On the nested p-center problem

Master Thesis Seminar

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Introduction

- *p*-center problem: Open p facilities such that the maximum distance between any customer and its nearest open facility is minimized
- Related to the set cover problem and the assignment problem
- Nesting: Open additional facilties subsequently
- Nested *p*-center problem: Open p_h facilities in time period h such that the sum of the maximum distances between any customer and its nearest open faciltiy in time period h is minimized.
- Use case: Ambulance/logistics stations, network design, screening/vaccination stations
- Optimizes expansion or retraction path

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Definition of the nested p-center problem

Definition:

- given a set of customer demand points $\mathcal{I}$, potential facility locations $\mathcal{J}$, time periods $\mathcal{H} = \{1, \ldots, H\}$ and a set of integers $\mathcal{P} = \{p^1, \ldots p^H\}$ where $p^h \leq p^{h+1}$ for $h = 1, \ldots, H-1$

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- open p^h-facilities at different locations $j \in \mathcal{J}$
- the sum of maximum distance of any customer demand point to its closest opened facility in time period h is minimized
- and facilities open in period h has to be open in period $h+1$ as well for $h = 1, \dots, H-1$

This problem is NP-hard (**kariv1979**)

Related work

p-center problem / TODO

- Hakimi (1964): first introduction of the p-center problem
- Daskin (2013): the standard textbook MILP formulation
- Contardo, Iori, and Kramer (2018): state-of-the-Art model based on the set cover problem
- Ales and Elloumi (2018): compact MILP formulations for the p-center problem
- Gaar and Sinnl (2022): projection-based branch-and-cut algorithm for the p-center problem

Nested facility location problems

- Roodman and Schwarz (1975): first introduction of the nesting constraint
- Roodman and Schwarz (1977): extension of the nesting to a phase-in and phase out
- McGarvey and Thorsen (2022): reintroduction of the nesting on the p-median problem
- Calogiuri et al. (2021): multi-period p-center problem with time-dependent travel times but no additional opening of facilities
    - uses multiple periods, but do not nest the facilities

Methodology / Mehtod is how I do things, Mehtodology is more about which questions I can answer

Mixed Integer Linear Programming (MILP) formulation of the nested p-center

$\mathcal{I}$ . . . set of customer demand points with $i \in \mathcal{I}$
$\mathcal{J}$ . . . set of potential facility locations with $j \in \mathcal{J}$
$\mathcal{H}$ . . . set of time periods, $\mathcal{H} = \{1, 2, ..., H\}$
p_h = number of facilities to be opened in time period $h \in \mathcal{H}$ with $p_h > p_{h-1}$

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p_h = number of facilities to be opened in time period $h \in \mathcal{H}$ with $p_h > p_{h-1}$
d_{ij} = distance between customer demand point i and potential facility location j

$$x_{ijh} = \begin{cases} 1 \ldots \text{ if customer i is assigned to facility j in time period h} \\ 0 \ldots \text{ otherwise} \end{cases}$$

$$y_{jh} = \begin{cases} 1 \ldots \text{ if a facility is opened at location j in time period h} \\ 0 \ldots \text{ otherwise} \end{cases}$$

z_h = maximum distance between any customer i and its nearest open facility in time period h.

First MILP formulation

$$
\begin{array}{llll}
(nPC1) & \min \displaystyle\sum_{h \in \mathcal{H}} z_h & & \text{(objective)} \\
\text{s.t.} & \displaystyle\sum_{j \in \mathcal{J}} y_{jh} = p_h & \forall h \in \mathcal{H} & \text{(facility)} \\
& \displaystyle\sum_{j \in \mathcal{J}} x_{ijh} = 1 & \forall i \in \mathcal{I}, h \in \mathcal{H} & \text{(assignment)} \\
& x_{ijh} \leq y_j h & \forall i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H} & \text{(open)} \\
& \displaystyle\sum_{j \in \mathcal{J}} d_{ij} x_{ijh} \leq z_h & \forall i \in \mathcal{I}, h \in \mathcal{H} & \text{(push)} \\
& y_{jh} \geq y_{jh-1} & \forall h \in \mathcal{H} \setminus \{1\} & \text{(nesting)} \\
& x_{ijh}, y_{jh} \in \{0, 1\} & \forall i \in \mathcal{I}, j \in \mathcal{J}, h \in H & \text{(binary)} \\
& z_h \in \mathbb{R}_{\geq 0} & \forall h \in \mathcal{H} & \text{(non-negativity)}
\end{array}
$$

Second MILP formulation

$$
\begin{aligned}
(nPC2) \quad & \min \sum_{h \in H} z_h & & \text{(objective-y)} \\
\text{s.t.} \quad & \sum_{j \in J} y_{jh} = p_h & \forall h \in H & \text{(facility-y)} \\
& z_h \geq d_{ij} - \sum_{j' : d_{ij'} < d_{ij}} (d_{ij} - d_{ij'}) y_{j'h} & \forall i \in I, j \in J, h \in H & \text{(push-y)} \\
& y_h \geq y_{h-1} & \forall j \in J, h \in H \setminus \{1\} & \text{(nesting-y)} \\
& y_{jh} \in \{0, 1\} & \forall j \in J, h \in H & \text{(binary-y)} \\
& z_h \in \mathbb{R}_{\geq 0} & \forall h \in H & \text{(non-negativity-y)}
\end{aligned}
$$

Third formualtion

Constraint lifting

Theorem [Brandstetter(2023)]

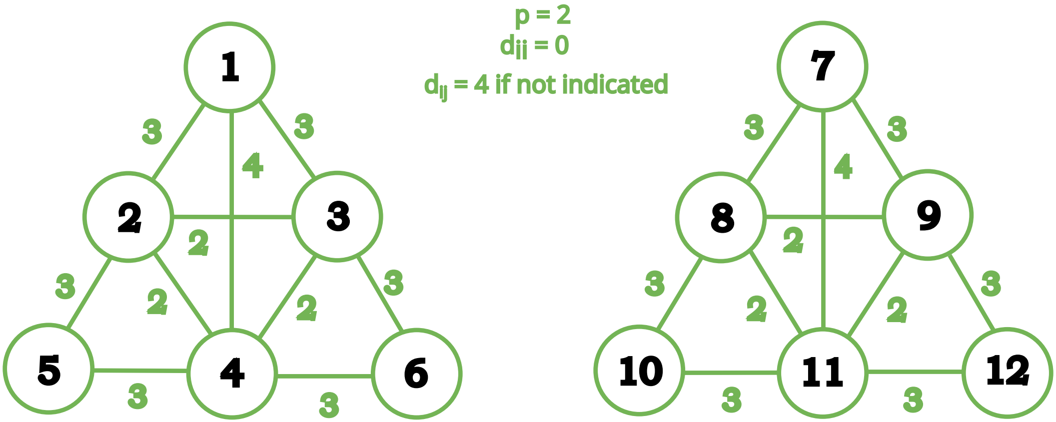
Let LB_h being a lower bound on the decision variable z_h of (nPC1) for every $i \in I, j \in J, h \in H$ then

$$\sum_{j \in J} \max\{LB_h, d_{ij}\} x_{ijh} \leq z_h \quad \forall i \in I, h \in H \tag{lift}$$

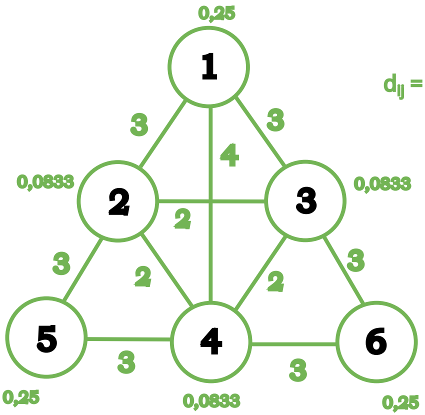
is a valid equality. Theorem is based on Gaar and Sinnl (2022).

- also valid for the y-formulation
- using various techniques to obtain LB_h

Constraint lifting / Delete



Constraint lifting / Delete

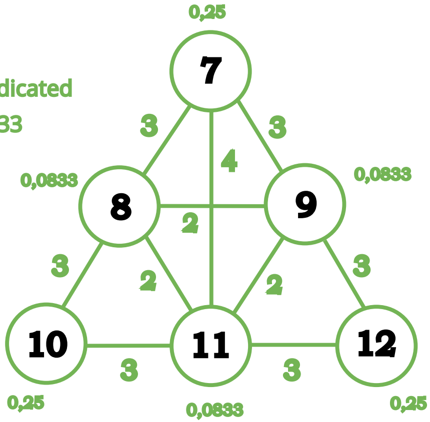


$p = 2$

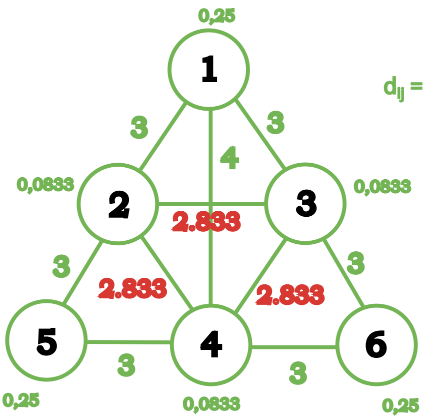
$d_{ii} = 0$

$d_{ij} = 4$ if not indicated

LB = 2.833



Constraint lifting / Delete



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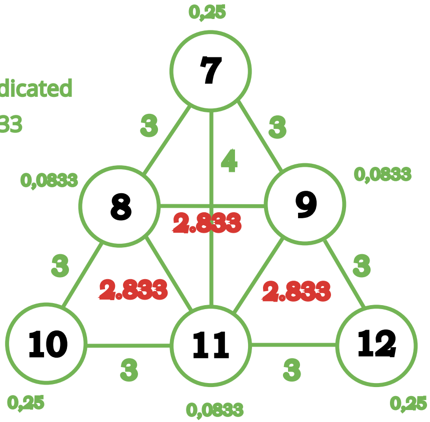

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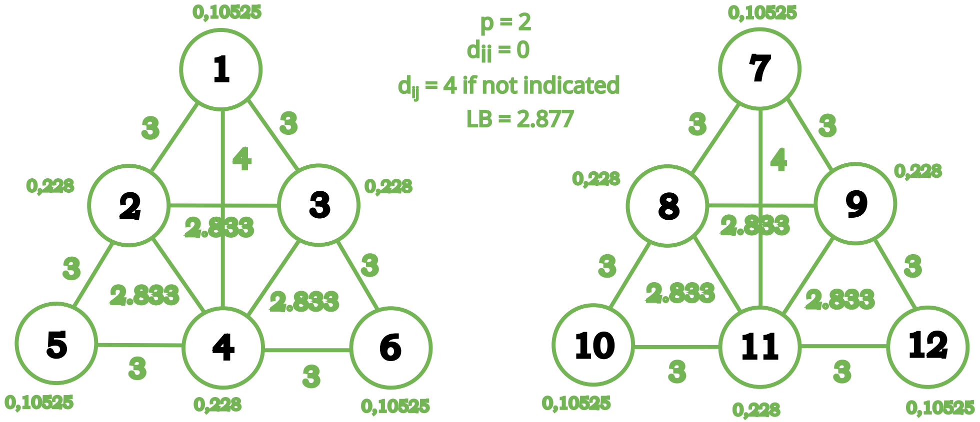

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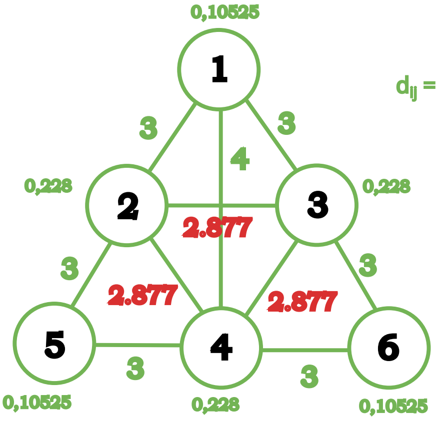
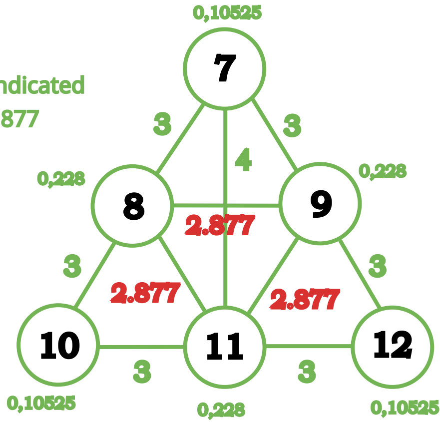


p = 2

$d_{ii} = 0$

$d_{ij} = 4$ if not indicated

LB = 2.877



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- adding cut(s) $\rightarrow$ branching $\rightarrow$ adding cut(s) $\rightarrow$ branching
- start with lower number of (push) constraints
- add (lift) inequalities
- allows for shorter solving times

Outline of the results / This should be the main topic for the final presentation

Data

- data set PMED from ”A note on solving large p-median problems” by Beasley (1985)
  - set of 40 instances
  - between 100 and 900 nodes

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    - set of 40 instances
    - between 100 and 900 nodes
- data set TSPLIB 2D-Euclidean distances from “TSPLIB — A Traveling Salesman Problem Library” by Reinelt (1991)
    - set of 80 instance
    - between 51 and 18512 nodes
    - rounded to the nearest integer value

Comparison

- comparing the results between (nPC1) and (nPC2)

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Comparison

- comparing the results between (nPC1) and (nPC2)
- comparing the different lifting methods
- analysing the solutions of the nested and comparing it to the standard p-center problem
- analysing the results regarding managerial insights

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Further research topics

- *u*-space model introduced by Elloumi, Labbé, and Pochet (2004)
- maximal regret as objective function
- trying to improve the performance of the nested p-median problem?

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