vals <= xs[i+1])
    y_vals[mask] = [float(S.subs('x', x)) for x in x_vals[mask]]

plt.plot(x_vals, y_vals, label='Cubic Spline')
plt.scatter(xs, ys, color='red', label='Data Points')
plt.legend()
plt.xlabel('x')
plt.ylabel('y')
plt.title('Cubic Spline Interpolation')
plt.show()

```

1 5 1.0 -4.5 1.5  
0 1 5.5 0.0 -1.5

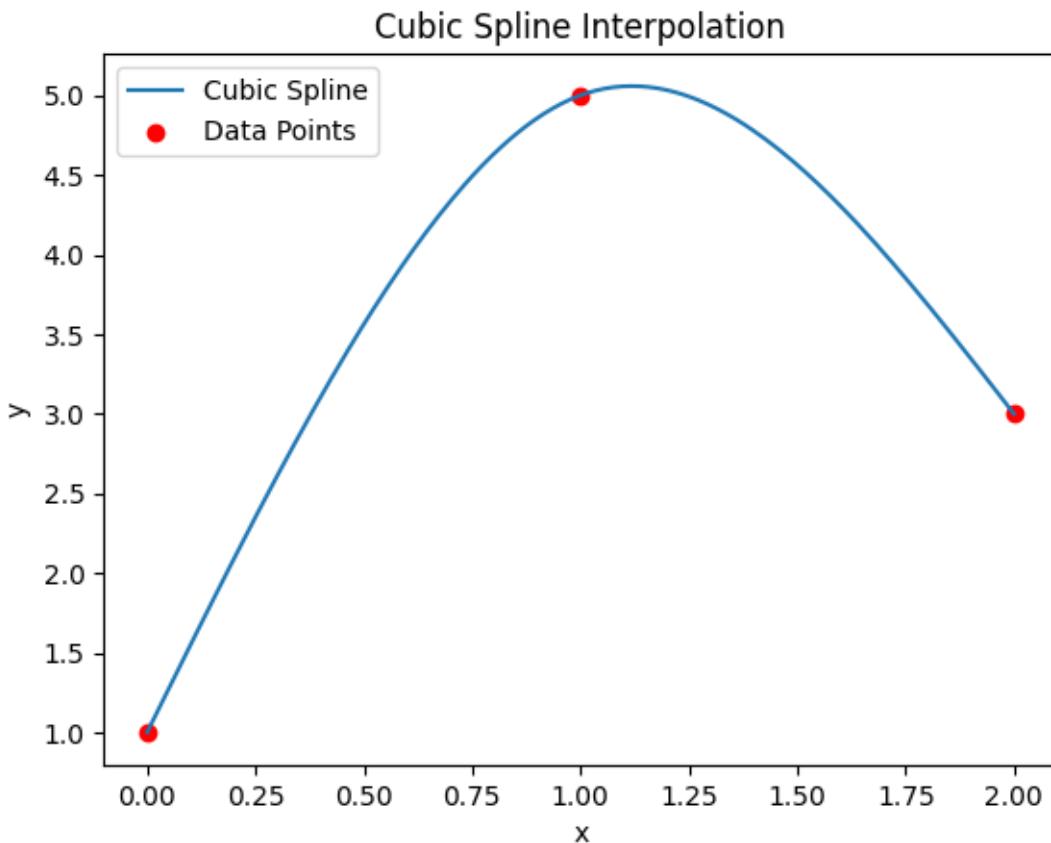
$$-1.5x^3 + 5.5x + 1$$

$$1.0x + 1.5(x - 1)^3 - 4.5(x - 1)^2 + 4.0$$

-----

$$-1.5x^3 + 5.5x + 1$$

$$1.5x^3 - 9.0x^2 + 14.5x - 2.0$$



**1.2 2. Dados los puntos  $(-1, 1), (1, 3)$ , determine el *spline* cúbico sabiendo que  $f'(x_0) = 1, f'(x_n) = 2$ .**

```

xs = [-1, 1]
ys = [1, 3]

splines = cubic_spline_clamped(xs=xs, ys=ys, d0=1, dn=2)

_ = [display(s) for s in splines]
print("_____")
_ = [display(s.expand()) for s in splines]

import matplotlib.pyplot as plt

x_vals = np.linspace(min(xs), max(xs), 200)
y_vals = np.zeros_like(x_vals)

for i, S in enumerate(splines):
    # Each spline is valid in [xs[i], xs[i+1]]

```

```

mask = (x_vals >= xs[i]) & (x_vals <= xs[i+1])
y_vals[mask] = [float(S.subs('x', x)) for x in x_vals[mask]]

plt.plot(x_vals, y_vals, label='Cubic Spline')
plt.scatter(xs, ys, color='red', label='Data Points')
plt.legend()
plt.xlabel('x')
plt.ylabel('y')
plt.title('Cubic Spline calmped Interpolation')
plt.show()

```

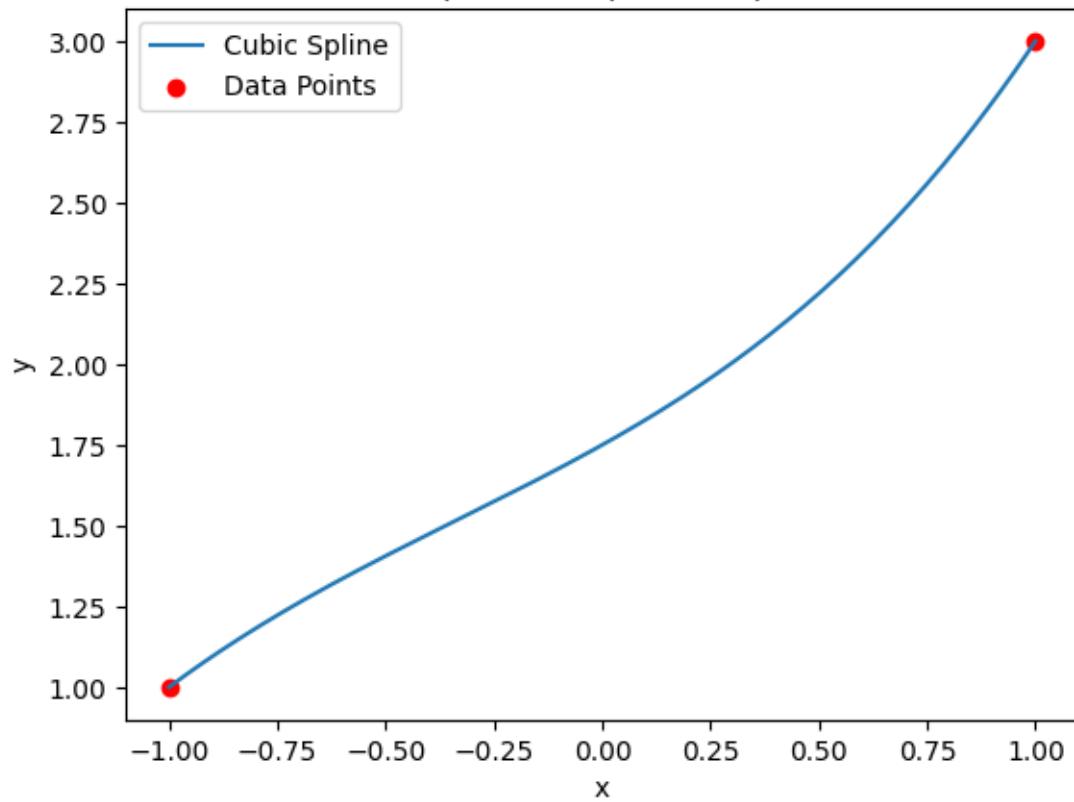
0 1 1.0 -0.5 0.25

$$1.0x + 0.25(x + 1)^3 - 0.5(x + 1)^2 + 2.0$$

-----

$$0.25x^3 + 0.25x^2 + 0.75x + 1.75$$

Cubic Spline calmped Interpolation



### 1.3 3. Diríjase al pseudocódigo del *spline* cúbico con frontera natural provisto en clase, en base a ese pseudocódigo complete la siguiente función:

```
import sympy as sym
from IPython.display import display

# ######
def cubic_spline(xs: list[float], ys: list[float]) -> list[sym.Symbol]:
    """
    Cubic spline interpolation ``S``. Every two points are interpolated by a cubic polynomial
    ``S_j`` of the form ``S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3``.

    xs must be different but not necessarily ordered nor equally spaced.

    ## Parameters
    - xs, ys: points to be interpolated

    ## Return
    - List of symbolic expressions for the cubic spline interpolation.
    """

    points = sorted(zip(xs, ys), key=lambda x: x[0]) # sort points by x

    xs = [x for x, _ in points]
    ys = [y for _, y in points]

    n = len(points) - 1 # number of splines

    h = [xs[i + 1] - xs[i] for i in range(n)] # distances between contiguous xs

    alpha = [0] * (n + 1) # coefficients for the tridiagonal system
    for i in range(1, n):
        alpha[i] = 3 / h[i] * (ys[i + 1] - ys[i]) - 3 / h[i - 1] * (ys[i] - ys[i - 1])

    l = [1]
    u = [0]
    z = [0]

    for i in range(1, n):
        l += [2 * (xs[i + 1] - xs[i - 1]) - h[i - 1] * u[i - 1]]
        u += [h[i] / l[i]]
```

```

z += [(alpha[i]-h[i-1]*z[i-1])/l[i]]

l.append(1)
z.append(0)
c = [0] * (n + 1)

x = sym.Symbol("x")
splines = []
for j in range(n - 1, -1, -1):
    c[j] = z[j] - u[j] * c[j + 1]
    b = (ys[j + 1] - ys[j]) / h[j] - h[j] * (c[j + 1] + 2 * c[j]) / 3
    d = (c[j + 1] - c[j]) / (3 * h[j])
    a = ys[j]

    print(j, a, b, c[j], d)
    S= a + b * (x - xs[j]) + c[j] * (x - xs[j])**2 + d * (x - xs[j])**3
    splines.append(S)
splines.reverse()
return splines

```

#### 1.4 4. Usando la función anterior, encuentre el spline cúbico para:

$$xs = [1, 2, 3]$$

$$ys = [2, 3, 5]$$

```

xs = [1, 2, 3]
ys = [2, 3, 5]

splines = cubic_spline(xs=xs, ys=ys)
_ = [display(s) for s in splines]
print("_____")
_ = [display(s.expand()) for s in splines]

import matplotlib.pyplot as plt

x_vals = np.linspace(min(xs), max(xs), 200)
y_vals = np.zeros_like(x_vals)

for i, S in enumerate(splines):
    # Each spline is valid in [xs[i], xs[i+1]]
    mask = (x_vals >= xs[i]) & (x_vals <= xs[i+1])

```

```

y_vals[mask] = [float(S.subs('x', x)) for x in x_vals[mask]]

plt.plot(x_vals, y_vals, label='Cubic Spline')
plt.scatter(xs, ys, color='red', label='Data Points')
plt.legend()
plt.xlabel('x')
plt.ylabel('y')
plt.title('Cubic Spline Interpolation')
plt.show()

```

1 3 1.5 0.75 -0.25

0 2 0.75 0.0 0.25

$$0.75x + 0.25(x-1)^3 + 1.25$$

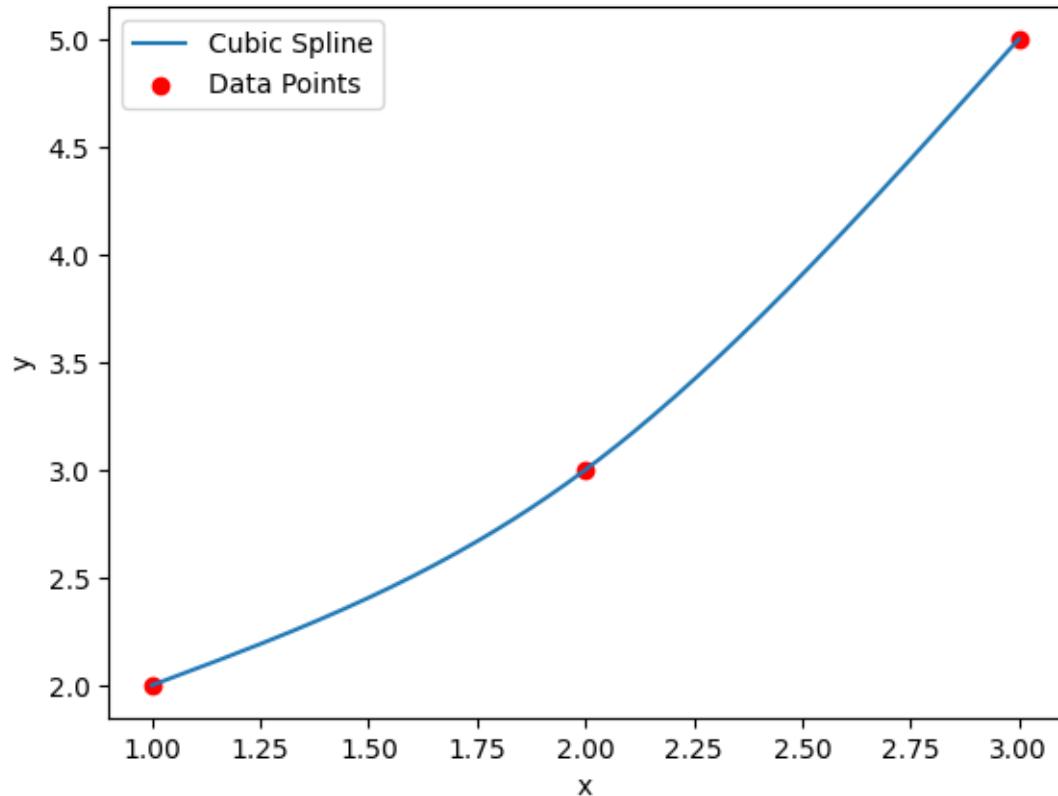
$$1.5x - 0.25(x-2)^3 + 0.75(x-2)^2$$

-----

$$0.25x^3 - 0.75x^2 + 1.5x + 1.0$$

$$-0.25x^3 + 2.25x^2 - 4.5x + 5.0$$

Cubic Spline Interpolation



## 1.5 5. Usando la función anterior, encuentre el spline cúbico para:

$$xs = [0, 1, 2, 3]$$

$$ys = [-1, 1, 5, 2]$$

```
xs = [0, 1, 2, 3]
ys = [-1, 1, 5, 2]

splines = cubic_spline(xs=xs, ys=ys)
_ = [display(s) for s in splines]
print("-----")
_ = [display(s.expand()) for s in splines]

import matplotlib.pyplot as plt

x_vals = np.linspace(min(xs), max(xs), 200)
y_vals = np.zeros_like(x_vals)

for i, S in enumerate(splines):
    # Each spline is valid in [xs[i], xs[i+1]]
    mask = (x_vals >= xs[i]) & (x_vals <= xs[i+1])
    y_vals[mask] = [float(S.subs('x', x)) for x in x_vals[mask]]

plt.plot(x_vals, y_vals, label='Cubic Spline')
plt.scatter(xs, ys, color='red', label='Data Points')
plt.legend()
plt.xlabel('x')
plt.ylabel('y')
plt.title('Cubic Spline Interpolation')
plt.show()
```

2 5 1.0 -6.0 2.0  
1 1 4.0 3.0 -3.0  
0 -1 1.0 0.0 1.0

$$1.0x^3 + 1.0x - 1$$

$$4.0x - 3.0(x - 1)^3 + 3.0(x - 1)^2 - 3.0$$

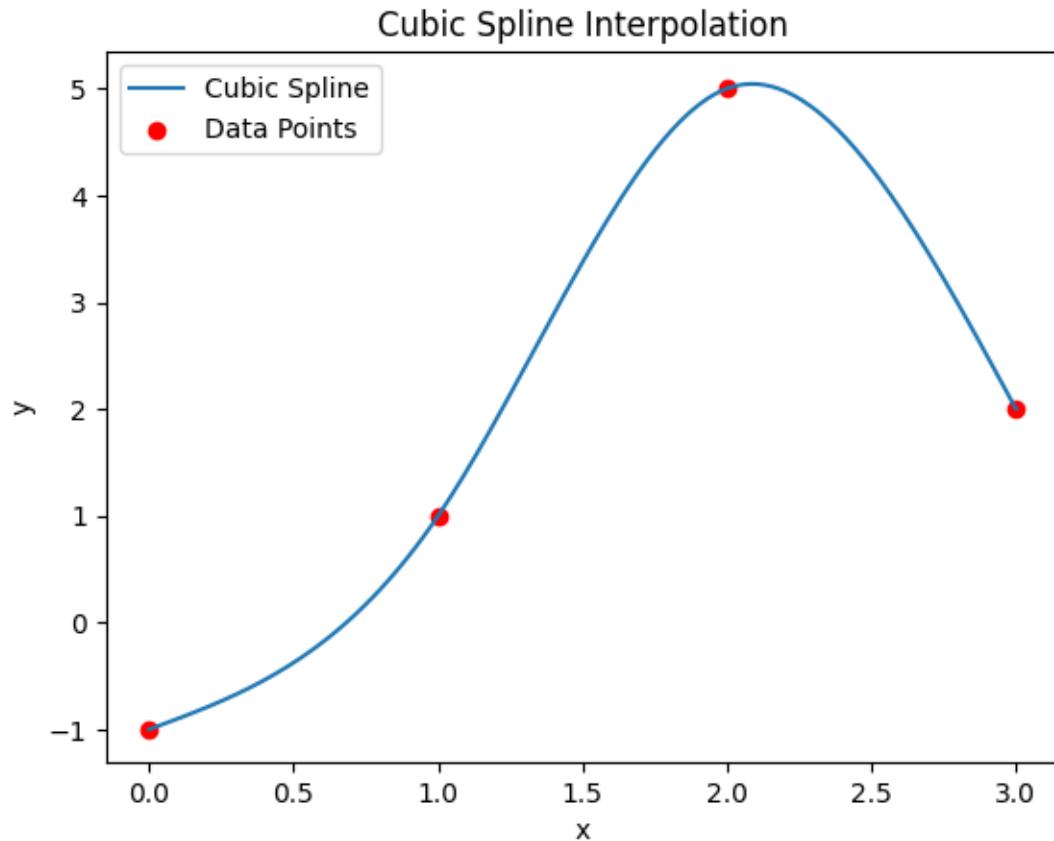
$$1.0x + 2.0(x - 2)^3 - 6.0(x - 2)^2 + 3.0$$

-----

$$1.0x^3 + 1.0x - 1$$

$$-3.0x^3 + 12.0x^2 - 11.0x + 3.0$$

$$2.0x^3 - 18.0x^2 + 49.0x - 37.0$$



## 1.6 6. Use la función `cubic_spline_clamped`, provista en el enlace de Github, para graficar los datos de la siguiente tabla.

Curva 1				Curva 2				Curva 3			
$i$	$x_i$	$f(x_i)$	$f'(x_i)$	$i$	$x_i$	$f(x_i)$	$f'(x_i)$	$i$	$x_i$	$f(x_i)$	$f'(x_i)$
0	1	3.0	1.0	0	17	4.5	3.0	0	27.7	4.1	0.33
1	2	3.7		1	20	7.0		1	28	4.3	
2	5	3.9		2	23	6.1		2	29	4.1	
3	6	4.2		3	24	5.6		3	30	3.0	-1.5
4	7	5.7		4	25	5.8					
5	8	6.6		5	27	5.2					
6	10	7.1		6	27.7	4.1	-4.0				
7	13	6.7									
8	17	4.5	-0.67								

Figura 1: TablaDeber.png

```

import sympy as sym
from IPython.display import display

# ######
def cubic_spline_clamped(
    xs: list[float], ys: list[float], d0: float, dn: float
) -> list[sym.Symbol]:
    """
        Cubic spline interpolation ``S``. Every two points are interpolated by a cubic polynomial ``S_j`` of the form ``S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3``.

        xs must be different but not necessarily ordered nor equally spaced.

        ## Parameters
        - xs, ys: points to be interpolated
        - d0, dn: derivatives at the first and last points

        ## Return
        - List of symbolic expressions for the cubic spline interpolation.
    """

    points = sorted(zip(xs, ys), key=lambda x: x[0]) # sort points by x
    xs = [x for x, _ in points]
    ys = [y for _, y in points]
    n = len(points) - 1 # number of splines

```

```

h = [xs[i + 1] - xs[i] for i in range(n)] # distances between contiguous xs

alpha = [0] * (n + 1) # prealloc
alpha[0] = 3 / h[0] * (ys[1] - ys[0]) - 3 * d0
alpha[-1] = 3 * dn - 3 / h[n - 1] * (ys[n] - ys[n - 1])

for i in range(1, n):
    alpha[i] = 3 / h[i] * (ys[i + 1] - ys[i]) - 3 / h[i - 1] * (ys[i] - ys[i - 1])

l = [2 * h[0]]
u = [0.5]
z = [alpha[0] / l[0]]

for i in range(1, n):
    l += [2 * (xs[i + 1] - xs[i - 1]) - h[i - 1] * u[i - 1]]
    u += [h[i] / l[i]]
    z += [(alpha[i] - h[i - 1] * z[i - 1]) / l[i]]

l.append(h[n - 1] * (2 - u[n - 1]))
z.append((alpha[n] - h[n - 1] * z[n - 1]) / l[n])
c = [0] * (n + 1) # prealloc
c[-1] = z[-1]

x = sym.Symbol("x")
splines = []
for j in range(n - 1, -1, -1):
    c[j] = z[j] - u[j] * c[j + 1]
    b = (ys[j + 1] - ys[j]) / h[j] - h[j] * (c[j + 1] + 2 * c[j]) / 3
    d = (c[j + 1] - c[j]) / (3 * h[j])
    a = ys[j]
    print(j, a, b, c[j], d)
    S = a + b * (x - xs[j]) + c[j] * (x - xs[j]) ** 2 + d * (x - xs[j]) ** 3

    splines.append(S)
splines.reverse()
return splines

```

### 1.6.1 Primera Curva

```

xs = [1, 2, 5, 6, 7, 8, 10, 13, 17]
ys = [3, 3.7, 3.9, 4.2, 5.7, 6.6, 7.1, 6.7, 4.5]

```

```

splines = cubic_spline_clamped(xs=xs, ys=ys, d0=1, dn=-0.67)
_ = [display(s) for s in splines]
print("-----")
_ = [display(s.expand()) for s in splines]

```

7 6.7 -0.3381314976116886 -0.07593425119415571 0.0057417813992694635  
 6 7.1 0.04846024164091059 -0.052929661890044014 -0.0025560654782346335  
 5 6.6 0.5472201929380908 -0.19645031375854607 0.023920108644750342  
 4 5.7 1.4091093003652708 -0.665438793668634 0.15632949330336265  
 3 4.2 1.0163426056008245 1.0582054884330803 -0.5745480940339047  
 2 3.9 -0.07447972276856785 0.03261683993631198 0.3418628828322561  
 1 3.7 0.4468099653460711 -0.20638006930785827 0.02655521213824114  
 0 3 1.0 -0.3468099653460706 0.046809965346070785

$$\begin{aligned}
 & 1.0x + 0.0468099653460708(x-1)^3 - 0.346809965346071(x-1)^2 + 2.0 \\
 & 0.446809965346071x + 0.0265552121382411(x-2)^3 - 0.206380069307858(x-2)^2 + \\
 & 2.80638006930786 \\
 & -0.0744797227685678x + 0.341862882832256(x-5)^3 + 0.032616839936312(x-5)^2 + \\
 & 4.27239861384284 \\
 & 1.01634260560082x - 0.574548094033905(x-6)^3 + 1.05820548843308(x-6)^2 - \\
 & 1.89805563360495 \\
 & 1.40910930036527x + 0.156329493303363(x-7)^3 - 0.665438793668634(x-7)^2 - \\
 & 4.1637651025569 \\
 & 0.547220192938091x + 0.0239201086447503(x-8)^3 - 0.196450313758546(x-8)^2 + \\
 & 2.22223845649527 \\
 & 0.0484602416409106x - 0.00255606547823463(x-10)^3 - 0.052929661890044(x-10)^2 + \\
 & 6.61539758359089 \\
 & -0.338131497611689x + 0.00574178139926946(x-13)^3 - 0.0759342511941557(x-13)^2 + \\
 & 11.095709468952
 \end{aligned}$$

-----

$$\begin{aligned}
 & 0.0468099653460708x^3 - 0.487239861384283x^2 + 1.83404982673035x + 1.60638006930786 \\
 & 0.0265552121382411x^3 - 0.365711342137305x^2 + 1.5909927882364x + 1.7684180949705 \\
 & 0.341862882832256x^3 - 5.09532640254753x^2 + 25.2390680902875x - 37.6450407417814 \\
 & -0.574548094033905x^3 + 11.4000711810434x^2 - 73.7333174112578x + 160.299730261309 \\
 & 0.156329493303363x^3 - 3.94835815303925x^2 + 33.7056879273205x - 90.3912821953733
 \end{aligned}$$

$$\begin{aligned}
& 0.0239201086447503x^3 - 0.770532921232554x^2 + 8.28308607286689x - 22.5976772501638 \\
& - 0.00255606547823463x^3 + 0.023752302456995x^2 + 0.340233835971401x + 3.87849687282113 \\
& 0.00574178139926946x^3 - 0.299863725765665x^2 + 4.54724220286598x - 14.3518727170554
\end{aligned}$$

## 1.6.2 Segunda Curva

```

xs = [17, 20, 23, 24, 25, 27, 27.7]
ys = [4.5, 7.0, 6.1, 5.6, 5.8, 5.2, 4.1]

splines = cubic_spline_clamped(xs=xs, ys=ys, d0=3, dn=-4.0)
_ = [display(s) for s in splines]
print("-----")
_ = [display(s.expand()) for s in splines]

```

5 5.2 -0.4011781849199465 0.1258152222202451 -2.568002126658778  
 4 5.8 0.1539868142803838 -0.4033977218204103 0.08820215734010924  
 3 5.6 -0.11137135038117751 0.6687558864819717 -0.35738453610079396  
 2 6.1 -0.6085014127556733 -0.17162582410747595 0.2801272368631492  
 1 7.0 -0.19787464681108174 0.03475023545927881 -0.022930673285194974  
 0 4.5 3.0 -1.1007084510629728 0.12616207628025017

$$\begin{aligned}
& 3.0x + 0.12616207628025(x - 17)^3 - 1.10070845106297(x - 17)^2 - 46.5 \\
& - 0.197874646811082x - 0.022930673285195(x - 20)^3 + 0.0347502354592788(x - 20)^2 + \\
& 10.9574929362216 \\
& - 0.608501412755673x + 0.280127236863149(x - 23)^3 - 0.171625824107476(x - 23)^2 + \\
& 20.0955324933805 \\
& - 0.111371350381178x - 0.357384536100794(x - 24)^3 + 0.668755886481972(x - 24)^2 + \\
& 8.27291240914826 \\
& 0.153986814280384x + 0.0882021573401092(x - 25)^3 - 0.40339772182041(x - 25)^2 + \\
& 1.9503296429904 \\
& - 0.401178184919947x - 2.56800212665878(x - 27)^3 + 0.125815222220245(x - 27)^2 + \\
& 16.0318109928386
\end{aligned}$$

-----

$$\begin{aligned}
& 0.12616207628025x^3 - 7.53497434135573x^2 + 149.806607471118x - 984.439023122068 \\
& - 0.022930673285195x^3 + 1.41059063257098x^2 - 29.1046920074162x + 208.302973401493 \\
& 0.280127236863149x^3 - 19.5004051676648x^2 + 451.848211398006x - 3479.00261937341 \\
& - 0.357384536100794x^3 + 26.4004424857391x^2 - 649.772132283688x + 5333.96013008014 \\
& 0.0882021573401092x^3 - 7.0185595223286x^2 + 185.702917918006x - 1628.33195493397 \\
& - 2.56800212665878x^3 + 208.133987481581x^2 - 5623.41585118756x + 50653.7369670161
\end{aligned}$$

### 1.6.3 Tercera Curva

```

xs = [27.7, 28, 29, 30]
ys = [4.1, 4.3, 4.1, 3.0]

splines = cubic_spline_clamped(xs=xs, ys=ys, d0=0.33, dn=-1.5)
_ = [display(s) for s in splines]
print("-----")
_ = [display(s.expand()) for s in splines]

```

$$\begin{aligned}
& 2 \ 4.1 \ -0.7653465346534649 \ -0.26930693069306927 \ -0.06534653465346556 \\
& 1 \ 4.3 \ 0.6613861386138599 \ -1.1574257425742556 \ 0.2960396039603954 \\
& 0 \ 4.1 \ 0.3299999999999999 \ 2.2620462046204524 \ -3.799413274660778
\end{aligned}$$

$$\begin{aligned}
& 0.33x - 3.79941327466078(x - 27.7)^3 + 2.26204620462045(x - 27.7)^2 - 5.041 \\
& 0.66138613861386x + 0.296039603960395(x - 28)^3 - 1.15742574257426(x - 28)^2 - \\
& 14.2188118811881 \\
& -0.765346534653465x - 0.0653465346534656(x - 29)^3 - 0.269306930693069(x - 29)^2 + \\
& 26.2950495049505
\end{aligned}$$

-----

$$\begin{aligned}
& -3.79941327466078x^3 + 317.993289328931x^2 - 8870.74279427938x + 82483.079611294 \\
& 0.296039603960395x^3 - 26.0247524752475x^2 + 761.762376237622x - 7420.30198019801 \\
& -0.0653465346534656x^3 + 5.41584158415843x^2 - 150.014851485149x + 1393.54455445545
\end{aligned}$$