



# The $p$ -center problem: Definition

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- we want to
  - open  $p$  locations of  $V$ , such that
  - the maximum distance of any location to its closest opened location is minimized.

# The nesting property





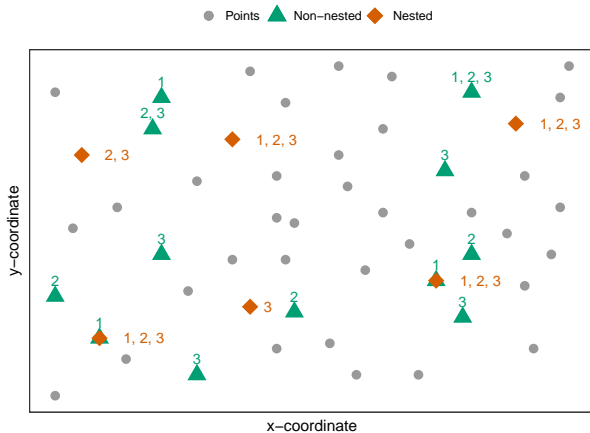
# The nested $p$ -center problem: Definition I

- given a set of customer demand points  $\mathcal{I}$ ,
- potential facility locations  $\mathcal{J}$ ,
- time periods  $\mathcal{H} = \{1, \dots, H\}$ ,
- integers  $\mathcal{P} = \{p^1, \dots, p^H\}$ 
  - where  $p^h \leq p^{h+1}$  for  $h = 1, \dots, H-1$  and
  - $p^H \leq |\mathcal{J}|$
- distances  $d_{ij} \geq 0$  between each  $i \in \mathcal{I}$  and  $j \in \mathcal{J}$

# The nested $p$ -center problem: Definition II

- a feasible solution to the nested  $p$ -center problem consists of a set  $\mathcal{J}^h \subseteq \mathcal{J}$ 
  - with  $|\mathcal{J}^h| = p^h$  for  $h \in \mathcal{H}$ ,
  - for which  $\mathcal{J}^h \subseteq \mathcal{J}^{h+1}$  for  $h = 1, \dots, H-1$  holds
- the goal is to find a feasible solution which minimizes  $\sum_{h=1}^H d_h(\mathcal{J}^h)$ ,
  - where  $d_h(\mathcal{J}^h) = \max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}^h} d_{ij}$  for  $h \in \mathcal{H}$

# $p$ -center problem vs nested $p$ -center problem



# The nested $p$ -center problem: Potential applications



$z^h$  ... maximum distance between any customer  $i$  and its nearest open facility (3)

in period  $h$

$y_j^h \dots \begin{cases} 1 \dots \text{if a facility is opened at location } j \text{ in period } h \\ 0 \dots \text{otherwise} \end{cases}$

$z^h$  ... maximum distance between any customer  $i$  and its nearest open facility in period  $h$





$$\begin{aligned}
 &< 1 - > x_{ij}^h \dots \begin{cases} 1 \dots \text{if location } i \text{ is assigned in time period } h \\ 0 \dots \text{otherwise} \end{cases} \\
 &y_j^h \dots \begin{cases} 1 \dots \text{if a facility is opened at location } j \text{ in period } h \\ 0 \dots \text{otherwise} \end{cases}
 \end{aligned}$$



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## $p$ -center problem

- First introduction of the  $p$ -center problem by Hakimi (1964)
- The standard textbook formulation of the  $p$ -center problem can be found in Daskin (2013)
- Solution approach based on the set cover problem by Contardo, Iori, and Kramer (2018)
- A compact formulation by Ales and Elloumi (2018) and Elloumi, Labbé, and Pochet (2004)
- A formulation with a projection-based branch-and-cut algorithm for the  $p$ -center problem by Gaar and Sinnl (2022)

# Nested facility location problems

- First introduction of the nesting property and constraint by Roodman and Schwarz (1975)
- Extension of the nesting to a phase-in and phase-out by Roodman and Schwarz (1977)
- Reintroduction of the nesting on the example of the  $p$ -median problem by McGarvey and Thorsen (2022)
- Other work on multi-period facility locations problems mainly focus on varying demand, distances, or cost over time f.e. Calogiuri et al. (2021)



# First MILP formulation

## Decision variables

$$x_{ij}^h \dots \begin{cases} 1 & \dots \text{if customer } i \text{ is assigned to facility location } j \text{ in period } h \\ 0 & \dots \text{otherwise} \end{cases}$$



# First MILP formulation

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$z^h \dots$  maximum distance between any customer  $i$  and its nearest open facility in period  $h$

# First MILP formulation

$$(nPC1) \quad \min \quad \sum_{h \in \mathcal{H}} z^h \quad (4a)$$

$$\text{s.t.} \quad \sum_{j \in \mathcal{J}} y_j^h = p^h \quad \forall h \in \mathcal{H} \quad (4b)$$

$$\sum_{j \in \mathcal{J}} x_{ij}^h = 1 \quad \forall i \in \mathcal{I}, h \in \mathcal{H} \quad (4c)$$

$$x_{ij}^h \leq y_j^h \quad \forall i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H} \quad (4d)$$

$$\sum_{j \in \mathcal{J}} d_{ij} x_{ij}^h \leq z^h \quad \forall i \in \mathcal{I}, h \in \mathcal{H} \quad (4e)$$

$$y_j^h \geq y_j^{h-1} \quad \forall h \in \mathcal{H} \setminus \{1\} \quad (4f)$$

$$x_{ij}^h, \in \{0, 1\} \quad \forall i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H} \quad (4g)$$

$$y_j^h, \in \{0, 1\} \quad \forall j \in \mathcal{J}, h \in \mathcal{H} \quad (4h)$$

$$z^h \in \mathbb{R}_{>0} \quad \forall h \in \mathcal{H} \quad (4i)$$

## Second MILP formulation

$$\text{(nPC2)} \quad \min \quad \sum_{h \in \mathcal{H}} z^h \quad (5a)$$

$$\text{s.t.} \quad \sum_{j \in \mathcal{J}} y_j^h = p^h \quad \forall h \in \mathcal{H} \quad (5b)$$

$$z^h \geq d_{ij} - \sum_{j': d_{ij'} < d_{ij}} (d_{ij} - d_{ij'}) y_{j'}^h \quad \forall i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H} \quad (5c)$$

$$y_j^h \geq y_j^{h-1} \quad \forall j \in \mathcal{J}, h \in \mathcal{H} \setminus \{1\} \quad (5d)$$

$$y_j^h \in \{0, 1\} \quad \forall j \in \mathcal{J}, h \in \mathcal{H} \quad (5e)$$

$$z^h \in \mathbb{R}_{\geq 0} \quad \forall h \in \mathcal{H} \quad (5f)$$

## Third MILP formulation

$\mathcal{D}$  ... set of distinct distances where  $D_0 \leq \dots \leq D_K$  are the values in  $\mathcal{D}$

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### Decision variables

$$y_j^h \dots \begin{cases} 1 & \dots \text{ if a facility is opened at location } j \text{ in period } h \\ 0 & \dots \text{ otherwise} \end{cases}$$



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$z^h$  ... maximum distance between any customer  $i$  and its nearest open facility in period  $h$

## Third formulation

$$\text{(nPC3) } \min \quad \sum_{h \in \mathcal{H}} z^h \quad (6a)$$

$$\text{s.t.} \quad \sum_{j \in \mathcal{J}} y_j^h = p^h \quad \forall h \in \mathcal{H} \quad (6b)$$

$$D_0 + \sum_{k=1}^K (D_k - D_{k-1}) u_k^h \leq z^h \quad \forall h \in \mathcal{H} \quad (6c)$$

$$u_k^h + \sum_{j: d_{ij} < D_k} y_j^h \geq 1 \quad \forall i \in \mathcal{I}, \forall h \in \mathcal{H}, \forall k \in \mathcal{S}_i \cup \{K\} \quad (6d)$$

$$u_k^h \geq u_{k+1}^h \quad \forall h \in \mathcal{H}, \forall k \in \mathcal{K} \setminus \{K\} \quad (6e)$$

$$y_j^h \leq y_j^{h-1} \quad \forall h \in \mathcal{H}, \forall j \in \mathcal{J} \quad (6f)$$

$$y_j^h \in \{0, 1\} \quad \forall h \in \mathcal{H}, \forall j \in \mathcal{J} \quad (6g)$$

$$u_k^h \in \{0, 1\} \quad \forall h \in \mathcal{H}, \forall k \in \mathcal{K} \quad (6h)$$

$$z^h \in \mathbb{R} \quad \forall h \in \mathcal{H} \quad (6i)$$

# Comparing formulations

Table: Comparison of complexity of the formulations

	(nPC1)	(nPC2)	(nPC3)
Variables	$\mathcal{O}( \mathcal{I}  \mathcal{J}  \mathcal{H} )$	$\mathcal{O}( \mathcal{J}  \mathcal{H} )$	$\mathcal{O}(( \mathcal{I}  +  \mathcal{K} ) \mathcal{H} )$
Constraints	$\mathcal{O}( \mathcal{I}  \mathcal{J}  \mathcal{H} )$	$\mathcal{O}( \mathcal{I}  \mathcal{J}  \mathcal{H} )$	$\mathcal{O}(\min( \mathcal{I}  \mathcal{J} ,  \mathcal{I}  \mathcal{K} ) \mathcal{H} )$

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The (pCP) version of formulation (nPC3) has the best known linear programming ( $\mathcal{LP}$ )-bounds for the (pCP), while the (pCP) versions of (nPC1) and (nPC2) have equally but worse  $\mathcal{LP}$ -bounds.

## Reducing set $\mathcal{K}$ in (nPC3)

### Lemma 1

Let  $\underline{z}^h$  be a valid lower bound and  $\overline{z}^h$  be a valid upper bound on the decision variable  $z^h$  for  $h \in \mathcal{H}$ , then the distinct distance  $D_k$  can only be the optimal distance for  $z^h$  if  $\underline{z}^h \leq D^k \leq \overline{z}^h$  holds.

Therefore let set  $S_i^h \subseteq S_i$  for  $h \in \mathcal{H}$ , where  $S_i^h$  contains only the indices  $k$  where  $\underline{z}^h \leq D^k \leq \overline{z}^h$  holds and constraint (??) can be replaced with

$$u_k^h + \sum_{j: d_{ij} < D_k} y_j^h \geq 1 \quad \forall i \in \mathcal{I}, \forall h \in \mathcal{H}, \forall k \in S_i^h \cup \{K\} \quad (7)$$

Depending on the bounds  $\underline{z}^h$  and  $\overline{z}^h$  the sets  $S_i^h$  can be much smaller than  $S_i$ .

# Strengthening constraints (nPC2)

## Lemma 2

Let  $LB_h$  be a lower bound on the decision variable  $z_h$  of (nPC2) for every  $i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H}$  then

$$z^h \geq \max\{LB^h, d_{ij}\} - \sum_{j': d_{ij'} < d_{ij}} (\max\{LB^h, d_{ij}\} - \max\{LB^h, d_{ij'}\}) y_{j'}^h \quad (\text{nL-OPT})$$

is a valid inequality for (nPC2), i.e., every feasible solution of (nPC2) fulfills (??).  
Theorem is based on Lemma 5 in Gaar and Sinnl (2022).

A similar strengthening can be done for (??) in formulation (nPC1).

# Obtaining bounds I

## Lemma 3

Let  $z'^{h*}$  be the optimal objective function value of (pCP) with  $p = p^h$  for  $h \in \mathcal{H} = \{1, 2, \dots, H\}$  where  $p^h > p^{h+1}$ , then  $UB = Hz'^{1*}$  is a valid upper bound on the optimal objective function value of (npCP).

Then

$$\overline{z^h} = \frac{UB - \sum_{x=h+1}^H z'^{x*}}{h+1}$$

where  $\overline{z^h}$  is a valid upper bound on the decision variable  $z^h$  of the (npCP) for  $h \in \mathcal{H}$ .



# Obtaining bounds II

## Lemma 4

*Let  $z'^*$  be the optimal objective function value of (pCP) for a certain  $p'$ . Then  $z'^*$  is a valid lower bound  $\underline{z}^h$  on the decision variable  $z^h$  of (npCP) with  $p^h = p'$ .*



# Implementation

- Implemented in C++ using the CPLEX API of CPLEX 20.1.0.0
- Formulation (nPC2) solved by branch-and-cut and separation based on the customers
- Preprocessing for all formulations
  - solving the (pCP) for  $p^h$ ,  $h \in \mathcal{H}$  starting with  $h = H$
  - $p^h$  is a valid lower bound for the (pCP) with  $p^{h-1}$
- Single core of an Intel Xeon X5570 machine
  - 2.93 GHz
  - 48 GB RAM
  - Each run limited to 9 GB RAM and 3600 sec

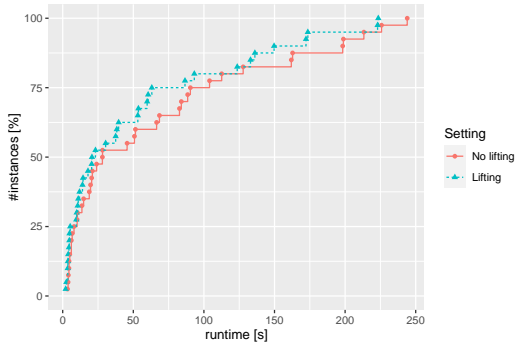
# Data

- data set **PMED** from "A note on solving large  $p$ -median problems" by Beasley (1985)
  - set of 40 test instances
  - the sets contain between 100 and 900 nodes
  - number of facilities to open initially ranging from 5 to 200
  - $\mathcal{P} = \{p, p + 1, p + 2\}$

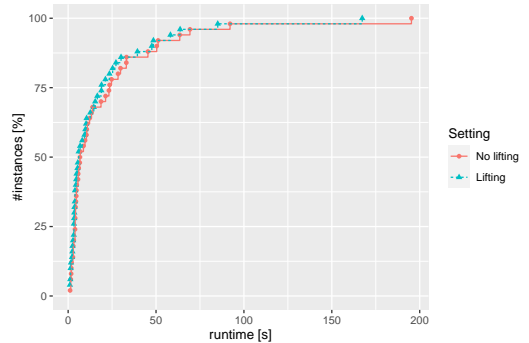
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- data set **TSPLIB** 2D-Euclidean distances from "TSPLIB—A Traveling Salesman Problem Library" by Reinelt (1991)
  - set of 80 test instances
  - the sets contain between 51 and 1002 nodes
  - rounded to the nearest integer value
  - $\mathcal{P} = \{4, 5, 6\}$

# Preprocessing



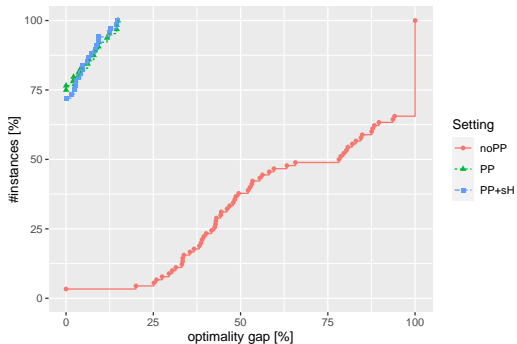
(a) Preprocessing PMED



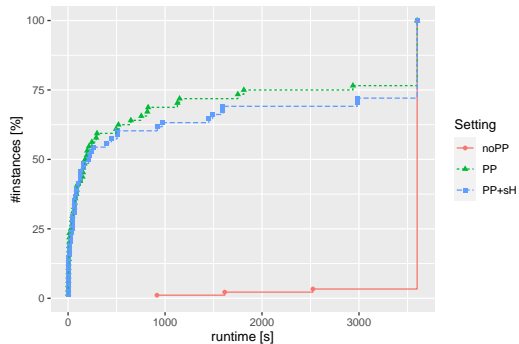
(b) Preprocessing TSPLIB

Figure: Preprocessing

# (nPC1)-results

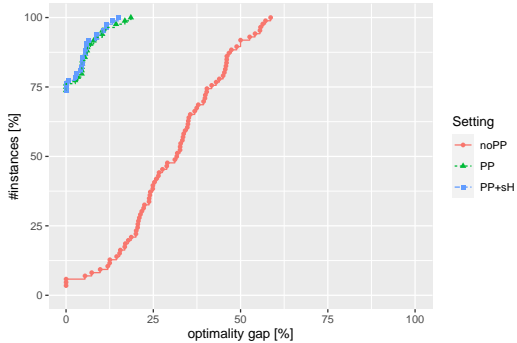


(a) (nPC1) optimality gap

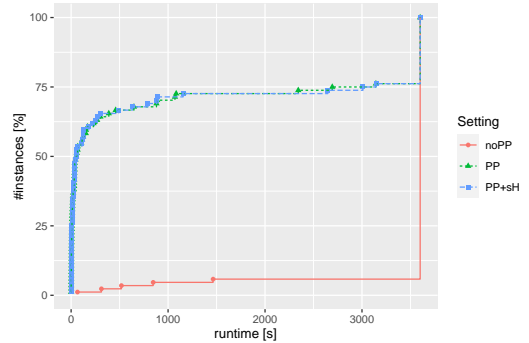


(b) (nPC1) runtime

# (nPC2)-results



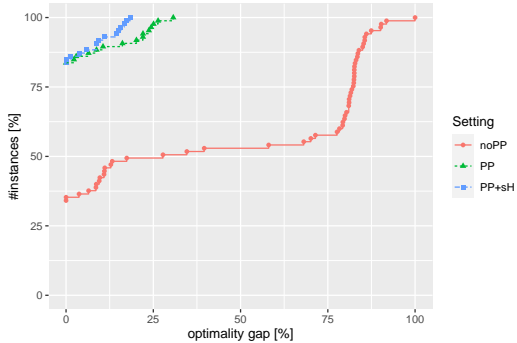
(c) (nPC2) optimality gap



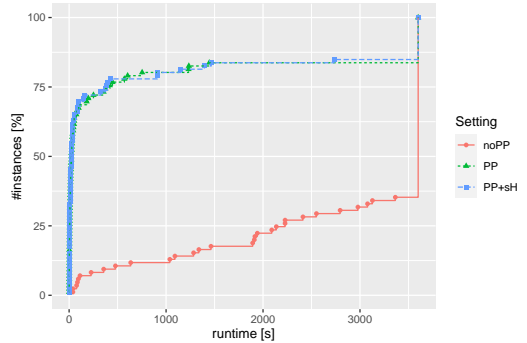
(d) (nPC2) runtime



# (nPC3)-results

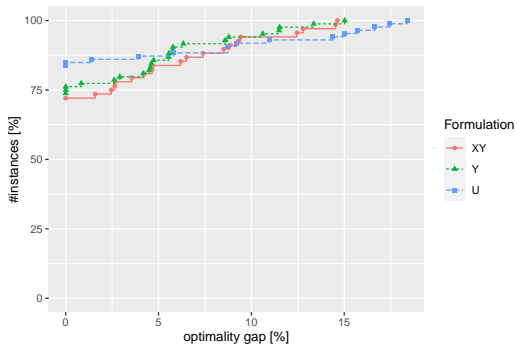


(e) (nPC3) optimality gap

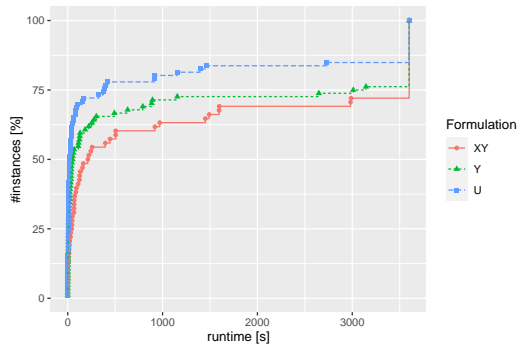


(f) (nPC3) runtime

# Formulation comparison

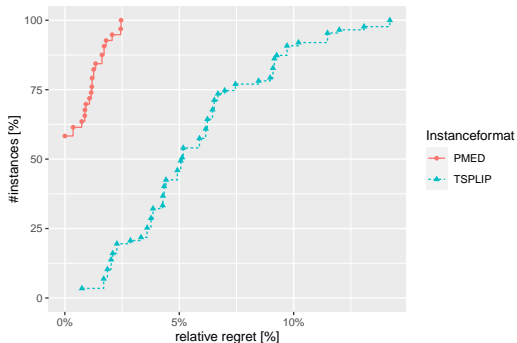


(g) models optimality gap

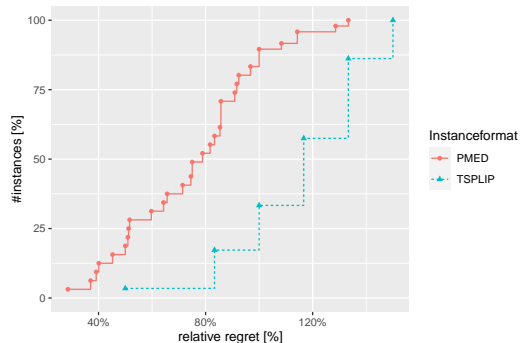


(h) models runtime

# Managerial insights



(i) Relative regret



(j) Relative regret of # of facilities

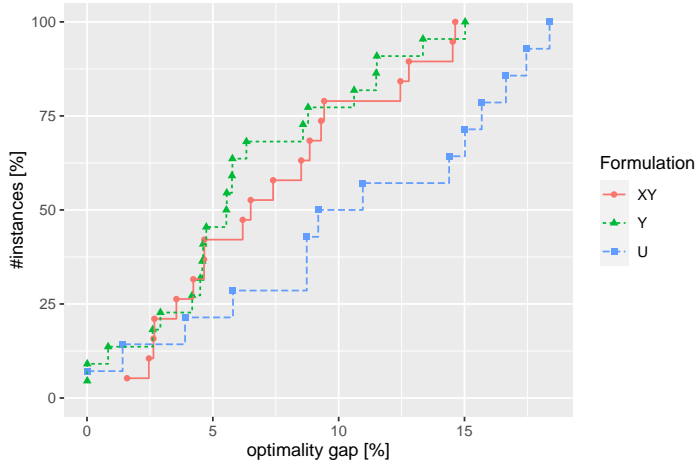
Figure: On a subset of instances: Only if the problem was solved to optimality

# Conclusion

- (nPC3) had the best runtimes over all instances
- For optimality gap the results are rather mixed
- The (nPC2) outperforms the (nPC1)
- Maximal relative regret of the objective value of 15%
- Maximal relative regret of number of facilities above 140%
- min-max regret as objective function was also analysed and can be found later in the thesis



# Formulations with non optimal instances



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