

On the nested p -center problem



Christof Brandstetter, Markus Sinnl
Institute of Business Analytics and Technology Transformation / JKU Business School,
Johannes Kepler University Linz
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The p -center problem: Definition

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 - a set V of locations,

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- we want to
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 - the maximum distance of any location to its closest opened location is minimized.

The nesting property

The nested p -center problem: Definition

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 - where $p^h \leq p^{h+1}$ for $h = 1, \dots, H-1$
and
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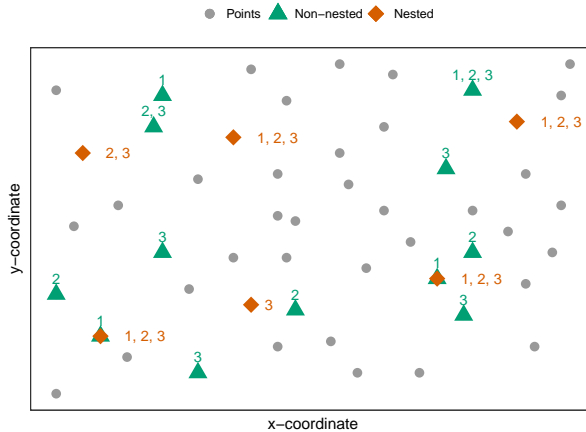
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 - with $|\mathcal{J}^h| = p^h$ for $h \in \mathcal{H}$,
 - for which $\mathcal{J}^h \subseteq \mathcal{J}^{h+1}$ for $h = 1, \dots, H-1$ holds
- the goal is to find a feasible solution which minimizes $\sum_{h=1}^H d_h(\mathcal{J}^h)$,
 - where $d_h(\mathcal{J}) = \max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}} d_{ij}$ for $h \in \mathcal{H}$

p -center problem vs nested p -center problem



The nested p -center problem: Potential applications

The nested p -center problem: Classical MILP formulation

Variables

$$x_{ij}^h \dots \begin{cases} 1 \dots \text{if location } i \text{ is assigned to} \\ \text{location } j \text{ in time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

The nested p -center problem: Classical MILP formulation

Variables

$$x_{ij}^h \dots \begin{cases} 1 \dots \text{if location } i \text{ is assigned to} \\ \text{location } j \text{ in time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

$$y_j^h \dots \begin{cases} 1 \dots \text{if location } j \text{ is open in} \\ \text{time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

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$z^h \dots \text{maximum distance between any} \\ \text{customer } i \text{ and its nearest open facility} \\ \text{in period } h$

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 $y_j^h \dots \begin{cases} 1 \dots \text{if location } j \text{ is open in} \\ \text{time period } h \\ 0 \dots \text{otherwise} \end{cases}$
 $z^h \dots \text{maximum distance between any} \\ \text{customer } i \text{ and its nearest open facility} \\ \text{in period } h$

(MILP) for nested p -center (based on Daskin 2013)

$$\text{(nPC1) min } \sum_{h \in \mathcal{H}} z^h \quad (1a)$$

$$\text{s.t. } \sum_{j \in V} y_j^h = p^h \quad \forall h \in \mathcal{H} \quad (1b)$$

$$\sum_{j \in V} x_{ij}^h = 1 \quad \forall i \in V, h \in \mathcal{H} \quad (1c)$$

$$x_{ij}^h \leq y_j^h \quad \forall i, j \in V, h \in \mathcal{H} \quad (1d)$$

$$\sum_{j \in V} d_{ij} x_{ij}^h \leq z^h \quad \forall i \in V, h \in \mathcal{H} \quad (1e)$$

$$y_j^h \geq y_j^{h-1} \quad \forall h \in \mathcal{H} \setminus \{1\} \quad (1f)$$

$$x_{ij}^h \in \{0, 1\} \quad \forall i, j \in V, h \in \mathcal{H} \quad (1g)$$

$$y_j^h \in \{0, 1\} \quad \forall j \in V, h \in \mathcal{H} \quad (1h)$$

Related work



p -center problem

- First introduction of the p -center problem by Hakimi (1964)
- The standard textbook formulation of the p -center problem can be found in Daskin (2013)
- Solution approach based on the set cover problem by Contardo, Iori, and Kramer (2018)
- A compact formulation by Ales and Elloumi (2018) and Elloumi, Labbé, and Pochet (2004)
- A formulation with a projection-based branch-and-cut algorithm for the p -center problem by Gaar and Sinnl (2022)

Nested facility location problems

- First introduction of the nesting property and constraint by Roodman and Schwarz (1975)
- Extension of the nesting to a phase-in and phase-out by Roodman and Schwarz (1977)
- Reintroduction of the nesting on the example of the p -median problem by McGarvey and Thorsen (2022)
- Other work on multi-period facility locations problems mainly focus on varying demand, distances, or cost over time f.e. Calogiuri et al. (2021)

Mixed Integer Linear Programming formulations



First MILP formulation

Decision variables

$$x_{ij}^h \dots \begin{cases} 1 & \dots \text{if customer } i \text{ is assigned to facility location } j \text{ in period } h \\ 0 & \dots \text{otherwise} \end{cases}$$

First MILP formulation

Decision variables

$$x_{ij}^h \dots \begin{cases} 1 & \dots \text{if customer } i \text{ is assigned to facility location } j \text{ in period } h \\ 0 & \dots \text{otherwise} \end{cases}$$

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$z^h \dots$ maximum distance between any customer i and its nearest open facility in period h

Second MILP formulation

$$\text{(nPC2)} \quad \min \quad \sum_{h \in \mathcal{H}} z^h \quad (2a)$$

$$\text{s.t.} \quad \sum_{j \in \mathcal{J}} y_j^h = p^h \quad \forall h \in \mathcal{H} \quad (2b)$$

$$z^h \geq d_{ij} - \sum_{j': d_{ij'} < d_{ij}} (d_{ij} - d_{ij'}) y_{j'}^h \quad \forall i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H} \quad (2c)$$

$$y_j^h \geq y_j^{h-1} \quad \forall j \in \mathcal{J}, h \in \mathcal{H} \setminus \{1\} \quad (2d)$$

$$y_j^h \in \{0, 1\} \quad \forall j \in \mathcal{J}, h \in \mathcal{H} \quad (2e)$$

$$z^h \in \mathbb{R}_{\geq 0} \quad \forall h \in \mathcal{H} \quad (2f)$$

Third MILP formulation

\mathcal{D} ... set of distinct distances where $D_0 \leq \dots \leq D_K$ are the values in \mathcal{D}

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z^h ... maximum distance between any customer i and its nearest open facility in period h

Third formulation

$$\text{(nPC3) } \min \quad \sum_{h \in \mathcal{H}} z^h \quad (3a)$$

$$\text{s.t.} \quad \sum_{j \in \mathcal{J}} y_j^h = p^h \quad \forall h \in \mathcal{H} \quad (3b)$$

$$D_0 + \sum_{k=1}^K (D_k - D_{k-1}) u_k^h \leq z^h \quad \forall h \in \mathcal{H} \quad (3c)$$

$$u_k^h + \sum_{j: d_{ij} < D_k} y_j^h \geq 1 \quad \forall i \in \mathcal{I}, \forall h \in \mathcal{H}, \forall k \in \mathcal{S}_i \cup \{K\} \quad (3d)$$

$$u_k^h \geq u_{k+1}^h \quad \forall h \in \mathcal{H}, \forall k \in \mathcal{K} \setminus \{K\} \quad (3e)$$

$$y_j^h \leq y_j^{h-1} \quad \forall h \in \mathcal{H}, \forall j \in \mathcal{J} \quad (3f)$$

$$y_j^h \in \{0, 1\} \quad \forall h \in \mathcal{H}, \forall j \in \mathcal{J} \quad (3g)$$

$$u_k^h \in \{0, 1\} \quad \forall h \in \mathcal{H}, \forall k \in \mathcal{K} \quad (3h)$$

$$z^h \in \mathbb{R} \quad \forall h \in \mathcal{H} \quad (3i)$$

Comparing formulations

Table: Comparison of complexity of the formulations

	(nPC1)	(nPC2)	(nPC3)
Variables	$\mathcal{O}(\mathcal{I} \mathcal{J} \mathcal{H})$	$\mathcal{O}(\mathcal{J} \mathcal{H})$	$\mathcal{O}((\mathcal{I} + \mathcal{K}) \mathcal{H})$
Constraints	$\mathcal{O}(\mathcal{I} \mathcal{J} \mathcal{H})$	$\mathcal{O}(\mathcal{I} \mathcal{J} \mathcal{H})$	$\mathcal{O}(\min(\mathcal{I} \mathcal{J} , \mathcal{I} \mathcal{K}) \mathcal{H})$

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The p CP version of formulation (nPC3) has the best known linear programming (\mathcal{LP})-bounds for the p CP, while the p CP versions of (nPC1) and (nPC2) have equally but worse \mathcal{LP} -bounds.

Reducing set \mathcal{K} in (nPC3)

Lemma 1

Let \underline{z}^h be a valid lower bound and \overline{z}^h be a valid upper bound on the decision variable z^h for $h \in \mathcal{H}$, then the distinct distance D_k can only be the optimal distance for z^h if $\underline{z}^h \leq D^k \leq \overline{z}^h$ holds.

Therefore let set $S_i^h \subseteq S_i$ for $h \in \mathcal{H}$, where S_i^h contains only the indices k where $\underline{z}^h \leq D^k \leq \overline{z}^h$ holds and constraint (3d) can be replaced with

$$u_k^h + \sum_{j: d_{ij} < D_k} y_j^h \geq 1 \quad \forall i \in \mathcal{I}, \forall h \in \mathcal{H}, \forall k \in S_i^h \cup \{K\} \quad (4)$$

Depending on the bounds \underline{z}^h and \overline{z}^h the sets S_i^h can be much smaller than S_i .

Strengthening constraints (nPC2)

Lemma 2

Let LB_h be a lower bound on the decision variable z_h of (nPC2) for every $i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H}$ then

$$z^h \geq \max\{LB^h, d_{ij}\} - \sum_{j': d_{ij'} < d_{ij}} (\max\{LB^h, d_{ij}\} - \max\{LB^h, d_{ij'}\}) y_{j'}^h \quad (\text{nL-OPT})$$

is a valid inequality for (nPC2), i.e., every feasible solution of (nPC2) fulfills (nL-OPT). Theorem is based on Lemma 5 in Gaar and Sinnl (2022).

A similar strengthening can be done for (??) in formulation (nPC1).

Obtaining bounds I

Lemma 3

Let z'^{h} be the optimal objective function value of pCP with $p = p^h$ for $h \in \mathcal{H} = \{1, 2, \dots, H\}$ where $p^h > p^{h+1}$, then $UB = Hz'^{1*}$ is a valid upper bound on the optimal objective function value of n-pCP.*

Then

$$\overline{z^h} = \frac{UB - \sum_{x=h+1}^H z'^{x*}}{h+1}$$

where $\overline{z^h}$ is a valid upper bound on the decision variable z^h of the n-pCP for $h \in \mathcal{H}$.

Obtaining bounds II

Lemma 4

Let z'^ be the optimal objective function value of pCP for a certain p' . Then z'^* is a valid lower bound \underline{z}^h on the decision variable z^h of n - pCP with $p^h = p'$.*

Implementation and outline of the results



Implementation

- Implemented in C++ using the CPLEX API of CPLEX 20.1.0.0
- Formulation (nPC2) solved by branch-and-cut and separation based on the customers
- Preprocessing for all formulations
 - solving the p CP for p^h , $h \in \mathcal{H}$ starting with $h = H$
 - p^h is a valid lower bound for the p CP with p^{h-1}
- Single core of an Intel Xeon X5570 machine
 - 2.93 GHz
 - 48 GB RAM
 - Each run limited to 9 GB RAM and 3600 sec

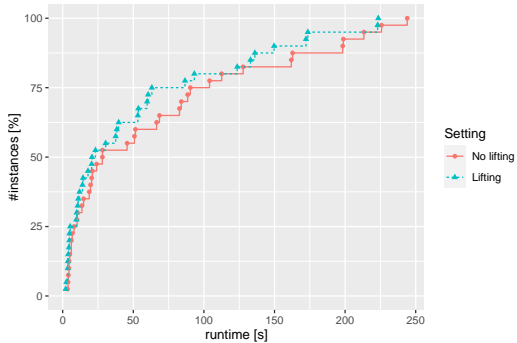
Data

- data set **PMED** from "A note on solving large p -median problems" by Beasley (1985)
 - set of 40 test instances
 - the sets contain between 100 and 900 nodes
 - number of facilities to open initially ranging from 5 to 200
 - $\mathcal{P} = \{p, p + 1, p + 2\}$

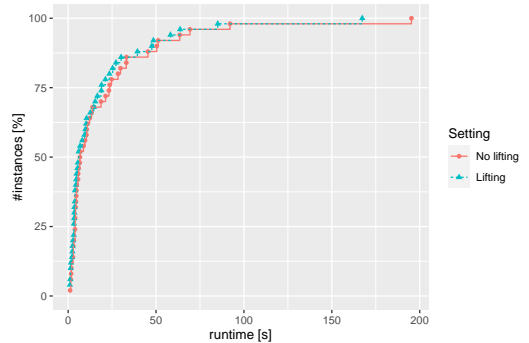
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- data set **TSPLIB** 2D-Euclidean distances from "TSPLIB—A Traveling Salesman Problem Library" by Reinelt (1991)
 - set of 80 test instances
 - the sets contain between 51 and 1002 nodes
 - rounded to the nearest integer value
 - $\mathcal{P} = \{4, 5, 6\}$

Preprocessing



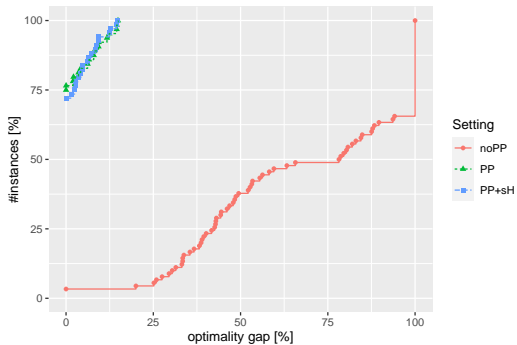
(a) Preprocessing PMED



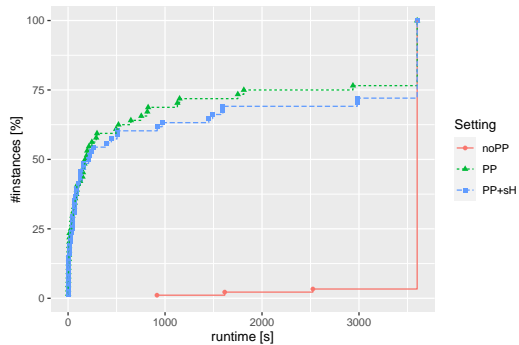
(b) Preprocessing TSPLIB

Figure: Preprocessing

(nPC1)-results

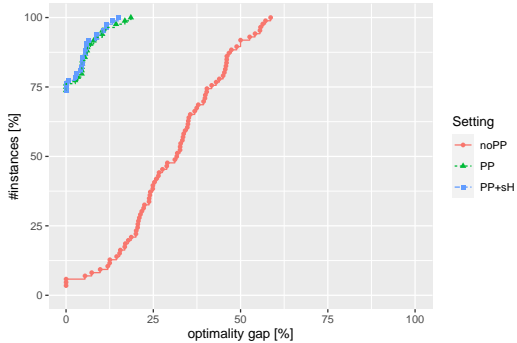


(a) (nPC1) optimality gap

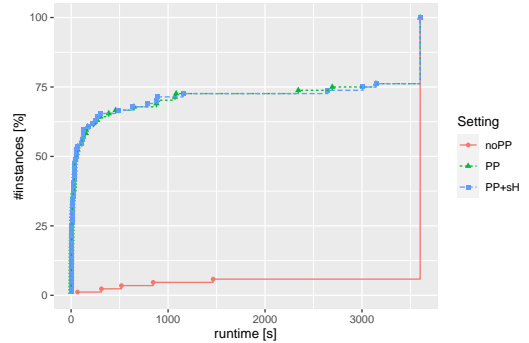


(b) (nPC1) runtime

(nPC2)-results

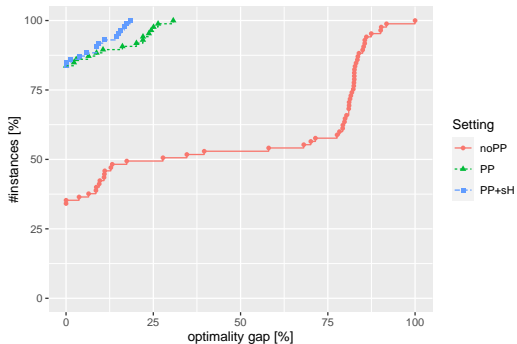


(c) (nPC2) optimality gap

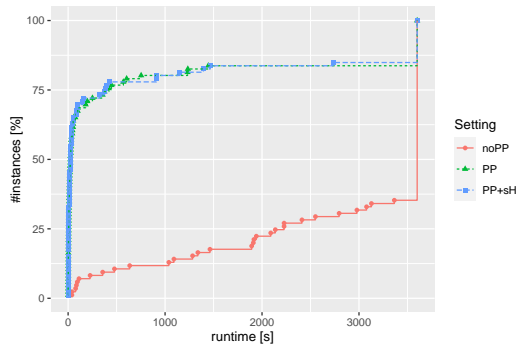


(d) (nPC2) runtime

(nPC3)-results

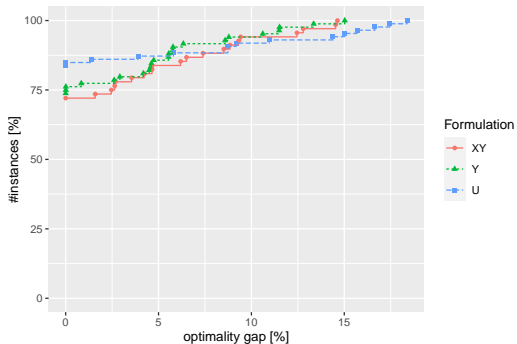


(e) (nPC3) optimality gap

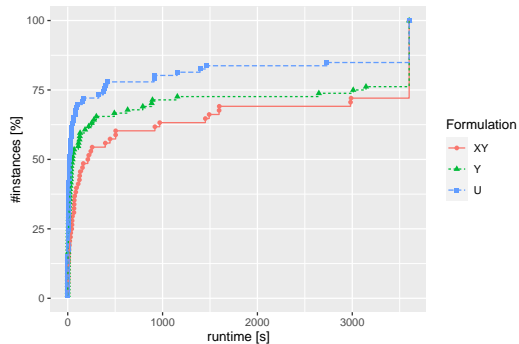


(f) (nPC3) runtime

Formulation comparison

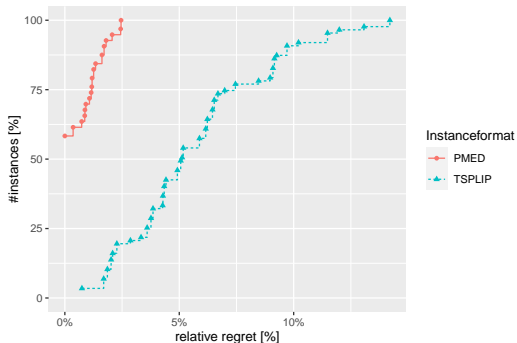


(g) models optimality gap

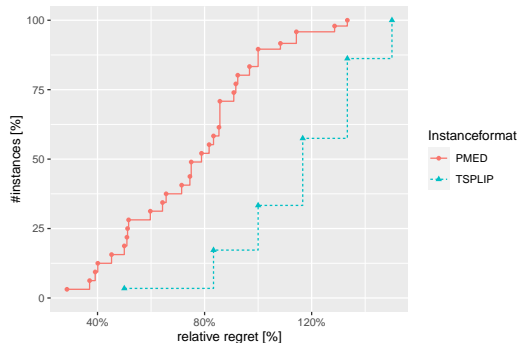


(h) models runtime

Managerial insights



(i) Relative regret



(j) Relative regret of # of facilities

Figure: On a subset of instances: Only if the problem was solved to optimality

Conclusion

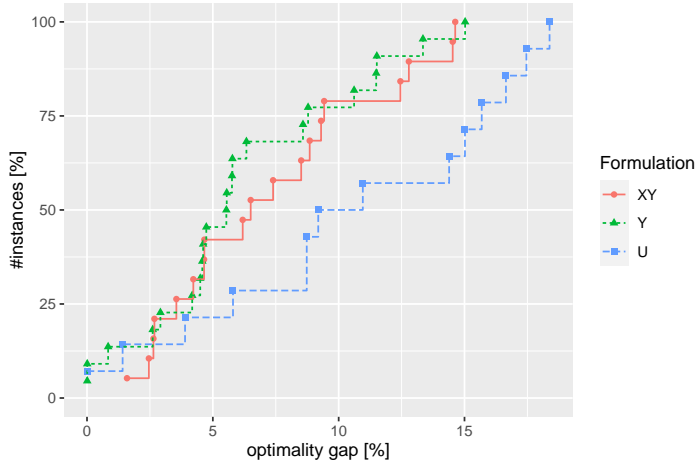
- (nPC3) had the best runtimes over all instances
- For optimality gap the results are rather mixed
- The (nPC2) outperforms the (nPC1)
- Maximal relative regret of the objective value of 15%
- Maximal relative regret of number of facilities above 140%
- min-max regret as objective function was also analysed and can be found later in the thesis

On the nested p -center problem



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Institute of Business Analytics and Technology Transformation / JKU Business School,
Johannes Kepler University Linz
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Formulations with non optimal instances



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