On the nested p-center problem

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  - the maximum distance of any location to its closest opened location is minimized.

The nesting property

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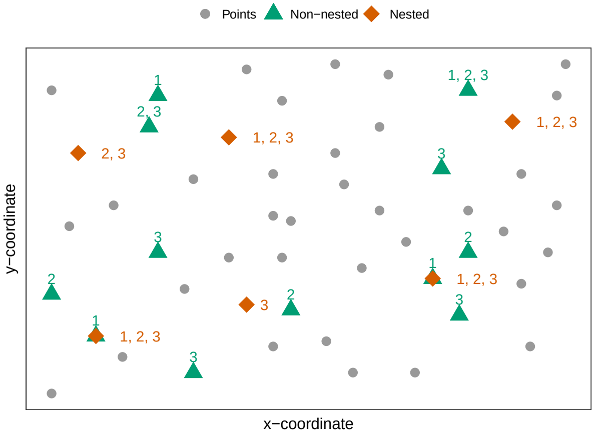
Introduced by Roodman and Schwarz 1975 and used in e.g., Albareda-Sambola et al. 2009; Bakker and Nickel 2024; Conforti et al. 2014 and reintroduced as nesting by McGarvey and Thorsen 2022

p-center problem vs nested p-center problem for $\mathcal{P} = \{4, 5, 6\}$



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  - for which $V^h \subseteq V^{h+1}$ for $h = 1, \dots, H-1$ holds
- the goal is to find a feasible solution which minimizes $\sum_{h=1}^{H} d(V^h)$,
  - where $d(V^h) = \max_{i \in V} \min_{j \in V^h} d_{ij}$ for $h \in \mathcal{H}$

Mixed Integer Linear Programming (MILP) Formulations

First formulation

Decision variables

$$y_j^h \ldots \begin{cases} 1 \ \ldots \text{ if location } j \text{ is open in} \\ \qquad \text{time period } h \\ 0 \ \ldots \text{ otherwise} \end{cases}$$

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(nPC2) based on Gaar and Sinnl 2022

$$\begin{aligned}
\min \quad & \sum_{h \in \mathcal{H}} z^h & & \text{(1a)} \\
\text{s.t.} \quad & \sum_{j \in V} y_j^h = p^h & \forall h \in \mathcal{H} & \text{(1b)} \\
& z^h \geq d_{ij} - \sum_{j' : d_{ij'} < d_{ij}} (d_{ij} - d_{ij'}) y_{j'}^h & \forall i, j \in V, h \in \mathcal{H} & \text{(1c)} \\
& y_j^h \geq y_j^{h-1} & \forall j \in V, \forall h \in \mathcal{H} \setminus \{1\} & \text{(1d)} \\
& (y, z) \in \mathbb{B}^{|V||\mathcal{H}|} \times \mathbb{R}_{\geq 0}^{|\mathcal{H}|} & & \text{(1e)}
\end{aligned}$$

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$$u_k^h \dots \begin{cases} 1 \dots \text{if } z^h \geq D_k \text{ in} \\ \qquad \text{time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

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(nPC3) based on Ales and Elloumi 2018

$$\begin{aligned}
\min \quad & \sum_{h \in \mathcal{H}} z^h \\
\text{s.t.} \quad & \sum_{j \in V} y_j^h = p^h && \forall h \in \mathcal{H} \\
& D_0 + \sum_{k=1}^{K} (D_k - D_{k-1}) u_k^h \leq z^h && \forall h \in \mathcal{H} \\
& u_k^h + \sum_{j : d_{ij} < D_k} y_j^h \geq 1 && \forall i \in V, \forall h \in \mathcal{H}, \forall k \in S_i \cup \{K\} \\
& u_k^h \geq u_{k+1}^h && \forall h \in \mathcal{H}, \forall k \in \mathcal{K} \setminus \{K\} \\
& y_j^h \geq y_j^{h-1} && \forall h \in \mathcal{H} \setminus \{1\} \\
& (u, y, z) \in \mathbb{B}^{|\mathcal{K}||\mathcal{H}|} \times \mathbb{B}^{|V||\mathcal{H}|} \times \mathbb{R}_{\geq 0}^{|\mathcal{H}|}
\end{aligned}$$

Improving the formulations

Strengthening constraints (nPC2)

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Let $\underline{z}^h$ be a lower bound on the decision variable z_h of (nPC2) for every $i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H}$ then constraints

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can be replaced by

$$z^h \geq \max\{\underline{z}^h, d_{ij}\} - \sum_{j': d_{ij'} < d_{ij}} \left(\max\{\underline{z}^h, d_{ij}\} - \max\{\underline{z}^h, d_{ij'}\} \right) y_{j'}^h \qquad \text{(nL-OPT)}$$

The lemma is based on Lemma 5 in Gaar and Sinnl (2022).

Reducing the number of variables u_k^h in (nPC3)

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Observation 1

Let $\underline{z^h}$ be a valid lower bound and $\overline{z^h}$ be a valid upper bound on the decision variable z^h for $h \in \mathcal{H}$, then the distinct distance D_k can only be the optimal distance for z^h if $\underline{z^h} \leq D^k \leq \overline{z^h}$ holds.

Observation 2

Following Observation 1, we observe that decision variables $u_h^k = 0$ for $k : D_k > \overline{z^h}$ and $u_h^k = 1$ for $k : D_k < \underline{z^h}$ in any optimal solution.

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The decision variables u_k^h which are zero or one following Observation 2, are redundant.

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Observation 3

For $\mathcal{H} = \{1\}$ the (n-pCP) reduces to the (pCP) where $p = p^1$, so the optimal objective value (z'^{h*}) of the (pCP), where $p = p^h$ is a lower bound $\underline{z}^h$ on the decision variable z^h of the (n-pCP).

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Proposition 1

Given an valid upper bound UB on the objective value of the (n-pCP) and valid lower bounds $\underline{z^h}$ on the variable z^h can be obtained by the following equation:

$$\overline{z^h} = \frac{UB - \sum_{x=h+1}^{H} \underline{z^h}}{h} \tag{3}$$

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  - Solving p-center problem with $p = p^H$
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  - Repeat for remaining p^h and calculate the upper bounds $\overline{z^h}$

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- `(nPC2)`
  - branch-and-cut algorithm
  - separation based on the fixedCustomer separation scheme from Gaar and Sinnl 2022

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- `PP`: with preprocessing
  - `(nPC2)`: set upper bounds on z^h and using lower bounds in (nL-OPT)
  - `(nPC3)`: problem initialized on reduced number of u_k^h
- `PH`: `PP` with starting heuristic

Results

Instance from literature

- instance set PMED
  - 40 instances
  - $\mathcal{P} = \{p, p+1, p+2\}$, p from 5 to 200, $\left|V\right|$ from 100 and 900 nodes

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- instance set TSPLIB
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  - $\mathcal{P} = \{4, 5, 6\}$, $|V|$ from 51 and 1002 nodes

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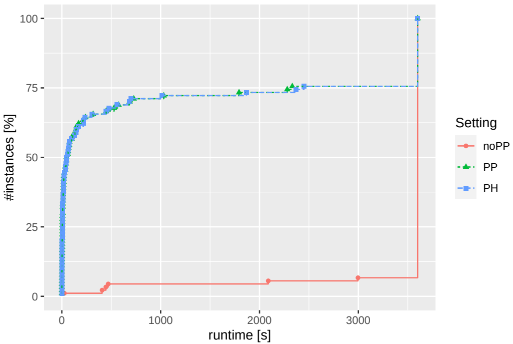
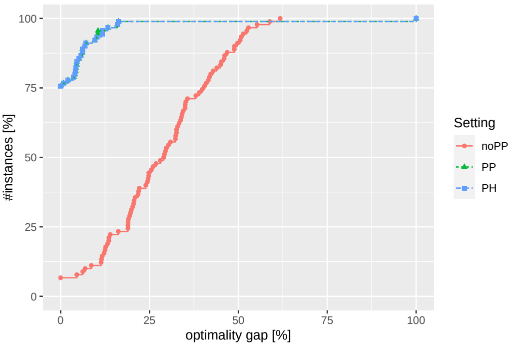
  - $\mathcal{P} = \{4, 5, 6\}$, $|V|$ from 51 and 1002 nodes
- computational setup
  - single core of Intel Xeon X5570 with 2.93 GHz and 8 GB RAM
  - timelimit of 3600 seconds

Setting comparison on formulation (nPC2)

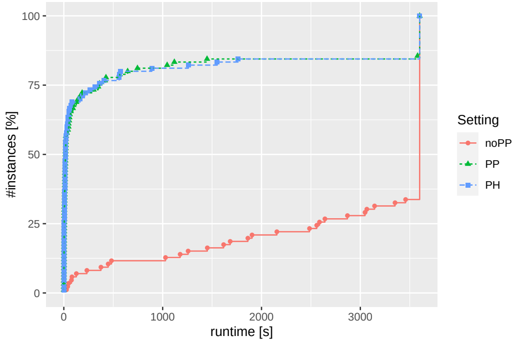


(a) Runtimes

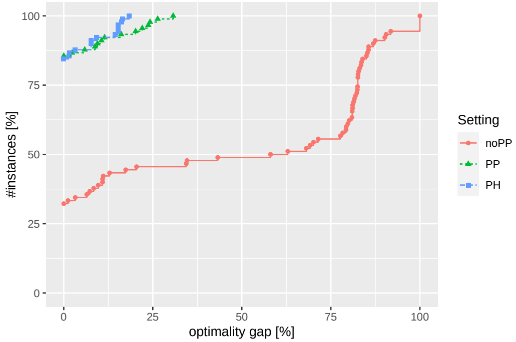


(b) Optimality gaps $\left[\frac{UB-LB}{UB}100\right]$

Setting comparison on formulation (nPC3)

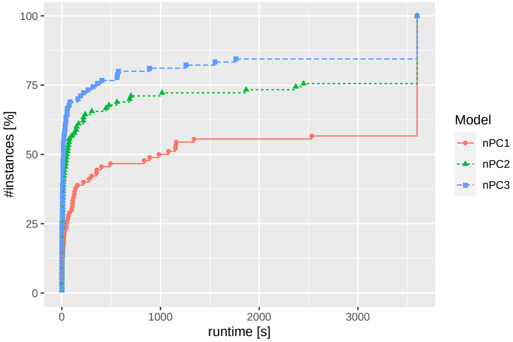


(c) Runtimes

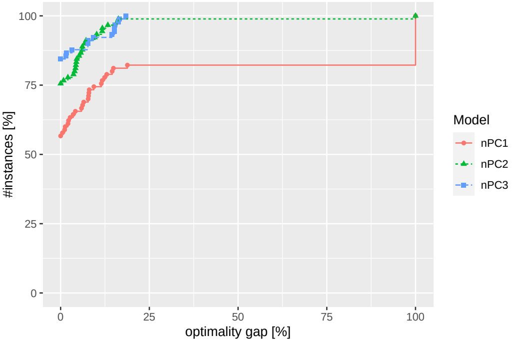


(d) Optimality gaps $\left[\frac{UB-LB}{UB}100\right]$

Formulation comparison on setting PH

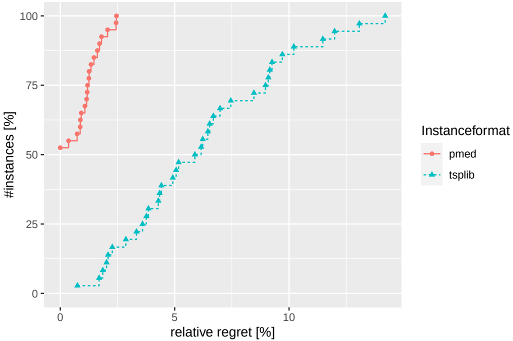


(e) Runtimes



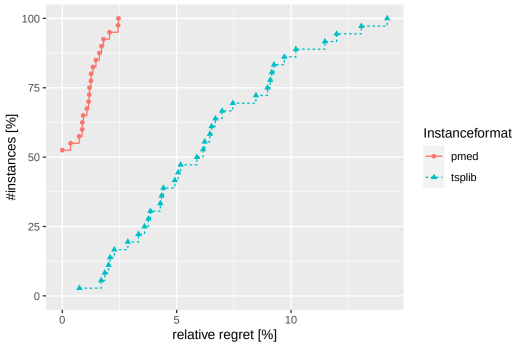
(f) Optimality gaps $\left[\frac{UB-LB}{UB}100\right]$

Managerial insights

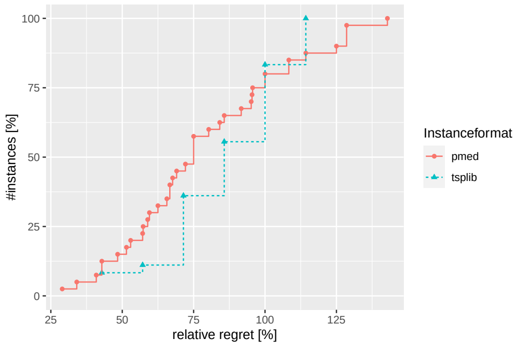


(g) Relative regrets of the optimal solution value

Managerial insights



(i) Relative regrets of the optimal solution value



(j) Relative regrets of # of opened facilities

Conclusion

- `(nPC3)` best performance in runtimes and optimality gap
- The `(nPC2)` outperforms the `(nPC1)`
- Preprocessing brings a large speed up on all formulations
- Starting heuristic little effect, shows good upper bound obtained in preprocessing
- Maximal relative regret of the optimal objective function value of 15%
- Maximal relative regret of number of opened facilities above 150%

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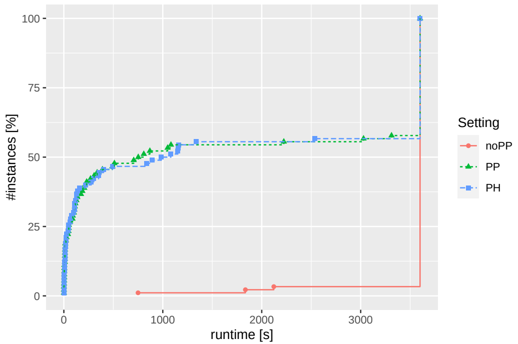
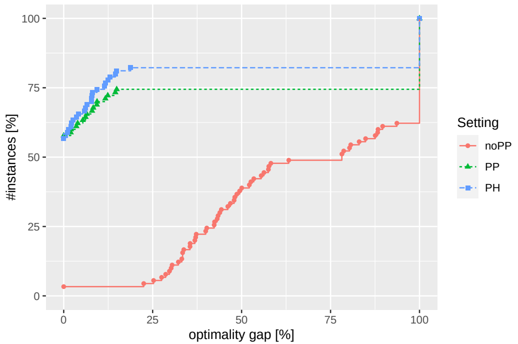
Setting comparison on formulation (nPC1)



(k) Runtimes



(l) Optimality gaps [$\frac{UB-LB}{UB}$]

The nested p-center problem: Classical MILP formulation

Decision variables

$$x_{ij}^h \dots \begin{cases} 1 \dots \text{if location } i \text{ is assigned to} \\ \qquad \text{location } j \text{ in time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

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$$y_j^h \dots \begin{cases} 1 \dots \text{if location } j \text{ is open in} \\ \qquad \text{time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

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Decision variables

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$$y_j^h \dots \begin{cases} 1 \dots \text{if location } j \text{ is open in} \\ \qquad \text{time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

$z^h \dots$ maximum distance between any customer i and its nearest open facility in period h

The nested p-center problem: Classical MILP formulation

Decision variables

$$x_{ij}^h \dots \begin{cases} 1 \dots \text{if location } i \text{ is assigned to} \\ \qquad \text{location } j \text{ in time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

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$z^h \dots$ maximum distance between any customer i and its nearest open facility in period h

(nPC1) based on Daskin 2013

$$\min \sum_{h \in \mathcal{H}} z^h \tag{4a}$$

$$\text{s.t.} \quad \sum_{j \in V} y_j^h = p^h \qquad \forall h \in \mathcal{H} \tag{4b}$$

$$\sum_{j \in V} x_{ij}^h = 1 \qquad \forall i \in V, h \in \mathcal{H} \tag{4c}$$

$$x_{ij}^h \le y_j^h \qquad \forall i, j \in V, h \in \mathcal{H} \tag{4d}$$

$$\sum_{j \in V} d_{ij} x_{ij}^h \le z^h \qquad \forall i \in V, h \in \mathcal{H} \tag{4e}$$

$$y_j^h \ge y_j^{h-1} \qquad \forall h \in \mathcal{H} \setminus \{1\} \tag{4f}$$

$$(x, y, z) \in |V|^2|\mathcal{H}| \times |V||\mathcal{H}| \times \mathbb{R}_{\ge 0} \tag{4g}$$