Techniques in Operations Research

Assignment 3 – 2018

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Due: 10am May 18, 2018

<u>(Question 2B from Assignment 2 at the end of this Assignment)</u>

Question 1: min $\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{c}^{\mathrm{T}}\mathbf{x}$

s.t $\mathbf{Ax} \leq \mathbf{b}$

where $\mathbf{x}, \mathbf{c} \in \mathbb{R}^{\boldsymbol{n}}, \mathbf{b} \in \mathbb{R}^{\boldsymbol{m}} \geq \mathbf{0}, \mathbf{A} \in \mathbb{R}^{\boldsymbol{m \times n}}$

Part A)

The linear constraints form a convex feasible domain. If the objective function is also convex, then we have a convex optimisation problem in which the KKT conditions become sufficient for the optimality of the problem.

Assume $Q = Q^T$, We start by writing the Lagrange function and its gradient:

$$\mathcal{L}(\mathrm{x}, \lambda) = \frac{1}{2}\mathrm{x}^{\mathrm{T}}\mathrm{Qx} + \mathrm{c}^{\mathrm{T}}\mathrm{x} + \lambda^T(\mathrm{Ax} - \mathrm{b})$$

$$\nabla_{\mathrm{x}}\mathcal{L}(\mathrm{x}, \lambda) = [\mathrm{Qx} + \mathrm{c}^{\mathrm{T}} + \mathrm{A}^{\mathrm{T}}\lambda]$$

To find the KKT points, we must find the solutions which satisfy all the KKT conditions, in particular:

KKTa:

$$\nabla_{\mathrm{x}}\mathcal{L}(\mathrm{x}, \lambda) = 0$$

$$(1) \Rightarrow \mathrm{Qx}^* + \mathrm{c} + \mathrm{A}^{\mathrm{T}}\lambda^* = 0$$

KKTb:

$$(2)\ \lambda^* \geq 0$$

$$(3)\ \mathrm{Ax}^* \leq \mathrm{b}$$

$$(4)\ \lambda^*(\mathrm{Ax}^* - \mathrm{b}) = 0$$

There are no equality constraints thus no need for KKTc condition

Therefore, all the points satisfying the KKT conditions must satisfy each of equation $(1), (2), (3)$, and (4).

Part B)

First, we note that the Lagrange function and gradient are:

$$\mathcal{L}(\mathrm{x}, \lambda) = \frac{1}{2}\mathrm{x}^{\mathrm{T}}\mathrm{Q}\mathrm{x} + \mathrm{c}^{\mathrm{T}}\mathrm{x} + \lambda^{T}(\mathrm{A}\mathrm{x} - \mathrm{b})$$

$$\nabla_{\mathrm{x}}\mathcal{L}(\mathrm{x}, \lambda) = [\mathrm{Q}\mathrm{x} + \mathrm{c}^{\mathrm{T}} + \mathrm{A}^{\mathrm{T}}\lambda]$$

The Hessian to the Lagrange function is:

$$\nabla^2_{\mathrm{xx}}\mathcal{L}(\mathrm{x}, \lambda) = \mathrm{Q}$$

Therefore, we can see that the matrix Q is in fact the Hessian matrix to the Lagrange function. By definition the matrix Q is positive definite, which implies that it is also positive semi-definite. It now follows that the minimising objective function is convex. Next, we can see immediately that since the inequality constraint is affine, it is also convex.

Now we can define the function:

$$\phi(x) = L(x, \lambda^*) = f(x) + \sum_i \lambda_i^* g_i(x)$$

This differs from the Lagrangian because we fix the KKT multipliers λ^*. Because $f(x)$ and $g(x)$ in our program are convex as shown above, and $\lambda_i^* \geq 0$, we know that $\phi(x)$ is convex function.

Therefore, for any feasible point x we can use the proof of the Global Minima Theorem:

$f(x) \geq \mathcal{L}(\mathrm{x}, \lambda^*) = \phi(x)$ since x is feasible.

$f(x) \geq \nabla\phi(x^*)^T(x - x^*)$ by the Lemma of Convex Functions.

$= \phi(x^*)$ since $\nabla\phi(x^*) = \nabla_{\mathrm{x}}\mathcal{L}(\mathrm{x}^*, \lambda^*) = 0$ from KKTa condition

$= \mathcal{L}(x^*, \lambda^*) = f(x^*)$ from KKTb conditions

Therefore, we have proven one direction of the Theorem for "Global minima of convex programs"

Hence, we can conclude that x^* is necessary and sufficient for a global minimiser when satisfying KKT conditions, as required.

Conversely, we can also use the Theorem for "Global minima of convex programs" which states that:

If $x^*$is a local or global minimiser of the NLP, and a constraint qualification holds, then $x^*$is also a stationary point.

Finishing off the proof.

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Question 2: min $f(x) = \frac{(x_1-2)^4}{4} + x_2^4 + 4$

s.t $\quad x_1 - x_2 \leq 8$

$x_1 - x_2^2 \geq 4$

Note the NLP can be written as:

$$\min f(x) = \frac{(x_1-2)^4}{4} + x_2^4 + 4 \ (a)$$

$$\text{s.t} \qquad g_1(x) = x_1 - x_2 - 8 \leq 0 \ (b)$$

$$g_2(x) = -x_1 + x_2^2 + 4 \leq 0 \ (c)$$

With no equality constraints.

Part A)

The Lagrange function is given by:

$$\mathcal{L}(x, \lambda, \eta) = f(x) + \langle \lambda, g(x) \rangle + \langle \eta, h(x) \rangle$$

And since there are no quality constraints we consider

$$\mathcal{L}(x, \lambda) = f(x) + \langle \lambda, g(x) \rangle$$

Now,

$$\mathcal{L}(x, \lambda, \eta) = \frac{(x_1 - 2)^4}{4} + x_2^4 + 4 + \lambda_1(x_1 - x_2 - 8) + \lambda_2(-x_1 + x_2^2 + 4)$$

$$\nabla_{\mathrm{x}} \mathcal{L}(\mathrm{x}, \lambda) = \begin{bmatrix} (x_1 - 2)^3 + \lambda_1 - \lambda_2 \\ 4x_2^3 - \lambda_1 + 2\lambda_2 x_2 \end{bmatrix}$$

To find the KKT points, we must find the solutions which satisfy all the KKT conditions:

KKTa:

$$\nabla_{\mathrm{x}} \mathcal{L}(\mathrm{x}, \lambda) = 0$$

$$(1) \Longrightarrow (x_1 - 2)^3 + \lambda_1 - \lambda_2 = 0$$

$$(2) \Longrightarrow 4x_2^3 - \lambda_1 + 2\lambda_2 x_2 = 0$$

KKTb:

$$(3)\ \lambda_1 \geq 0$$

$$(4)\ \lambda_2 \geq 0$$

$$(5)\ x_1 - x_2 - 8 \leq 0$$

$$(6)\ -x_1 + x_2^2 + 4 \leq 0$$

$$(7)\ \lambda_1(x_1 - x_2 - 8) = 0$$

$$(8) \quad \lambda_2(-x_1 + x_2^2 + 4) = 0$$

No KKTc conditions as no equality constraints.

Now, we consider all possible combinations of λ satisfying the conditions (3), (4) with equality. We know there are two $\lambda's$ so we have 2^2cases to test:

<table>
<tr><th>Case</th><th>Test</th><th>Result</th></tr>
<tr><td>$\lambda_1 > 0, \lambda_2 > 0$</td><td>From (5) and (6) we have:
$x_1 - x_2 - 8 = 0$
$-x_1 + x_2^2 + 4 = 0$
Using Algebra, from these two equations we find:
$$x_1 = \frac{17-\sqrt{17}}{2} \, and \ x_2 = \frac{1-\sqrt{17}}{2}$$
OR
$$x_1 = \frac{17+\sqrt{17}}{2} \, and \ x_2 = \frac{1+\sqrt{17}}{2}$$
We now sub in the first point into equations (1) and (2)
$$(1) \Longrightarrow \left(\frac{17-\sqrt{17}}{2} - 2\right)^3 + \lambda_1 - \lambda_2 = 0$$
$$(2) \Longrightarrow 4\left(\frac{1-\sqrt{17}}{2}\right)^3 - \lambda_1 + 2\lambda_2\left(\frac{1-\sqrt{17}}{2}\right) = 0$$
Now if we rearrange the first equation for λ_1 and sub into the second equation here:
$$\Longrightarrow 4(1-\sqrt{17})^3 - \left(\frac{2\lambda_2 + 131\sqrt{17} - 715}{2}\right) + \lambda_2(1-\sqrt{17}) = 0$$
Using algebra, solve for λ_2:
$$\lambda_2 = \frac{767}{2\sqrt{17}} - \frac{151}{2}$$
Meaning for λ_1:
$$\Longrightarrow \lambda_1 = \frac{1497}{\sqrt{17}} - 433 \approx -69.924 < 0$$
A contradiction to the original assumption.</td><td>Since both λ_1 and $\lambda_2$must always be positive in this case.

This is <u>not</u> a KKT point</td></tr>
</table>

$\lambda_1 = 0, \lambda_2 > 0$	From (6), (8) we have: $\implies -x_1 + x_2^2 + 4 = 0$ $\implies x_1 = x_2^2 + 4$ Now subbing x_1 into (1) and (2): (1) $\implies (2 + x_2^2)^3 + 0 - \lambda_2 = 0$ (2) $\implies 4x_2^3 - 0 + 2\lambda_2 x_2 = 0$ Using algebra, from these two equations we find that: $x_2 = 0 \text{ and } \lambda_2 = 8 > 0$ And if $x_2 = 0 \implies x_1 = 4$ Looking at equations (5), (6), (7), and (8): (5) $4 - 0 - 8 = -4 < 0$ (6) $-4 + 0 + 4 = 0 \leq 0$ (7) $\lambda_1(x_1 - x_2 - 8) = 0 \implies 0(x_1 - x_2 - 8) = 0$ (8) $\lambda_2(-4 + 0 + 4) = \lambda_2(0) = 0$ Every condition is satisfied	$x = [4, 0]^T$ $\lambda = [0,8]^T$ Is a solution of the KKT conditions
$\lambda_1 > 0, \lambda_2 = 0$	From (5), (7) we have: $\implies x_1 - x_2 - 8 = 0$ $\implies x_1 = x_2 + 8$ Now subbing x_1 into (1) and (2): (1) $\implies (6 + x_2)^3 + \lambda_1 - 0 = 0$ (2) $\implies 4x_2^3 - \lambda_1 + 0 = 0$ Using algebra, from these two equations we find that: $x_2 \approx -2.31893 \text{ and } \lambda_1 \approx -49.8795 < 0$ A contradiction to the original assumption.	Since λ_1 must always be positive in this case. This is <u>not</u> a KKT point

$\lambda_1 = 0, \lambda_2 = 0$	From (1) and (2) we have: (1) $\Longrightarrow (x_1 - 2)^3 + 0 - 0 = 0$ (2) $\Longrightarrow 4x_2^3 - 0 + 0 = 0$ Using algebra, from these two equations we find that: $x_1 = 0 \; and \; x_2 = 0$ Now subbing $x_1 \; and \; x_2$ into (5) and (6): (5) $0 - 0 - 8 = -8 \leq 0$ (6) $-0 + 0 + 4 = 4 \nleq 0$ A contradiction to one of the KKT conditions.	Since all KKT conditions must always be satisfied. This is not a KKT point

A single solution from above is found, that is the point:

$$x^* = [4, 0]^T$$

$$\lambda^* = [0,8]^T$$

Part B)

To check that one constraint qualification holds, we check the Linear Independence Constraint Qualification:

We know that the active constraint is $g_2(x) = -x_1 + x_2^2 + 4 \leq 0$, thus:

$$\nabla g(x) = \nabla g(x_1, x_2) = \begin{bmatrix} -1 \\ 2x_2 \end{bmatrix}$$

$$\nabla g(x^*) = \nabla g(x_1^*, x_2^*) = \begin{bmatrix} -1 \\ 2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Now use Gaussian Jordan elimination to obtain the rank of this matrix to determine linear independence:

$$\Longrightarrow \nabla g(x^*) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Therefore, full Rank has been obtained and thus the Linear Independence Constraint Qualification has been satisfied.

Since the linear independence constraint qualification has been satisfied, this implies that the Mangasarian-Fromowitz Constraint Qualification also holds since:

$$LICQ \Rightarrow MFCQ$$

However, we shall check if this is true in this case:

No equality constraints$\Longrightarrow$ $d\epsilon\mathbb{R}^2$

g_2: $\nabla g_2(x^*)^T d = [-1 \quad 0]\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \Longrightarrow -d_1 < 0$

Now we find a d that satisfies both inequalities, and since $d\epsilon\mathbb{R}^2$, we can choose:

$d = [1]$

Which now satisfies the Mangasarian-Fromowitz Constraint Qualification as assumed.

We have checked that one of the constraint qualifications holds, which implies that another constraint qualification is held, that is as expected:

$$LICQ \Rightarrow MFCQ$$

Part C)

We now check whether the point found in Part A) is a local minimizer.

$$x^* = [4, 0]^T$$

$$\lambda^* = [0{,}8]^T$$

The active constraint is $g_2(x) = -x_1 + x_2^2 + 4 \le 0$, therefore the critical cone is given by:

$$\boldsymbol{C}(x^*, \lambda^*) = \{d\epsilon\mathbb{R}^2 : \nabla g_2(4{,}0)^T d = 0\}$$

$$= \left\{d\epsilon\mathbb{R}^2 : (4{,}0)\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = 0\right\}$$

$$= \{d\epsilon\mathbb{R}^2 : 4d_1 = 0\}$$

$$= \{(d_1, d_2)\epsilon\mathbb{R}^2 : d_1 = 0, d_2 = d_2\}$$

We now see the Hessian of the Lagrange function is:

$$\nabla^2_{\mathrm{xx}}\mathcal{L}(\mathrm{x}, \lambda) = \begin{bmatrix} 3(x_1 - 2)^2 & 0 \\ 0 & 12x_2^2 + 2\lambda_2 \end{bmatrix}$$

Since our only point was $x^* = [4, 0]^T and\ \lambda^* = [0{,}8]^T$

We see that the Hessian becomes:

$$\nabla^2_{\mathrm{xx}}\mathcal{L}(\mathrm{x}^*, \lambda^*) = \begin{bmatrix} 3(4 - 2)^2 & 0 \\ 0 & 12(0) + 2(8) \end{bmatrix}$$

$$\nabla^2_{xx}\mathcal{L}(x^*,\lambda^*) = \begin{bmatrix} 12 & 0 \\ 0 & 16 \end{bmatrix}$$

Now, for $d\epsilon\boldsymbol{C}(x^*,\lambda^*)$ we have:

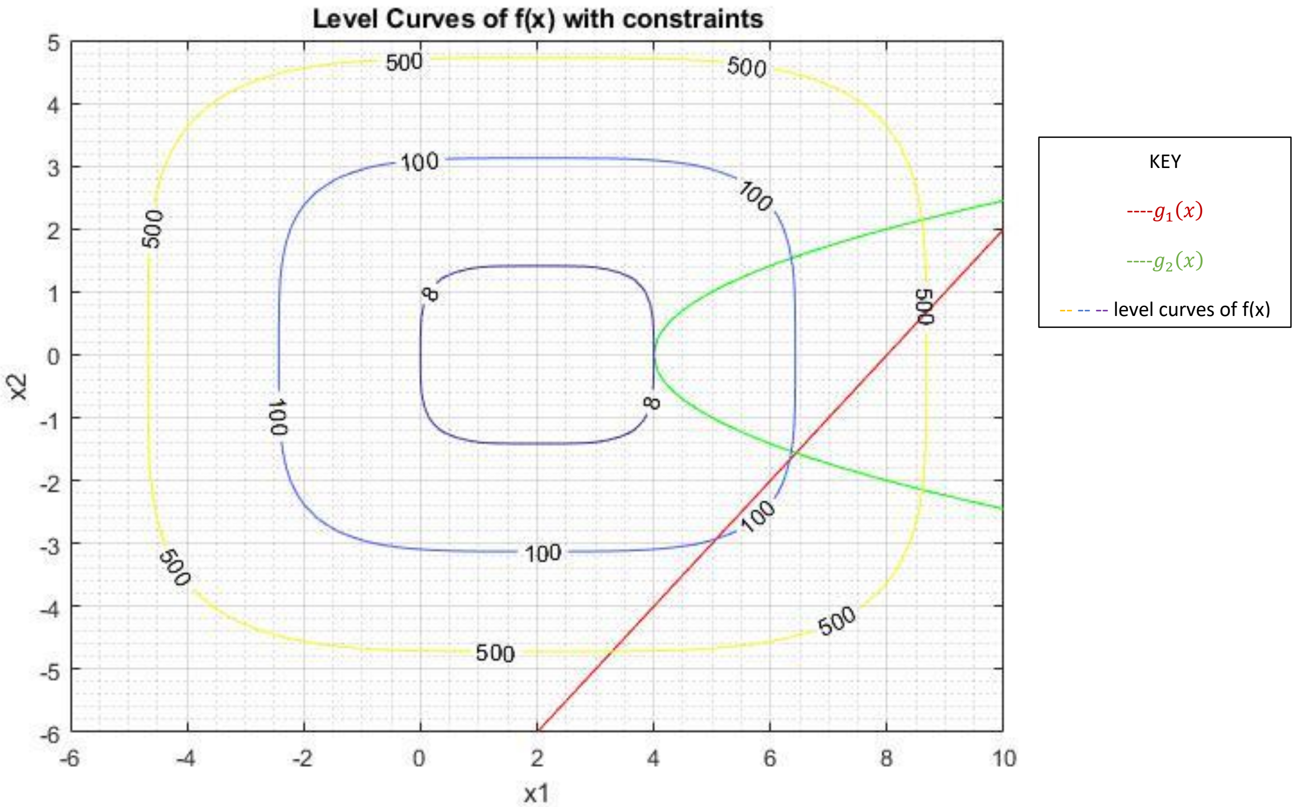
$$[d_1 \quad d_2]\begin{bmatrix} 12 & 0 \\ 0 & 16 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = [0 \quad d_2]\begin{bmatrix} 12 & 0 \\ 0 & 16 \end{bmatrix}\begin{bmatrix} 0 \\ d_2 \end{bmatrix} = [0 \quad 16d_2]\begin{bmatrix} 0 \\ d_2 \end{bmatrix} = 16d_2^2 > 0$$

Thus, $\nabla^2_{xx}\mathcal{L}(x^*,\lambda^*)$ is positive definite on the critical cone. Therefore $x^* = [4,0]^T$ is a local minimum. In fact, since $x^*$was the only point found, $x^* = [4,0]^T$is a global minimum.

Part D)

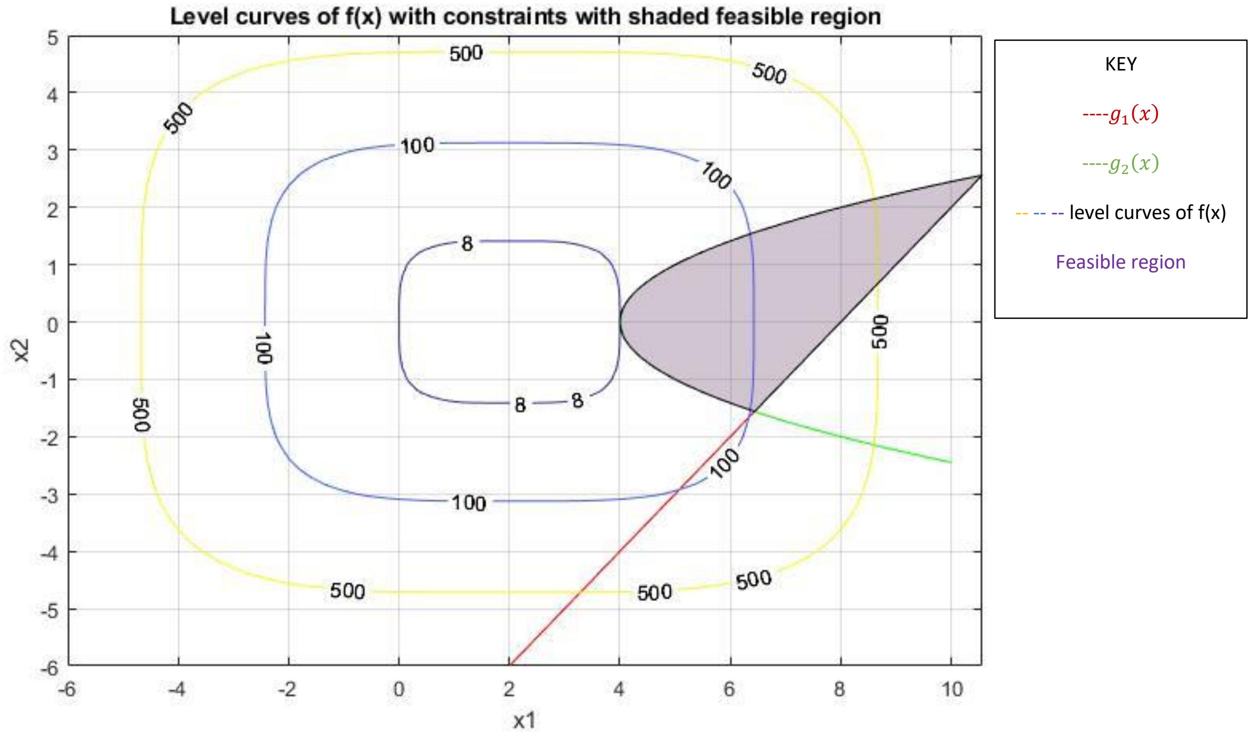
Using MATLAB to plot the level curves of our function with constraints (MATLAB code for FIGURE 1 and FIGURE 2 provided at end of assignment (after question 2B from last assignment):

FIGURE 1



Now we observe the feasible region:

FIGURE 2:



From this graph and seeing the feasible region, it is easy to observe by inspection that $g_2(x) = -x_1 + x_2^2 + 4 \leq 0$ is the only active constraint, and minimising our function means we are going as far to the left on the x_1 axis as possible. Thus confirming the point we found using the KKT conditions: $x^* = [4, 0]^T$ is the correct minimiser for this function.

Part E)

To show that the objective function is convex for all x in the constraint set, we must show that using the Lemma for Convex Functions for C^2 functions:

- If a function is C^2 then the function is convex if and only if the Hessian$\nabla^2 f(x)$ is positive semi-definite $\forall x$

We look at $f(x) = \frac{(x_1-2)^4}{4} + x_2^4 + 4$

Since $f(x)$ is C^2 we need to apply this Lemma, thus:

$$f(x) = \frac{(x_1 - 2)^4}{4} + x_2^4 + 4$$

The gradient and Hessian are:

$$\nabla_x f(x) = \begin{bmatrix} 3(x_1 - 2)^2 \\ 4x_2^3 \end{bmatrix}$$

$$\nabla^2_{xx} f(x) = \begin{bmatrix} 3(x_1 - 2)^2 & 0 \\ 0 & 12x_2^2 \end{bmatrix}$$

Since this is a square matrix with strictly positive values down the diagonal for all values of x, it is always positive definite and thus implies that it is also positive semi-definite, which satisfies the convex condition in the lemma, thus $f(x)$ is convex.

Since we have proven the convexity of the objective function for all values of x, this implies that the objective function is also convex for all x in the constraint set, as required.

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Part F)

We have already seen that there is a KKT point at $x^* = [4, 0]^T$, $\lambda^* = [0,8]^T$, now we consider the Lagrangian at x^*, considered as a function of λ only:

$$\mathcal{L}(x^*, \lambda) = \frac{(4-2)^4}{4} + 0 + 4 + \lambda_1(4 - 0 - 8) + \lambda_2(-4 + 0 + 4)$$

$$\mathcal{L}(x^*, \lambda) = 8 + \lambda_1(-4) + \lambda_2(0)$$

Maximising this over all $\lambda_1, \lambda_2 \geq 0$ is clearly $\lambda_1 = 0$ and λ_2 can take any value greater than or equal to zero.

We can note from here that for any value of λ_2, the $\mathcal{L}(x^*, \lambda)$ does not change, so it follows that:

$$\mathcal{L}(x^*, \lambda) \leq \mathcal{L}(x^*, (0,8))$$

That is, the saddle inequality holds for all x in the constraint set, as required.

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Question 3: min $\boldsymbol{f(x) = x_2^3 - x_1 - 2x_2}$

s.t $\boldsymbol{x_1 + x_2 \leq 1}$

$\boldsymbol{x_2^2 \geq 0}$

Part A)

Note that the NLP can be written as:

$$\min f(x) = x_2^3 - x_1 - 2x_2$$

$$\text{s.t} \quad x_1 + x_2 - 1 \leq 0$$

$$-x_2^2 \leq 0$$

Now, the log barrier penalty function:

$$P_k(x) = x_2^3 - x_1 - 2x_2 - \frac{1}{k}\log(1 - x_1 - x_2) - \frac{1}{k}\log(x_2)$$

Part B)

To find the stationary points $x^k = (x_1^k, x_2^k)$ for $P_k(x)$, we need to solve $\nabla P_k(x) = 0$, thus:

$$\nabla P_k(x) = \begin{bmatrix} -1 - \frac{1}{k}\left(\frac{1}{x_1^k + x_2^k - 1}\right) \\ 3x_2^2 - 2 - \frac{1}{k}\left(\frac{1}{x_1^k + x_2^k - 1}\right) - -\frac{1}{k}\left(\frac{1}{x_2^k}\right) \end{bmatrix}$$

$$\nabla P_k(x) = 0$$

$$(1) \Longrightarrow -1 - \frac{1}{k}\left(\frac{1}{x_1^k + x_2^k - 1}\right) = 0$$

$$(2) \Longrightarrow 3\left(x_2^k\right)^2 - 2 - \frac{1}{k}\left(\frac{1}{x_1^k + x_2^k - 1}\right) - -\frac{1}{k}\left(\frac{1}{x_2^k}\right) = 0$$

Rearrange (1) in terms of x_1:

$$(1a) \Longrightarrow x_1^k = 1 - \frac{1}{k} - x_2^k$$

Now sub (1a) into equation (2):

$$(2) \Longrightarrow 3\left(x_2^k\right)^2 - 2 - \frac{1}{k}\left(\frac{1}{(1 - \frac{1}{k} - x_2^k) + x_2^k - 1}\right) - -\frac{1}{k}\left(\frac{1}{x_2^k}\right) = 0$$

Cancel out like terms:

$$\Longrightarrow 3\left(x_2^k\right)^2 - 2 - \frac{1}{k}\left(\frac{1}{(-\frac{1}{k})}\right) - -\frac{1}{k}\left(\frac{1}{x_2^k}\right) = 0$$

Rearrange equation using algebra:

$$\Longrightarrow 3\left(x_2^k\right)^2 - 1 - \frac{1}{k}\left(\frac{1}{x_2^k}\right) = 0$$

$$\Longrightarrow 3\left(x_2^k\right)^3 - x_2^k - \frac{1}{k} = 0$$

$$\Longrightarrow \left(x_2^k\right)^3 - \left(\frac{1}{3}\right)x_2^k - \frac{1}{3k} = 0$$

This is now in the form of a depressed cubic function, using the hint provided, we know for a depressed cubic function in the form: $t^3 + pt + q = 0$

Then: $t_n = 2\sqrt{-\frac{p}{3}}\cos(\frac{1}{3}\arccos\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right) - \frac{2\pi n}{3})\ for\ n = 0, 1, 2, 3, \dots \dots$

Therefore, our equation becomes:

$$\Rightarrow \left(x_2^k\right)^3 + \left(-\frac{1}{3}\right)x_2^k + \left(-\frac{1}{3k}\right) = 0$$

$$\Rightarrow p = \left(-\frac{1}{3}\right)\ and\ q = \left(-\frac{1}{3k}\right) = 0$$

Now using the depressed cubic formula:

$$\Rightarrow (x_2^k)_n = 2\sqrt{-\frac{\left(-\frac{1}{3}\right)}{3}}\cos(\frac{1}{3}\arccos\left(\frac{3\left(-\frac{1}{3k}\right)}{2\left(-\frac{1}{3}\right)}\sqrt{\frac{-3}{\left(-\frac{1}{3}\right)}}\right) - \frac{2\pi n}{3})\ for\ n = 0, 1, 2, 3, \ldots$$

$$\Rightarrow (x_2^k)_n = 2\sqrt{\frac{1}{9}}\cos(\frac{1}{3}\arccos\left(\frac{\left(-\frac{1}{k}\right)}{\left(-\frac{2}{3}\right)}\sqrt{9}\right) - \frac{2\pi n}{3})\ for\ n = 0, 1, 2, 3, \ldots \ldots$$

$$\Rightarrow (x_2^k)_n = \frac{2}{3}\cos(\frac{1}{3}\arccos\left(\frac{9}{2k}\right) - \frac{2\pi n}{3})\ for\ n = 0, 1, 2$$

Due to the domain of arccos which is defined $x \in [-1,1]$, we need to split this solution into different values of k, namely where $\left|\frac{9}{2k}\right| \leq 1$, and since we are only considering solution for a large enough k, we assess $k \geq \frac{9}{2}$, this way we can assess x^k as $k \to \infty$.

Now, for $k = \frac{9}{2}$, we obtain the result for $(x_2^k)_n$ as:

$$x_2^k = \begin{Bmatrix} \frac{2}{3}, & n = 0 \\ -\frac{1}{3}, & n = 1 \\ -\frac{1}{3}, & n = 2 \end{Bmatrix}$$

Therefore for $k = \frac{9}{2}$, we know $x_2^k \geq 0$, thus we can disregard the values of $n = 1$ and $n = 2$ and only consider $(x_2^k)_{n=0} = \frac{2}{3}$.

For $k > \frac{9}{2}$, we obtain the 3 results for $(x_2^k)_n$ as:

$$0 < \frac{9}{2k} < 1$$

$$\Rightarrow \frac{\pi}{6} > \frac{1}{3}\arccos\left(\frac{9}{2k}\right) > 0 \text{ for n=0}$$

$$\Rightarrow -\frac{\pi}{2} > \frac{1}{3}\arccos\left(\frac{9}{2k}\right) - \frac{2\pi}{3} > -\frac{2\pi}{3} \text{ for n=1}$$

$$\Rightarrow -\frac{7\pi}{6} > \frac{1}{3}\arccos\left(\frac{9}{2k}\right) - \frac{4\pi}{3} > -\frac{4\pi}{3} \text{ for n=2}$$

From these three results, we can see that only n=0 solution is within the first quadrant where cosine is positive, the n=1 and n=2 solutions are in the 2nd and 3rd quadrant respectively, where cosine is negative.

Therefore for x_2^k we have:

$$x_2^k = \begin{Bmatrix} positive, & n=0 \\ negative, & n=1 \\ negative, & n=2 \end{Bmatrix}$$

Since we know $x_2^k \geq 0$, thus we can disregard the values of $n = 1$ and $n = 2$ and only consider $x_2^k = \frac{2}{3}\cos(\frac{1}{3}\arccos\left(\frac{9}{2k}\right))$>0.

Therefore, combining the results of $k = \frac{9}{2}, k > \frac{9}{2}$ we are left with:

$$\Longrightarrow (x_2^k)_{n=0} = \frac{2}{3}\cos(\frac{1}{3}\arccos\left(\frac{9}{2k}\right)) \text{ only}$$

Now, the stationary point within the constraints is:

$$x^k = \left(x_1^k, x_2^k\right) = (1 - \frac{1}{k} - x_2^k, \quad \frac{2}{3}\cos(\frac{1}{3}\arccos\left(\frac{9}{2k}\right)) \ )$$

$$\Longrightarrow \left(x_1^k, x_2^k\right) = (1 - \frac{1}{k} - \frac{2}{3}\cos(\frac{1}{3}\arccos\left(\frac{9}{2k}\right)) \ , \quad \frac{2}{3}\cos(\frac{1}{3}\arccos\left(\frac{9}{2k}\right)) \ )$$

The final x^k is given as:

$$x^k = (1 - \frac{1}{k} - \frac{2}{3}\cos(\frac{1}{3}\arccos\left(\frac{9}{2k}\right)) \ , \quad \frac{2}{3}\cos(\frac{1}{3}\arccos\left(\frac{9}{2k}\right)) \ )$$

Part C)

Find the limit as $k \to \infty$, that is, to find $x^* = \lim_{k\to\infty} x^k$:

$$x^* = \lim_{k\to\infty} x^k$$

$$x^* = \lim_{k\to\infty} (1 - \frac{1}{k} - \frac{2}{3}\cos(\frac{1}{3}\arccos\left(\frac{9}{2k}\right)) \ , \quad \frac{2}{3}\cos(\frac{1}{3}\arccos\left(\frac{9}{2k}\right)) \ )$$

By inspection, and limit laws, we know $\lim_{k\to\infty} \frac{1}{k} = 0$, therefore:

$$x^* = (1 - 0 - \frac{2}{3}\cos(\frac{1}{3}\arccos(0)) \;, \qquad \frac{2}{3}\cos(\frac{1}{3}\arccos(0)) \;)$$

$$x^* = (1 - 0 - \frac{2}{3}\cos\left(\frac{1}{3}\times\frac{\pi}{2}\right), \qquad \frac{2}{3}\cos\left(\frac{1}{3}\times\frac{\pi}{2}\right))$$

$$x^* = (1 - 0 - \frac{2}{3}\cos\left(\frac{\pi}{6}\right), \qquad \frac{2}{3}\cos\left(\frac{\pi}{6}\right))$$

$$x^* = (1 - 0 - (\frac{2}{3}\times\frac{\sqrt{3}}{2}), \qquad (\frac{2}{3}\times\frac{\sqrt{3}}{2}))$$

$$x^* = (1 - \frac{\sqrt{3}}{3}, \qquad \frac{\sqrt{3}}{3})$$

Part D)

The estimate $\lambda^k$to the optimal Lagrange multiplier vector, given $x_1 + x_2 - 1 \leq 0, -x_2^2 \leq 0$

is given by:

$$\lambda^k = \left[-\frac{1}{kg_1(x^k)}, -\frac{1}{kg_2(x^k)}\right]^T$$

$$\lambda^k = \left[-\frac{1}{k\left(x_1^k + x_2^k - 1\right)}, -\frac{1}{k\left(x_2^k\right)}\right]^T$$

Now to find $\lambda^* = \lim_{k\to\infty} \lambda^k$:

$$\lambda^* = \lim_{k\to\infty} \lambda^k = \lim_{k\to\infty} \left[-\frac{1}{k\left(x_1^k + x_2^k - 1\right)}, -\frac{1}{k\left(x_2^k\right)}\right]^T$$

$$\Rightarrow \lim_{k\to\infty} \left[-\frac{1}{k\left((1 - \frac{1}{k} - \frac{2}{3}\cos(\frac{1}{3}\arccos\left(\frac{9}{2k}\right)) \;) + \frac{2}{3}\cos(\frac{1}{3}\arccos\left(\frac{9}{2k}\right)) \;\; - 1\right)}, -\frac{1}{k\left(\frac{2}{3}\cos(\frac{1}{3}\arccos\left(\frac{9}{2k}\right)) \;\right)}\right]^T$$

$$\Rightarrow \lim_{k\to\infty} \left[-\frac{1}{\left(\cancel{k} - 1 - \cancel{k(\frac{2}{3}\cos(\frac{1}{3}\arccos\left(\frac{9}{2k}\right))} \;) + \cancel{k(\frac{2}{3}\cos(\frac{1}{3}\arccos\left(\frac{9}{2k}\right))} \;) \; - \cancel{k}\right)}, -\frac{1}{k\left(\frac{2}{3}\cos(\frac{1}{3}\arccos\left(\frac{9}{2k}\right)) \;\right)}\right]^T$$

After cancelling like terms:

$$\Longrightarrow \lim_{k\to\infty}\left[-\frac{1}{(-1)}, -\frac{1}{k\left(\frac{\sqrt{3}}{3}\right)}\right]^T = [1,0]^T$$

Hence for the point $x^* = (1 - \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$, the corresponding $\lambda^* = \lim_{k\to\infty} \lambda^k = [1,0]^T$

FROM ASSIGNMENT 2

Question 2: min $\boldsymbol{f(x) = x_1^2 + 3x_2^2 + x_3}$

s.t $\boldsymbol{x_1^2 + x_2^2 + x_3^2 - 4 = 0}$

Part B)

We use Theorem 6 on lecture slide 334 (Theorem: conditions on an equality constrained NLP for local minimality)

If $\nabla_{xx}^2 L(x^*, \eta^*)$ is positive definite on $C(x^*)$, that is:

If $0 \neq d \in \mathbb{R}^n, \nabla h(x^*)^T d = 0, and\ d^T \nabla_{xx}^2 L(x^*, \eta^*) d > 0$

Then x^* is a local minimum

Having checked in assignment 2 that the constraint qualifications hold for all stationary points

Now $f(x)$ and $h(x)$ are clearly C^2 and the Lagrange function and its gradient are:

$$L(x,\eta) = x_1^2 + 3x_2^2 + x_3 + \eta(x_1^2 + x_2^2 + x_3^2 - 4)$$

$$\nabla_x L(x,\eta) = \begin{bmatrix} 2x_1 + 2\eta x_1 \\ 6x_2 + 2\eta x_2 \\ 1 + 2\eta x_3 \end{bmatrix}$$

Taking the Hessian of this:

$$\nabla_{xx}^2 L(x^*, \eta^*) = \begin{bmatrix} 2+2\eta & 0 & 0 \\ 0 & 6+2\eta & 0 \\ 0 & 0 & 2\eta \end{bmatrix}$$

Now,

$$\nabla_{xx}^2\, L\left(\begin{bmatrix} \sqrt{\frac{15}{4}} \\ 0 \\ \frac{1}{2} \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\nabla_{xx}^2\, L\left(\begin{bmatrix} -\sqrt{\frac{15}{4}} \\ 0 \\ \frac{1}{2} \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\nabla_{xx}^2\, L\left(\begin{bmatrix} 0 \\ \sqrt{\frac{143}{36}} \\ \frac{1}{6} \end{bmatrix}\right) = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

$$\nabla_{xx}^2\, L\left(\begin{bmatrix} 0 \\ -\sqrt{\frac{143}{36}} \\ \frac{1}{6} \end{bmatrix}\right) = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

$$\nabla_{xx}^2\, L\left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{11}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

$$\nabla_{xx}^2\, L\left(\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} \frac{5}{2} & 0 & 0 \\ 0 & \frac{13}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

We wish to know whether each of these matrices are positive definite with respect to the directions which maintain feasibility, thus using Theorem 6, we compute the null space of the transpose of the Jacobian:

$$\nabla g(x) = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

Now,

$$\nabla g\left(\begin{bmatrix}\sqrt{\frac{15}{4}}\\0\\\frac{1}{2}\end{bmatrix}\right)^T\left(\begin{bmatrix}d_1\\d_2\end{bmatrix}\right)=0$$

$$\Rightarrow\begin{pmatrix}\sqrt{\frac{15}{4}} & 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}d_1\\d_2\end{pmatrix}=0$$

$$\Rightarrow\sqrt{15}d_1+d_3=0$$

$$\boldsymbol{d}=\begin{pmatrix}d_1\\d_2\\-\sqrt{15}d_1\end{pmatrix}$$

$$\nabla g\left(\begin{bmatrix}-\sqrt{\frac{15}{4}}\\0\\\frac{1}{2}\end{bmatrix}\right)^T\left(\begin{bmatrix}d_1\\d_2\end{bmatrix}\right)=0$$

$$\Rightarrow\begin{pmatrix}-\sqrt{\frac{15}{4}} & 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}d_1\\d_2\end{pmatrix}=0$$

$$\Rightarrow-\sqrt{15}d_1+d_3=0$$

$$\boldsymbol{d}=\begin{pmatrix}d_1\\d_2\\\sqrt{15}d_1\end{pmatrix}$$

$$\nabla g\left(\begin{bmatrix} 0 \\ \sqrt{\frac{143}{36}} \\ \frac{1}{6} \end{bmatrix}\right)^{T}\left(\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}\right) = 0$$

$$\Rightarrow \begin{pmatrix} 0 & \sqrt{\frac{143}{36}} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = 0$$

$$\Rightarrow \sqrt{\frac{143}{36}}d_2 + \frac{1}{6}d_3 = 0$$

$$\boldsymbol{d} = \begin{pmatrix} d_1 \\ d_2 \\ -\sqrt{143}d_1 \end{pmatrix}$$

$$\nabla g\left(\begin{bmatrix} 0 \\ -\sqrt{\frac{143}{36}} \\ \frac{1}{6} \end{bmatrix}\right)^{T}\left(\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}\right) = 0$$

$$\Rightarrow \begin{pmatrix} 0 & -\sqrt{\frac{143}{36}} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = 0$$

$$\Rightarrow -\sqrt{\frac{143}{36}}d_2 + \frac{1}{6}d_3 = 0$$

$$\boldsymbol{d} = \begin{pmatrix} d_1 \\ d_2 \\ \sqrt{143}d_1 \end{pmatrix}$$

$$\nabla g\left(\begin{bmatrix}0\\0\\2\end{bmatrix}\right)^T\left(\begin{bmatrix}d_1\\d_2\end{bmatrix}\right)=0$$

$$\Longrightarrow \begin{pmatrix}0 & 0 & 2\end{pmatrix}\begin{pmatrix}d_1\\d_2\end{pmatrix}=0$$

$$\Longrightarrow 2d_3 = 0$$

$$\boldsymbol{d} = \begin{pmatrix}d_1\\d_2\\0\end{pmatrix}$$

$$\nabla g\left(\begin{bmatrix}0\\0\\-2\end{bmatrix}\right)^T\left(\begin{bmatrix}d_1\\d_2\end{bmatrix}\right)=0$$

$$\Longrightarrow \begin{pmatrix}0 & 0 & -2\end{pmatrix}\begin{pmatrix}d_1\\d_2\end{pmatrix}=0$$

$$\Longrightarrow -2d_3 = 0$$

$$\boldsymbol{d} = \begin{pmatrix}d_1\\d_2\\0\end{pmatrix}$$

For $\boldsymbol{x}^* = \begin{pmatrix}\sqrt{\frac{15}{4}} & 0 & \frac{1}{2}\end{pmatrix}^T$,

$$\begin{pmatrix}d_1 & d_2 & -\sqrt{15}d_1\end{pmatrix}\begin{bmatrix}0 & 0 & 0\\0 & 4 & 0\\0 & 0 & -2\end{bmatrix}\begin{pmatrix}d_1\\d_2\\-\sqrt{15}d_1\end{pmatrix} = -30d_1^2+4d_2^2$$

For $\boldsymbol{x}^* = \begin{pmatrix}-\sqrt{\frac{15}{4}} & 0 & \frac{1}{2}\end{pmatrix}^T$,

$$\begin{pmatrix}d_1 & d_2 & \sqrt{15}d_1\end{pmatrix}\begin{bmatrix}0 & 0 & 0\\0 & 4 & 0\\0 & 0 & -2\end{bmatrix}\begin{pmatrix}d_1\\d_2\\\sqrt{15}d_1\end{pmatrix} = -30d_1^2+4d_2^2$$

For $\boldsymbol{x}^* = \begin{pmatrix}0 & \sqrt{\frac{143}{36}} & \frac{1}{6}\end{pmatrix}^T$,

$$\begin{pmatrix}d_1 & d_2 & -\sqrt{143}d_2\end{pmatrix}\begin{bmatrix}-4 & 0 & 0\\0 & 0 & 0\\0 & 0 & -6\end{bmatrix}\begin{pmatrix}d_1\\d_2\\-\sqrt{143}d_2\end{pmatrix} = -4d_1^2-858d_2^2<0$$

For $\boldsymbol{x}^* = \begin{pmatrix} 0 & -\sqrt{\frac{143}{36}} & \frac{1}{6} \end{pmatrix}^T$,

$$\begin{pmatrix} d_1 & d_2 & \sqrt{143}d_2 \end{pmatrix} \begin{bmatrix} -4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{pmatrix} d_1 \\ d_2 \\ \sqrt{143}d_2 \end{pmatrix} = -4d_1^2 - 858d_2^2 < 0$$

For $\boldsymbol{x}^* = (0 \quad 0 \quad 2)^T$,

$$(d_1 \quad d_2 \quad 0) \begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{11}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \begin{pmatrix} d_1 \\ d_2 \\ 0 \end{pmatrix} = \frac{3}{2}d_1^2 + \frac{11}{2}d_2^2 > 0$$

For $\boldsymbol{x}^* = (0 \quad 0 \quad -2)^T$,

$$(d_1 \quad d_2 \quad 0) \begin{bmatrix} \frac{5}{2} & 0 & 0 \\ 0 & \frac{13}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{pmatrix} d_1 \\ d_2 \\ 0 \end{pmatrix} = \frac{5}{2}d_1^2 + \frac{13}{2}d_2^2 > 0$$

In Conclusion, the second order sufficient condition implies that the two minima are $(0 \quad 0 \quad 2)^T$ and $(0 \quad 0 \quad -2)^T$. In addition, we can also conclude that the two maxima are $\begin{pmatrix} 0 & \sqrt{\frac{143}{36}} & \frac{1}{6} \end{pmatrix}^T$ and $\begin{pmatrix} 0 & -\sqrt{\frac{143}{36}} & \frac{1}{6} \end{pmatrix}^T$.

MATLAB CODE FOR FIGURE 1:

```
x = linspace(2,10);
y = (x-8);
plot(x,y,'Color',[1,0,0]); hold on;

x = linspace(4,10);
y = (-sqrt(x-4));
plot(x,y,'Color',[0,1,0]); hold on;

x = linspace(4,10);
y = (sqrt(x-4));
plot(x,y,'Color',[0,1,0]); hold on;

x1 = -6:0.2:10;
x2 = -5:0.2:5;
[X1,X2] = meshgrid(x1,x2);
Z = (((X1-2).^4)/4)+(X2.^4)+4;


[c,h]=contour(X1,X2,Z,[8 100 500])
clabel(c,h)
grid on
grid minor
```

MATLAB CODE FOR FIGURE 2:

```
x = linspace(2,10);
y = (x-8);
plot(x,y,'Color',[1,0,0]); hold on;

x = linspace(4,10);
y = (-sqrt(x-4));
plot(x,y,'Color',[0,1,0]); hold on;

x = linspace(4,10);
y = (sqrt(x-4));
plot(x,y,'Color',[0,1,0]); hold on;

x1 = -6:0.2:10;
x2 = -5:0.2:5;
[X1,X2] = meshgrid(x1,x2);
Z = (((X1-2).^4)/4)+(X2.^4)+4;

[c,h]=contour(X1,X2,Z,[8 100 500])
clabel(c,h)
grid on
grid minor

x1_col = linspace((1-sqrt(17))/2 + 8,(1-sqrt(17))/2 + 8);
y1_col = x1_col-8;
y2_col = linspace((1-sqrt(17))/2,(1+sqrt(17))/2);
x2_col = 4+y2_col.^2;

reg_bw = [x1_col, fliplr(x2_col)];
inBetween = [y1_col, fliplr(y2_col)];
h = fill(reg_bw, inBetween, [0.5, 0.25, 0.5]);
set(h,'facealpha',.2);
```