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On the nested *p*-center problem

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The p-center problem pCP: Definition

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  - the maximum distance of any location to its closest opened location is minimized.

The nesting property

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  - where $p^h \leq p^{h+1}$ for $h = 1, \dots, H-1$

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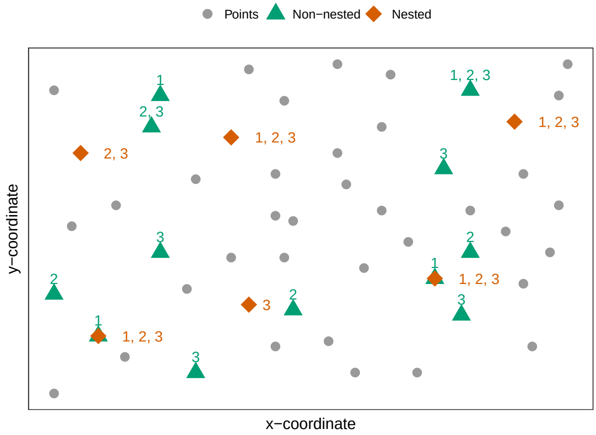
Introduced by Roodman and Schwarz 1975 and used in e.g., Albareda-Sambola et al. 2009; Bakker and Nickel 2024; Conforti et al. 2014 and reintroduced as nesting by McGarvey and Thorsen 2022

p-center problem vs nested p-center problem for $\mathcal{P} = \{4, 5, 6\}$



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  - for which $\mathcal{J}^h \subseteq \mathcal{J}^{h+1}$ for $h = 1, \dots, H-1$ holds
- the goal is to find a feasible solution which minimizes $\sum_{h=1}^{H} d_h(\mathcal{J}^h)$,
  - where $d_h(\mathcal{J}) = \max_{i\in\mathcal{I}} \min_{j\in\mathcal{J}} d_{ij}$ for $h \in \mathcal{H}$

Mixed Integer Linear Programming (MILP) Formulations

The nested p-center problem: MILP formulation

Decision variables

$$y_j^h \ldots \begin{cases} 1 \ldots \text{if location } j \text{ is open in} \\ \qquad \text{time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

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`(nPC2)` based on Gaar and Sinnl 2022

$$\min \quad \sum_{h \in \mathcal{H}} z^h \tag{1a}$$

$$\text{s.t.} \quad \sum_{j \in V} y_j^h = p^h \qquad \forall h \in \mathcal{H} \tag{1b}$$

$$z^h \geq d_{ij} - \sum_{j' : d_{ij'} < d_{ij}} (d_{ij} - d_{ij'}) y_{j'}^h \qquad \forall i, j \in V, h \in \mathcal{H} \tag{1c}$$

$$y_j^h \geq y_j^{h-1} \qquad \forall j \in V, \forall h \in \mathcal{H} \setminus \{1\} \tag{1d}$$

$$(y, z) \in \left|V\right| \left|\mathcal{H}\right| \times \mathbb{R}_{\geq 0} \tag{1e}$$

The nested p-center problem: Compact MILP formulation

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(nPC3) based on Ales and Elloumi 2018

$$
\begin{aligned}
\min \quad & \sum_{h \in \mathcal{H}} z^h \\
\text{s.t.} \quad & \sum_{j \in V} y_j^h = p^h && \forall h \in \mathcal{H} \\
& D_0 + \sum_{k=1}^{K} (D_k - D_{k-1}) u_k^h \leq z^h && \forall h \in \mathcal{H} \\
& u_k^h + \sum_{j : d_{ij} < D_k} y_j^h \geq 1 && \forall i \in V, \forall h \in \mathcal{H}, \forall k \in S_i \cup \{K\} \\
& u_k^h \geq u_{k+1}^h && \forall h \in \mathcal{H}, \forall k \in \mathcal{K} \setminus \{K\} \\
& y_j^h \geq y_j^{h-1} && \forall h \in \mathcal{H} \setminus \{1\} \\
& (u, y, z) \in \quad |V| \;\; |K| \times |V| \, |\mathcal{H}| \times \mathbb{R}_{\geq 0}
\end{aligned}
$$

Comparing formulations

Table: Comparison of complexity of the formulations

	(nPC1)	(nPC2)	(nPC3)
Variables	$\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}(\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}((\lvert\mathcal{I}\rvert + \lvert\mathcal{K}\rvert)\lvert\mathcal{H}\rvert)$
Constraints	$\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}(\min(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert, \lvert\mathcal{I}\rvert\lvert\mathcal{K}\rvert)\lvert\mathcal{H}\rvert)$

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The *p*CP version of formulation `(nPC3)` has the best known linear programming *LP*-bounds for the *p*CP, while the *p*CP versions of `(nPC1)` and `(nPC2)` have equally but worse *LP*-bounds.

Improving the formulations

Strengthening constraints `(nPC2)`

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Lemma 1

Let LB_h be a lower bound on the decision variable z_h of (nPC2) for every $i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H}$ then constraints

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can be replaced by

$$z^h \geq max\{LB^h, d_{ij}\} - \sum_{j':d_{ij'}<d_{ij}} \left(\max\{LB^h, d_{ij}\} - \max\{LB^h, d_{ij'}\} \right) y_{j'}^h \qquad \text{(nL-OPT)}$$

The theorem is based on Lemma 5 in Gaar and Sinnl (2022).

Reducing variables u_k^h in `(nPC3)`

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Observation 1

Let $\underline{z^h}$ be a valid lower bound and $\overline{z^h}$ be a valid upper bound on the decision variable z^h for $h \in \mathcal{H}$, then the distinct distance D_k can only be the optimal distance for z^h if $\underline{z^h} \leq D^k \leq \overline{z^h}$ holds.

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Observation 2

Following Observation 1, we observe that decision variables $u_h^k = 0$ for $k : D_k > \overline{z^h}$ and $u_h^k = 1$ for $k : D_k < \underline{z^h}$ in any optimal solution.

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Proposition 1

The decision variables u_k^h which are zero or one following Observation 2, are redundant.

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Observation 3

For $\mathcal{H} = \{1\}$ the n-pCP reduces to the pCP where $p = p^1$, so the optimal objective value (z'^{h*}) of the pCP, where $p = p^h$ is a lower bound $\underline{z}^h$ on the decision variable z^h of the n-pCP.

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Proposition 2

Given an valid upper bound UB on the objective value of the n-pCP and valid lower bounds $\underline{z^h}$ on the variable z^h can be obtained by the following equation:

$$\overline{z^h} = \frac{UB - \sum_{x=h+1}^{H} \underline{z^h}}{h} \tag{3}$$

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- `PH`: `PP` with starting heuristic

Results

Instance from literature

- instance set PMED
  - 40 instances
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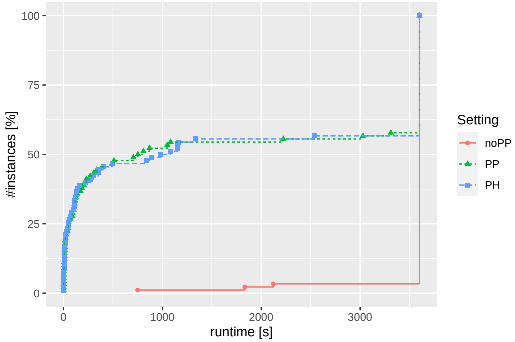
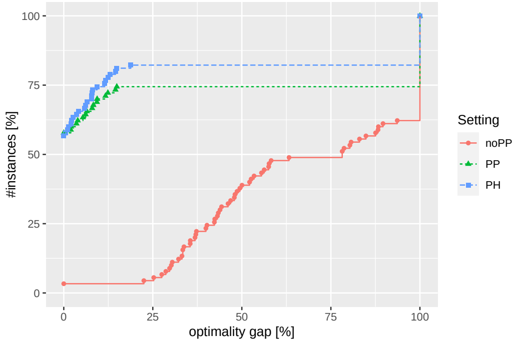
  - $\mathcal{P} = \{4, 5, 6\}$, $|V|$ from 51 and 1002 nodes
- computational setup
  - single core of Intel Xeon X5570 with 2.93 GHz and 8 GB RAM
  - timelimit of 3600 seconds

Setting comparison on formulation (`nPC1`)

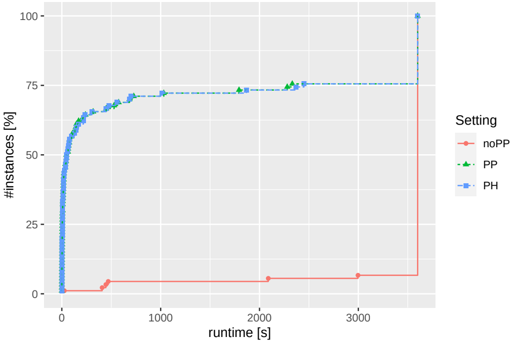


(a) runtime

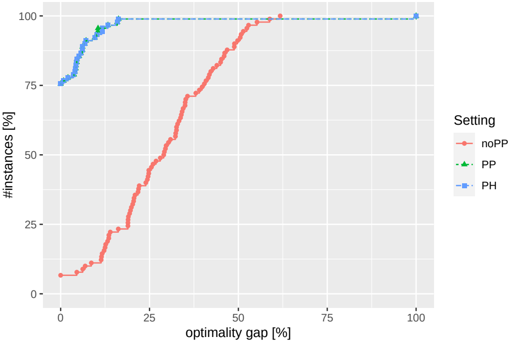


(b) optimality gap $[\frac{UB-LB}{UB}]$

Setting comparison on formulation (nPC2)

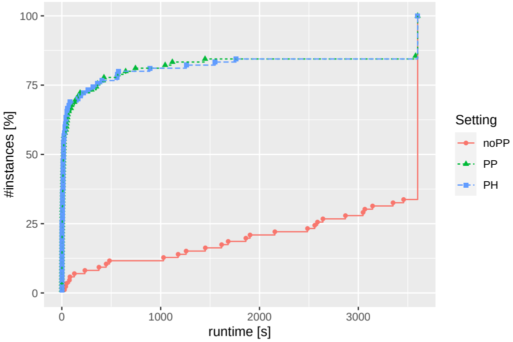


(c) runtime

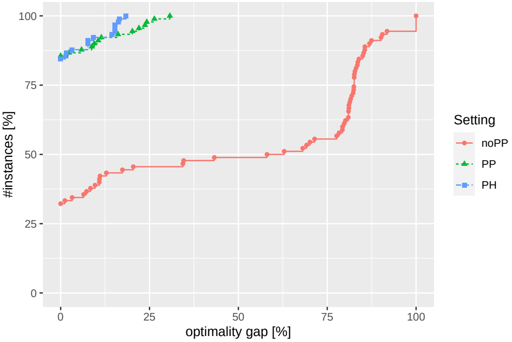


(d) optimality gap

Setting comparison on formulation (nPC3)

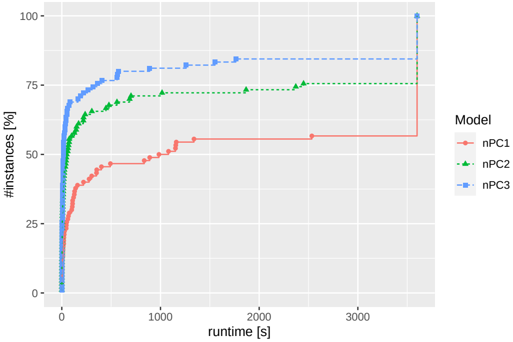


(e) runtime

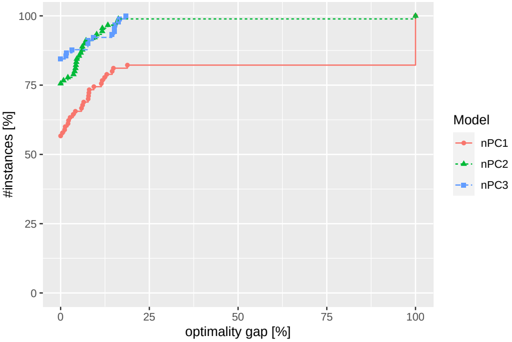


(f) optimality gap

Formulation comparison on setting PH

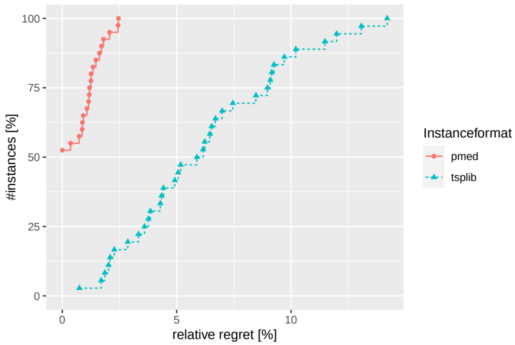


(g) runtime



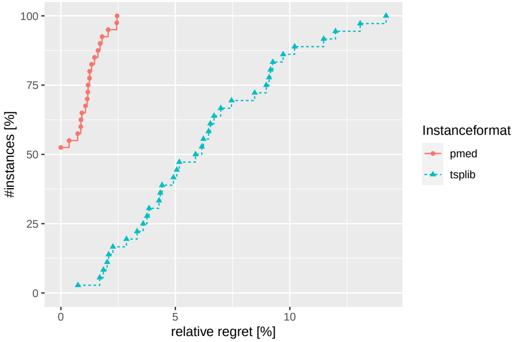
(h) optimality gap

Managerial insights

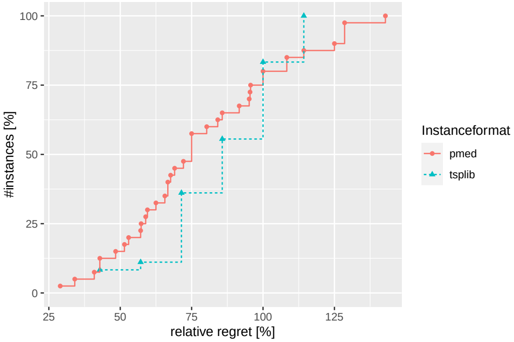


(i) Relative regret of the optimal solution value

Managerial insights



(k) Relative regret of the optimal solution value



(l) Relative regret of # of opened facilities

Conclusion

- `(nPC3)` best performance in runtimes and optimality gap
- The `(nPC2)` outperforms the `(nPC1)`
- Preprocessing brings a large speed up on all formulations
- Starting heuristic little effect, shows good upper bound obtained in preprocessing
- Maximal relative regret of the optimal objective function value of 15%
- Maximal relative regret of number of opened facilities above 150%

On the nested p-center problem

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The nested p-center problem: Classical MILP formulation

Decision variables

$$x_{ij}^h \ldots \begin{cases} 1 \ldots \text{if location } i \text{ is assigned to} \\ \qquad \text{location } j \text{ in time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

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(nPC1) based on Daskin 2013

$$\min \sum_{h \in \mathcal{H}} z^h \tag{4a}$$

$$\text{s.t.} \quad \sum_{j \in V} y_j^h = p^h \qquad \forall h \in \mathcal{H} \tag{4b}$$

$$\sum_{j \in V} x_{ij}^h = 1 \qquad \forall i \in V, h \in \mathcal{H} \tag{4c}$$

$$x_{ij}^h \leq y_j^h \qquad \forall i, j \in V, h \in \mathcal{H} \tag{4d}$$

$$\sum_{j \in V} d_{ij} x_{ij}^h \leq z^h \qquad \forall i \in V, h \in \mathcal{H} \tag{4e}$$

$$y_j^h \geq y_j^{h-1} \qquad \forall h \in \mathcal{H} \setminus \{1\} \tag{4f}$$

$$(x, y, z) \in |V|^2|\mathcal{H}| \times |V||\mathcal{H}| \times \mathbb{R}_{\geq 0} \tag{4g}$$