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On the nested *p*-center problem

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  - the maximum distance of any location to its closest opened location is minimized.

Ales and Elloumi (2018), Contardo, Iori, and Kramer (2019), Gaar and Sinnl (2022), and Hakimi (1964)

The nesting property for location problems

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Introduced by Roodman and Schwarz (1975) and used in e.g., Albareda-Sambola et al. (2009) and Bakker and Nickel (2024) and reintroduced as nesting by McGarvey and Thorsen (2022)

p-center problem vs nested p-center problem

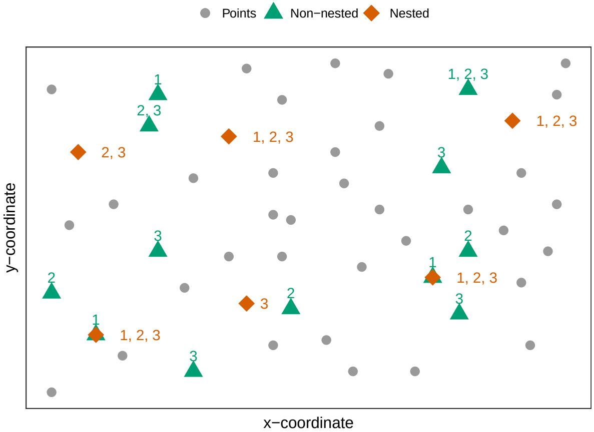

Figure: Optimal solution of (pCP) with $p = 4, 5, 6$ and (n-pCP) with $\mathcal{P} = \{4, 5, 6\}$

The nested *p*-center problem

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  - for which $V^h \subseteq V^{h+1}$ for $h = 1, \ldots, H-1$ holds
- the goal is to find a feasible solution which minimizes $\sum_{h=1}^{H} d(V^h)$,
  - where $d(V^h) = \max_{i \in V} \min_{j \in V^h} d_{ij}$ for $h \in \mathcal{H}$
- objective function can be seen as sum of absolute regrets of nestedness over time periods

Mixed Integer Linear Programming (MILP) formulations

Formulation `(nPC2)`

Decision variables

$$y_j^h \ldots \begin{cases} 1 \ldots \text{if location } j \text{ is open in} \\ \qquad \text{time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

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(nPC2) based on Gaar and Sinnl (2022)

$$
\begin{aligned}
\min \quad & \sum_{h \in \mathcal{H}} z^h & & \text{(1a)} \\
\text{s.t.} \quad & \sum_{j \in V} y_j^h = p^h & \forall h \in \mathcal{H} \quad & \text{(1b)} \\
& z^h \geq d_{ij} - \sum_{j' : d_{ij'} < d_{ij}} (d_{ij} - d_{ij'}) y_{j'}^h & \forall i, j \in V, h \in \mathcal{H} \quad & \text{(1c)} \\
& y_j^h \geq y_j^{h-1} & \forall j \in V, \forall h \in \mathcal{H} \setminus \{1\} \quad & \text{(1d)} \\
& (y, z) \in \mathbb{B}^{|V||\mathcal{H}|} \times \mathbb{R}_{\geq 0}^{|\mathcal{H}|} & & \text{(1e)}
\end{aligned}
$$

Formulation `(nPC3)`

Variables and sets

$$u_k^h \ldots \begin{cases} 1 \ldots \text{if } z^h \geq D_k \text{ in} \\ \qquad\quad \text{time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

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(nPC3) based on Ales and Elloumi (2018)

$$\begin{aligned}
\min \quad & \sum_{h \in \mathcal{H}} z^h \\
\text{s.t.} \quad & \sum_{j \in V} y_j^h = p^h & \forall h \in \mathcal{H} \\
& D_0 + \sum_{k=1}^{K} (D_k - D_{k-1}) u_k^h \leq z^h & \forall h \in \mathcal{H} \\
& u_k^h + \sum_{j: d_{ij} < D_k} y_j^h \geq 1 & \forall i \in V, \forall h \in \mathcal{H}, \forall k \in S_i \cup \{K\} \\
& u_k^h \geq u_{k+1}^h & \forall h \in \mathcal{H}, \forall k \in \mathcal{K} \setminus \{K\} \\
& y_j^h \geq y_j^{h-1} & \forall h \in \mathcal{H} \setminus \{1\} \\
& (u, y, z) \in \mathbb{B}^{|\mathcal{K}||\mathcal{H}|} \times \mathbb{B}^{|V||\mathcal{H}|} \times \mathbb{R}_{\geq 0}^{|\mathcal{H}|}
\end{aligned}$$

Improving the formulations

Strengthening constraints of `(nPC2)`

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Lemma 1

Let $\underline{z}^h$ be a lower bound on the decision variable z_h of (nPC2) for a given h then for every $i \in \mathcal{I}, j \in \mathcal{J}$ constraints

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can be replaced by

$$z^h \geq max\{\underline{z}^h, d_{ij}\} - \sum_{j': d_{ij'} < d_{ij}} \left(\max\{\underline{z}^h, d_{ij}\} - \max\{\underline{z}^h, d_{ij'}\}\right) y_{j'}^h \qquad \text{(nL-OPT)}$$

The lemma is based on Lemma 5 in Gaar and Sinnl (2022) for the (pCP).

Reducing the number of variables u_k^h in `(nPC3)`

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Observation 1

Let $\underline{z^h}$ be a valid lower bound and $\overline{z^h}$ be a valid upper bound on the decision variable z^h for $h \in \mathcal{H}$, then the distinct distance D_k can only be the optimal distance for z^h if $\underline{z^h} \leq D^k \leq \overline{z^h}$ holds.

Observation 2

Following Observation 1, we observe that decision variables $u_h^k = 0$ for $k : D_k > \overline{z^h}$ and $u_h^k = 1$ for $k : D_k < \underline{z^h}$ in any optimal solution.

$\implies$ these variables can be removed

Obtaining bounds

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Observation 3

For $\mathcal{H} = \{1\}$ the (n-pCP) reduces to the (pCP) where $p = p^1$, so the optimal objective value $({z'}^{h*})$ of the (pCP), where $p = p^h$ is a lower bound $\underline{z^h}$ on the decision variable z^h of the (n-pCP).

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Proposition 1

Given an valid upper bound UB on the objective value of the (n-pCP) and valid lower bounds $\underline{z^h}$ on the variable z^h can be obtained by the following equation:

$$\overline{z^h} = \frac{UB - \sum_{h'=h+1}^{H} \underline{z^{h'}}}{h} \tag{3}$$

Computational results

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- C++, CPLEX 20.1

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  - branch-and-cut separating of (1c)/(nL-OPT)
  - separation based on the fixedCustomer separation scheme from Gaar and Sinnl (2022)

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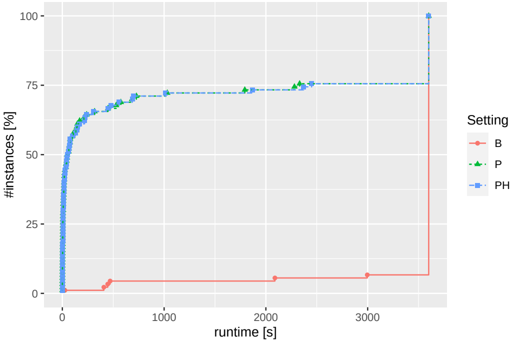
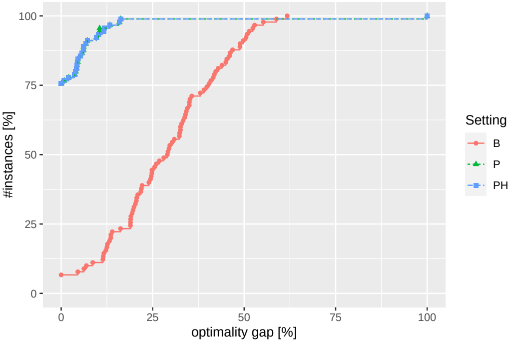
- `PH`: preprocessing, cut lifting/variable removing, starting heuristic

Setting comparison on formulation (nPC2)

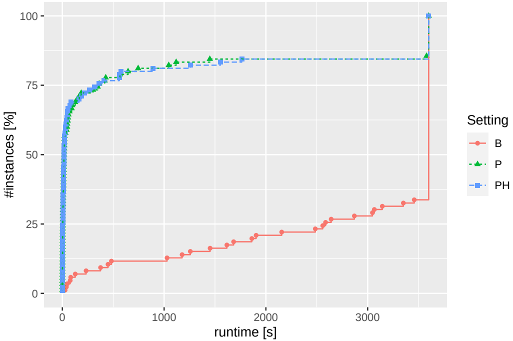


(a) Runtimes

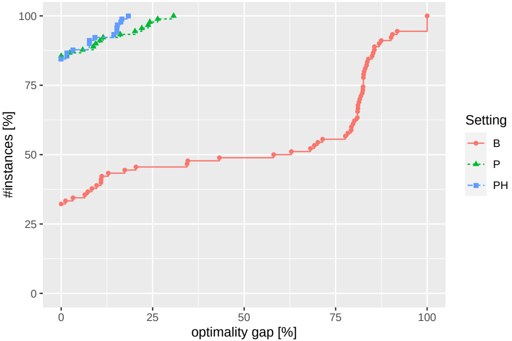


(b) Optimality gaps $\left[\frac{UB-LB}{UB}100\right]$

Setting comparison on formulation (nPC3)

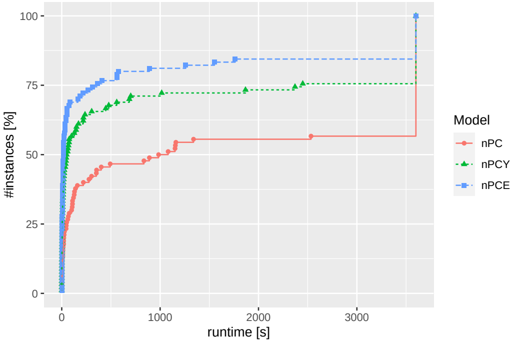


(c) Runtimes

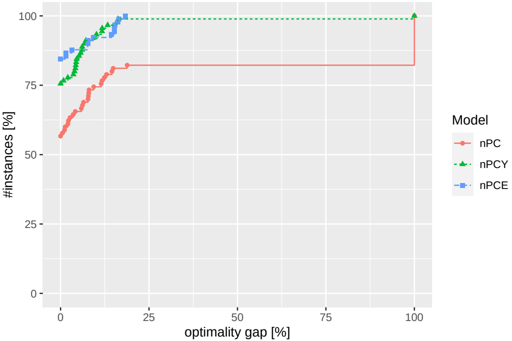


(d) Optimality gaps $\left[\frac{UB-LB}{UB}100\right]$

Formulation comparison on setting PH

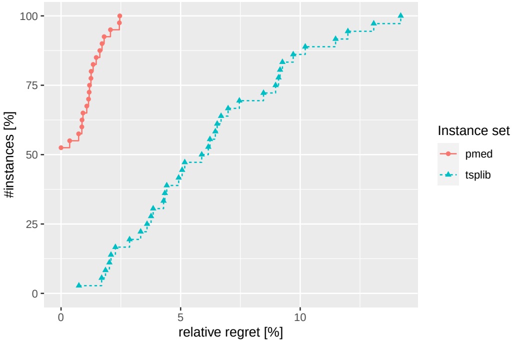


(e) Runtimes



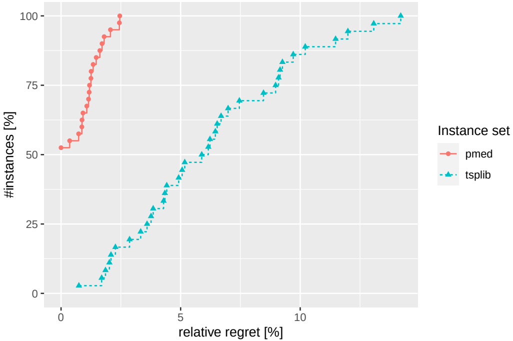
(f) Optimality gaps $\left[\frac{UB-LB}{UB}100\right]$

Managerial insights

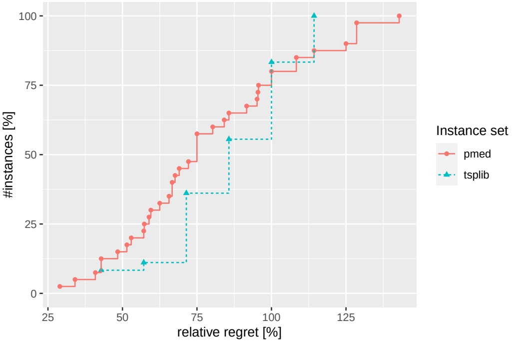


(g) Relative regrets of the optimal solution value

Managerial insights



(i) Relative regrets of the optimal solution value



(j) Relative regrets of # of opened facilities

Conclusion

- introduced nested p-center problem
- three mixed integer formulations
- improvement of formulations
- preprocessing brings a large speed up on all formulations
- starting heuristic little effect, shows good upper bound obtained in preprocessing
- nested facility location with uncertainty interesting for future work
- or nested maximum coverage problem

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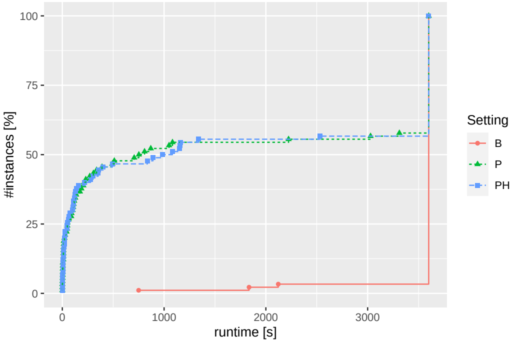
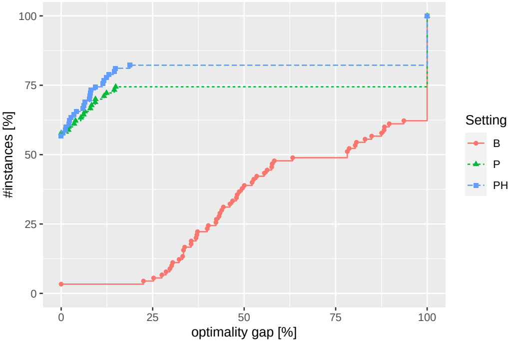
Setting comparison on formulation (nPC1)



(k) Runtimes



(l) Optimality gaps $[\frac{UB-LB}{UB}]$

Formulation (nPC1)

Decision variables

$$x_{ij}^{h} \ldots \begin{cases} 1 \ldots \text{if location } i \text{ is assigned to} \\ \qquad \text{location } j \text{ in time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

Formulation (nPC1)

Decision variables

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$$y_j^h \dots \begin{cases} 1 \dots \text{if location } j \text{ is open in} \\ \qquad \text{time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

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(nPC1) based on Daskin (2013)

$$
\begin{aligned}
\min \quad & \sum_{h \in \mathcal{H}} z^h && && \text{(4a)} \\
\text{s.t.} \quad & \sum_{j \in V} y_j^h = p^h && \forall h \in \mathcal{H} && \text{(4b)} \\
& \sum_{j \in V} x_{ij}^h = 1 && \forall i \in V, h \in \mathcal{H} && \text{(4c)} \\
& x_{ij}^h \leq y_j^h && \forall i, j \in V, h \in \mathcal{H} && \text{(4d)} \\
& \sum_{j \in V} d_{ij} x_{ij}^h \leq z^h && \forall i \in V, h \in \mathcal{H} && \text{(4e)} \\
& y_j^h \geq y_j^{h-1} && \forall h \in \mathcal{H} \setminus \{1\} && \text{(4f)} \\
& (x, y, z) \in |V|^2|\mathcal{H}| \times |V||\mathcal{H}| \times \mathbb{R}_{\geq 0} && && \text{(4g)}
\end{aligned}
$$