On the nested p-center problem

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Introduction

- *p*-center problem (pCP): open *p* number of facilities, such that the maximum distance between a customer and its closest open facility minimized

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- nested: considering more than one time period in which facilities are opened, facilities once open cannot be closed
- nested *p*-center problem (npCP): open p^h number of nested facilities in period h, such that the sum of the maximum distances between a customer and its closest open facility in this time period is minimized

p-center problem vs nested p-center problem

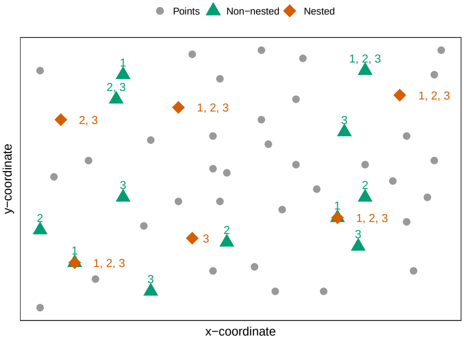

Figure: Feasible solution for the nested and the non-nested p-center problem with $p = p^h = 4, 5, 6$

Research question

How can the nesting concept be applied to the p-center problem?

- proposed by McGarvey and Thorsen (2022)

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How can the nesting concept be applied to the p-center problem?

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- Which mixed integer programming (MILP) formulations can be used?
- Which MILP formulations have the best runtime?
- How can the nested p-center problem affect managerial decisions?

Definition of the nested *p*-center problem I

- given a set of customer demand points $\mathcal{I}$,
- potential facility locations $\mathcal{J}$,
- time periods $\mathcal{H} = \{1, \ldots, H\}$,
- integers $\mathcal{P} = \left\{p^1, \ldots p^H\right\}$
  - where $p^h \leq p^{h+1}$ for $h = 1, \ldots, H - 1$ and
  - $p^H \leq |\mathcal{J}|$
- distances $d_{ij} \geq 0$ between each $i \in \mathcal{I}$ and $j \in \mathcal{J}$

Definition of the nested p-center problem II

- a feasible solution to the nested p-center problem consists of a set $\mathcal{J}^h \subseteq \mathcal{J}$
  - with $\left|\mathcal{J}^h\right| = p^h$ for $h \in \mathcal{H}$,
  - for which $\mathcal{J}^h \subseteq \mathcal{J}^{h+1}$ for $h = 1, \ldots, H-1$ holds
- the goal is to find a feasible solution which minimizes $\sum_{h=1}^{H} d_h(\mathcal{J}^h)$,
  - where $d_h(\mathcal{J}^h) = \max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}^h} d_{ij}$ for $h \in \mathcal{H}$

Related work

p-center problem

- First introduction of the p-center problem by Hakimi (1964)
- The standard textbook formulation of the p-center problem can be found in Daskin (2013)
- Solution approach based on the set cover problem by Contardo, Iori, and Kramer (2018)
- A compact formulation by Ales and Elloumi (2018) and Elloumi, Labbé, and Pochet (2004)
- A formulation with a projection-based branch-and-cut algorithm for the p-center problem by Gaar and Sinnl (2022)

Nested facility location problems

- First introduction of the nesting property and constraint by Roodman and Schwarz (1975)
- Extension of the nesting to a phase-in and phase-out by Roodman and Schwarz (1977)
- Reintroduction of the nesting on the example of the p-median problem by McGarvey and Thorsen (2022)
- Other work on multi-period facility locations problems mainly focus on varying demand, distances, or cost over time f.e. Calogiuri et al. (2021)

Mixed Integer Linear Programming formulations

First MILP formulation

Decision variables

$$x_{ij}^{h} \ldots \begin{cases} 1 \ldots \text{ if customer } i \text{ is assigned to facility location } j \text{ in period } h \\ 0 \ldots \text{ otherwise} \end{cases}$$

First MILP formulation

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$z^h \ldots$ maximum distance between any customer i and its nearest open facility in period h

First MILP formulation

$$
\begin{aligned}
\texttt{(nPC1)} \qquad & \min && \sum_{h\in\mathcal{H}} z^h && && \text{(1a)}\\
& \text{s.t.} && \sum_{j\in\mathcal{J}} y_j^h = p^h && \forall h \in \mathcal{H} && \text{(1b)}\\
& && \sum_{j\in\mathcal{J}} x_{ij}^h = 1 && \forall i \in \mathcal{I}, h \in \mathcal{H} && \text{(1c)}\\
& && x_{ij}^h \leq y_j^h && \forall i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H} && \text{(1d)}\\
& && \sum_{j\in\mathcal{J}} d_{ij} x_{ij}^h \leq z^h && \forall i \in \mathcal{I}, h \in \mathcal{H} && \text{(1e)}\\
& && y_j^h \geq y_j^{h-1} && \forall h \in \mathcal{H} \setminus \{1\} && \text{(1f)}\\
& && x_{ij}^h, \in \{0, 1\} && \forall i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H} && \text{(1g)}\\
& && y_j^h, \in \{0, 1\} && \forall j \in \mathcal{J}, h \in \mathcal{H} && \text{(1h)}\\
& && z^h \in \mathbb{R}_{>0} && \forall h \in \mathcal{H} && \text{(1i)}
\end{aligned}
$$

Second MILP formulation

$$
\begin{aligned}
(\texttt{nPC2}) \qquad \min \quad & \sum_{h \in \mathcal{H}} z^h && && \text{(2a)} \\
\text{s.t.} \quad & \sum_{j \in \mathcal{J}} y_j^h = p^h && \forall h \in \mathcal{H} && \text{(2b)} \\
& z^h \geq d_{ij} - \sum_{j' : d_{ij'} < d_{ij}} (d_{ij} - d_{ij'}) y_{j'}^h && \forall i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H} && \text{(2c)} \\
& y_j^h \geq y_j^{h-1} && \forall j \in \mathcal{J}, h \in \mathcal{H} \setminus \{1\} && \text{(2d)} \\
& y_j^h \in \{0, 1\} && \forall j \in \mathcal{J}, h \in \mathcal{H} && \text{(2e)} \\
& z^h \in \mathbb{R}_{\geq 0} && \forall h \in \mathcal{H} && \text{(2f)}
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$$

Third MILP formualtion

$\mathcal{D} \dots$ set of distinct distances where $D_0 \leq \cdots \leq D_K$ are the values in $\mathcal{D}$

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Third formulation

$$
\begin{aligned}
\text{(nPC3)} \quad \min \quad & \sum_{h \in \mathcal{H}} z^h && && \text{(3a)} \\
\text{s.t.} \quad & \sum_{j \in \mathcal{J}} y_j^h = p^h && \forall h \in \mathcal{H} && \text{(3b)} \\
& D_0 + \sum_{k=1}^{K} (D_k - D_{k-1}) u_k^h \leq z^h && \forall h \in \mathcal{H} && \text{(3c)} \\
& u_k^h + \sum_{j : d_{ij} < D_k} y_j^h \geq 1 && \forall i \in \mathcal{I}, \forall h \in \mathcal{H}, \forall k \in S_i \cup \{K\} && \text{(3d)} \\
& u_k^h \geq u_{k+1}^h && \forall h \in \mathcal{H}, \forall k \in \mathcal{K} \setminus \{K\} && \text{(3e)} \\
& y_j^h \leq y_j^{h-1} && \forall h \in \mathcal{H}, \forall j \in \mathcal{J} && \text{(3f)} \\
& y_j^h \in \{0, 1\} && \forall h \in \mathcal{H}, \forall j \in \mathcal{J} && \text{(3g)} \\
& u_k^h \in \{0, 1\} && \forall h \in \mathcal{H}, \forall k \in \mathcal{K} && \text{(3h)} \\
& z^h \in \mathbb{R} && \forall h \in \mathcal{H} && \text{(3i)}
\end{aligned}
$$

Comparing formulations

Table: Comparison of complexity of the formulations

	(nPC1)	(nPC2)	(nPC3)
Variables	$\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}(\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}((\lvert\mathcal{I}\rvert + \lvert\mathcal{K}\rvert)\lvert\mathcal{H}\rvert)$
Constraints	$\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}(\min(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert, \lvert\mathcal{I}\rvert\lvert\mathcal{K}\rvert)\lvert\mathcal{H}\rvert)$

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The (pCP) version of formulation (nPC3) has the best known linear programming ($\mathcal{LP}$)-bounds for the (pCP), while the (pCP) versions of (nPC1) and (nPC2) have equally but worse $\mathcal{LP}$-bounds.

Reducing set $\mathcal{K}$ in `(nPC3)`

Lemma 1

Let $\underline{z^h}$ be a valid lower bound and $\overline{z^h}$ be a valid upper bound on the decision variable z^h for $h \in \mathcal{H}$, then the distinct distance D_k can only be the optimal distance for z^h if $\underline{z^h} \leq D^k \leq \overline{z^h}$ holds.
Therfore let set $S_i^h \subseteq S_i$ for $h \in \mathcal{H}$, where S_i^h contains only the indices k where $\underline{z^h} \leq D^k \leq \overline{z^h}$ holds and constraint (3d) *can be replaced with*

$$u_k^h + \sum_{j:d_{ij}<D_k} y_j^h \geq 1 \qquad \forall i \in \mathcal{I}, \forall h \in \mathcal{H}, \forall k \in S_i^h \cup \{K\} \tag{4}$$

Depending on the bounds $\underline{z^h}$ and $\overline{z^h}$ the sets S_i^h can be much smaller than S_i.

Strengthening constraints (nPC2)

Lemma 2

Let LB_h be a lower bound on the decision variable z_h of (nPC2) for every $i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H}$ then

$$z^h \geq max\{LB^h, d_{ij}\} - \sum_{j':d_{ij'}<d_{ij}} (\max\{LB^h, d_{ij}\} - \max\{LB^h, d_{ij'}\})y^h_{j'} \qquad \text{(nL-OPT)}$$

is a valid inequality for (nPC2), i.e., every feasible solution of (nPC2) fulfills (nL-OPT). *Theorem is based on Lemma 5 in Gaar and Sinnl (2022).*

A similar strengthening can be done for (1e) in fromulation (nPC1).

Obtaining bounds I

Lemma 3

Let z'^{h*} be the optimal objective function value of (pCP) with $p = p^h$ for $h \in \mathcal{H} = \{1, 2, \ldots, H\}$ where $p^h > p^{h+1}$, then $UB = Hz'^{1*}$ is a valid upper bound on the optimal objective function value of (npCP).
Then

$$\overline{z^h} = \frac{UB - \sum_{x=h+1}^{H} z'^{x*}}{h+1}$$

where $\overline{z^h}$ is a valid upper bound on the decision variable z^h of the (npCP) for $h \in \mathcal{H}$.

Obtaining bounds II

Lemma 4

Let z'^* be the optimal objective function value of (pCP) for a certain p'. Then z'^* is a valid lower bound $\underline{z}^h$ on the decision variable z^h of (npCP) with $p^h = p'$.

Implementation and outline of the results

Implementation

- Implemented in C++ using the CPLEX API of CPLEX 20.1.0.0
- Formulation `(nPC2)` solved by branch-and-cut and seperation based on the customers
- Preprocessing for all formulations
  - solving the (pCP) for p^h, $h \in \mathcal{H}$ starting with $h = H$
  - p^h is a valid lower bound for the (pCP) with p^{h-1}
- Single core of an Intel Xeon X5570 machine
  - 2.93 GHz
  - 48 GB RAM
  - Each run limited to 9 GB RAM and 3600 sec

Data

- data set PMED from "A note on solving large p-median problems” by Beasley (1985)
  - set of 40 test instances
  - the sets contain between 100 and 900 nodes
  - number of facilities to open initially ranging from 5 to 200
  - $\mathcal{P} = \{p, p+1, p+2\}$

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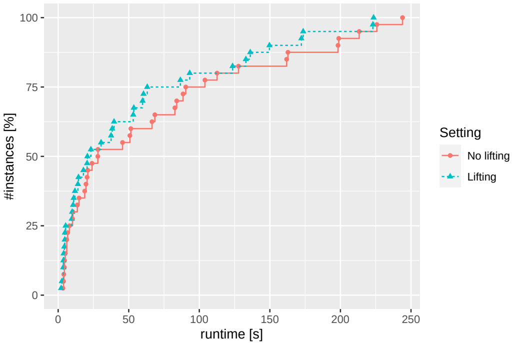
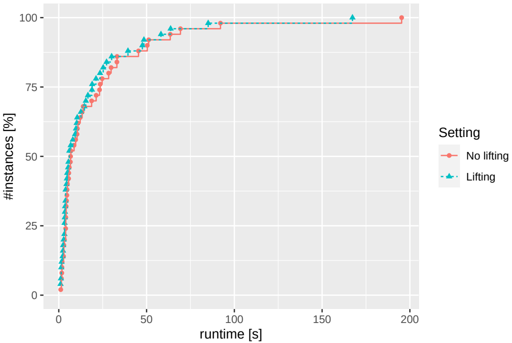
- data set TSPLIB 2D-Euclidean distances from “TSPLIB—A Traveling Salesman Problem Library” by Reinelt (1991)
  - set of 80 test instances
  - the sets contain between 51 and 1002 nodes
  - rounded to the nearest integer value
  - $\mathcal{P} = \{4, 5, 6\}$

Preprocessing



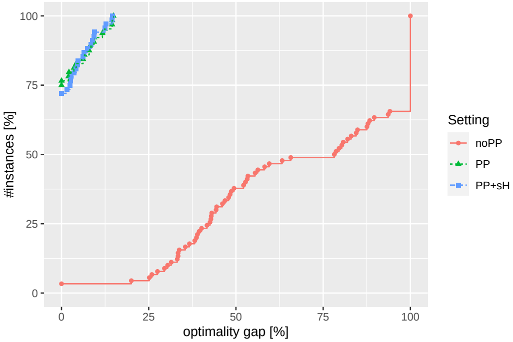
(a) Preprocessing PMED



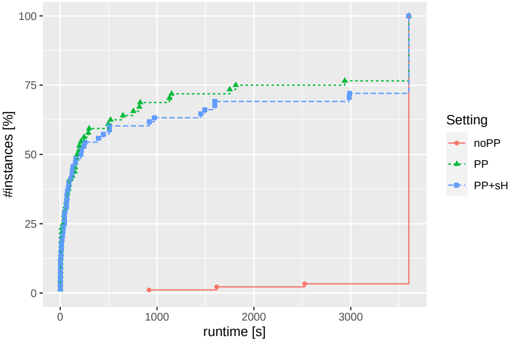
(b) Preprocessing TSPLIB

Figure: Preprocessing

(nPC1)-results

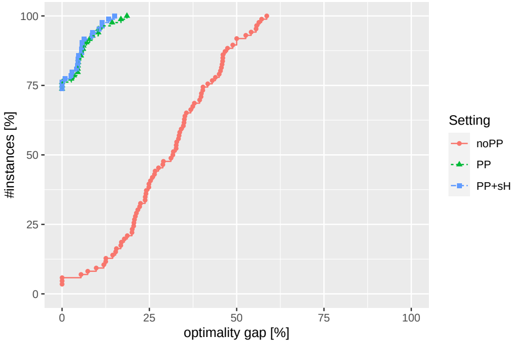


(a) (nPC1) optimality gap

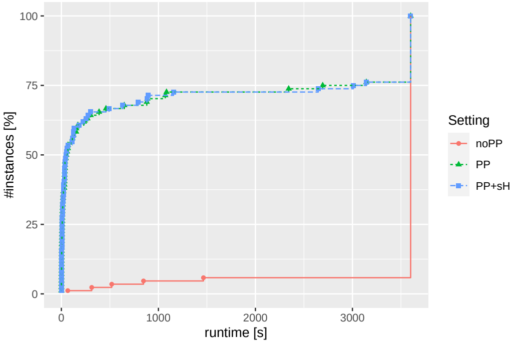


(b) (nPC1) runtime

(nPC2)-results

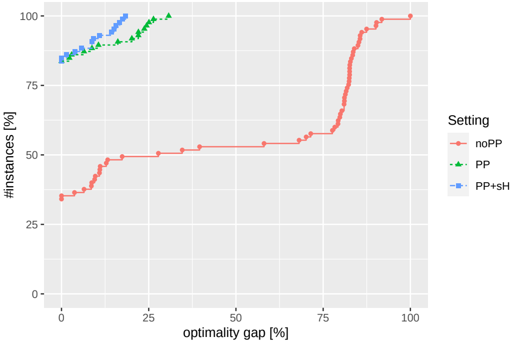


(c) (nPC2) optimality gap

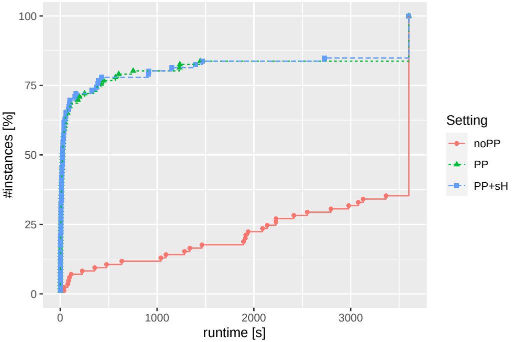


(d) (nPC2) runtime

(nPC3)-results

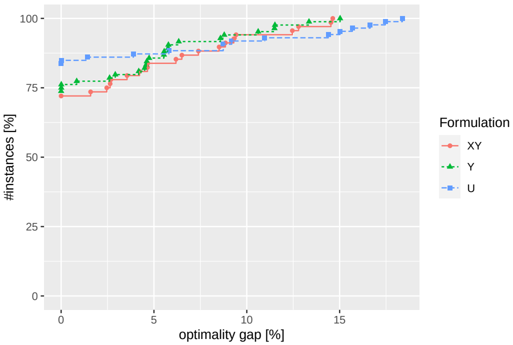


(e) (nPC3) optimality gap

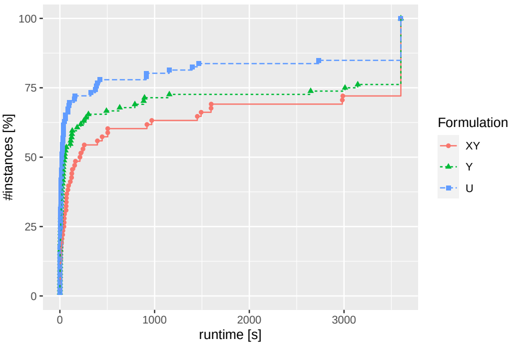


(f) (nPC3) runtime

Formulation comparison

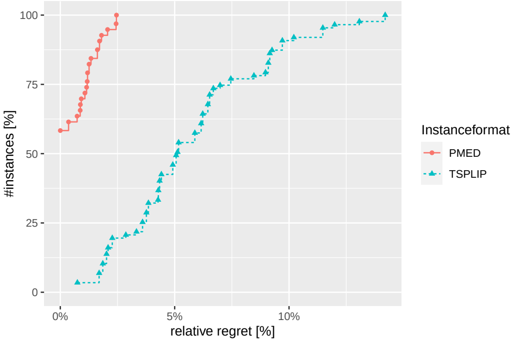


(g) models optimality gap

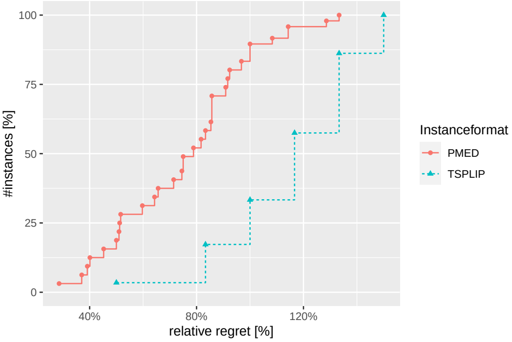


(h) models runtime

Managerial insights



(i) Relative regret



(j) Relative regret of # of facilities

Figure: On a subset of instances: Only if the problem was solved to optimality

Conclusion

- `(nPC3)` had the best runtimes over all instances
- For optimality gap the results are rather mixed
- The `(nPC2)` outperforms the `(nPC1)`
- Maximal relative regret of the objective value of 15%
- Maximal relative regret of number of facilities above 140%
- min-max regret as objective function was also analysed and can be found later in the thesis

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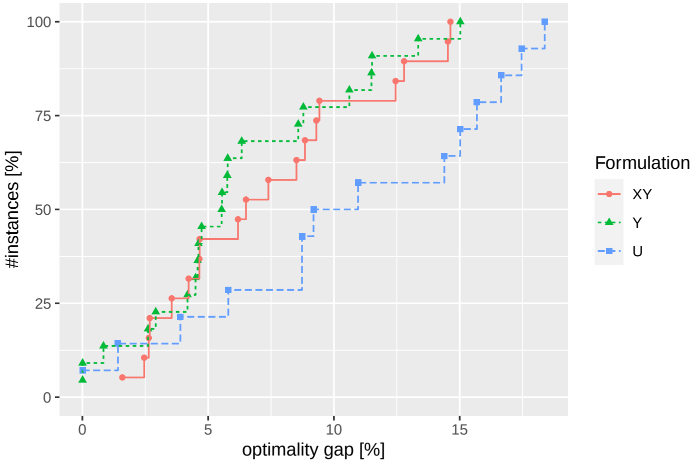
Formulations with non optimal instances



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