Introduction

- *p*-center problem: Choose a set of *p* facilities such that the maximum distance between a demand point and its closest facility belonging to that set is minimized.
- there exist approaches based on the set cover problem and the assignment problem
- Nested *p*-center problem: Open p_h facilities in period h such that the sum of the maximum distance between any customer and its nearest open facility in period h is minimized.
- Choose a set of p^h*facilitiesoverdifferenttimeperiodsh, suchthatthesumofthemaximumdistancebetwe*
  setsoflaterperiodscontainmorefacilitiesandhavetocontainthefacilitiesofsetsofprevio

- Use case: Ambulance/logistics stations, network design, screening/vaccination

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Definition of the nested p-center problem

Definition:

- given a set of customer demand points $\mathcal{I}$, potential facility locations $\mathcal{J}$, time periods $\mathcal{H} = \{1, \ldots, H\}$, integers $\mathcal{P} = \{p^1, \ldots p^H\}$ where $p^h \leq p^{h+1}$ for $h = 1, \ldots, H-1$ and distances $d_{ij} \geq 0$ between each $i \in \mathcal{I}$ and $j \in \mathcal{J}$

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- let $\mathcal{J}^h \subseteq \mathcal{J}$ with $|\mathcal{J}| = p^h$ for $h \in \mathcal{H}$ be a feasible solution to the nested p-center problem
- that fulfills $\mathcal{J}^h \subseteq \mathcal{J}^{h+1}$ for $h = 1, \ldots, H - 1$
- for a given time period $h \in \mathcal{H}$ and set $\mathcal{J}^h$, let $d_h(\mathcal{J}^h) = \max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}^h} d_{ij}$
- the goal is to minimize $\sum_{h=1}^{H} d_h(\mathcal{J}^h)$

p-center problem / TODO

- First introduction of the p-center problem by Hakimi (1964)
- The standard textbook formulation of the p-center problem can be found in Daskin (2013)
- Solution approach based on the set cover problem by Contardo, Iori, and Kramer (2018)
- A compact formulation by Ales and Elloumi (2018) and Elloumi, Labbé, and Pochet (2004)
- A fast projection-based branch-and-cut algorithm by Gaar and Sinnl (2022)

Nested facility location problems

- First introduction of the nesting property and constraint by Roodman and Schwarz (1975)
- Extension of the nesting to a phase-in and phase-out by Roodman and Schwarz (1977)
- Reintroduction of the nesting on the example of the p-median problem by McGarvey and Thorsen (2022)
- Recent work on multi-period facility locations problems mainly focus on varying demand, distances, or cost over time f.e. Calogiuri et al. (2021)

First MILP formulation

$\mathcal{I}$ ... set of cusotmer demand points i

$\mathcal{J}$ ... set of potential facility locations j

$\mathcal{H}$ ... set of time periods, $\mathcal{H} = \{1, 2, \ldots, H\}$

$\mathcal{P}$ ... set of integers of facilities to open, $\mathcal{P} = \{p^1, \ldots, p^H\}$ where $p^h > p^{h+1}$ for $h = 1, \ldots, H - 1$

d_{ij} = distance between customer demand point i and potential facility location j

$$x_{ij}^h = \begin{cases} 1 \ldots \text{ if customer demand point } i \text{ is assigned to facility location } j \text{ in period } h \\ 0 \ldots \text{ otherwise} \end{cases}$$

$$y_j^h = \begin{cases} 1 \ldots \text{ if a facility is opened at location } j \text{ in period } h \\ 0 \ldots \text{ otherwise} \end{cases}$$

z^h ... maximum distance between any customer i and its nearest open facility in period h

First MILP formulation

$$
\begin{aligned}
(nPC1)\quad \min\ & \sum_{h\in\mathcal{H}} z^h && && (1a)\\
\text{s.t.}\ & \sum_{j\in\mathcal{J}} y_j^h = p_h && \forall h \in \mathcal{H} && (1b)\\
& \sum_{j\in\mathcal{J}} x_{ij}^h = 1 && \forall i \in \mathcal{I}, h \in \mathcal{H} && (1c)\\
& x_{ij}^h \leq y_j^h && \forall i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H} && (1d)\\
& \sum_{j\in\mathcal{J}} d_{ij} x_{ij}^h \leq z_h && \forall i \in \mathcal{I}, h \in \mathcal{H} && (1e)\\
& y_j^h \geq y_j^{h-1} && \forall h \in \mathcal{H} \setminus \{1\} && (1f)\\
& x_{ij}^h, \in \{0, 1\} && \forall i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H} && (1g)\\
& y_j^h, \in \{0, 1\} && \forall j \in \mathcal{J}, h \in \mathcal{H} && (1h)\\
& z^h \in \mathbb{R}_{>0} && \forall h \in \mathcal{H} && (1i)
\end{aligned}
$$

Second MILP formulation

$\mathcal{I}$ ... set of cusotmer demand points i

$\mathcal{J}$ ... set of potential facility locations j

$\mathcal{H}$ ... set of time periods, $\mathcal{H} = \{1, 2, \ldots, H\}$

$\mathcal{P}$ ... set of integers of facilities to open, $\mathcal{P} = \{p^1, \ldots, p^H\}$ where $p^h > p^{h+1}$ for $h = 1, \ldots, H-1$

d_{ij} = distance between customer demand point i and potential facility location j

$$y_j^h = \begin{cases} 1 \ldots \text{ if a facility is opened at location } j \text{ in period } h \\ 0 \ldots \text{ otherwise} \end{cases}$$

z^h ... maximum distance between any customer i and its nearest open facility in period h

Second MILP formulation

$$
\begin{aligned}
(nPC2) \quad \min \quad & \sum_{h\in\mathcal{H}} z^h && && (2a)\\
\text{s.t.} \quad & \sum_{j\in\mathcal{J}} y_j^h = p^h h && \forall h \in \mathcal{H} && (2b)\\
& z^h \geq d_{ij} - \sum_{j':d_{ij'}<d_{ij}} (d_{ij} - d_{ij'}) y_{j'h} && \forall i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H} && (2c)\\
& y^h \geq y^{h-1} && \forall j \in \mathcal{J}, h \in \mathcal{H} \setminus \{1\} && (2d)\\
& y_j^h \in \{0, 1\} && \forall j \in \mathcal{J}, h \in \mathcal{H} && (2e)\\
& z^h \in \mathbb{R}_{\geq 0} && \forall h \in \mathcal{H} && (2f)
\end{aligned}
$$

Constraint lifting

Theorem [Brandstetter(2023)]

Let LB_h being a lower bound on the decision variable z_h of (nPC1) for every $i \in I, j \in J, h \in H$ then

$$\sum_{j \in J} \max\{LB_h, d_{ij}\} x_{ijh} \leq z_h \quad \forall i \in I, h \in H \tag{lift}$$

is a valid equality. Theorem is based on Gaar and Sinnl (2022).

- also valid for the y-formulation
- using various techniques to obtain LB_h

Third MILP formualtion

$\mathcal{I} \ldots$ set of cusotmer demand points i, $\mathcal{J} \ldots$ set of potential facility locations j

$\mathcal{H} \ldots$ set of time periods, $\mathcal{H} = \{1, 2, \ldots, H\}$

$\mathcal{P} \ldots$ set of integers of facilities to open, $\mathcal{P} = \{p^1, \ldots, p^H\}$ where $p^h > p^{h+1}$ for $h = 1, \ldots, H-1$

$\mathcal{D} \ldots$ set of distinct distances where $D_1 \leq \cdots \leq D_K$ are the values in $\mathcal{D}$

$\mathcal{K} \ldots$ set of indices in $\mathcal{D}$

$S_i \ldots$ set of indices $k \in \mathcal{K}$ for which there exists a facility $j \in \mathcal{J}$ with $d_{ij} = D_k$

$$y_j^h = \begin{cases} 1 \ldots \text{ if a facility is opened at location } j \text{ in period } h \\ 0 \ldots \text{ otherwise} \end{cases}$$

$$u_k^h = \begin{cases} 1 \ldots \text{ if objective function value in time period } h \text{ is less or equal than } D_k \\ 0 \ldots \text{ otherwise} \end{cases}$$

$z^h \ldots$ maximum distance between any customer i and its nearest open facility

Third formulation

$$
\begin{aligned}
(nPC3)\quad \min \quad & \sum_{h\in\mathcal{H}} z^h && && (3a)\\
\text{s.t.} \quad & \sum_{j\in\mathcal{J}} y_j^h = p^h && \forall h \in \mathcal{H} && (3b)\\
& D_0 + \sum_{k=1}^{K} (D_k - D_{k-1}) u_k^h \leq z^h && \forall h \in \mathcal{H} && (3c)\\
& u_k^h + \sum_{j:d_{ij}<D_k} y_j^h \geq 1 && \forall i \in \mathcal{I}, \forall h \in \mathcal{H}, \forall k \in S_i \cup \{K\} && (3d)\\
& u_k^h \geq u_{k+1}^h && \forall h \in \mathcal{H}, \forall k \in \mathcal{K} \setminus \{K\} && (3e)\\
& y_j^h \leq y_j^{h-1} && \forall h \in \mathcal{H}, \forall j \in \mathcal{J} && (3f)\\
& y_j^h \in \{0,1\} && \forall h \in \mathcal{H}, \forall j \in \mathcal{J} && (3g)\\
& u_k^h \in \{0,1\} && \forall h \in \mathcal{H}, \forall k \in \mathcal{K} && (3h)\\
& z^h \in \mathbb{R} && \forall h \in \mathcal{H} && (3i)
\end{aligned}
$$

Initial bounds

- Every optimal solution to the p-center problem is a lower bound to the respective z^h
- We can use this by calculating the solution to the p-center problem for all $p \in \mathcal{P}$
- and using the optimal values as lower bounds for the z^h.
- Every optimal solution to the p-center problem is a upper bound to the $p + 1$-center problem

Upper bound third formulation

- The third formulation performance is very dependent on the size of $\mathcal{D}$
- The lower bounds obtained through the p-center solutions can be used to to reduce the size of $\mathcal{D}$
- Furthermore, we can reduce the size of $\mathcal{D}$ by obtaining a upper bound on the z^h
- We get this upper bound by finding a feasible solution to the nested p-center problem, subtracting $\sum_{h=2}^{H} \bar{z^h}$
- This works because we know that the optimal solution cannot be larger than any feasible solution we found
- and we know that the z^h cannot be smaller than $\bar{z^h}$. So we assume that all but z^1 are optimal.
- This can be applied every time a new best feasible solution is found.

Lifting for the first and second formulation

Data

- data set PMED from ”A note on solving large p-median problems” by Beasley (1985)
  - set of 40 test instances
  - the sets contain between 100 and 900 nodes

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  - set of 40 test instances
  - the sets contain between 100 and 900 nodes

- data set TSPLIB 2D-Euclidean distances from “TSPLIB — A Traveling Salesman Problem Library” by Reinelt (1991)
  - set of 80 instances
  - the sets contain between 51 and 18512 nodes
  - rounded to the nearest integer value

Comparison

- comparing the results between (nPC1) and (nPC2) and (nPC3)

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- comparing the results between (nPC1) and (nPC2) and (nPC3)
- comparing the different lifting methods
- analysing the solutions of the nested and comparing it to the standard *p*-center problem
- analysing the results regarding managerial insights

Further research topics

- *u*-space model introduced by Elloumi, Labbé, and Pochet (2004)
- maximal regret as objective function
- trying to improve the performance of the nested p-median problem?

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