On the nested p-center problem

Master Thesis Seminar

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2024-01-12

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Introduction

- *p*-center problem: Open p facilities such that the maximum distance between any customer and its nearest open facility is minimized
- Related to the set cover problem and the assignment problem
- Nesting: Open additional facilities subsequently
- Nested *p*-center problem: Open p_h facilities in period h such that the sum of the maximum distances between any customer and its nearest open facility in period h is minimized.
- Use case: Ambulance/logistics stations, network design, screening/vaccination stations
- Optimizes expansion or retraction path

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Definition of the nested p-center problem

Definition:

- given a set of customer demand points $\mathcal{I}$, potential facility locations $\mathcal{J}$, time periods $\mathcal{H} = \{1, \ldots, H\}$, integers $\mathcal{P} = \{p^1, \ldots p^H\}$ where $p^h \leq p^{h+1}$ for $h = 1, \ldots, H-1$ and distances $d_{ij} \geq 0$ between each $i \in \mathcal{I}$ and $j \in \mathcal{J}$

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- open p^h-facilities at different locations $j \in \mathcal{J}$
- the sum of maximum distance of any customer demand i point to its closest opened facility in time period h is minimized
- and facilities open in period h has to be open in period $h + 1$ as well for $h = 1, \ldots, H-1$

Related work

p-center problem / TODO

- First introduction of the p-center problem by Hakimi (1964)
- The standard textbook formulation of the p-center problem can be found in Daskin (2013)
- Solution approach based on the set cover problem by Contardo, Iori, and Kramer (2018)
- A compact formulation by Ales and Elloumi (2018) and Elloumi, Labbé, and Pochet (2004)
- A fast projection-based branch-and-cut algorithm by Gaar and Sinnl (2022)

Nested facility location problems

- First introduction of the nesting property and constraint by Roodman and Schwarz (1975)
- Extension of the nesting to a phase-in and phase-out by Roodman and Schwarz (1977)
- Reintroduction of the nesting on the example of the p-median problem by McGarvey and Thorsen (2022)
- Recent work on multi-period facility locations problems mainly focus on varying demand, distances, or cost over time f.e. Calogiuri et al. (2021)

Mixed Integer Linear Programming formulations

First MILP formulation

$\mathcal{I}$ ... set of customer demand points with $i \in \mathcal{I}$
$\mathcal{J}$ ... set of potential facility locations with $j \in \mathcal{J}$
$\mathcal{H}$ ... set of time periods, $\mathcal{H} = \{1, 2, ..., H\}$
p_h = number of facilities to be opened in time period $h \in \mathcal{H}$ with $p_h > p_{h-1}$

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p_h = number of facilities to be opened in time period $h \in \mathcal{H}$ with $p_h > p_{h-1}$

d_{ij} = distance between customer demand point i and potential facility location j

$$x_{ijh} = \begin{cases} 1 \ldots \text{if customer i is assigned to facility j in time period h} \\ 0 \ldots \text{otherwise} \end{cases}$$

$$y_{jh} = \begin{cases} 1 \ldots \text{if a facility is opened at location j in time period h} \\ 0 \ldots \text{otherwise} \end{cases}$$

z_h = maximum distance between any customer i and its nearest open facility in period h

FirstMILPformulation$\mathcal{I}$ ... set of cusotmer demand points i

[–1*mm*]$\mathcal{J}$ ... set of potential facility locations j

$\mathcal{H}$ ... set of time periods,$\mathcal{H} = \{1, 2, \ldots, H\}$

$\mathcal{P}$ ... set of integers of facilities to open, $\mathcal{P} = \{p^1, \ldots, p^H\}$ where p
for $h = 1, \ldots, H - 1$

$d_{ij} =$ distance between customer demand point i and potential f

$$x_{ij}^h = \begin{cases} 1 \ldots \text{ if a customer demand point } i \text{ is assigned to facility} \\ 0 \ldots \text{ otherwise} \end{cases}$$

$$y_j^h = \begin{cases} 1 \ldots \text{ if a facility is opened at location } j \text{ in period } h \\ 0 \ldots \text{ otherwise} \end{cases}$$

z^h ... maximum distance between any customer i and its neares
in period h

First MILP formulation

$$
\begin{array}{llllr}
(nPC1) & \min & \sum_{h \in \mathcal{H}} z_h & & \text{(objective)} \\
& \text{s.t.} & \sum_{j \in \mathcal{J}} y_{jh} = p_h & \forall h \in \mathcal{H} & \text{(facility)} \\
& & \sum_{j \in \mathcal{J}} x_{ijh} = 1 & \forall i \in \mathcal{I}, h \in \mathcal{H} & \text{(assignment)} \\
& & x_{ijh} \leq y_j h & \forall i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H} & \text{(open)} \\
& & \sum_{j \in \mathcal{J}} d_{ij} x_{ijh} \leq z_h & \forall i \in \mathcal{I}, h \in \mathcal{H} & \text{(push)} \\
& & y_{jh} \geq y_{jh-1} & \forall h \in \mathcal{H} \setminus \{1\} & \text{(nesting)} \\
& & x_{ijh}, y_{jh} \in \{0, 1\} & \forall i \in \mathcal{I}, j \in \mathcal{J}, h \in H & \text{(binary)} \\
& & z_h \in \mathbb{R}_{\geq 0} & \forall h \in \mathcal{H} & \text{(non-negativity)}
\end{array}
$$

Second MILP formulation

$\mathcal{I}$ ... set of cusotmer demand points i

$\mathcal{J}$ ... set of potential facility locations j

$\mathcal{H}$ ... set of time periods, $\mathcal{H} = \{1, 2, \dots, H\}$

$\mathcal{P}$ ... set of integers of facilities to open, $\mathcal{P} = \{p^1, \dots, p^H\}$ where $p^h > p^{h+1}$ for $h = 1, \dots, H-1$

d_{ij} = distance between customer demand point i and potential facility location j

$$y_j^h = \begin{cases} 1 \dots \text{ if a facility is opened at location } j \text{ in period } h \\ 0 \dots \text{ otherwise} \end{cases}$$

z^h ... maximum distance between any customer i and its nearest open facility in period h

Second MILP formulation

$$
\begin{aligned}
(nPC2) \quad \min \quad & \sum_{h\in H} z_h & & \text{(objective-y)} \\
\text{s.t.} \quad & \sum_{j\in J} y_{jh} = p_h & \forall h \in H & \quad \text{(facility-y)} \\
& z_h \geq d_{ij} - \sum_{j':d_{ij'}<d_{ij}} (d_{ij} - d_{ij'}) y_{j'h} & \forall i \in I, j \in J, h \in H & \quad \text{(push-y)} \\
& y_h \geq y_{h-1} & \forall j \in J, h \in H \setminus \{1\} & \quad \text{(nesting-y)} \\
& y_{jh} \in \{0, 1\} & \forall j \in J, h \in H & \quad \text{(binary-y)} \\
& z_h \in \mathbb{R}_{\geq 0} & \forall h \in H & \quad \text{(non-negativity-y)}
\end{aligned}
$$

Third formualtion

Constraint lifting

Theorem [Brandstetter(2023)]

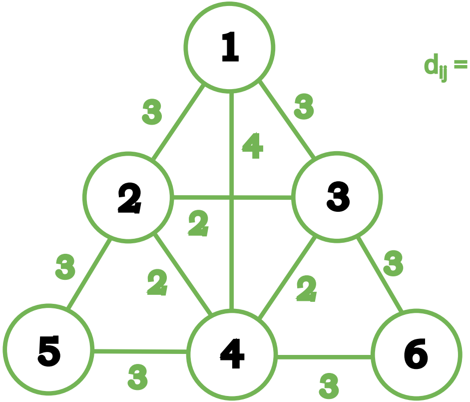
Let LB_h being a lower bound on the decision variable z_h of (nPC1) for every $i \in I, j \in J, h \in H$ then

$$\sum_{j \in J} \max\{LB_h, d_{ij}\} x_{ijh} \leq z_h \quad \forall i \in I, h \in H \tag{lift}$$

is a valid equality. Theorem is based on Gaar and Sinnl (2022).

- also valid for the y-formulation
- using various techniques to obtain LB_h

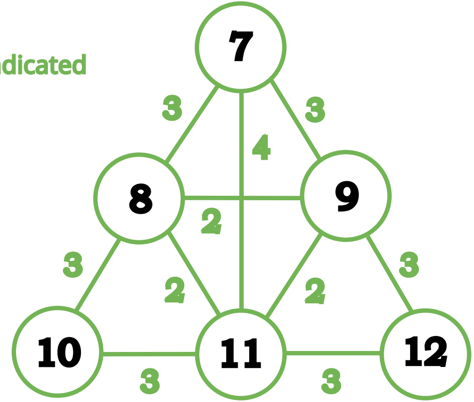
Constraint lifting / Delete



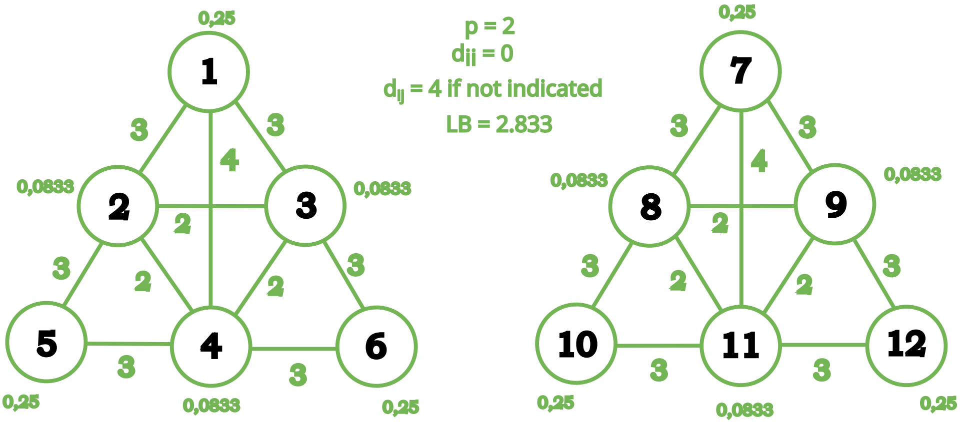
$p = 2$

$d_{ii} = 0$

$d_{ij} = 4$ if not indicated



Constraint lifting / Delete



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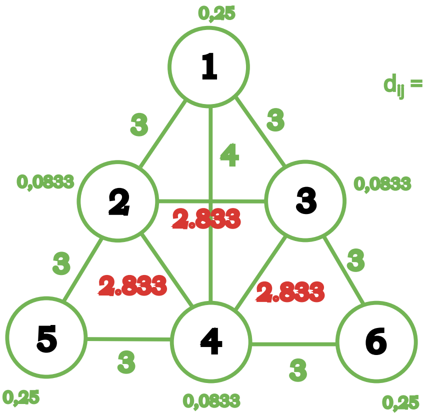
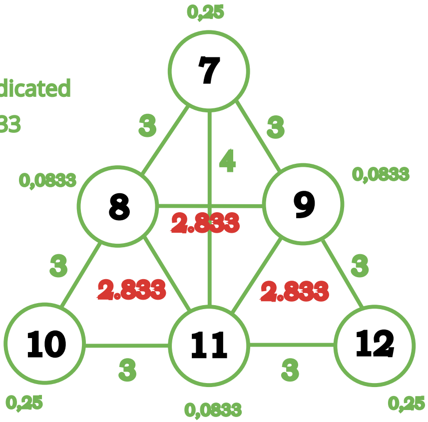
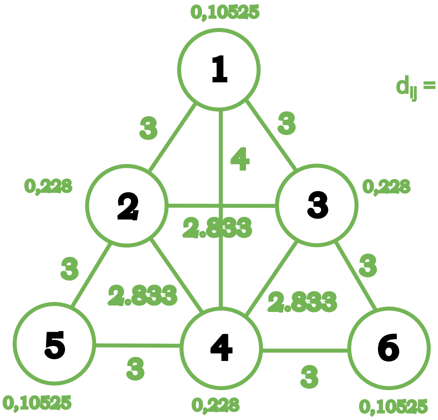


$p = 2$
$d_{ii} = 0$
$d_{ij} = 4$ if not indicated
LB = 2.833



Constraint lifting / Delete

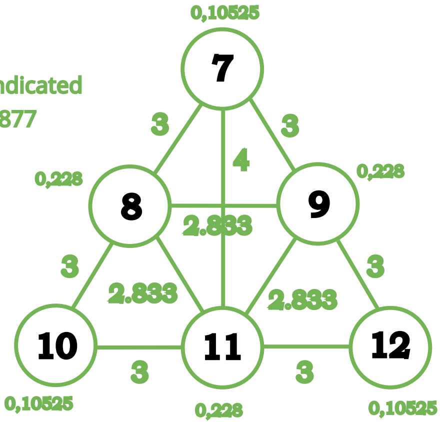


p = 2

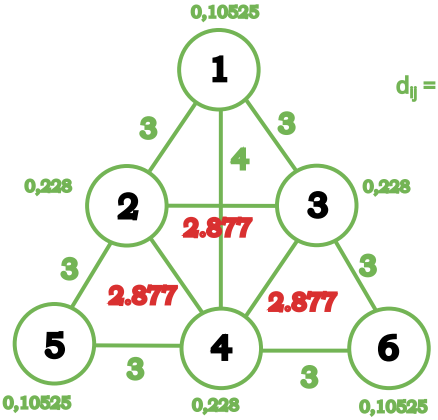
$d_{ii} = 0$

$d_{ij} = 4$ if not indicated

LB = 2.877



Constraint lifting / Delete



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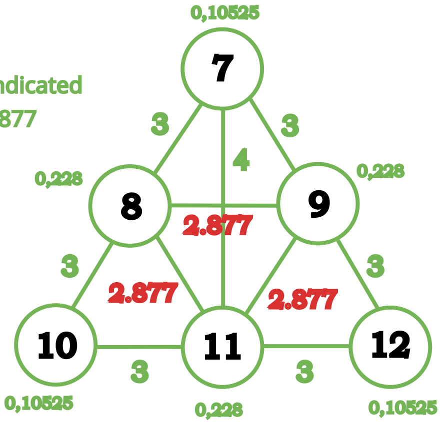
$d_{ii} = 0$

$d_{ij} = 4$ if not indicated

LB = 2.877



Branch-and-cut / Delete

- adds the cutting plane method

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- adding cut(s) $\longrightarrow$ branching $\longrightarrow$ adding cut(s) $\longrightarrow$ branching
- start with lower number of (push) constraints
- add (lift) inequalities
- allows for shorter solving times

Outline of the results / This should be the main topic for the final presentation

Data

- data set PMED from ”A note on solving large p-median problems” by Beasley (1985)
  - set of 40 instances
  - between 100 and 900 nodes

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  - set of 40 instances
  - between 100 and 900 nodes

- data set TSPLIB 2D-Euclidean distances from “TSPLIB — A Traveling Salesman Problem Library” by Reinelt (1991)
  - set of 80 instance
  - between 51 and 18512 nodes
  - rounded to the nearest integer value

Comparison

- comparing the results between (nPC1) and (nPC2)

Comparison

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- comparing the different lifting methods

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- analysing the solutions of the nested and comparing it to the standard p-center problem

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- comparing the results between (nPC1) and (nPC2)
- comparing the different lifting methods
- analysing the solutions of the nested and comparing it to the standard p-center problem
- analysing the results regarding managerial insights

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Further research topics

- u-space model introduced by Elloumi, Labbé, and Pochet (2004)
- maximal regret as objective function
- trying to improve the performance of the nested p-median problem?

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