On the nested *p*-center problem

Christof Brandstetter, Markus Sinnl
Institute of Business Analytics and Technology Transformation / JKU Business School,
Johannes Kepler University Linz

This research was funded in whole, or in part, by the Austrian Science Fund (FWF) [P 35160-N].

2024-03-06

INOC - 2024 Dublin

JOHANNES KEPLER
UNIVERSITY LINZ
Altenberger Straße 69
4040 Linz, Austria
jku.at

The p-center problem (pCP):

Definition

The p-center problem (pCP):

Definition

- given
  - a set V of locations,

The p-center problem (pCP):

Definition

- given
  - a set V of locations,
  - $p \in \mathbb{Z}$, and

The p-center problem (pCP):

Definition

- given
  - a set V of locations,
  - $p \in \mathbb{Z}$, and
  - distances d_{ij} from location $i \in V$ to $j \in V$

The p-center problem (pCP):

Definition

- given
  - a set V of locations,
  - $p \in \mathbb{Z}$, and
  - distances d_{ij} from location $i \in V$ to $j \in V$
- we want to
  - open p locations of V, such that

The p-center problem (pCP):

Definition

- given
  - a set V of locations,
  - $p \in \mathbb{Z}$, and
  - distances d_{ij} from location $i \in V$ to $j \in V$
- we want to
  - open p locations of V, such that
  - the maximum distance of any location to its closest opened location is minimized.

The nesting property

Definition

- given
  - a set of locations V

The nesting property

Definition

- given
  - a set of locations V
  - a set of time periods $\mathcal{H} = \{1, \ldots, H\}$ and

The nesting property

Definition

- given
  - a set of locations V
  - a set of time periods $\mathcal{H} = \{1, \dots, H\}$ and
  - a set of integers $\mathcal{P} = \left\{p^1, \dots, p^H\right\}$ indicating the number of facilities to open

The nesting property

Definition

- given
  - a set of locations V
  - a set of time periods $\mathcal{H} = \{1, \ldots, H\}$ and
  - a set of integers $\mathcal{P} = \left\{p^1, \ldots, p^H\right\}$ indicating the number of facilities to open
  - where $p^h \leq p^{h+1}$ for $h = 1, \ldots, H-1$

The nesting property

Definition

- given
  - a set of locations V
  - a set of time periods $\mathcal{H} = \{1, \ldots, H\}$ and
  - a set of integers $\mathcal{P} = \left\{p^1, \ldots, p^H\right\}$ indicating the number of facilities to open
  - where $p^h \leq p^{h+1}$ for $h = 1, \ldots, H-1$
- the subsets $V^h \subseteq V$ with $|V^h| = p^h$

The nesting property

Definition

- given
  - a set of locations V
  - a set of time periods $\mathcal{H} = \{1, \ldots, H\}$ and
  - a set of integers $\mathcal{P} = \left\{p^1, \ldots, p^H\right\}$ indicating the number of facilities to open
  - where $p^h \leq p^{h+1}$ for $h = 1, \ldots, H-1$
- the subsets $V^h \subseteq V$ with $|V^h| = p^h$
- are called nested iff $V^h \subseteq V^{h+1}$ for $h = 1, \ldots, H-1$

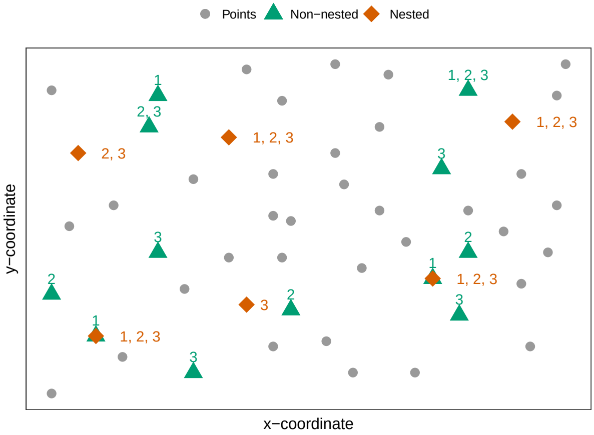
The nesting property

Definition

- given
  - a set of locations V
  - a set of time periods $\mathcal{H} = \{1, \ldots, H\}$ and
  - a set of integers $\mathcal{P} = \left\{p^1, \ldots, p^H\right\}$ indicating the number of facilities to open
  - where $p^h \leq p^{h+1}$ for $h = 1, \ldots, H-1$
- the subsets $V^h \subseteq V$ with $|V^h| = p^h$
- are called nested iff $V^h \subseteq V^{h+1}$ for $h = 1, \ldots, H-1$

Introduced by Roodman and Schwarz 1975 and used in e.g., Albareda-Sambola et al. 2009; Bakker and Nickel 2024; Conforti et al. 2014 and reintroduced as nesting by McGarvey and Thorsen 2022

p-center problem vs nested p-center problem for $\mathcal{P} = \{4, 5, 6\}$



The nested p-center problem

Definition part I

- given a set of locations V,

The nested p-center problem

Definition part I

- given a set of locations V,
- time periods $\mathcal{H} = \{1, \ldots, H\}$,

The nested p-center problem

Definition part I

- given a set of locations V,
- time periods $\mathcal{H} = \{1, \ldots, H\}$,
- integers $\mathcal{P} = \left\{p^1, \ldots p^H\right\}$

The nested p-center problem

Definition part I

- given a set of locations V,
- time periods $\mathcal{H} = \{1, \dots, H\}$,
- integers $\mathcal{P} = \left\{p^1, \dots p^H\right\}$
  - where $p^h \leq p^{h+1}$ for $h = 1, \dots, H-1$ and
  - $p^H \leq |V|$

The nested *p*-center problem

Definition part I

- given a set of locations V,
- time periods $\mathcal{H} = \{1, \dots, H\}$,
- integers $\mathcal{P} = \left\{p^1, \dots p^H\right\}$
  - where $p^h \leq p^{h+1}$ for $h = 1, \dots, H-1$ and
  - $p^H \leq |V|$
- distances $d_{ij} \geq 0$ between each $i \in V$ and $j \in V$

The nested p-center problem

Definition part I

- given a set of locations V,
- time periods $\mathcal{H} = \{1, \dots, H\}$,
- integers $\mathcal{P} = \left\{p^1, \dots p^H\right\}$
  - where $p^h \leq p^{h+1}$ for $h = 1, \dots, H-1$ and
  - $p^H \leq |V|$
- distances $d_{ij} \geq 0$ between each $i \in V$ and $j \in V$

Definition part II

- a feasible solution to the nested p-center problem consists of a set $\mathcal{J}^h \subseteq V$

The nested p-center problem

Definition part I

- given a set of locations V,
- time periods $\mathcal{H} = \{1, \dots, H\}$,
- integers $\mathcal{P} = \left\{p^1, \dots p^H\right\}$
  - where $p^h \leq p^{h+1}$ for $h = 1, \dots, H-1$ and
  - $p^H \leq |V|$
- distances $d_{ij} \geq 0$ between each $i \in V$ and $j \in V$

Definition part II

- a feasible solution to the nested p-center problem consists of a set $\mathcal{J}^h \subseteq V$
  - with $\left|V^h\right| = p^h$ for $h \in \mathcal{H}$,
  - for which $V^h \subseteq V^{h+1}$ for $h = 1, \dots, H-1$ holds

The nested p-center problem

Definition part I

- given a set of locations V,
- time periods $\mathcal{H} = \{1, \dots, H\}$,
- integers $\mathcal{P} = \left\{p^1, \dots p^H\right\}$
  - where $p^h \leq p^{h+1}$ for $h = 1, \dots, H-1$ and
  - $p^H \leq |V|$
- distances $d_{ij} \geq 0$ between each $i \in V$ and $j \in V$

Definition part II

- a feasible solution to the nested p-center problem consists of a set $\mathcal{J}^h \subseteq V$
  - with $\left|V^h\right| = p^h$ for $h \in \mathcal{H}$,
  - for which $V^h \subseteq V^{h+1}$ for $h = 1, \dots, H-1$ holds
- the goal is to find a feasible solution which minimizes $\sum_{h=1}^{H} d(V^h)$,
  - where $d(V^h) = \max_{i \in V} \min_{j \in V^h} d_{ij}$ for $h \in \mathcal{H}$

Mixed Integer Linear Programming (MILP) Formulations

First formulation

Decision variables

$$y_j^h \ldots \begin{cases} 1 \ldots \text{if location } j \text{ is open in} \\ \qquad \text{time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

First formulation

Decision variables

$$y_j^h \ldots \begin{cases} 1 \ldots \text{if location } j \text{ is open in} \\ \qquad \text{time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

$z^h \ldots$ maximum distance between any customer i and its nearest open facility in period h

First formulation

Decision variables

$$y_j^h \dots \begin{cases} 1 \dots \text{if location } j \text{ is open in} \\ \qquad \text{time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

$z^h \dots$ maximum distance between any customer i and its nearest open facility in period h

(nPC2) based on Gaar and Sinnl 2022

$$
\begin{aligned}
\min \quad & \sum_{h \in \mathcal{H}} z^h && && \text{(1a)}\\
\text{s.t.} \quad & \sum_{j \in V} y_j^h = p^h && \forall h \in \mathcal{H} && \text{(1b)}\\
& z^h \geq d_{ij} - \sum_{j' : d_{ij'} < d_{ij}} (d_{ij} - d_{ij'}) y_{j'}^h && \forall i, j \in V, h \in \mathcal{H} && \text{(1c)}\\
& y_j^h \geq y_j^{h-1} && \forall j \in V, \forall h \in \mathcal{H} \setminus \{1\} && \text{(1d)}\\
& (y, z) \in \mathbb{B}^{|V||\mathcal{H}|} \times \mathbb{R}_{\geq 0}^{|\mathcal{H}|} && && \text{(1e)}
\end{aligned}
$$

Second formulation

Decision variables

$$y_j^h \ldots \begin{cases} 1 \ldots \text{if location } j \text{ is open in} \\ \qquad \text{time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

$z^h \ldots$ maximum distance between any customer i and its nearest open facility in period h

Second formulation

Decision variables

$$y_j^h \ldots \begin{cases} 1 \ldots \text{if location } j \text{ is open in} \\ \qquad\quad \text{time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

$z^h \ldots$ maximum distance between any customer i and its nearest open facility in period h

$$u_k^h \ldots \begin{cases} 1 \ldots \text{if } z^h \geq D_k \text{ in} \\ \qquad\quad \text{time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

Second formulation

Decision variables

$$y_j^h \dots \begin{cases} 1 \dots \text{if location } j \text{ is open in time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

$z^h \dots$ maximum distance between any customer i and its nearest open facility in period h

$$u_k^h \dots \begin{cases} 1 \dots \text{if } z^h \geq D_k \text{ in time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

(nPC3) based on Ales and Elloumi 2018

$$\begin{aligned}
\min \quad & \sum_{h \in \mathcal{H}} z^h \\
\text{s.t.} \quad & \sum_{j \in V} y_j^h = p^h && \forall h \in \mathcal{H} \\
& D_0 + \sum_{k=1}^{K} (D_k - D_{k-1}) u_k^h \leq z^h && \forall h \in \mathcal{H} \\
& u_k^h + \sum_{j: d_{ij} < D_k} y_j^h \geq 1 && \forall i \in V, \forall h \in \mathcal{H}, \forall k \in S_i \cup \{K\} \\
& u_k^h \geq u_{k+1}^h && \forall h \in \mathcal{H}, \forall k \in \mathcal{K} \setminus \{K\} \\
& y_j^h \geq y_j^{h-1} && \forall h \in \mathcal{H} \setminus \{1\} \\
& (u, y, z) \in \mathbb{B}^{|\mathcal{K}||\mathcal{H}|} \times \mathbb{B}^{|V||\mathcal{H}|} \times \mathbb{R}_{\geq 0}^{|\mathcal{H}|}
\end{aligned}$$

Improving the formulations

Strengthening constraints `(nPC2)`

Strengthening constraints (nPC2)

Lemma 1

Let $\underline{z}^h$ be a lower bound on the decision variable z_h of (nPC2) for every $i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H}$ then constraints

Strengthening constraints (nPC2)

Lemma 1

Let $\underline{z}^h$ be a lower bound on the decision variable z_h of (nPC2) for every $i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H}$ then constraints

$$z^h \geq d_{ij} - \sum_{j': d_{ij'} < d_{ij}} (d_{ij} - d_{ij'}) y_{j'}^h$$

can be replaced by

Strengthening constraints (nPC2)

Lemma 1

Let $\underline{z}^h$ be a lower bound on the decision variable z_h of (nPC2) for every $i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H}$ then constraints

$$z^h \geq d_{ij} - \sum_{j':d_{ij'}<d_{ij}} (d_{ij} - d_{ij'}) y_{j'}^h$$

can be replaced by

$$z^h \geq max\{\underline{z}^h, d_{ij}\} - \sum_{j':d_{ij'}<d_{ij}} \left(\max\{\underline{z}^h, d_{ij}\} - \max\{\underline{z}^h, d_{ij'}\}\right) y_{j'}^h \qquad \text{(nL-OPT)}$$

The lemma is based on Lemma 5 in Gaar and Sinnl (2022).

Reducing the number of variables u_k^h in `(nPC3)`

Reducing the number of variables u_k^h in (nPC3)

Observation 1

Let $\underline{z^h}$ be a valid lower bound and $\overline{z^h}$ be a valid upper bound on the decision variable z^h for $h \in \mathcal{H}$, then the distinct distance D_k can only be the optimal distance for z^h if $\underline{z^h} \leq D^k \leq \overline{z^h}$ holds.

Observation 2

Following Observation 1, we observe that decision variables $u_h^k = 0$ for $k : D_k > \overline{z^h}$ and $u_h^k = 1$ for $k : D_k < \underline{z^h}$ in any optimal solution.

Reducing the number of variables u_k^h in (nPC3)

Observation 1

Let $\underline{z^h}$ be a valid lower bound and $\overline{z^h}$ be a valid upper bound on the decision variable z^h for $h \in \mathcal{H}$, then the distinct distance D_k can only be the optimal distance for z^h if $\underline{z^h} \leq D^k \leq \overline{z^h}$ holds.

Observation 2

Following Observation 1, we observe that decision variables $u_h^k = 0$ for $k : D_k > \overline{z^h}$ and $u_h^k = 1$ for $k : D_k < \underline{z^h}$ in any optimal solution.

The decision variables u_k^h which are zero or one following Observation 2, are redundant.

Obtaining bounds

Obtaining bounds

Observation 3

For $\mathcal{H} = \{1\}$ the (n-pCP) reduces to the (pCP) where $p = p^1$, so the optimal objective value (z'^{h*}) of the (pCP), where $p = p^h$ is a lower bound $\underline{z^h}$ on the decision variable z^h of the (n-pCP).

Obtaining bounds

Observation 3

For $\mathcal{H} = \{1\}$ the (n-pCP) reduces to the (pCP) where $p = p^1$, so the optimal objective value (z'^{h*}) of the (pCP), where $p = p^h$ is a lower bound $\underline{z^h}$ on the decision variable z^h of the (n-pCP).

Proposition 1

Given an valid upper bound UB on the objective value of the (n-pCP) and valid lower bounds $\underline{z^h}$ on the variable z^h can be obtained by the following equation:

$$\overline{z^h} = \frac{UB - \sum_{x=h+1}^{H} \underline{z^h}}{h} \tag{3}$$

Implementation

Implementation

- C++, CPLEX 20.1

Implementation

- C++, CPLEX 20.1
- preprocessing
  - Solving p-center problem with $p = p^H$
  - Use optimal solution value as lower bound for p-center problem with $p = p^{H-1}$
  - Repeat for remaining p^h and calculate the upper bounds $\overline{z^h}$

Implementation

- C++, CPLEX 20.1
- preprocessing
  - Solving p-center problem with $p = p^H$
  - Use optimal solution value as lower bound for p-center problem with $p = p^{H-1}$
  - Repeat for remaining p^h and calculate the upper bounds $\overline{z^h}$
- `(nPC2)`
  - branch-and-cut algorithm
  - separation based on the fixedCustomer separation scheme from Gaar and Sinnl 2022

Implementation

- C++, CPLEX 20.1
- preprocessing
  - Solving p-center problem with $p = p^H$
  - Use optimal solution value as lower bound for p-center problem with $p = p^{H-1}$
  - Repeat for remaining p^h and calculate the upper bounds $\overline{z^h}$
- `(nPC2)`
  - branch-and-cut algorithm
  - separation based on the fixedCustomer separation scheme from Gaar and Sinnl 2022
- `(nPC3)`

Implementation

- C++, CPLEX 20.1
- preprocessing
  - Solving p-center problem with $p = p^H$
  - Use optimal solution value as lower bound for p-center problem with $p = p^{H-1}$
  - Repeat for remaining p^h and calculate the upper bounds $\overline{z^h}$
- `(nPC2)`
  - branch-and-cut algorithm
  - separation based on the fixedCustomer separation scheme from Gaar and Sinnl 2022
- `(nPC3)`
  - branch-and-bound algorithm

Settings

- noPP: without preprocessing and without lifting

Settings

- noPP: without preprocessing and without lifting
- PP: with preprocessing

Settings

- `noPP`: without preprocessing and without lifting
- `PP`: with preprocessing
  - `(nPC2)`: set upper bounds on z^h and using lower bounds in (nL-OPT)
  - `(nPC3)`: problem initialized on reduced number of u_k^h

Settings

- `noPP`: without preprocessing and without lifting
- `PP`: with preprocessing
  - `(nPC2)`: set upper bounds on z^h and using lower bounds in (nL-OPT)
  - `(nPC3)`: problem initialized on reduced number of u_k^h
- `PH`: `PP` with starting heuristic

Results

Instance from literature

- instance set PMED
  - 40 instances
  - $\mathcal{P} = \{p, p+1, p+2\}$, p from 5 to 200, $|V|$ from 100 and 900 nodes

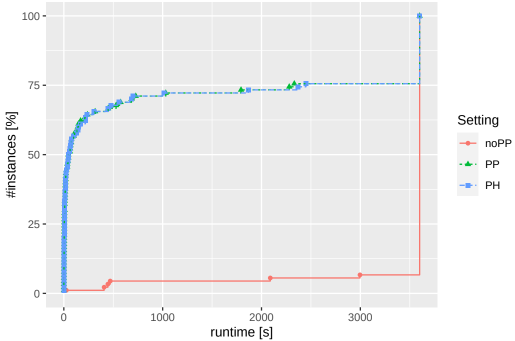
Instance from literature

- instance set PMED
  - 40 instances
  - $\mathcal{P} = \{p, p+1, p+2\}$, p from 5 to 200, $|V|$ from 100 and 900 nodes
- instance set TSPLIB
  - 50 instances
  - $\mathcal{P} = \{4, 5, 6\}$, $|V|$ from 51 and 1002 nodes

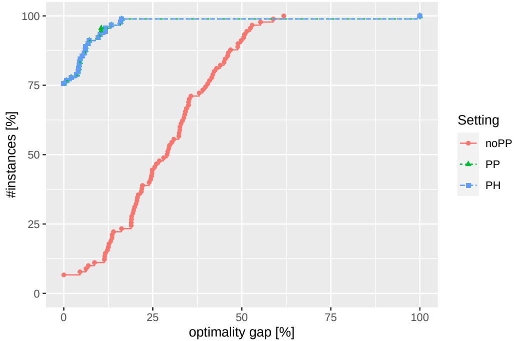
Instance from literature

- instance set PMED
  - 40 instances
  - $\mathcal{P} = \{p, p+1, p+2\}$, p from 5 to 200, $|V|$ from 100 and 900 nodes
- instance set TSPLIB
  - 50 instances
  - $\mathcal{P} = \{4, 5, 6\}$, $|V|$ from 51 and 1002 nodes
- computational setup
  - single core of Intel Xeon X5570 with 2.93 GHz and 8 GB RAM
  - timelimit of 3600 seconds

Setting comparison on formulation (nPC2)

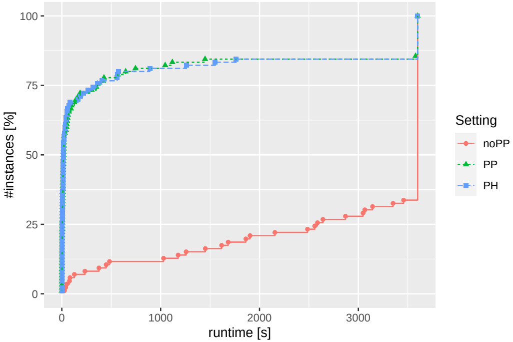


(a) runtime

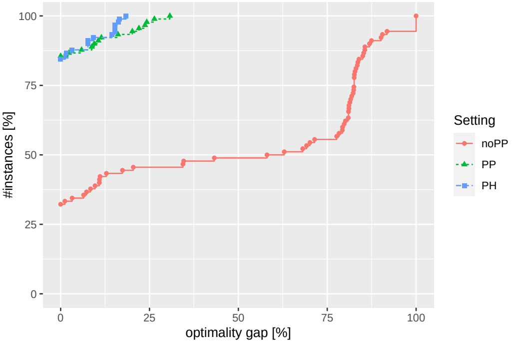


(b) optimality gap

Setting comparison on formulation (nPC3)

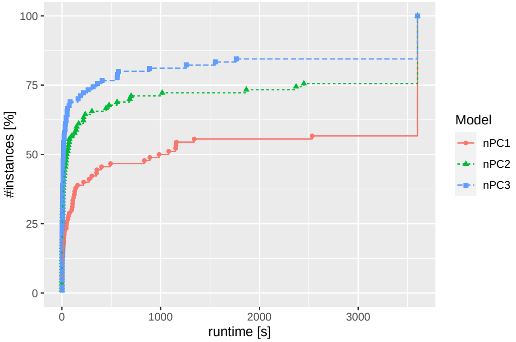


(c) runtime

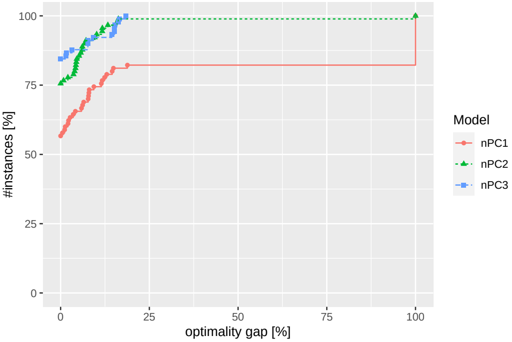


(d) optimality gap

Formulation comparison on setting PH

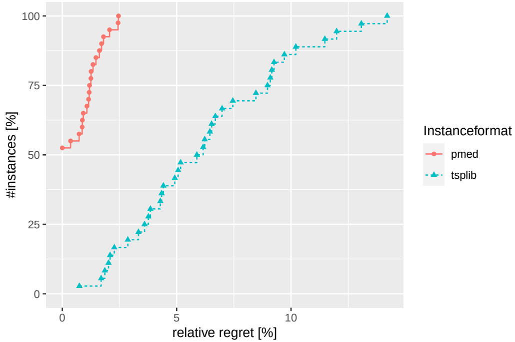


(e) runtime



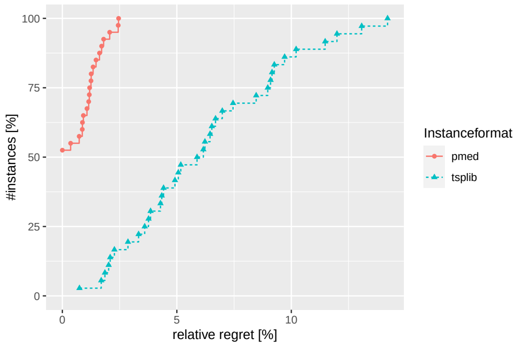
(f) optimality gap

Managerial insights

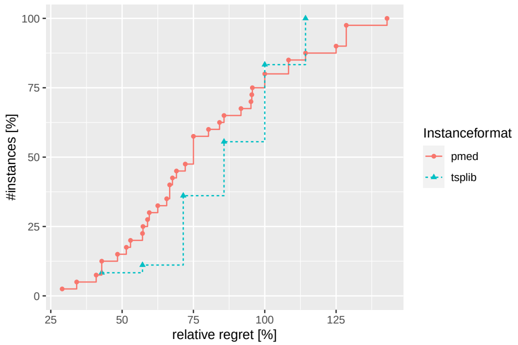


(g) Relative regret of the optimal solution value

Managerial insights



(i) Relative regret of the optimal solution value



(j) Relative regret of # of opened facilities

Conclusion

- `(nPC3)` best performance in runtimes and optimality gap
- The `(nPC2)` outperforms the `(nPC1)`
- Preprocessing brings a large speed up on all formulations
- Starting heuristic little effect, shows good upper bound obtained in preprocessing
- Maximal relative regret of the optimal objective function value of 15%
- Maximal relative regret of number of opened facilities above 150%

JOHANNES KEPLER
UNIVERSITY LINZ

On the nested *p*-center problem

Christof Brandstetter, Markus Sinnl
Institute of Business Analytics and Technology Transformation / JKU Business School,
Johannes Kepler University Linz

This research was funded in whole, or in part, by the Austrian Science Fund (FWF) [P 35160-N].

2024-03-06

Dublin
INOC - 2024

JOHANNES KEPLER
UNIVERSITY LINZ
Altenberger Straße 69
4040 Linz, Austria
jku.at

References I

[1] M. Albareda-Sambola et al. The multi-period incremental service facility location problem. In: Computers & Operations Research 36.5 (2009), pp. 1356–1375.

[2] Z. Ales and S. Elloumi. Compact MILP formulations for the p-center problem. In: Combinatorial Optimization: 5th International Symposium, ISCO 2018, Marrakesh, Morocco, April 11–13, 2018, Revised Selected Papers 5. 2018, pp. 14–25.

[3] H. Bakker and S. Nickel. The Value of the Multi-period Solution revisited: When to model time in capacitated location problems. In: Computers & Operations Research 161 (2024), p. 106428.

[4] M. Conforti et al. Integer programming models. Springer, 2014.

References II

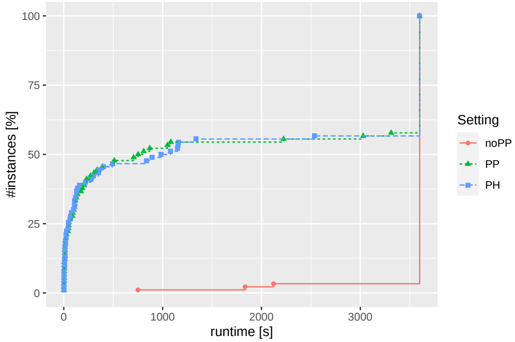
[5] M. S. Daskin. Network and Discrete Location: Models, Algorithms, and Applications, Second Edition. John Wiley & Sons, Ltd, 2013.

[6] E. Gaar and M. Sinnl. A scaleable projection-based branch-and-cut algorithm for the p-center problem. In: European Journal of Operational Research 303.1 (2022), pp. 78–98.

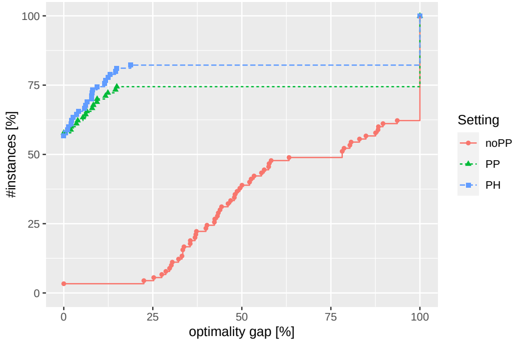
[7] R. G. McGarvey and A. Thorsen. Nested-solution facility location models. In: Optimization letters 16.2 (2022), pp. 497–514.

[8] G. M. Roodman and L. B. Schwarz. Optimal and heuristic facility phase-out strategies. In: AIIE transactions 7.2 (1975), pp. 177–184.

Setting comparison on formulation (nPC1)



(k) runtime



(l) optimality gap [$\frac{UB-LB}{UB}$]

The nested p-center problem: Classical MILP formulation

Decision variables

$$x_{ij}^{h} \ldots \begin{cases} 1 \ldots \text{if location } i \text{ is assigned to} \\ \qquad \text{location } j \text{ in time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

The nested p-center problem: Classical MILP formulation

Decision variables

$$x_{ij}^h \ldots \begin{cases} 1 \ldots \text{if location } i \text{ is assigned to} \\ \qquad \text{location } j \text{ in time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

$$y_j^h \ldots \begin{cases} 1 \ldots \text{if location } j \text{ is open in} \\ \qquad \text{time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

The nested p-center problem: Classical MILP formulation

Decision variables

$$x_{ij}^h \ldots \begin{cases} 1 \ldots \text{if location } i \text{ is assigned to} \\ \qquad \text{location } j \text{ in time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

$$y_j^h \ldots \begin{cases} 1 \ldots \text{if location } j \text{ is open in} \\ \qquad \text{time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

$z^h \ldots$ maximum distance between any customer i and its nearest open facility in period h

The nested p-center problem: Classical MILP formulation

Decision variables

$$x_{ij}^h \dots \begin{cases} 1 \dots \text{if location } i \text{ is assigned to location } j \text{ in time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

$$y_j^h \dots \begin{cases} 1 \dots \text{if location } j \text{ is open in time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

$z^h \dots$ maximum distance between any customer i and its nearest open facility in period h

(nPC1) based on Daskin 2013

$$\min \sum_{h \in \mathcal{H}} z^h \tag{4a}$$

$$\text{s.t.} \sum_{j \in V} y_j^h = p^h \qquad \forall h \in \mathcal{H} \tag{4b}$$

$$\sum_{j \in V} x_{ij}^h = 1 \qquad \forall i \in V, h \in \mathcal{H} \tag{4c}$$

$$x_{ij}^h \leq y_j^h \qquad \forall i, j \in V, h \in \mathcal{H} \tag{4d}$$

$$\sum_{j \in V} d_{ij} x_{ij}^h \leq z^h \qquad \forall i \in V, h \in \mathcal{H} \tag{4e}$$

$$y_j^h \geq y_j^{h-1} \qquad \forall h \in \mathcal{H} \setminus \{1\} \tag{4f}$$

$$(x, y, z) \in |V|^2|\mathcal{H}| \times |V||\mathcal{H}| \times \mathbb{R}_{\geq 0} \tag{4g}$$