# On the nested $p$-center problem

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- we want to
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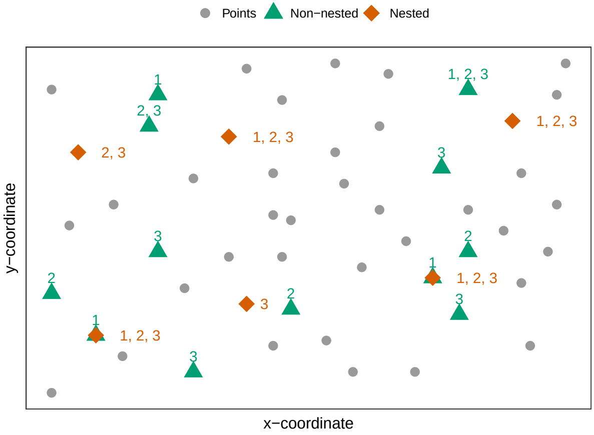
  - the maximum distance of any location to its closest opened location is minimized.

# The nesting property

# $p$-center problem vs nested $p$-center problem



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  - for which $\mathcal{J}^h \subseteq \mathcal{J}^{h+1}$ for $h = 1, \ldots, H-1$ holds
- the goal is to find a feasible solution which minimizes $\sum_{h=1}^{H} d_h(\mathcal{J}^h)$,
  - where $d_h(\mathcal{J}) = \max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}} d_{ij}$ for $h \in \mathcal{H}$

# The nested $p$-center problem: Potential applications

# *p*-center problem

- First introduction of the *p*-center problem by Hakimi (1964)
- The standard textbook formulation of the *p*-center problem can be found in Daskin (2013)
- Solution approach based on the set cover problem by Contardo, Iori, and Kramer (2019a)
- A compact formulation by Ales and Elloumi (2018) and Elloumi, Labbé, and Pochet (2004)
- A formulation with a projection-based branch-and-cut algorithm for the *p*-center problem by Gaar and Sinnl (2022)

# Nested facility location problems

- First introduction of the nesting property and constraint by Roodman and Schwarz (1975)
- Extension of the nesting to a phase-in and phase-out by Roodman and Schwarz (1977)
- Reintroduction of the nesting on the example of the $p$-median problem by McGarvey and Thorsen (2022)
- Other work on multi-period facility locations problems mainly focus on varying demand, distances, or cost over time f.e. Calogiuri et al. (2021)

# MILP formulations

# The nested $p$-center problem: MILP formulation

## Decision variables

$$y_j^h \ldots \begin{cases} 1 \ldots \text{if location } j \text{ is open in} \\ \qquad \text{time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

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## `(nPC2)` based on Gaar and Sinnl 2022

$$
\begin{aligned}
\min \quad & \sum_{h \in \mathcal{H}} z^h & & \text{(1a)} \\
\text{s.t.} \quad & \sum_{j \in V} y_j^h = p^h & \forall h \in \mathcal{H} \quad & \text{(1b)} \\
& z^h \geq d_{ij} - \sum_{j' : d_{ij'} < d_{ij}} (d_{ij} - d_{ij'}) y_{j'}^h & \forall i, j \in V, h \in \mathcal{H} \quad & \text{(1c)} \\
& y_j^h \geq y_j^{h-1} & \forall j \in V, \forall h \in \mathcal{H} \setminus \{1\} \quad & \text{(1d)} \\
& (y, z) \in \left|V\right| \left|\mathcal{H}\right| \times \mathbb{R}_{\geq 0} & & \text{(1e)}
\end{aligned}
$$

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$$u_k^h \dots \begin{cases} 1 \dots \text{if } z^h \geq D_k \text{ in} \\ \qquad \text{time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

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## (nPC3) based on Ales and Elloumi 2018

$$\begin{aligned}
\min \quad & \sum_{h \in \mathcal{H}} z^h \\
\text{s.t.} \quad & \sum_{j \in V} y_j^h = p^h && \forall h \in \mathcal{H} \\
& D_0 + \sum_{k=1}^{K} (D_k - D_{k-1}) u_k^h \leq z^h && \forall h \in \mathcal{H} \\
& u_k^h + \sum_{j: d_{ij} < D_k} y_j^h \geq 1 && \forall i \in V, \forall h \in \mathcal{H}, \forall k \in S_i \cup \{K\} \\
& u_k^h \geq u_{k+1}^h && \forall h \in \mathcal{H}, \forall k \in \mathcal{K} \setminus \{K\} \\
& y_j^h \geq y_j^{h-1} && \forall h \in \mathcal{H} \setminus \{1\}
\end{aligned}$$

# Comparing formulations

Table: Comparison of complexity of the formulations

| | (nPC1) | (nPC2) | (nPC3) |
|---|---|---|---|
| Variables | $\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$ | $\mathcal{O}(\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$ | $\mathcal{O}((\lvert\mathcal{I}\rvert + \lvert\mathcal{K}\rvert)\lvert\mathcal{H}\rvert)$ |
| Constraints | $\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$ | $\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$ | $\mathcal{O}(\min(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert, \lvert\mathcal{I}\rvert\lvert\mathcal{K}\rvert)\lvert\mathcal{H}\rvert)$ |

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The *p*CP version of formulation `(nPC3)` has the best known linear programming ($\mathcal{LP}$)-bounds for the *p*CP, while the *p*CP versions of `(nPC1)` and `(nPC2)` have equally but worse $\mathcal{LP}$-bounds.

# Improving the formulations

# Strengthening constraints `(nPC2)`

**Lemma 1**

*Let $LB_h$ be a lower bound on the decision variable $z_h$ of (nPC2) for every $i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H}$ then*

$$z^h \geq max\{LB^h, d_{ij}\} - \sum_{j':d_{ij'}<d_{ij}} (\max\{LB^h, d_{ij}\} - \max\{LB^h, d_{ij'}\})y^h_{j'} \qquad \text{(nL-OPT)}$$

*is a valid inequality for (nPC2), i.e., every feasible solution of (nPC2) fulfills (nL-OPT). Theorem is based on Lemma 5 in Gaar and Sinnl (2022).*

A similar strengthening can be done for `(nPC1)` which is based on Daskin 2013.

# Reducing set $\mathcal{K}$ in `(nPC3)`

*Lemma 2*

*Let $\underline{z}^h$ be a valid lower bound and $\overline{z}^h$ be a valid upper bound on the decision variable $z^h$ for $h \in \mathcal{H}$, then the distinct distance $D_k$ can only be the optimal distance for $z^h$ if $\underline{z}^h \leq D^k \leq \overline{z}^h$ holds.*
*Therfore let set $S_i^h \subseteq S_i$ for $h \in \mathcal{H}$, where $S_i^h$ contains only the indices $k$ where $\underline{z}^h \leq D^k \leq \overline{z}^h$ holds.*

Depending on the bounds $\underline{z}^h$ and $\overline{z}^h$ the sets $S_i^h$ can be much smaller than $S_i$.

# Obtaining bounds

*Lemma 3*

*Let $z'^{h*}$ be the optimal objective function value of pCP with $p = p^h$ for $h \in \mathcal{H} = \{1, 2, \ldots, H\}$ where $p^h > p^{h+1}$, then $UB = Hz'^{1*}$ is a valid upper bound on the optimal objective function value of n-pCP.*
*Then*

$$\overline{z^h} = \frac{UB - \sum_{x=h+1}^{H} z'^{x*}}{h+1}$$

*where $\overline{z^h}$ is a valid upper bound on the decision variable $z^h$ of the n-pCP for $h \in \mathcal{H}$.*

# Implementation

# Implementation

- C++, CPLEX 20.1
- preprocessing
- `(nPC2)`
  - branch-and-cut algorithm
  - separation similar to fixedCustomer from Gaar and Sinnl 2022
- `(nPC3)`
  - branch-and-bound algorithm
  - starting with the reduced set $\mathcal{K}$

# Settings

- `noPP`: without preprocessing or any lifting
- `PP`: with preprocessing a lifting
  - `(nPC1)` and `(nPC2)`: set upper bounds on $z^h$ and using lower bounds in (nL-OPT)
  - `(nPC3)`: problem initialized on reduced set $\mathcal{K}$
- `PH`: `PP` with an additional starting heuristic

# Results

# Instance from literature

- computational setup
  - single core of Intel Xeon X5570 with 2.93 GHz and 8 GB RAM
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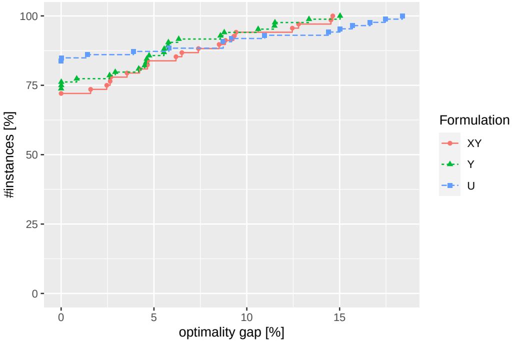
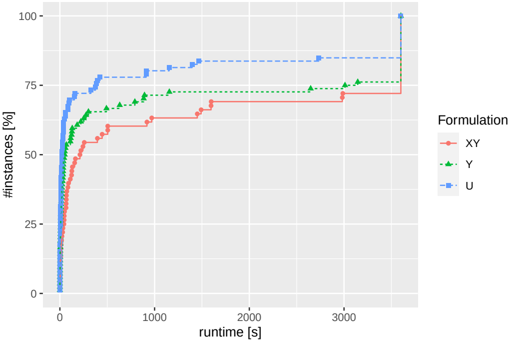
- instance set TSPLIB Contardo, Iori, and Kramer 2019b; Gaar and Sinnl 2022
  - 80 instances
  - $\mathcal{P} = \{4, 5, 6\}$, $|V|$ from 100 and 900 nodes

# Formulation comparison

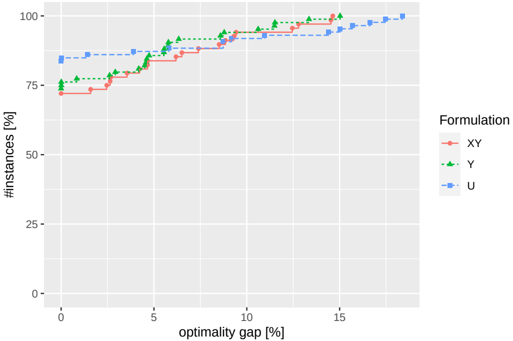



(a) models optimality gap

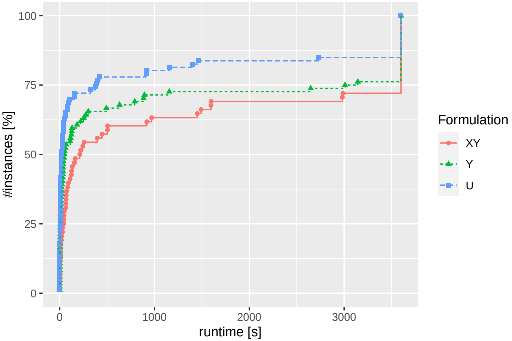



(b) models runtime

# Setting comparison

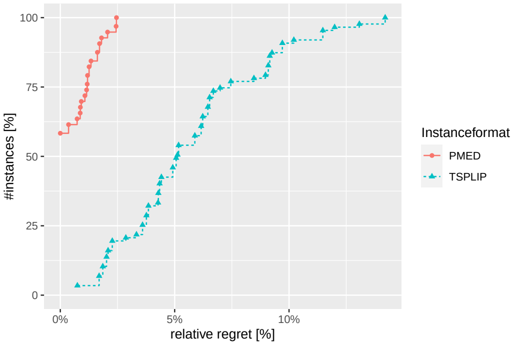



(c) models optimality gap

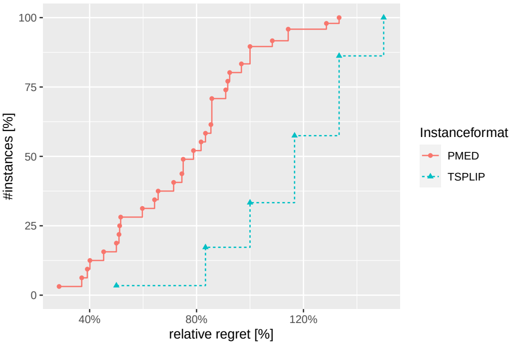



(d) models runtime

# Managerial insights




(e) Relative regret




(f) Relative regret of # of facilities

Figure: On a subset of instances: Only if the problem was solved to optimality

# Conclusion

- `(nPC3)` had the best runtimes over all instances
- For optimality gap the results are rather mixed
- The `(nPC2)` outperforms the `(nPC1)`
- Maximal relative regret of the objective value of 15%
- Maximal relative regret of number of facilities above 140%
- min-max regret as objective function was also analysed and can be found later in the thesis

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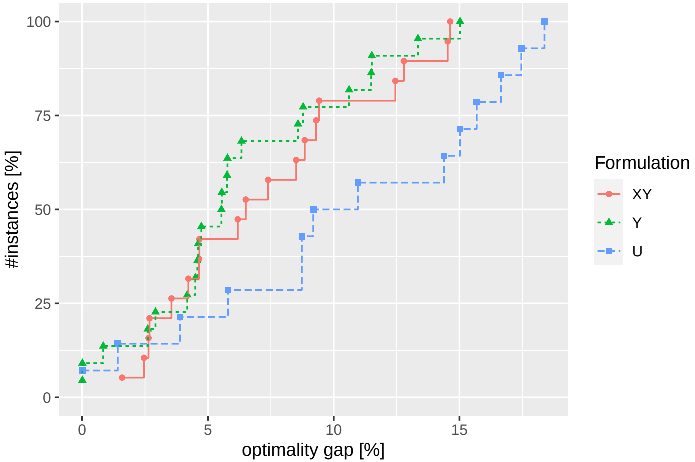
# Formulations with non optimal instances



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# The nested $p$-center problem: Classical MILP formulation

## Decision variables

$$x_{ij}^h \ldots \begin{cases} 1 \ldots \text{if location } i \text{ is assigned to} \\ \qquad \text{location } j \text{ in time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

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## (nPC1) based on Daskin 2013

$$\min \sum_{h \in \mathcal{H}} z^h \tag{3a}$$

$$\text{s.t.} \quad \sum_{j \in V} y_j^h = p^h \qquad \forall h \in \mathcal{H} \tag{3b}$$

$$\sum_{j \in V} x_{ij}^h = 1 \qquad \forall i \in V, h \in \mathcal{H} \tag{3c}$$

$$x_{ij}^h \leq y_j^h \qquad \forall i, j \in V, h \in \mathcal{H} \tag{3d}$$

$$\sum_{j \in V} d_{ij} x_{ij}^h \leq z^h \qquad \forall i \in V, h \in \mathcal{H} \tag{3e}$$

$$y_j^h \geq y_j^{h-1} \qquad \forall h \in \mathcal{H} \setminus \{1\} \tag{3f}$$

$$(x, y, z) \in |V|^2|\mathcal{H}| \times |V||\mathcal{H}| \times \mathbb{R}_{\geq 0} \tag{3g}$$