On the nested *p*-center problem

Christof Brandstetter, Markus Sinnl
Institute of Business Analytics and Technology Transformation / JKU Business School,
Johannes Kepler University Linz

2024-02-28

JOHANNES KEPLER
UNIVERSITY LINZ
Altenberger Straße 69
4040 Linz, Austria
jku.at

The p-center problem: Definition

Definition

- Given

The p-center problem: Definition

Definition

- Given
  - a set V of locations,

The p-center problem: Definition

Definition

- Given
  - a set V of locations,
  - $p \in \mathbb{Z}$, and

The p-center problem: Definition

Definition

- Given
  - a set V of locations,
  - $p \in \mathbb{Z}$, and
  - distances d_{ij} from location $i \in V$ to $j \in V$

The p-center problem: Definition

Definition

- Given
  - a set V of locations,
  - $p \in \mathbb{Z}$, and
  - distances d_{ij} from location $i \in V$ to $j \in V$
- we want to
  - open p locations of V, such that

The p-center problem: Definition

Definition

- Given
  - a set V of locations,
  - $p \in \mathbb{Z}$, and
  - distances d_{ij} from location $i \in V$ to $j \in V$
- we want to
  - open p locations of V, such that
  - the maximum distance of any location to its closest opened location is minimized.

The nesting property

The nested p-center problem: Definition

Definition I

- given a set of locations V,

The nested p-center problem: Definition

Definition I

- given a set of locations V,
- time periods $\mathcal{H} = \{1, \ldots, H\}$,

The nested p-center problem: Definition

Definition I

- given a set of locations V,
- time periods $\mathcal{H} = \{1, \ldots, H\}$,
- integers $\mathcal{P} = \left\{p^1, \ldots p^H\right\}$

The nested p-center problem: Definition

Definition I

- given a set of locations V,
- time periods $\mathcal{H} = \{1, \dots, H\}$,
- integers $\mathcal{P} = \left\{p^1, \dots p^H\right\}$
  - where $p^h \leq p^{h+1}$ for $h = 1, \dots, H - 1$ and
  - $p^H \leq |\mathcal{J}|$

The nested p-center problem: Definition

Definition I

- given a set of locations V,
- time periods $\mathcal{H} = \{1, \ldots, H\}$,
- integers $\mathcal{P} = \left\{p^1, \ldots p^H\right\}$
  - where $p^h \leq p^{h+1}$ for $h = 1, \ldots, H-1$ and
  - $p^H \leq |\mathcal{J}|$
- distances $d_{ij} \geq 0$ between each $i \in V$ and $j \in V$

The nested p-center problem: Definition

Definition I

- given a set of locations V,
- time periods $\mathcal{H} = \{1, \ldots, H\}$,
- integers $\mathcal{P} = \left\{p^1, \ldots p^H\right\}$
  - where $p^h \leq p^{h+1}$ for $h = 1, \ldots, H - 1$ and
  - $p^H \leq |\mathcal{J}|$
- distances $d_{ij} \geq 0$ between each $i \in V$ and $j \in V$

Definition II

- a feasible solution to the nested p-center problem consists of a set $\mathcal{J}^h \subseteq V$

The nested p-center problem: Definition

Definition I

- given a set of locations V,
- time periods $\mathcal{H} = \{1, \dots, H\}$,
- integers $\mathcal{P} = \left\{p^1, \dots p^H\right\}$
  - where $p^h \leq p^{h+1}$ for $h = 1, \dots, H-1$ and
  - $p^H \leq |\mathcal{J}|$
- distances $d_{ij} \geq 0$ between each $i \in V$ and $j \in V$

Definition II

- a feasible solution to the nested p-center problem consists of a set $\mathcal{J}^h \subseteq V$
  - with $\left|\mathcal{J}^h\right| = p^h$ for $h \in \mathcal{H}$,
  - for which $\mathcal{J}^h \subseteq \mathcal{J}^{h+1}$ for $h = 1, \dots, H-1$ holds

The nested p-center problem: Definition

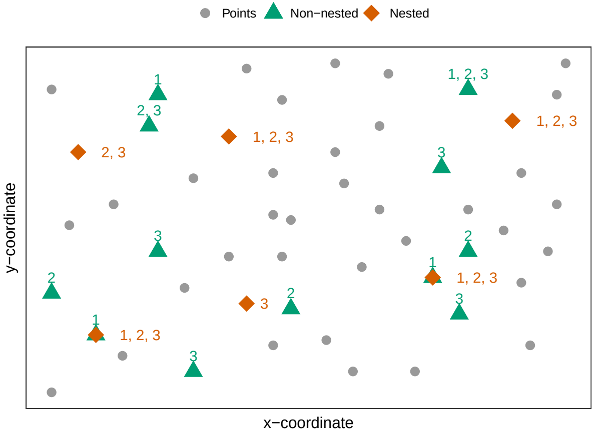
Definition I

- given a set of locations V,
- time periods $\mathcal{H} = \{1, \ldots, H\}$,
- integers $\mathcal{P} = \left\{p^1, \ldots p^H\right\}$
  - where $p^h \leq p^{h+1}$ for $h = 1, \ldots, H-1$ and
  - $p^H \leq |\mathcal{J}|$
- distances $d_{ij} \geq 0$ between each $i \in V$ and $j \in V$

Definition II

- a feasible solution to the nested p-center problem consists of a set $\mathcal{J}^h \subseteq V$
  - with $\left|\mathcal{J}^h\right| = p^h$ for $h \in \mathcal{H}$,
  - for which $\mathcal{J}^h \subseteq \mathcal{J}^{h+1}$ for $h = 1, \ldots, H-1$ holds
- the goal is to find a feasible solution which minimizes $\sum_{h=1}^{H} d_h(\mathcal{J}^h)$,
  - where $d_h(\mathcal{J}) = \max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}} d_{ij}$ for $h \in \mathcal{H}$

p-center problem vs nested p-center problem



The nested p-center problem: Potential applications

The nested *p*-center problem: Classical MILP formulation

Variables

$$x_{ij}^h \ldots \begin{cases} 1 \ldots \text{if location } i \text{ is assigned to} \\ \qquad \text{location } j \text{ in time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

The nested p-center problem: Classical MILP formulation

Variables

$$x_{ij}^h \ldots \begin{cases} 1 \ldots \text{if location } i \text{ is assigned to} \\ \qquad \text{location } j \text{ in time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

$$y_j^h \ldots \begin{cases} 1 \ldots \text{if location } j \text{ is open in} \\ \qquad \text{time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

The nested p-center problem: Classical MILP formulation

Variables

$$x_{ij}^h \dots \begin{cases} 1 \dots \text{if location } i \text{ is assigned to} \\ \qquad \text{location } j \text{ in time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

$$y_j^h \dots \begin{cases} 1 \dots \text{if location } j \text{ is open in} \\ \qquad \text{time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

$z^h \dots$ maximum distance between any customer i and its nearest open facility in period h

The nested p-center problem: Classical MILP formulation

Variables

$$x_{ij}^h \dots \begin{cases} 1 \dots \text{if location } i \text{ is assigned to} \\ \qquad \text{location } j \text{ in time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

$$y_j^h \dots \begin{cases} 1 \dots \text{if location } j \text{ is open in} \\ \qquad \text{time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

$z^h \dots$ maximum distance between any customer i and its nearest open facility in period h

Formulation based on Daskin 2013

$$\min \sum_{h \in \mathcal{H}} z^h \tag{1a}$$

$$\text{s.t.} \quad \sum_{j \in V} y_j^h = p^h \qquad \forall h \in \mathcal{H} \tag{1b}$$

$$\sum_{j \in V} x_{ij}^h = 1 \qquad \forall i \in V, h \in \mathcal{H} \tag{1c}$$

$$x_{ij}^h \leq y_j^h \qquad \forall i, j \in V, h \in \mathcal{H} \tag{1d}$$

$$\sum_{j \in V} d_{ij} x_{ij}^h \leq z^h \qquad \forall i \in V, h \in \mathcal{H} \tag{1e}$$

$$y_j^h \geq y_j^{h-1} \qquad \forall h \in \mathcal{H} \setminus \{1\} \tag{1f}$$

$$(x, y, z) \in |V|^2 |\mathcal{H}| \times |V||\mathcal{H}| \times \mathbb{R}_{\geq 0} \tag{1g}$$

$$
\begin{aligned}
\min \quad & \sum_{h\in\mathcal{H}} z^h && && \text{(2a)}\\
\text{s.t.} \quad & \sum_{j\in V} y_j^h = p^h && \forall h \in \mathcal{H} && \text{(2b)}\\
& \sum_{j\in V} x_{ij}^h = 1 && \forall i \in V, h \in \mathcal{H} && \text{(2c)}\\
& x_{ij}^h \leq y_j^h && \forall i, j \in V, h \in \mathcal{H} && \text{(2d)}\\
& \sum_{j\in V} d_{ij} x_{ij}^h \leq z^h && \forall i \in V, h \in \mathcal{H} && \text{(2e)}\\
& y_j^h \geq y_j^{h-1} && \forall h \in \mathcal{H} \setminus \{1\} && \text{(2f)}\\
& (x, y, z) \in |V|^2|\mathcal{H}| \times |V||\mathcal{H}| \times \mathbb{R}_{\geq 0} && && \text{(2g)}
\end{aligned}
$$

Related work

p-center problem

- First introduction of the *p*-center problem by Hakimi (1964)
- The standard textbook formulation of the *p*-center problem can be found in Daskin (2013)
- Solution approach based on the set cover problem by Contardo, Iori, and Kramer (2018)
- A compact formulation by Ales and Elloumi (2018) and Elloumi, Labbé, and Pochet (2004)
- A formulation with a projection-based branch-and-cut algorithm for the *p*-center problem by Gaar and Sinnl (2022)

Nested facility location problems

- First introduction of the nesting property and constraint by Roodman and Schwarz (1975)
- Extension of the nesting to a phase-in and phase-out by Roodman and Schwarz (1977)
- Reintroduction of the nesting on the example of the p-median problem by McGarvey and Thorsen (2022)
- Other work on multi-period facility locations problems mainly focus on varying demand, distances, or cost over time f.e. Calogiuri et al. (2021)

Mixed Integer Linear Programming formulations

First MILP formulation

Decision variables

$$x_{ij}^{h} \ldots \begin{cases} 1 \ldots \text{ if customer } i \text{ is assigned to facility location } j \text{ in period } h \\ 0 \ldots \text{ otherwise} \end{cases}$$

First MILP formulation

Decision variables

$$x_{ij}^{h} \ldots \begin{cases} 1 \ldots \text{if customer } i \text{ is assigned to facility location } j \text{ in period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

$$y_{j}^{h} \ldots \begin{cases} 1 \ldots \text{if a facility is opened at location } j \text{ in period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

First MILP formulation

Decision variables

$$x_{ij}^h \ldots \begin{cases} 1 \ldots \text{ if customer } i \text{ is assigned to facility location } j \text{ in period } h \\ 0 \ldots \text{ otherwise} \end{cases}$$

$$y_j^h \ldots \begin{cases} 1 \ldots \text{ if a facility is opened at location } j \text{ in period } h \\ 0 \ldots \text{ otherwise} \end{cases}$$

$z^h \ldots$ maximum distance between any customer i and its nearest open facility in period h

Second MILP formulation

$$
\begin{aligned}
\texttt{(nPC2)} \qquad & \min \quad \sum_{h \in \mathcal{H}} z^h && && \text{(3a)}\\
& \text{s.t.} \quad \sum_{j \in \mathcal{J}} y_j^h = p^h && \forall h \in \mathcal{H} && \text{(3b)}\\
& \qquad z^h \geq d_{ij} - \sum_{j' : d_{ij'} < d_{ij}} (d_{ij} - d_{ij'}) y_{j'}^h && \forall i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H} && \text{(3c)}\\
& \qquad y_j^h \geq y_j^{h-1} && \forall j \in \mathcal{J}, h \in \mathcal{H} \setminus \{1\} && \text{(3d)}\\
& \qquad y_j^h \in \{0, 1\} && \forall j \in \mathcal{J}, h \in \mathcal{H} && \text{(3e)}\\
& \qquad z^h \in \mathbb{R}_{\geq 0} && \forall h \in \mathcal{H} && \text{(3f)}
\end{aligned}
$$

Third MILP formualtion

$\mathcal{D} \ldots$ set of distinct distances where $D_0 \leq \cdots \leq D_K$ are the values in $\mathcal{D}$

Third MILP formualtion

$\mathcal{D}$ ... set of distinct distances where $D_0 \leq \cdots \leq D_K$ are the values in $\mathcal{D}$

$\mathcal{K}$ ... set of indices in $\mathcal{D}$

Third MILP formualtion

$\mathcal{D}\ldots$ set of distinct distances where $D_0 \leq \cdots \leq D_K$ are the values in $\mathcal{D}$

$\mathcal{K}\ldots$ set of indices in $\mathcal{D}$

$S_i\ldots$ set of indices $k \in \mathcal{K}$ for which there exists a facility $j \in \mathcal{J}$ with $d_{ij} = D_k$

Third MILP formualtion

$\mathcal{D}$ ... set of distinct distances where $D_0 \leq \cdots \leq D_K$ are the values in $\mathcal{D}$

$\mathcal{K}$ ... set of indices in $\mathcal{D}$

S_i ... set of indices $k \in \mathcal{K}$ for which there exists a facility $j \in \mathcal{J}$ with $d_{ij} = D_k$

Decision variables

$$y_j^h \ldots \begin{cases} 1 \ \ldots \text{ if a facility is opened at location } j \text{ in period } h \\ 0 \ \ldots \text{ otherwise} \end{cases}$$

Third MILP formualtion

$\mathcal{D}\ldots$set of distinct distances where $D_0 \leq \cdots \leq D_K$ are the values in $\mathcal{D}$

$\mathcal{K}\ldots$set of indices in $\mathcal{D}$

$S_i\ldots$set of indices $k \in \mathcal{K}$ for which there exists a facility $j \in \mathcal{J}$ with $d_{ij} = D_k$

Decision variables

$$y_j^h \ldots \begin{cases} 1 \ldots \text{ if a facility is opened at location } j \text{ in period } h \\ 0 \ldots \text{ otherwise} \end{cases}$$

$$u_k^h \ldots \begin{cases} 1 \ldots \text{ if the optimal value of } z^h \text{ is greater or equal than } D_k \\ 0 \ldots \text{ otherwise} \end{cases}$$

Third MILP formualtion

$\mathcal{D} \ldots$ set of distinct distances where $D_0 \leq \cdots \leq D_K$ are the values in $\mathcal{D}$

$\mathcal{K} \ldots$ set of indices in $\mathcal{D}$

$S_i \ldots$ set of indices $k \in \mathcal{K}$ for which there exists a facility $j \in \mathcal{J}$ with $d_{ij} = D_k$

Decision variables

$$y_j^h \ldots \begin{cases} 1 \ldots \text{ if a facility is opened at location } j \text{ in period } h \\ 0 \ldots \text{ otherwise} \end{cases}$$

$$u_k^h \ldots \begin{cases} 1 \ldots \text{ if the optimal value of } z^h \text{ is greater or equal than } D_k \\ 0 \ldots \text{ otherwise} \end{cases}$$

$z^h \ldots$ maximum distance between any customer i and its nearest open facility in period h

Third formulation

$$
\begin{aligned}
\texttt{(nPC3)} \quad \min \quad & \sum_{h \in \mathcal{H}} z^h && && \text{(4a)}\\
\text{s.t.} \quad & \sum_{j \in \mathcal{J}} y_j^h = p^h && \forall h \in \mathcal{H} && \text{(4b)}\\
& D_0 + \sum_{k=1}^{K} (D_k - D_{k-1}) u_k^h \leq z^h && \forall h \in \mathcal{H} && \text{(4c)}\\
& u_k^h + \sum_{j : d_{ij} < D_k} y_j^h \geq 1 && \forall i \in \mathcal{I}, \forall h \in \mathcal{H}, \forall k \in S_i \cup \{K\} && \text{(4d)}\\
& u_k^h \geq u_{k+1}^h && \forall h \in \mathcal{H}, \forall k \in \mathcal{K} \setminus \{K\} && \text{(4e)}\\
& y_j^h \leq y_j^{h-1} && \forall h \in \mathcal{H}, \forall j \in \mathcal{J} && \text{(4f)}\\
& y_j^h \in \{0, 1\} && \forall h \in \mathcal{H}, \forall j \in \mathcal{J} && \text{(4g)}\\
& u_k^h \in \{0, 1\} && \forall h \in \mathcal{H}, \forall k \in \mathcal{K} && \text{(4h)}\\
& z^h \in \mathbb{R} && \forall h \in \mathcal{H} && \text{(4i)}
\end{aligned}
$$

Comparing formulations

Table: Comparison of complexity of the formulations

	`(nPC1)`	`(nPC2)`	`(nPC3)`
Variables	$\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}(\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}((\lvert\mathcal{I}\rvert + \lvert\mathcal{K}\rvert)\lvert\mathcal{H}\rvert)$
Constraints	$\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}(\min(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert, \lvert\mathcal{I}\rvert\lvert\mathcal{K}\rvert)\lvert\mathcal{H}\rvert)$

Comparing formulations

Table: Comparison of complexity of the formulations

	`(nPC1)`	`(nPC2)`	`(nPC3)`
Variables	$\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}(\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}((\lvert\mathcal{I}\rvert + \lvert\mathcal{K}\rvert)\lvert\mathcal{H}\rvert)$
Constraints	$\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}(\min(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert, \lvert\mathcal{I}\rvert\lvert\mathcal{K}\rvert)\lvert\mathcal{H}\rvert)$

The *p*CP version of formulation `(nPC3)` has the best known linear programming ($\mathcal{LP}$)-bounds for the *p*CP, while the *p*CP versions of `(nPC1)` and `(nPC2)` have equally but worse $\mathcal{LP}$-bounds.

Reducing set $\mathcal{K}$ in `(nPC3)`

Lemma 1

Let $\underline{z^h}$ be a valid lower bound and $\overline{z^h}$ be a valid upper bound on the decision variable z^h for $h \in \mathcal{H}$, then the distinct distance D_k can only be the optimal distance for z^h if $\underline{z^h} \leq D^k \leq \overline{z^h}$ holds.
Therfore let set $S_i^h \subseteq S_i$ for $h \in \mathcal{H}$, where S_i^h contains only the indices k where $\underline{z^h} \leq D^k \leq \overline{z^h}$ holds and constraint (4d) can be replaced with

$$u_k^h + \sum_{j:d_{ij}<D_k} y_j^h \geq 1 \qquad \forall i \in \mathcal{I}, \forall h \in \mathcal{H}, \forall k \in S_i^h \cup \{K\} \tag{5}$$

Depending on the bounds $\underline{z^h}$ and $\overline{z^h}$ the sets S_i^h can be much smaller than S_i.

Strengthening constraints (nPC2)

Lemma 2

Let LB_h be a lower bound on the decision variable z_h of (nPC2) for every $i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H}$ then

$$z^h \geq max\{LB^h, d_{ij}\} - \sum_{j':d_{ij'}<d_{ij}} (\max\{LB^h, d_{ij}\} - \max\{LB^h, d_{ij'}\})y_{j'}^h \tag{nL-OPT}$$

is a valid inequality for (nPC2), i.e., every feasible solution of (nPC2) fulfills (nL-OPT)*. Theorem is based on Lemma 5 in Gaar and Sinnl (2022).*

A similar strengthening can be done for (**??**) in fromulation (nPC1).

Obtaining bounds I

Lemma 3

Let z'^{h*} *be the optimal objective function value of pCP with* $p = p^h$ *for* $h \in \mathcal{H} = \{1, 2, \ldots, H\}$ *where* $p^h > p^{h+1}$, *then* $UB = Hz'^{1*}$ *is a valid upper bound on the optimal objective function value of n-pCP.*
Then

$$\overline{z^h} = \frac{UB - \sum_{x=h+1}^{H} z'^{x*}}{h+1}$$

where $\overline{z^h}$ *is a valid upper bound on the decision variable* z^h *of the n-pCP for* $h \in \mathcal{H}$.

Obtaining bounds II

Lemma 4

Let z'^* be the optimal objective function value of pCP for a certain p'. Then z'^* is a valid lower bound $\underline{z}^h$ on the decision variable z^h of n-pCP with $p^h = p'$.

Implementation and outline of the results

Implementation

- Implemented in C++ using the CPLEX API of CPLEX 20.1.0.0
- Formulation `(nPC2)` solved by branch-and-cut and seperation based on the customers
- Preprocessing for all formulations
  - solving the pCP for p^h, $h \in \mathcal{H}$ starting with $h = H$
  - p^h is a valid lower bound for the pCP with p^{h-1}
- Single core of an Intel Xeon X5570 machine
  - 2.93 GHz
  - 48 GB RAM
  - Each run limited to 9 GB RAM and 3600 sec

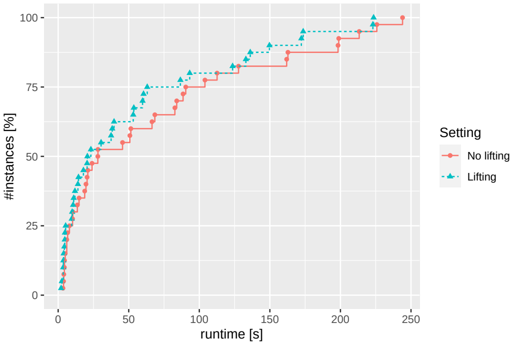
Data

- data set PMED from "A note on solving large p-median problems” by Beasley (1985)
  - set of 40 test instances
  - the sets contain between 100 and 900 nodes
  - number of facilities to open initially ranging from 5 to 200
  - $\mathcal{P} = \{p, p+1, p+2\}$

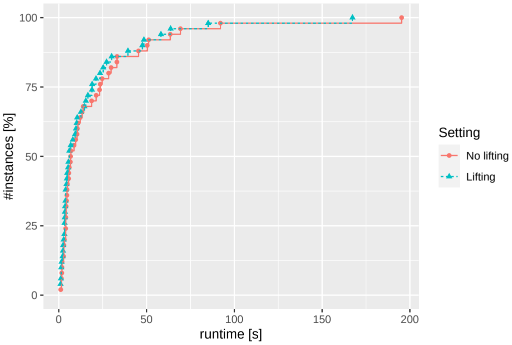
Data

- data set PMED from "A note on solving large p-median problems” by Beasley (1985)
  - set of 40 test instances
  - the sets contain between 100 and 900 nodes
  - number of facilities to open initially ranging from 5 to 200
  - $\mathcal{P} = \{p, p+1, p+2\}$
- data set TSPLIB 2D-Euclidean distances from “TSPLIB—A Traveling Salesman Problem Library” by Reinelt (1991)
  - set of 80 test instances
  - the sets contain between 51 and 1002 nodes
  - rounded to the nearest integer value
  - $\mathcal{P} = \{4, 5, 6\}$

Preprocessing



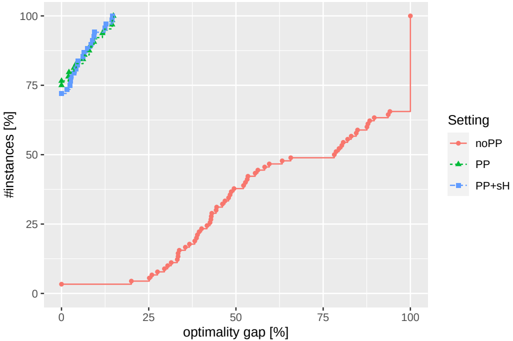
(a) Preprocessing PMED



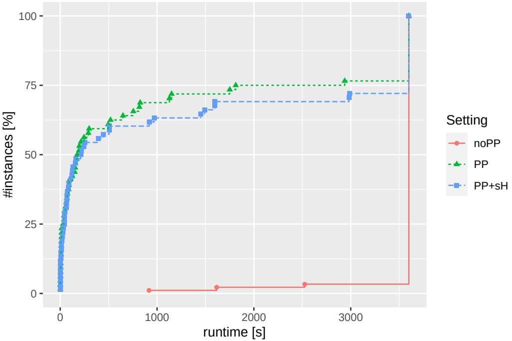
(b) Preprocessing TSPLIB

Figure: Preprocessing

(nPC1)-results

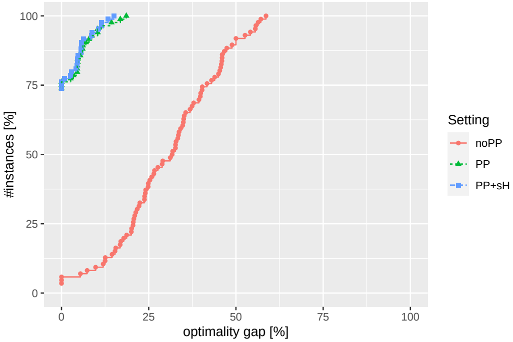


(a) (nPC1) optimality gap

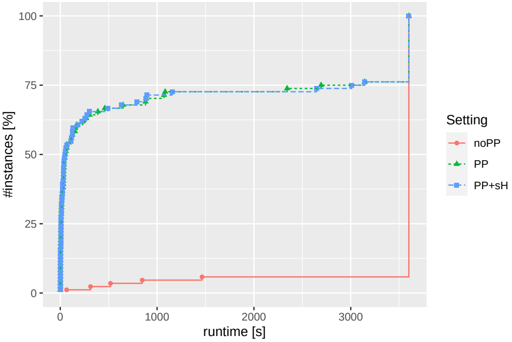


(b) (nPC1) runtime

(nPC2)-results

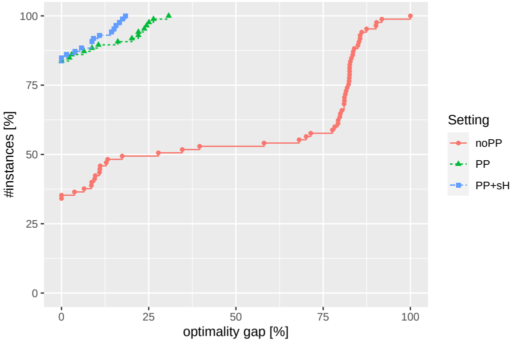


(c) (nPC2) optimality gap

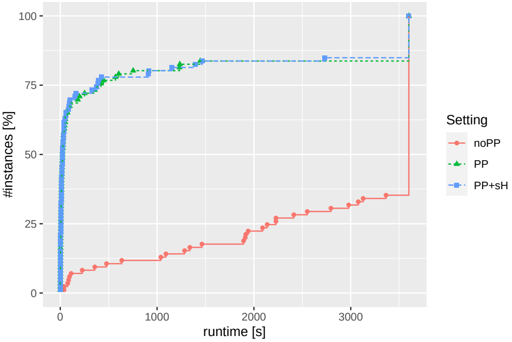


(d) (nPC2) runtime

(nPC3)-results

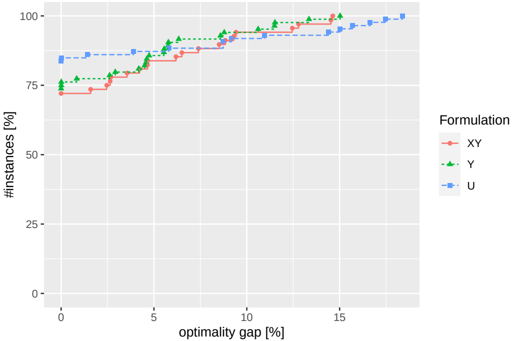


(e) (nPC3) optimality gap

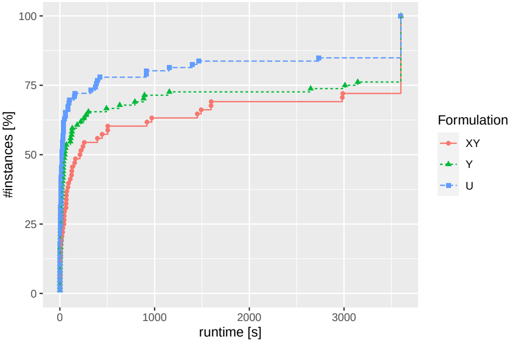


(f) (nPC3) runtime

Formulation comparison

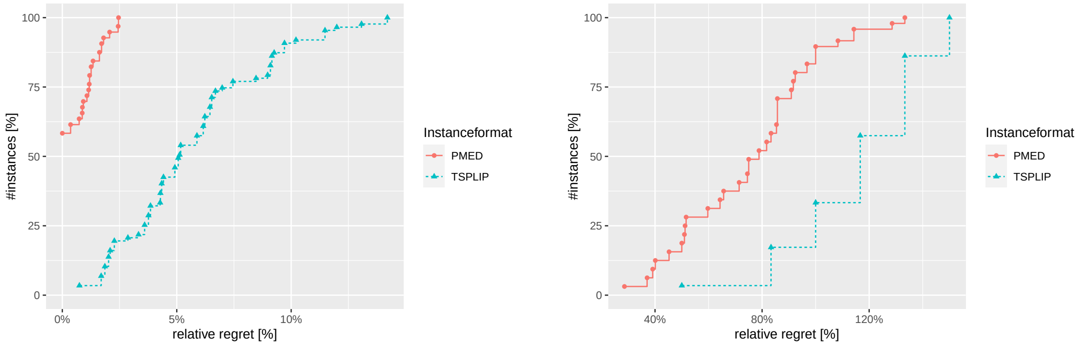


(g) models optimality gap



(h) models runtime

Managerial insights



(i) Relative regret

(j) Relative regret of # of facilities

Figure: On a subset of instances: Only if the problem was solved to optimality

Conclusion

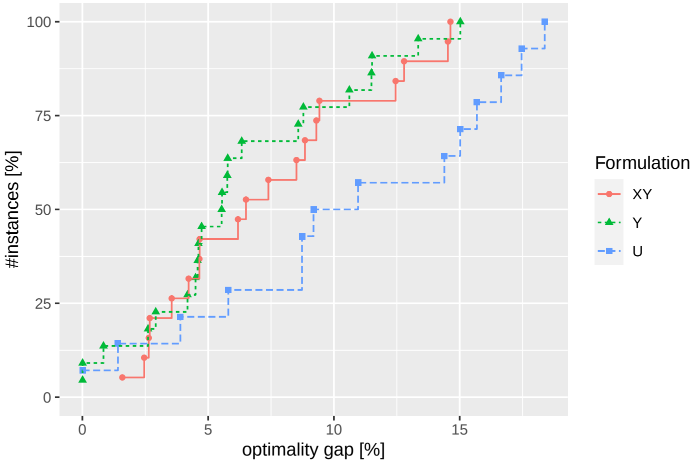
- `(nPC3)` had the best runtimes over all instances
- For optimality gap the results are rather mixed
- The `(nPC2)` outperforms the `(nPC1)`
- Maximal relative regret of the objective value of 15%
- Maximal relative regret of number of facilities above 140%
- min-max regret as objective function was also analysed and can be found later in the thesis

On the nested *p*-center problem

Christof Brandstetter, Markus Sinnl
Institute of Business Analytics and Technology Transformation / JKU Business School,
Johannes Kepler University Linz
2024-02-28

JOHANNES KEPLER
UNIVERSITY LINZ
Altenberger Straße 69
4040 Linz, Austria
jku.at

Formulations with non optimal instances



References I

[1] Zacharie Ales and Sourour Elloumi. Compact MILP Formulations for the p-Center Problem. In: Combinatorial Optimization. Ed. by Jon Lee, Giovanni Rinaldi, and A. Ridha Mahjoub. Cham: Springer International Publishing, 2018, pp. 14–25. ISBN: 978-3-319-96151-4.

[2] J.E. Beasley. A note on solving large p-median problems. In: European Journal of Operational Research 21.2 (1985), pp. 270–273. ISSN: 0377-2217. DOI: https://doi.org/10.1016/0377-2217(85)90040-2.

References II

[3] Tobia Calogiuri et al. The multi-period p-center problem with time-dependent travel times. In: Computers & Operations Research 136 (2021), p. 105487. ISSN: 0305-0548. DOI: https://doi.org/10.1016/j.cor.2021.105487. URL: https://www.sciencedirect.com/science/article/pii/S0305054821002343.

[4] Claudio Contardo, Manuel Iori, and Raphael Kramer. A scalable exact algorithm for the vertex p-center problem. In: Computers & Operations Research 103 (Mar. 2018). DOI: 10.1016/j.cor.2018.11.006.

References III

[5] Mark S. Daskin. Center Problems. In: Network and Discrete Location: Models, Algorithms, and Applications, Second Edition. John Wiley & Sons, Ltd, 2013. Chap. 5, pp. 193–234. ISBN: 9781118537015. DOI: https://doi.org/10.1002/9781118537015.ch05.

[6] Sourour Elloumi, Martine Labbé, and Yves Pochet. A New Formulation and Resolution Method for the p-Center Problem. In: INFORMS Journal on Computing 16 (Feb. 2004), pp. 83–94. DOI: 10.1287/ijoc.1030.0028.

References IV

[7] Elisabeth Gaar and Markus Sinnl. A scaleable projection-based branch-and-cut algorithm for the p-center problem. In: European Journal of Operational Research 303.1 (2022), pp. 78–98. ISSN: 0377-2217. DOI: https://doi.org/10.1016/j.ejor.2022.02.016.

[8] S. L. Hakimi. Optimum Locations of Switching Centers and the Absolute Centers and Medians of a Graph. In: Operations Research 12.3 (1964), pp. 450–459. DOI: 10.1287/opre.12.3.450.

[9] Ronald McGarvey and Andreas Thorsen. Nested-Solution Facility Location Models. In: Optimization Letters 16 (Mar. 2022). DOI: 10.1007/s11590-021-01759-4.

References V

[10] Gerhard Reinelt. TSPLIB—A Traveling Salesman Problem Library. In: ORSA Journal on Computing 3.4 (1991), pp. 376–384. DOI: 10.1287/ijoc.3.4.376.

[11] Gary M. Roodman and Leroy B. Schwarz. Extensions of the Multi-Period Facility Phase-Out Model: New Procedures and Application to a Phase-In/Phase-Out Problem. In: A I I E Transactions 9.1 (1977), pp. 103–107. DOI: 10.1080/05695557708975128.

[12] Gary M. Roodman and Leroy B. Schwarz. Optimal and Heuristic Facility Phase-out Strategies. In: A I I E Transactions 7.2 (1975), pp. 177–184. DOI: 10.1080/05695557508975000.