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On the nested *p*-center problem

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2024-03-01

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The p-center problem: Definition

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  - a set V of locations,

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  - the maximum distance of any location to its closest opened location is minimized.

The nesting property

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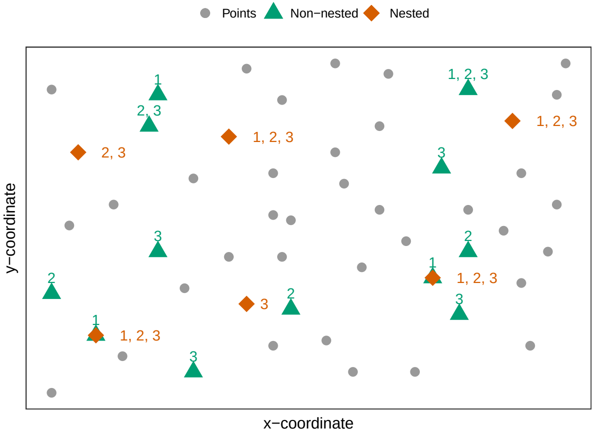
Introduced by Roodman and Schwarz 1975 and used in e.g., Albareda-Sambola et al. 2009; Bakker and Nickel 2024; Conforti et al. 2014 and reintroduced as nesting by McGarvey and Thorsen 2022

p-center problem vs nested p-center problem



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  - for which $\mathcal{J}^h \subseteq \mathcal{J}^{h+1}$ for $h = 1, \dots, H - 1$ holds
- the goal is to find a feasible solution which minimizes $\sum_{h=1}^{H} d_h(\mathcal{J}^h)$,
  - where $d_h(\mathcal{J}) = \max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}} d_{ij}$ for $h \in \mathcal{H}$

The nested *p*-center problem: Potential applications

- variant of the *p*-center problem
- part of the multi-period facility locations
- emergency medical services
- relief actions in humanitarian crisis
- expansion into new markets/areas

MILP formulations

The nested p-center problem: MILP formulation

Decision variables

$$y_j^h \ldots \begin{cases} 1 \ldots \text{if location } j \text{ is open in} \\ \qquad\ \ \text{time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

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(nPC2) based on Gaar and Sinnl 2022

$$
\begin{aligned}
\min \quad & \sum_{h \in \mathcal{H}} z^h && && \text{(1a)} \\
\text{s.t.} \quad & \sum_{j \in V} y_j^h = p^h && \forall h \in \mathcal{H} && \text{(1b)} \\
& z^h \geq d_{ij} - \sum_{j' : d_{ij'} < d_{ij}} (d_{ij} - d_{ij'}) y_{j'}^h && \forall i, j \in V, h \in \mathcal{H} && \text{(1c)} \\
& y_j^h \geq y_j^{h-1} && \forall j \in V, \forall h \in \mathcal{H} \setminus \{1\} && \text{(1d)} \\
& (y, z) \in \left|V\right| \left|\mathcal{H}\right| \times \mathbb{R}_{\geq 0} && && \text{(1e)}
\end{aligned}
$$

The nested *p*-center problem: Compact MILP formulation

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$$u_k^h \ldots \begin{cases} 1 \ldots \text{if } z^h \geq D_k \text{ in} \\ \qquad \text{time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

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(nPC3) based on Ales and Elloumi 2018

$$\begin{aligned}
\min \quad & \sum_{h \in \mathcal{H}} z^h \\
\text{s.t.} \quad & \sum_{j \in V} y_j^h = p^h && \forall h \in \mathcal{H} \\
& D_0 + \sum_{k=1}^{K} (D_k - D_{k-1}) u_k^h \leq z^h && \forall h \in \mathcal{H} \\
& u_k^h + \sum_{j : d_{ij} < D_k} y_j^h \geq 1 && \forall i \in V, \forall h \in \mathcal{H}, \forall k \in S_i \cup \{K\} \\
& u_k^h \geq u_{k+1}^h && \forall h \in \mathcal{H}, \forall k \in \mathcal{K} \setminus \{K\} \\
& y_j^h \geq y_j^{h-1} && \forall h \in \mathcal{H} \setminus \{1\}
\end{aligned}$$

Comparing formulations

Table: Comparison of complexity of the formulations

	`(nPC1)`	`(nPC2)`	`(nPC3)`
Variables	$\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}(\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}((\lvert\mathcal{I}\rvert + \lvert\mathcal{K}\rvert)\lvert\mathcal{H}\rvert)$
Constraints	$\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}(\min(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert, \lvert\mathcal{I}\rvert\lvert\mathcal{K}\rvert)\lvert\mathcal{H}\rvert)$

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The *p*CP version of formulation `(nPC3)` has the best known linear programming ($\mathcal{LP}$)-bounds for the *p*CP, while the *p*CP versions of `(nPC1)` and `(nPC2)` have equally but worse $\mathcal{LP}$-bounds.

Improving the formulations

Strengthening constraints `(nPC2)`

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Lemma 1

Let LB_h be a lower bound on the decision variable z_h of (nPC2) for every $i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H}$ then

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$$z^h \geq max\{LB^h, d_{ij}\} - \sum_{j' : d_{ij'} < d_{ij}} (\max\{LB^h, d_{ij}\} - \max\{LB^h, d_{ij'}\}) y^h_{j'} \qquad \text{(nL-OPT)}$$

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is a valid inequality for (nPC2), i.e., every feasible solution of (nPC2) fulfills (nL-OPT)*. The theorem is based on Lemma 5 in Gaar and Sinnl (2022).*

Reducing set $\mathcal{K}$ in `(nPC3)`

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Lemma 2

Let $\underline{z^h}$ be a valid lower bound and $\overline{z^h}$ be a valid upper bound on the decision variable z^h for $h \in \mathcal{H}$, then the distinct distance D_k can only be the optimal distance for z^h if $\underline{z^h} \leq D^k \leq \overline{z^h}$ holds.

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Therfore let set $S_i^h \subseteq S_i$ for $h \in \mathcal{H}$, where S_i^h contains only the indices k where $\underline{z^h} \leq D^k \leq \overline{z^h}$ holds.

Depending on the bounds $\underline{z^h}$ and $\overline{z^h}$ the sets S_i^h can be much smaller than S_i.

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$$\overline{z^h} = \frac{UB - \sum_{x=h+1}^{H} z'^{x*}}{h+1}$$

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- `PH`: `PP` with starting heuristic

Results

Instance from literature

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  - single core of Intel Xeon X5570 with 2.93 GHz and 8 GB RAM
  - timelimit of 3600 seconds

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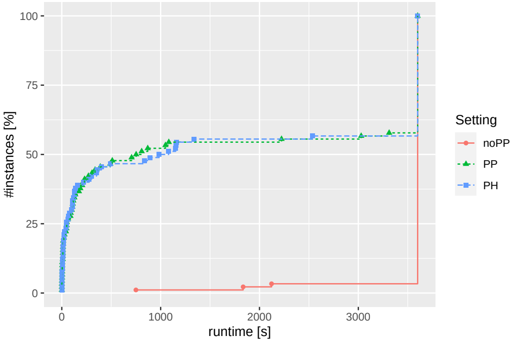
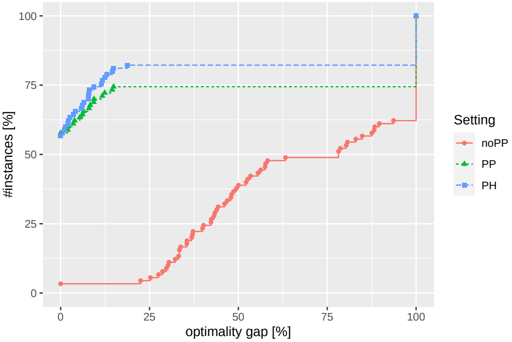
- instance set TSPLIB Contardo, Iori, and Kramer 2019; Gaar and Sinnl 2022
  - 80 instances
  - $\mathcal{P} = \{4, 5, 6\}$, $\left|V\right|$ from 100 and 900 nodes

Setting comparison on formulation (nPC1)

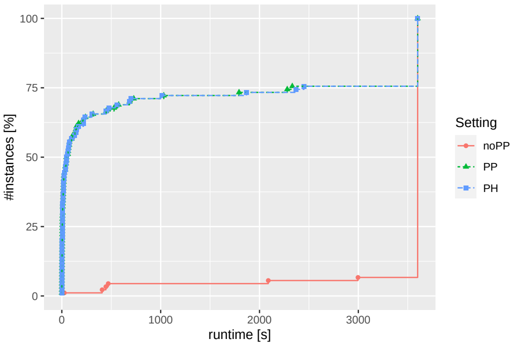


(a) models runtime

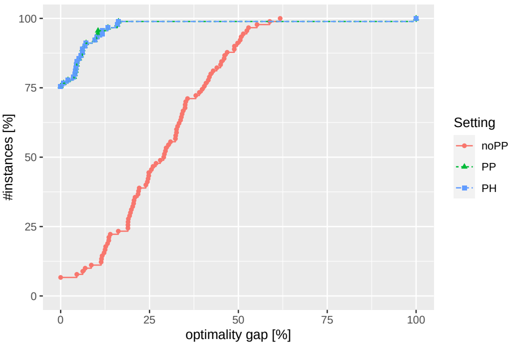


(b) models optimality gap

Setting comparison on formulation (nPC2)

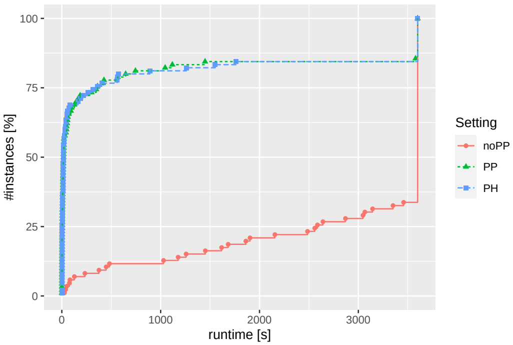


(c) models runtime

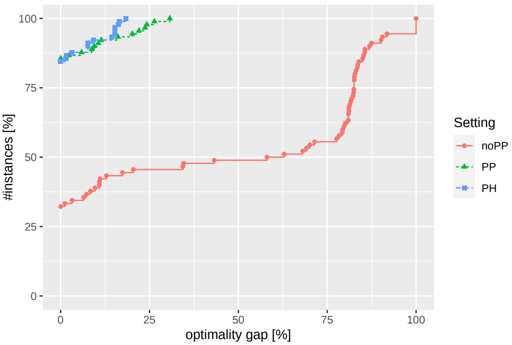


(d) models optimality gap

Setting comparison on formulation (nPC3)

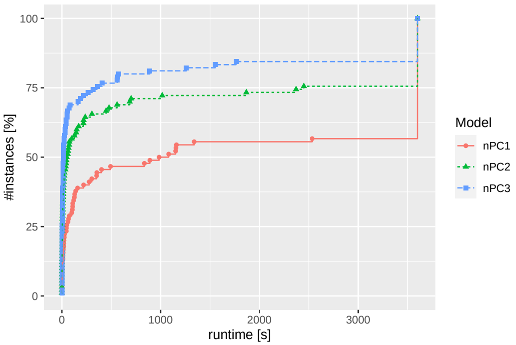


(e) models runtime



(f) models optimality gap

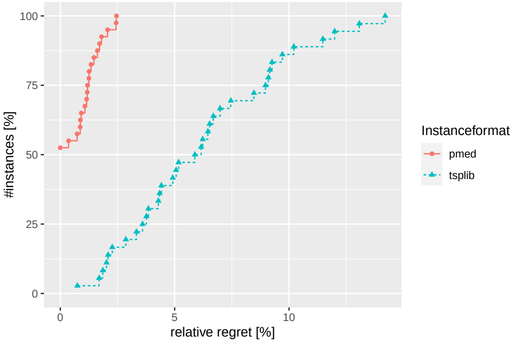
Formulation comparison on setting PH



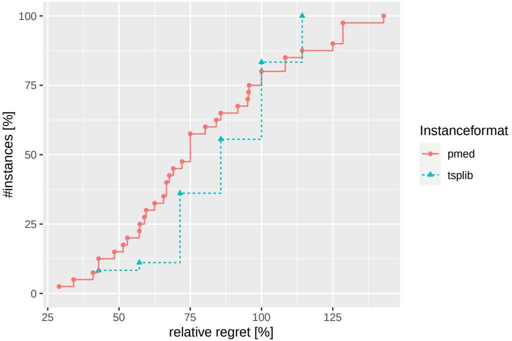
(g) models runtime

(h) models optimality gap

Managerial insights



(i) Relative regret of the optimal solution value



(j) Relative regret of # of opened facilities

Conclusion

- `(nPC3)` best performance in runtimes and optimality gap
- The `(nPC2)` outperforms the `(nPC1)`
- Preprocessing brings a large speed up on all formulations
- Starting heuristic improves optimality gap
- Maximal relative regret of the optimal objective function value of 15%
- Maximal relative regret of number of opened facilities above 140%

On the nested p-center problem

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The nested p-center problem: Classical MILP formulation

Decision variables

$$x_{ij}^h \ldots \begin{cases} 1 \ldots \text{if location } i \text{ is assigned to} \\ \qquad \text{location } j \text{ in time period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

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$z^h \dots$ maximum distance between any customer i and its nearest open facility in period h

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(nPC1) based on Daskin 2013

$$\min \sum_{h \in \mathcal{H}} z^h \tag{3a}$$

$$\text{s.t.} \quad \sum_{j \in V} y_j^h = p^h \qquad \forall h \in \mathcal{H} \tag{3b}$$

$$\sum_{j \in V} x_{ij}^h = 1 \qquad \forall i \in V, h \in \mathcal{H} \tag{3c}$$

$$x_{ij}^h \leq y_j^h \qquad \forall i, j \in V, h \in \mathcal{H} \tag{3d}$$

$$\sum_{j \in V} d_{ij} x_{ij}^h \leq z^h \qquad \forall i \in V, h \in \mathcal{H} \tag{3e}$$

$$y_j^h \geq y_j^{h-1} \qquad \forall h \in \mathcal{H} \setminus \{1\} \tag{3f}$$

$$(x, y, z) \in |V|^2|\mathcal{H}| \times |V||\mathcal{H}| \times \mathbb{R}_{\geq 0} \tag{3g}$$