On the nested p-center problem

Christof Brandstetter, Markus Sinnl
Institute of Business Analytics and Technology Transformation / JKU Business School,
Johannes Kepler University Linz

2024-03-03

JOHANNES KEPLER
UNIVERSITY LINZ
Altenberger Straße 69
4040 Linz, Austria
jku.at

The p-center problem pCP: Definition

Definition

- Given

The p-center problem pCP: Definition

Definition

- Given
  - a set V of locations,

The p-center problem pCP: Definition

Definition

- Given
  - a set V of locations,
  - $p \in \mathbb{Z}$, and

The p-center problem pCP: Definition

Definition

- Given
  - a set V of locations,
  - $p \in \mathbb{Z}$, and
  - distances d_{ij} from location $i \in V$ to $j \in V$

The p-center problem pCP: Definition

Definition

- Given
  - a set V of locations,
  - $p \in \mathbb{Z}$, and
  - distances d_{ij} from location $i \in V$ to $j \in V$
- we want to
  - open p locations of V, such that

The p-center problem pCP: Definition

Definition

- Given
  - a set V of locations,
  - $p \in \mathbb{Z}$, and
  - distances d_{ij} from location $i \in V$ to $j \in V$
- we want to
  - open p locations of V, such that
  - the maximum distance of any location to its closest opened location is minimized.

The nesting property

Definition

- Given

The nesting property

Definition

- Given
  - a set of locations V
  - a set of time periods $\mathcal{H} = \{1, \ldots, H\}$ and

The nesting property

Definition

- Given
  - a set of locations V
  - a set of time periods $\mathcal{H} = \{1, \ldots, H\}$ and
  - a set of integers $\mathcal{P} = \left\{p^1, \ldots, p^H\right\}$ indicating the number of facilities to open

The nesting property

Definition

- Given
  - a set of locations V
  - a set of time periods $\mathcal{H} = \{1, \ldots, H\}$ and
  - a set of integers $\mathcal{P} = \left\{p^1, \ldots, p^H\right\}$ indicating the number of facilities to open
  - where $p^h \leq p^{h+1}$ for $h = 1, \ldots, H - 1$

The nesting property

Definition

- Given
  - a set of locations V
  - a set of time periods $\mathcal{H} = \{1, \ldots, H\}$ and
  - a set of integers $\mathcal{P} = \left\{p^1, \ldots, p^H\right\}$ indicating the number of facilities to open
  - where $p^h \leq p^{h+1}$ for $h = 1, \ldots, H - 1$
- the subsets $V^h \subseteq V$ with $|V^h| = p^h$

The nesting property

Definition

- Given
  - a set of locations V
  - a set of time periods $\mathcal{H} = \{1, \ldots, H\}$ and
  - a set of integers $\mathcal{P} = \left\{p^1, \ldots, p^H\right\}$ indicating the number of facilities to open
  - where $p^h \leq p^{h+1}$ for $h = 1, \ldots, H-1$
- the subsets $V^h \subseteq V$ with $|V^h| = p^h$
- are called nested iff $V^h \subset V^{h+1}$ for $h = 1, \ldots, H-1$

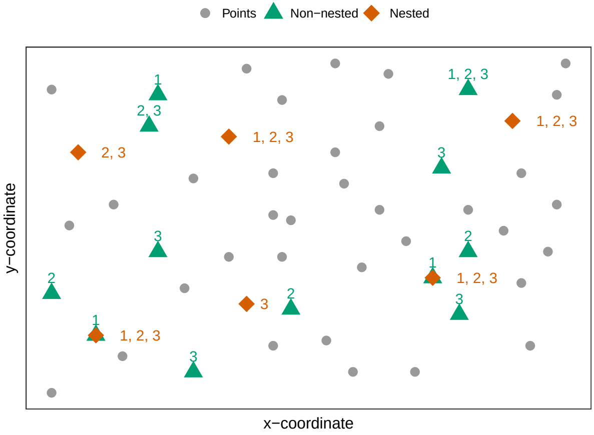
The nesting property

Definition

- Given
  - a set of locations V
  - a set of time periods $\mathcal{H} = \{1, \ldots, H\}$ and
  - a set of integers $\mathcal{P} = \left\{p^1, \ldots, p^H\right\}$ indicating the number of facilities to open
  - where $p^h \leq p^{h+1}$ for $h = 1, \ldots, H-1$
- the subsets $V^h \subseteq V$ with $|V^h| = p^h$
- are called nested iff $V^h \subset V^{h+1}$ for $h = 1, \ldots, H-1$

Introduced by Roodman and Schwarz 1975 and used in e.g., Albareda-Sambola et al. 2009; Bakker and Nickel 2024; Conforti et al. 2014 and reintroduced as nesting by McGarvey and Thorsen 2022

p-center problem vs nested p-center problem for $\mathcal{P} = \{4, 5, 6\}$



The nested p-center problem: Definition

Definition part I

- given a set of locations V,

The nested p-center problem: Definition

Definition part I

- given a set of locations V,
- time periods $\mathcal{H} = \{1, \ldots, H\}$,

The nested p-center problem: Definition

Definition part I

- given a set of locations V,
- time periods $\mathcal{H} = \{1, \dots, H\}$,
- integers $\mathcal{P} = \left\{p^1, \dots p^H\right\}$

The nested *p*-center problem: Definition

Definition part I

- given a set of locations V,
- time periods $\mathcal{H} = \{1, \dots, H\}$,
- integers $\mathcal{P} = \left\{p^1, \dots p^H\right\}$
  - where $p^h \leq p^{h+1}$ for $h = 1, \dots, H-1$ and
  - $p^H \leq |V|$

The nested p-center problem: Definition

Definition part I

- given a set of locations V,
- time periods $\mathcal{H} = \{1, \dots, H\}$,
- integers $\mathcal{P} = \left\{p^1, \dots p^H\right\}$
  - where $p^h \leq p^{h+1}$ for $h = 1, \dots, H-1$ and
  - $p^H \leq |V|$
- distances $d_{ij} \geq 0$ between each $i \in V$ and $j \in V$

The nested p-center problem: Definition

Definition part I

- given a set of locations V,
- time periods $\mathcal{H} = \{1, \dots, H\}$,
- integers $\mathcal{P} = \left\{p^1, \dots p^H\right\}$
  - where $p^h \leq p^{h+1}$ for $h = 1, \dots, H-1$ and
  - $p^H \leq |V|$
- distances $d_{ij} \geq 0$ between each $i \in V$ and $j \in V$

Definition part II

- a feasible solution to the nested p-center problem consists of a set $\mathcal{J}^h \subseteq V$

The nested p-center problem: Definition

Definition part I

- given a set of locations V,
- time periods $\mathcal{H} = \{1, \dots, H\}$,
- integers $\mathcal{P} = \left\{p^1, \dots p^H\right\}$
  - where $p^h \leq p^{h+1}$ for $h = 1, \dots, H-1$ and
  - $p^H \leq |V|$
- distances $d_{ij} \geq 0$ between each $i \in V$ and $j \in V$

Definition part II

- a feasible solution to the nested p-center problem consists of a set $\mathcal{J}^h \subseteq V$
  - with $\left|\mathcal{J}^h\right| = p^h$ for $h \in \mathcal{H}$,
  - for which $\mathcal{J}^h \subseteq \mathcal{J}^{h+1}$ for $h = 1, \dots, H-1$ holds

The nested p-center problem: Definition

Definition part I

- given a set of locations V,
- time periods $\mathcal{H} = \{1, \dots, H\}$,
- integers $\mathcal{P} = \left\{p^1, \dots p^H\right\}$
  - where $p^h \leq p^{h+1}$ for $h = 1, \dots, H-1$ and
  - $p^H \leq |V|$
- distances $d_{ij} \geq 0$ between each $i \in V$ and $j \in V$

Definition part II

- a feasible solution to the nested p-center problem consists of a set $\mathcal{J}^h \subseteq V$
  - with $\left|\mathcal{J}^h\right| = p^h$ for $h \in \mathcal{H}$,
  - for which $\mathcal{J}^h \subseteq \mathcal{J}^{h+1}$ for $h = 1, \dots, H-1$ holds
- the goal is to find a feasible solution which minimizes $\sum_{h=1}^{H} d_h(\mathcal{J}^h)$,
  - where $d_h(\mathcal{J}) = \max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}} d_{ij}$ for $h \in \mathcal{H}$

The nested p-center problem: Potential applications

- variant of the p-center problem

The nested p-center problem: Potential applications

- variant of the p-center problem
- part of the multi-period facility locations

The nested *p*-center problem: Potential applications

- variant of the *p*-center problem
- part of the multi-period facility locations
- emergency medical services

The nested p-center problem: Potential applications

- variant of the p-center problem
- part of the multi-period facility locations
- emergency medical services
- relief actions in humanitarian crisis

The nested *p*-center problem: Potential applications

- variant of the *p*-center problem
- part of the multi-period facility locations
- emergency medical services
- relief actions in humanitarian crisis
- expansion into new markets/areas

Mixed Integer Linear Programming (MILP) Formulations

The nested p-center problem: MILP formulation

Decision variables

$$y_j^h \ldots \begin{cases} 1 \ldots \text{if location } j \text{ is open in} \\ \qquad \text{time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

The nested p-center problem: MILP formulation

Decision variables

$$y_j^h \ldots \begin{cases} 1 \ldots \text{if location } j \text{ is open in} \\ \qquad \text{time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

$z^h \ldots$ maximum distance between any customer i and its nearest open facility in period h

The nested p-center problem: MILP formulation

Decision variables

$$y_j^h \dots \begin{cases} 1 \dots \text{if location } j \text{ is open in} \\ \qquad \text{time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

$z^h \dots$ maximum distance between any customer i and its nearest open facility in period h

`(nPC2)` based on Gaar and Sinnl 2022

$$\begin{aligned} \min \quad & \sum_{h \in \mathcal{H}} z^h && && \text{(1a)} \\ \text{s.t.} \quad & \sum_{j \in V} y_j^h = p^h && \forall h \in \mathcal{H} && \text{(1b)} \\ & z^h \geq d_{ij} - \sum_{j' : d_{ij'} < d_{ij}} (d_{ij} - d_{ij'}) y_{j'}^h && \forall i, j \in V, h \in \mathcal{H} && \text{(1c)} \\ & y_j^h \geq y_j^{h-1} && \forall j \in V, \forall h \in \mathcal{H} \setminus \{1\} && \text{(1d)} \\ & (y, z) \in |V| \, |\mathcal{H}| \times \mathbb{R}_{\geq 0} && && \text{(1e)} \end{aligned}$$

The nested p-center problem: Compact MILP formulation

Decision variables

$$y_j^h \dots \begin{cases} 1 \dots \text{if location } j \text{ is open in} \\ \qquad \text{time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

The nested p-center problem: Compact MILP formulation

Decision variables

$$y_j^h \dots \begin{cases} 1 \dots \text{if location } j \text{ is open in time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

$z^h \dots$ maximum distance between any customer i and its nearest open facility in period h

The nested p-center problem: Compact MILP formulation

Decision variables

$$y_j^h \dots \begin{cases} 1 \dots \text{if location } j \text{ is open in} \\ \qquad \text{time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

$z^h \dots$ maximum distance between any customer i and its nearest open facility in period h

$$u_k^h \dots \begin{cases} 1 \dots \text{if } z^h \geq D_k \text{ in} \\ \qquad \text{time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

The nested p-center problem: Compact MILP formulation

Decision variables

$$y_j^h \dots \begin{cases} 1 \dots \text{if location } j \text{ is open in time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

$z^h \dots$ maximum distance between any customer i and its nearest open facility in period h

$$u_k^h \dots \begin{cases} 1 \dots \text{if } z^h \geq D_k \text{ in time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

(nPC3) based on Ales and Elloumi 2018

$$\begin{aligned}
\min \quad & \sum_{h \in \mathcal{H}} z^h \\
\text{s.t.} \quad & \sum_{j \in V} y_j^h = p^h && \forall h \in \mathcal{H} \\
& D_0 + \sum_{k=1}^{K} (D_k - D_{k-1}) u_k^h \leq z^h && \forall h \in \mathcal{H} \\
& u_k^h + \sum_{j: d_{ij} < D_k} y_j^h \geq 1 && \forall i \in V, \forall h \in \mathcal{H}, \forall k \in S_i \cup \{K\} \\
& u_k^h \geq u_{k+1}^h && \forall h \in \mathcal{H}, \forall k \in \mathcal{K} \setminus \{K\} \\
& y_j^h \geq y_j^{h-1} && \forall h \in \mathcal{H} \setminus \{1\}
\end{aligned}$$

Comparing formulations

Table: Comparison of complexity of the formulations

	`(nPC1)`	`(nPC2)`	`(nPC3)`
Variables	$\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}(\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}((\lvert\mathcal{I}\rvert + \lvert\mathcal{K}\rvert)\lvert\mathcal{H}\rvert)$
Constraints	$\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}(\min(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert, \lvert\mathcal{I}\rvert\lvert\mathcal{K}\rvert)\lvert\mathcal{H}\rvert)$

Comparing formulations

Table: Comparison of complexity of the formulations

	(nPC1)	(nPC2)	(nPC3)
Variables	$\mathcal{O}(\vert\mathcal{I}\vert\vert\mathcal{J}\vert\vert\mathcal{H}\vert)$	$\mathcal{O}(\vert\mathcal{J}\vert\vert\mathcal{H}\vert)$	$\mathcal{O}((\vert\mathcal{I}\vert + \vert\mathcal{K}\vert)\vert\mathcal{H}\vert)$
Constraints	$\mathcal{O}(\vert\mathcal{I}\vert\vert\mathcal{J}\vert\vert\mathcal{H}\vert)$	$\mathcal{O}(\vert\mathcal{I}\vert\vert\mathcal{J}\vert\vert\mathcal{H}\vert)$	$\mathcal{O}(\min(\vert\mathcal{I}\vert\vert\mathcal{J}\vert, \vert\mathcal{I}\vert\vert\mathcal{K}\vert)\vert\mathcal{H}\vert)$

The *p*CP version of formulation `(nPC3)` has the best known linear programming *LP*-bounds for the *p*CP, while the *p*CP versions of `(nPC1)` and `(nPC2)` have equally but worse *LP*-bounds.

Improving the formulations

Strengthening constraints `(nPC2)`

Strengthening constraints (nPC2)

Lemma 1

Let LB_h be a lower bound on the decision variable z_h of (nPC2) for every $i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H}$ then

Strengthening constraints (nPC2)

Lemma 1

Let LB_h be a lower bound on the decision variable z_h of (nPC2) for every $i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H}$ then

$$z^h \geq max\{LB^h, d_{ij}\} - \sum_{j' : d_{ij'} < d_{ij}} \left(\max\{LB^h, d_{ij}\} - \max\{LB^h, d_{ij'}\}\right) y_{j'}^h \qquad \text{(nL-OPT)}$$

Strengthening constraints (nPC2)

Lemma 1

Let LB_h be a lower bound on the decision variable z_h of (nPC2) for every $i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H}$ then

$$z^h \geq max\{LB^h, d_{ij}\} - \sum_{j':d_{ij'}<d_{ij}} \left(\max\{LB^h, d_{ij}\} - \max\{LB^h, d_{ij'}\}\right) y^h_{j'} \tag{nL-OPT}$$

is a valid inequality for (nPC2), i.e., every feasible solution of (nPC2) fulfills (nL-OPT). The theorem is based on Lemma 5 in Gaar and Sinnl (2022).

Reducing set $\mathcal{K}$ in (nPC3)

Reducing set $\mathcal{K}$ in `(nPC3)`

Lemma 2

Let $\underline{z^h}$ be a valid lower bound and $\overline{z^h}$ be a valid upper bound on the decision variable z^h for $h \in \mathcal{H}$, then the distinct distance D_k can only be the optimal distance for z^h if $\underline{z^h} \leq D^k \leq \overline{z^h}$ holds.

Reducing set $\mathcal{K}$ in `(nPC3)`

Lemma 2

Let $\underline{z}^h$ be a valid lower bound and $\overline{z}^h$ be a valid upper bound on the decision variable z^h for $h \in \mathcal{H}$, then the distinct distance D_k can only be the optimal distance for z^h if $\underline{z}^h \leq D^k \leq \overline{z}^h$ holds.
Therfore let set $S_i^h \subseteq S_i$ for $h \in \mathcal{H}$, where S_i^h contains only the indices k where $\underline{z}^h \leq D^k \leq \overline{z}^h$ holds.

Depending on the bounds $\underline{z}^h$ and $\overline{z}^h$ the sets S_i^h can be much smaller than S_i.

Obtaining bounds

Obtaining bounds

Lemma 3

Let z'^{h*} *be the optimal objective function value of pCP with* $p = p^h$ *for* $h \in \mathcal{H} = \{1, 2, \dots, H\}$ *where* $p^h > p^{h+1}$,

Obtaining bounds

Lemma 3

Let z'^{h*} be the optimal objective function value of pCP with $p = p^h$ for $h \in \mathcal{H} = \{1, 2, \ldots, H\}$ where $p^h > p^{h+1}$, then $UB = Hz'^{1*}$ is a valid upper bound on the optimal objective function value of n-pCP.

Obtaining bounds

Lemma 3

Let z'^{h*} be the optimal objective function value of pCP with $p = p^h$ for $h \in \mathcal{H} = \{1, 2, \ldots, H\}$ where $p^h > p^{h+1}$, then $UB = Hz'^{1*}$ is a valid upper bound on the optimal objective function value of n-pCP.
Then

$$\overline{z^h} = \frac{UB - \sum_{x=h+1}^{H} z'^{x*}}{h}$$

Obtaining bounds

Lemma 3

Let z'^{h*} *be the optimal objective function value of pCP with* $p = p^h$ *for* $h \in \mathcal{H} = \{1, 2, \ldots, H\}$ *where* $p^h > p^{h+1}$*, then* $UB = Hz'^{1*}$ *is a valid upper bound on the optimal objective function value of n-pCP.*
Then

$$\overline{z^h} = \frac{UB - \sum_{x=h+1}^{H} z'^{x*}}{h}$$

where $\overline{z^h}$ *is a valid upper bound on the decision variable* z^h *of the n-pCP for* $h \in \mathcal{H}$*.*

Implementation

Implementation

- C++, CPLEX 20.1

Implementation

- C++, CPLEX 20.1
- preprocessing

Implementation

- C++, CPLEX 20.1
- preprocessing
  - Solving p-center problem with $p = p^H$
  - Use optimal solution value for p-center problem with $p = p^{H-1}$
  - Repeat for remaining p^h
- `(nPC2)`

Implementation

- C++, CPLEX 20.1
- preprocessing
  - Solving p-center problem with $p = p^H$
  - Use optimal solution value for p-center problem with $p = p^{H-1}$
  - Repeat for remaining p^h
- `(nPC2)`
  - branch-and-cut algorithm
  - separation similar to fixedCustomer from Gaar and Sinnl 2022

Implementation

- C++, CPLEX 20.1
- preprocessing
  - Solving p-center problem with $p = p^H$
  - Use optimal solution value for p-center problem with $p = p^{H-1}$
  - Repeat for remaining p^h
- `(nPC2)`
  - branch-and-cut algorithm
  - separation similar to fixedCustomer from Gaar and Sinnl 2022
- `(nPC3)`

Implementation

- C++, CPLEX 20.1
- preprocessing
  - Solving p-center problem with $p = p^H$
  - Use optimal solution value for p-center problem with $p = p^{H-1}$
  - Repeat for remaining p^h
- (nPC2)
  - branch-and-cut algorithm
  - separation similar to fixedCustomer from Gaar and Sinnl 2022
- (nPC3)
  - branch-and-bound algorithm
  - starting with the reduced set $\mathcal{K}$

Settings

- noPP: without preprocessing or any lifting

Settings

- `noPP`: without preprocessing or any lifting
- `PP`: with preprocessing

Settings

- `noPP`: without preprocessing or any lifting
- `PP`: with preprocessing
  - `(nPC2)`: set upper bounds on z^h and using lower bounds in (nL-OPT)
  - `(nPC3)`: problem initialized on reduced set $\mathcal{K}$

Settings

- `noPP`: without preprocessing or any lifting
- `PP`: with preprocessing
  - `(nPC2)`: set upper bounds on z^h and using lower bounds in (nL-OPT)
  - `(nPC3)`: problem initialized on reduced set $\mathcal{K}$
- `PH`: `PP` with starting heuristic

Results

Instance from literature

- computational setup
  - single core of Intel Xeon X5570 with 2.93 GHz and 8 GB RAM
  - timelimit of 3600 seconds

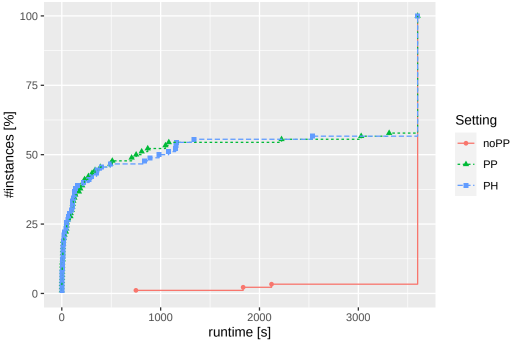
Instance from literature

- computational setup
  - single core of Intel Xeon X5570 with 2.93 GHz and 8 GB RAM
  - timelimit of 3600 seconds
- instance set PMED Çalık and Tansel 2013; D. Chen and R. Chen 2009; Contardo, Iori, and Kramer 2019
  - 40 instances
  - $\mathcal{P} = \{p, p+1, p+2\}$, p from 5 to 200, $\left|V\right|$ from 100 and 900 nodes

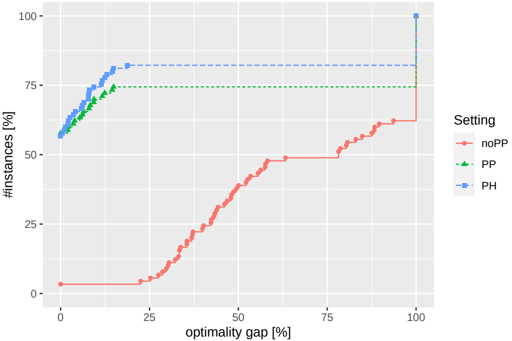
Instance from literature

- computational setup
  - single core of Intel Xeon X5570 with 2.93 GHz and 8 GB RAM
  - timelimit of 3600 seconds
- instance set PMED Çalık and Tansel 2013; D. Chen and R. Chen 2009; Contardo, Iori, and Kramer 2019
  - 40 instances
  - $\mathcal{P} = \{p, p+1, p+2\}$, p from 5 to 200, $|V|$ from 100 and 900 nodes
- instance set TSPLIB Contardo, Iori, and Kramer 2019; Gaar and Sinnl 2022
  - 80 instances
  - $\mathcal{P} = \{4, 5, 6\}$, $|V|$ from 51 and 1002 nodes

Setting comparison on formulation (nPC1)

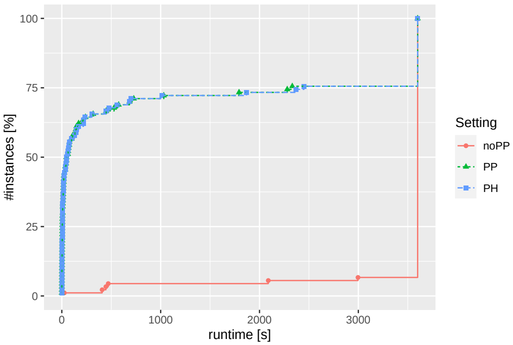


(a) runtime

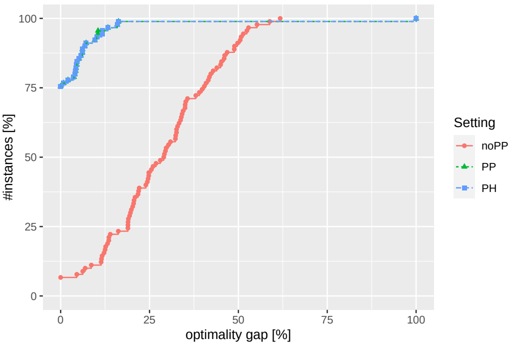


(b) optimality gap [$\frac{UB-LB}{UB}$]

Setting comparison on formulation (nPC2)

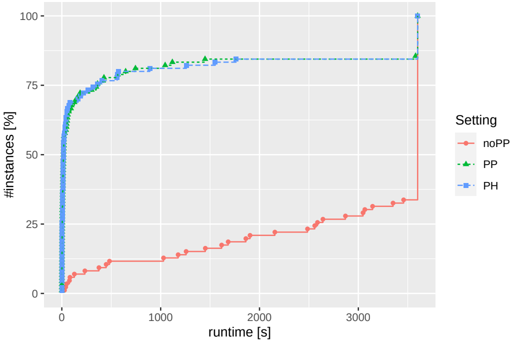


(c) runtime

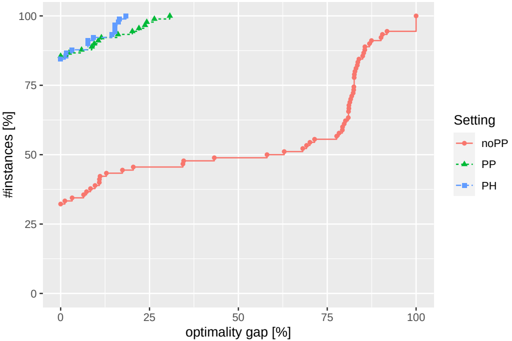


(d) optimality gap

Setting comparison on formulation (nPC3)

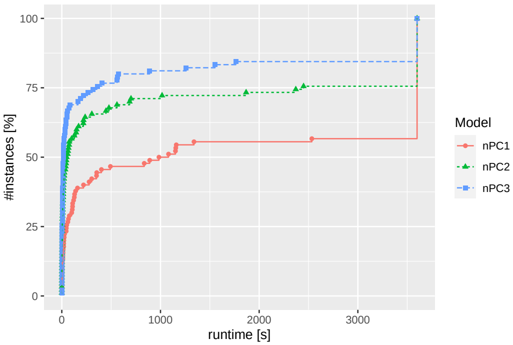


(e) runtime

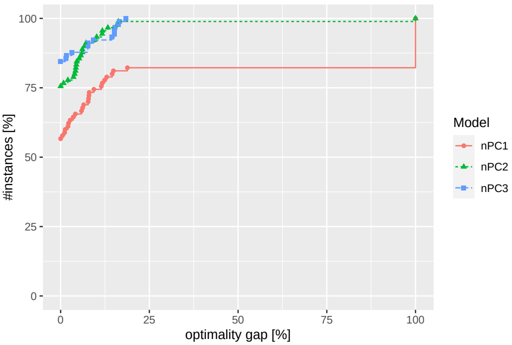


(f) optimality gap

Formulation comparison on setting PH

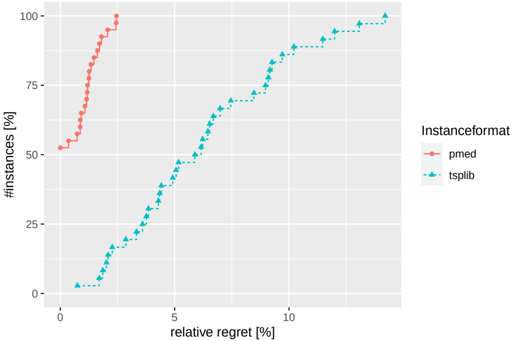


(g) runtime



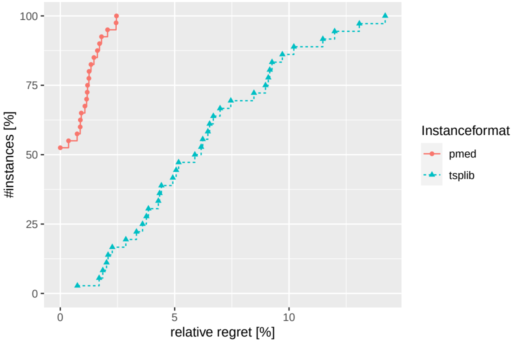
(h) optimality gap

Managerial insights

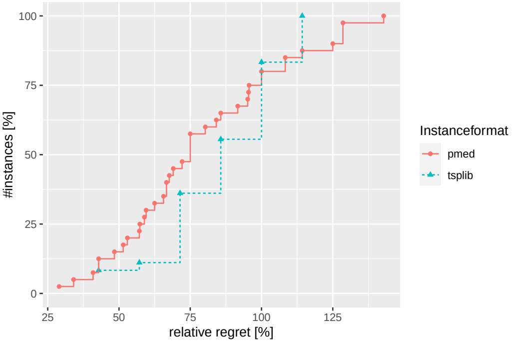


(i) Relative regret of the optimal solution value

Managerial insights



(k) Relative regret of the optimal solution value



(l) Relative regret of # of opened facilities

Conclusion

- `(nPC3)` best performance in runtimes and optimality gap
- The `(nPC2)` outperforms the `(nPC1)`
- Preprocessing brings a large speed up on all formulations
- Starting heuristic little effect, shows good upper bound obtained in preprocessing
- Maximal relative regret of the optimal objective function value of 15%
- Maximal relative regret of number of opened facilities above 150%

On the nested p-center problem

Christof Brandstetter, Markus Sinnl
Institute of Business Analytics and Technology Transformation / JKU Business School,
Johannes Kepler University Linz
2024-03-03

JOHANNES KEPLER
UNIVERSITY LINZ
Altenberger Straße 69
4040 Linz, Austria
jku.at

References I

[1] M. Albareda-Sambola et al. The multi-period incremental service facility location problem. In: Computers & Operations Research 36.5 (2009), pp. 1356–1375.

[2] Z. Ales and S. Elloumi. Compact MILP formulations for the p-center problem. In: Combinatorial Optimization: 5th International Symposium, ISCO 2018, Marrakesh, Morocco, April 11–13, 2018, Revised Selected Papers 5. 2018, pp. 14–25.

[3] H. Bakker and S. Nickel. The Value of the Multi-period Solution revisited: When to model time in capacitated location problems. In: Computers & Operations Research 161 (2024), p. 106428.

References II

[4] H. Çalık and B. C. Tansel. Double bound method for solving the p-center location problem. In: Computers & Operations Research 40.12 (2013), pp. 2991–2999.

[5] D. Chen and R. Chen. New relaxation-based algorithms for the optimal solution of the continuous and discrete p-center problems. In: Computers & Operations Research 36.5 (2009), pp. 1646–1655.

[6] M. Conforti et al. Integer programming models. Springer, 2014.

[7] C. Contardo, M. Iori, and R. Kramer. A scalable exact algorithm for the vertex p-center problem. In: Computers & Operations Research 103 (2019), pp. 211–220.

References III

[8] M. S. Daskin. Network and Discrete Location: Models, Algorithms, and Applications, Second Edition. John Wiley & Sons, Ltd, 2013.

[9] E. Gaar and M. Sinnl. A scaleable projection-based branch-and-cut algorithm for the p-center problem. In: European Journal of Operational Research 303.1 (2022), pp. 78–98.

[10] R. G. McGarvey and A. Thorsen. Nested-solution facility location models. In: Optimization letters 16.2 (2022), pp. 497–514.

[11] G. M. Roodman and L. B. Schwarz. Optimal and heuristic facility phase-out strategies. In: AIIE transactions 7.2 (1975), pp. 177–184.

The nested p-center problem: Classical MILP formulation

Decision variables

$$x_{ij}^h \ldots \begin{cases} 1 \ldots \text{if location } i \text{ is assigned to} \\ \qquad \text{location } j \text{ in time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

The nested p-center problem: Classical MILP formulation

Decision variables

$$x_{ij}^h \dots \begin{cases} 1 \dots \text{if location } i \text{ is assigned to} \\ \qquad \text{location } j \text{ in time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

$$y_j^h \dots \begin{cases} 1 \dots \text{if location } j \text{ is open in} \\ \qquad \text{time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

The nested p-center problem: Classical MILP formulation

Decision variables

$$x_{ij}^h \dots \begin{cases} 1 \dots \text{if location } i \text{ is assigned to} \\ \qquad \text{location } j \text{ in time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

$$y_j^h \dots \begin{cases} 1 \dots \text{if location } j \text{ is open in} \\ \qquad \text{time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

$z^h \dots$ maximum distance between any customer i and its nearest open facility in period h

The nested p-center problem: Classical MILP formulation

Decision variables

$$x_{ij}^h \dots \begin{cases} 1 \dots \text{if location } i \text{ is assigned to location } j \text{ in time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

$$y_j^h \dots \begin{cases} 1 \dots \text{if location } j \text{ is open in time period } h \\ 0 \dots \text{otherwise} \end{cases}$$

$z^h \dots$ maximum distance between any customer i and its nearest open facility in period h

(nPC1) based on Daskin 2013

$$\min \sum_{h \in \mathcal{H}} z^h \tag{3a}$$

$$\text{s.t.} \quad \sum_{j \in V} y_j^h = p^h \qquad \forall h \in \mathcal{H} \tag{3b}$$

$$\sum_{j \in V} x_{ij}^h = 1 \qquad \forall i \in V, h \in \mathcal{H} \tag{3c}$$

$$x_{ij}^h \leq y_j^h \qquad \forall i, j \in V, h \in \mathcal{H} \tag{3d}$$

$$\sum_{j \in V} d_{ij} x_{ij}^h \leq z^h \qquad \forall i \in V, h \in \mathcal{H} \tag{3e}$$

$$y_j^h \geq y_j^{h-1} \qquad \forall h \in \mathcal{H} \setminus \{1\} \tag{3f}$$

$$(x, y, z) \in |V|^2|\mathcal{H}| \times |V||\mathcal{H}| \times \mathbb{R}_{\geq 0} \tag{3g}$$