On the nested p-center problem

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  - the maximum distance of any location to its closest opened location is minimized.

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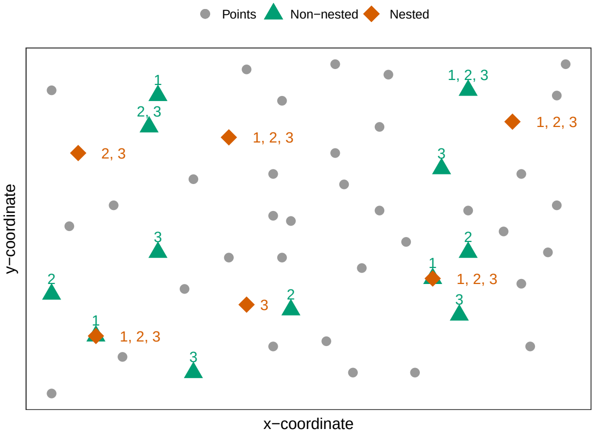
Introduced by Roodman and Schwarz 1975 and used in e.g., Albareda-Sambola et al. 2009; Bakker and Nickel 2024; Conforti et al. 2014 and reintroduced as nesting by McGarvey and Thorsen 2022

p-center problem vs nested p-center problem for $\mathcal{P} = \{4, 5, 6\}$



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  - for which $V^h \subseteq V^{h+1}$ for $h = 1, \dots, H-1$ holds
- the goal is to find a feasible solution which minimizes $\sum_{h=1}^{H} d(V^h)$,
  - where $d(V^h) = \max_{i \in V} \min_{j \in V^h} d_{ij}$ for $h \in \mathcal{H}$

Mixed Integer Linear Programming (MILP) Formulations

First formulation

Decision variables

$$y_j^h \ldots \begin{cases} 1 \ldots \text{if location } j \text{ is open in} \\ \qquad \text{time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

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(nPC2) based on Gaar and Sinnl 2022

$$
\begin{aligned}
\min \quad & \sum_{h \in \mathcal{H}} z^h && && \text{(1a)}\\
\text{s.t.} \quad & \sum_{j \in V} y_j^h = p^h && \forall h \in \mathcal{H} && \text{(1b)}\\
& z^h \geq d_{ij} - \sum_{j': d_{ij'} < d_{ij}} (d_{ij} - d_{ij'}) y_{j'}^h && \forall i, j \in V, h \in \mathcal{H} && \text{(1c)}\\
& y_j^h \geq y_j^{h-1} && \forall j \in V, \forall h \in \mathcal{H} \setminus \{1\} && \text{(1d)}\\
& (y, z) \in \mathbb{B}^{|V||\mathcal{H}|} \times \mathbb{R}_{\geq 0}^{|\mathcal{H}|} && && \text{(1e)}
\end{aligned}
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(nPC3) based on Ales and Elloumi 2018

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\min \quad & \sum_{h \in \mathcal{H}} z^h \\
\text{s.t.} \quad & \sum_{j \in V} y_j^h = p^h && \forall h \in \mathcal{H} \\
& D_0 + \sum_{k=1}^{K} (D_k - D_{k-1}) u_k^h \leq z^h && \forall h \in \mathcal{H} \\
& u_k^h + \sum_{j: d_{ij} < D_k} y_j^h \geq 1 && \forall i \in V, \forall h \in \mathcal{H}, \forall k \in S_i \cup \{K\} \\
& u_k^h \geq u_{k+1}^h && \forall h \in \mathcal{H}, \forall k \in \mathcal{K} \setminus \{K\} \\
& y_j^h \geq y_j^{h-1} && \forall h \in \mathcal{H} \setminus \{1\} \\
& (u, y, z) \in \mathbb{B}^{|\mathcal{K}||\mathcal{H}|} \times \mathbb{R}_{\geq 0}^{|\mathcal{H}|}
\end{aligned}
$$

Improving the formulations

Strengthening constraints `(nPC2)`

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Lemma 1

Let $\underline{z}^h$ be a lower bound on the decision variable z_h of (nPC2) for every $i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H}$ then constraints

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$$z^h \geq max\{\underline{z}^h, d_{ij}\} - \sum_{j':d_{ij'}<d_{ij}} \left(\max\{\underline{z}^h, d_{ij}\} - \max\{\underline{z}^h, d_{ij'}\}\right) y_{j'}^h \qquad \text{(nL-OPT)}$$

The lemma is based on Lemma 5 in Gaar and Sinnl (2022).

Reducing the number of variables u_k^h in (nPC3)

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Observation 1

Let $\underline{z^h}$ be a valid lower bound and $\overline{z^h}$ be a valid upper bound on the decision variable z^h for $h \in \mathcal{H}$, then the distinct distance D_k can only be the optimal distance for z^h if $\underline{z^h} \leq D^k \leq \overline{z^h}$ holds.

Observation 2

Following Observation 1, we observe that decision variables $u_h^k = 0$ for $k : D_k > \overline{z^h}$ and $u_h^k = 1$ for $k : D_k < \underline{z^h}$ in any optimal solution.

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The decision variables u_k^h which are zero or one following Observation 2, are redundant.

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Observation 3

For $\mathcal{H} = \{1\}$ the (n-pCP) reduces to the (pCP) where $p = p^1$, so the optimal objective value (z'^{h*}) of the (pCP), where $p = p^h$ is a lower bound $\underline{z^h}$ on the decision variable z^h of the (n-pCP).

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Proposition 1

Given an valid upper bound UB on the objective value of the (n-pCP) and valid lower bounds $\underline{z^h}$ on the variable z^h can be obtained by the following equation:

$$\overline{z^h} = \frac{UB - \sum_{x=h+1}^{H} \underline{z^h}}{h} \tag{3}$$

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  - Solving p-center problem with $p = p^H$
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  - Repeat for remaining p^h and calculate the upper bounds $\overline{z^h}$

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- `(nPC2)`
  - branch-and-cut algorithm
  - separation based on the fixedCustomer separation scheme from Gaar and Sinnl 2022

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  - branch-and-bound algorithm

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  - `(nPC3)`: problem initialized on reduced number of u_k^h
- `PH`: `PP` with starting heuristic

Results

Instance from literature

- instance set PMED
  - 40 instances
  - $\mathcal{P} = \{p, p+1, p+2\}$, p from 5 to 200, $\left|V\right|$ from 100 and 900 nodes

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  - $\mathcal{P} = \{4, 5, 6\}$, $|V|$ from 51 and 1002 nodes

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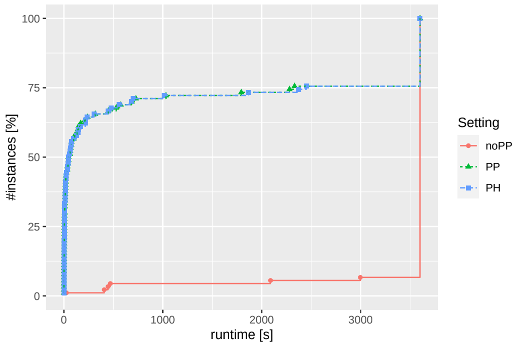
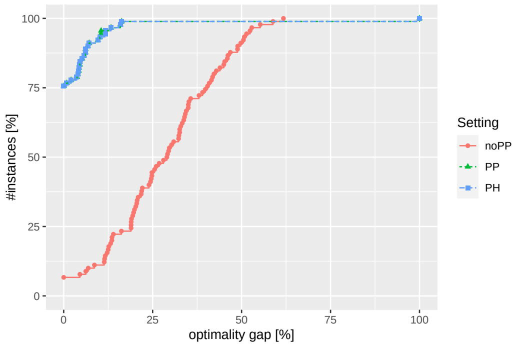
  - $\mathcal{P} = \{4, 5, 6\}$, $|V|$ from 51 and 1002 nodes
- computational setup
  - single core of Intel Xeon X5570 with 2.93 GHz and 8 GB RAM
  - timelimit of 3600 seconds

Setting comparison on formulation (nPC2)

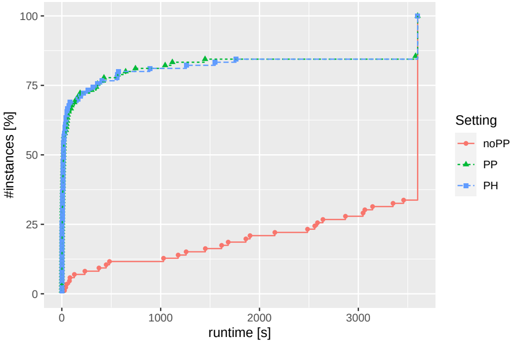


(a) runtime

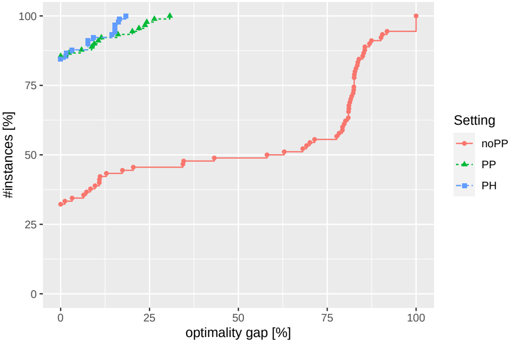


(b) optimality gap

Setting comparison on formulation (nPC3)

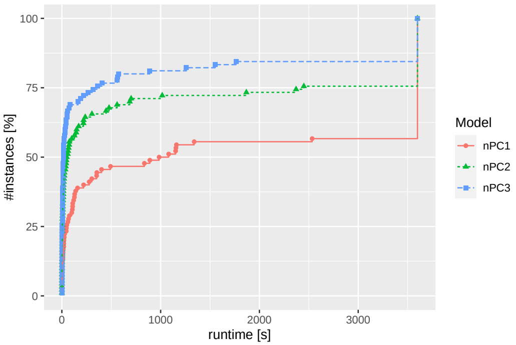


(c) runtime

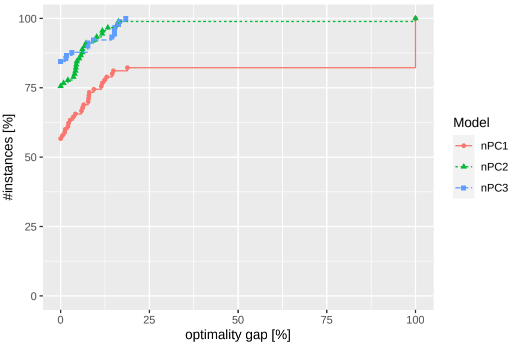


(d) optimality gap

Formulation comparison on setting PH

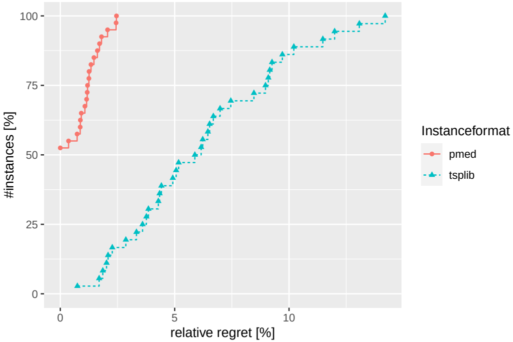


(e) runtime



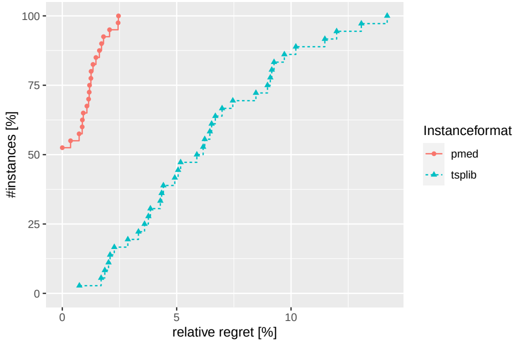
(f) optimality gap

Managerial insights

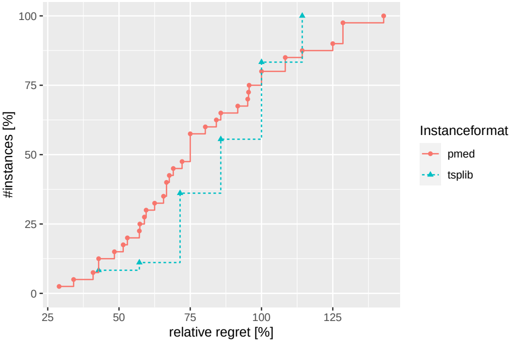


(g) Relative regret of the optimal solution value

Managerial insights



(i) Relative regret of the optimal solution value



(j) Relative regret of # of opened facilities

Conclusion

- `(nPC3)` best performance in runtimes and optimality gap
- The `(nPC2)` outperforms the `(nPC1)`
- Preprocessing brings a large speed up on all formulations
- Starting heuristic little effect, shows good upper bound obtained in preprocessing
- Maximal relative regret of the optimal objective function value of 15%
- Maximal relative regret of number of opened facilities above 150%

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References I

[1] M. Albareda-Sambola et al. The multi-period incremental service facility location problem. In: Computers & Operations Research 36.5 (2009), pp. 1356–1375.

[2] Z. Ales and S. Elloumi. Compact MILP formulations for the p-center problem. In: Combinatorial Optimization: 5th International Symposium, ISCO 2018, Marrakesh, Morocco, April 11–13, 2018, Revised Selected Papers 5. 2018, pp. 14–25.

[3] H. Bakker and S. Nickel. The Value of the Multi-period Solution revisited: When to model time in capacitated location problems. In: Computers & Operations Research 161 (2024), p. 106428.

[4] M. Conforti et al. Integer programming models. Springer, 2014.

References II

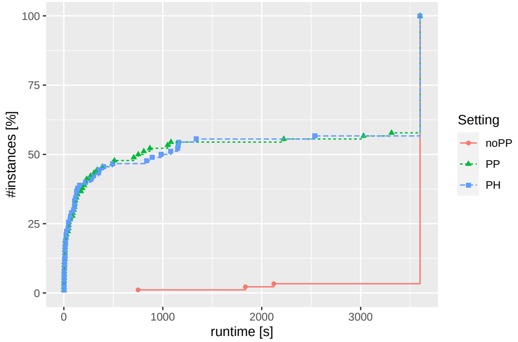
[5] M. S. Daskin. Network and Discrete Location: Models, Algorithms, and Applications, Second Edition. John Wiley & Sons, Ltd, 2013.

[6] E. Gaar and M. Sinnl. A scaleable projection-based branch-and-cut algorithm for the p-center problem. In: European Journal of Operational Research 303.1 (2022), pp. 78–98.

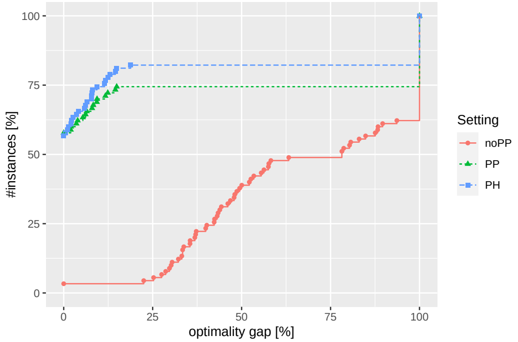
[7] R. G. McGarvey and A. Thorsen. Nested-solution facility location models. In: Optimization letters 16.2 (2022), pp. 497–514.

[8] G. M. Roodman and L. B. Schwarz. Optimal and heuristic facility phase-out strategies. In: AIIE transactions 7.2 (1975), pp. 177–184.

Setting comparison on formulation (nPC1)



(k) runtime



(l) optimality gap $[\frac{UB-LB}{UB}]$

The nested p-center problem: Classical MILP formulation

Decision variables

$$x_{ij}^h \ldots \begin{cases} 1 \ldots \text{if location } i \text{ is assigned to} \\ \qquad \text{location } j \text{ in time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

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The nested p-center problem: Classical MILP formulation

Decision variables

$$x_{ij}^h \dots \begin{cases} 1 \dots \text{if location } i \text{ is assigned to} \\ \qquad \text{location } j \text{ in time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

$$y_j^h \dots \begin{cases} 1 \dots \text{if location } j \text{ is open in} \\ \qquad \text{time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

$z^h \dots$ maximum distance between any customer i and its nearest open facility in period h

(nPC1) based on Daskin 2013

$$\min \sum_{h \in \mathcal{H}} z^h \tag{4a}$$

$$\text{s.t.} \sum_{j \in V} y_j^h = p^h \qquad \forall h \in \mathcal{H} \tag{4b}$$

$$\sum_{j \in V} x_{ij}^h = 1 \qquad \forall i \in V, h \in \mathcal{H} \tag{4c}$$

$$x_{ij}^h \leq y_j^h \qquad \forall i, j \in V, h \in \mathcal{H} \tag{4d}$$

$$\sum_{j \in V} d_{ij} x_{ij}^h \leq z^h \qquad \forall i \in V, h \in \mathcal{H} \tag{4e}$$

$$y_j^h \geq y_j^{h-1} \qquad \forall h \in \mathcal{H} \setminus \{1\} \tag{4f}$$

$$(x, y, z) \in |V|^2|\mathcal{H}| \times |V||\mathcal{H}| \times \mathbb{R}_{\geq 0} \tag{4g}$$