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On the nested p-center problem

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The *p*-center problem: Definition

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The p-center problem: Definition

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  - a set V of locations,

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  - the maximum distance of any location to its closest opened location is minimized.

The nesting property

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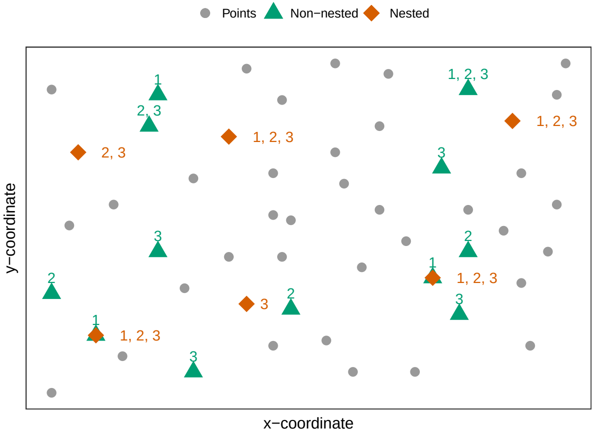
  - for which $\mathcal{J}^h \subseteq \mathcal{J}^{h+1}$ for $h = 1, \ldots, H-1$ holds
- the goal is to find a feasible solution which minimizes $\sum_{h=1}^{H} d_h(\mathcal{J}^h)$,
  - where $d_h(\mathcal{J}^h) = \max_{i \in \mathcal{I}} \min_{j \in \mathcal{J}^h} d_{ij}$ for $h \in \mathcal{H}$

p-center problem vs nested p-center problem



The nested p-center problem: Potential applications

The nested p-center problem: Classical MILP formulation

Variables

$$x_{ij}^h \dots \begin{cases} 1 & \dots \text{ if location } i \text{ is assigned to location } j \text{ in time period } h \\ 0 & \dots \text{ otherwise} \end{cases}$$

The nested p-center problem: Classical MILP formulation

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(MILP) for nested p-center (based on Daskin 2013)

$$\texttt{(nPC1)} \quad \min \sum_{h \in \mathcal{H}} z^h \tag{1a}$$

$$\text{s.t.} \quad \sum_{j \in V} y_j^h = p^h \qquad \forall h \in \mathcal{H} \tag{1b}$$

$$\sum_{j \in V} x_{ij}^h = 1 \qquad \forall i \in V, h \in \mathcal{H} \tag{1c}$$

$$x_{ij}^h \leq y_j^h \qquad \forall i, j \in V, h \in \mathcal{H} \tag{1d}$$

$$\sum_{j \in V} d_{ij} x_{ij}^h \leq z^h \qquad \forall i \in V, h \in \mathcal{H} \tag{1e}$$

$$y_j^h \geq y_j^{h-1} \qquad \forall h \in \mathcal{H} \setminus \{1\} \tag{1f}$$

$$x_{ij}^h, \in \{0, 1\} \qquad \forall i, j \in V, h \in \mathcal{H} \tag{1g}$$

$$y_j^h, \in \{0, 1\} \qquad \forall j \in V, h \in \mathcal{H} \tag{1h}$$

$$z^h \in \mathbb{R}_{\geq 0} \qquad \forall h \in \mathcal{H} \tag{1i}$$

Related work

p-center problem

- First introduction of the *p*-center problem by Hakimi (1964)
- The standard textbook formulation of the *p*-center problem can be found in Daskin (2013)
- Solution approach based on the set cover problem by Contardo, Iori, and Kramer (2018)
- A compact formulation by Ales and Elloumi (2018) and Elloumi, Labbé, and Pochet (2004)
- A formulation with a projection-based branch-and-cut algorithm for the *p*-center problem by Gaar and Sinnl (2022)

Nested facility location problems

- First introduction of the nesting property and constraint by Roodman and Schwarz (1975)
- Extension of the nesting to a phase-in and phase-out by Roodman and Schwarz (1977)
- Reintroduction of the nesting on the example of the p-median problem by McGarvey and Thorsen (2022)
- Other work on multi-period facility locations problems mainly focus on varying demand, distances, or cost over time f.e. Calogiuri et al. (2021)

Mixed Integer Linear Programming formulations

First MILP formulation

Decision variables

$$x_{ij}^h \ldots \begin{cases} 1 \ldots \text{if customer } i \text{ is assigned to facility location } j \text{ in period } h \\ 0 \ldots \text{otherwise} \end{cases}$$

First MILP formulation

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$$y_j^h \ldots \begin{cases} 1 \ldots \text{ if a facility is opened at location } j \text{ in period } h \\ 0 \ldots \text{ otherwise} \end{cases}$$

$z^h \ldots$ maximum distance between any customer i and its nearest open facility in period h

Second MILP formulation

$$
\begin{aligned}
\texttt{(nPC2)} \qquad & \min && \sum_{h \in \mathcal{H}} z^h && && \text{(2a)}\\
& \text{s.t.} && \sum_{j \in \mathcal{J}} y_j^h = p^h && \forall h \in \mathcal{H} && \text{(2b)}\\
& && z^h \geq d_{ij} - \sum_{j' : d_{ij'} < d_{ij}} (d_{ij} - d_{ij'}) y_{j'}^h && \forall i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H} && \text{(2c)}\\
& && y_j^h \geq y_j^{h-1} && \forall j \in \mathcal{J}, h \in \mathcal{H} \setminus \{1\} && \text{(2d)}\\
& && y_j^h \in \{0, 1\} && \forall j \in \mathcal{J}, h \in \mathcal{H} && \text{(2e)}\\
& && z^h \in \mathbb{R}_{\geq 0} && \forall h \in \mathcal{H} && \text{(2f)}
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Third MILP formualtion

$\mathcal{D}\ldots$ set of distinct distances where $D_0 \leq \cdots \leq D_K$ are the values in $\mathcal{D}$

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$S_i \ldots$ set of indices $k \in \mathcal{K}$ for which there exists a facility $j \in \mathcal{J}$ with $d_{ij} = D_k$

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z^h ... maximum distance between any customer i and its nearest open facility in period h

Third formulation

$$
\begin{aligned}
\texttt{(nPC3)} \quad \min \quad & \sum_{h\in\mathcal{H}} z^h && && \text{(3a)}\\
\text{s.t.} \quad & \sum_{j\in\mathcal{J}} y_j^h = p^h && \forall h \in \mathcal{H} && \text{(3b)}\\
& D_0 + \sum_{k=1}^{K} (D_k - D_{k-1}) u_k^h \leq z^h && \forall h \in \mathcal{H} && \text{(3c)}\\
& u_k^h + \sum_{j:d_{ij}<D_k} y_j^h \geq 1 && \forall i \in \mathcal{I}, \forall h \in \mathcal{H}, \forall k \in S_i \cup \{K\} && \text{(3d)}\\
& u_k^h \geq u_{k+1}^h && \forall h \in \mathcal{H}, \forall k \in \mathcal{K} \setminus \{K\} && \text{(3e)}\\
& y_j^h \leq y_j^{h-1} && \forall h \in \mathcal{H}, \forall j \in \mathcal{J} && \text{(3f)}\\
& y_j^h \in \{0, 1\} && \forall h \in \mathcal{H}, \forall j \in \mathcal{J} && \text{(3g)}\\
& u_k^h \in \{0, 1\} && \forall h \in \mathcal{H}, \forall k \in \mathcal{K} && \text{(3h)}\\
& z^h \in \mathbb{R} && \forall h \in \mathcal{H} && \text{(3i)}
\end{aligned}
$$

Comparing formulations

Table: Comparison of complexity of the formulations

	(nPC1)	(nPC2)	(nPC3)
Variables	$\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}(\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}((\lvert\mathcal{I}\rvert + \lvert\mathcal{K}\rvert)\lvert\mathcal{H}\rvert)$
Constraints	$\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}(\min(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert, \lvert\mathcal{I}\rvert\lvert\mathcal{K}\rvert)\lvert\mathcal{H}\rvert)$

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Constraints	$\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{H}\rvert)$	$\mathcal{O}(\min(\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert, \lvert\mathcal{I}\rvert\lvert\mathcal{K}\rvert)\lvert\mathcal{H}\rvert)$

The *p*CP version of formulation `(nPC3)` has the best known linear programming ($\mathcal{LP}$)-bounds for the *p*CP, while the *p*CP versions of `(nPC1)` and `(nPC2)` have equally but worse $\mathcal{LP}$-bounds.

Reducing set $\mathcal{K}$ in `(nPC3)`

Lemma 1

Let $\underline{z}^h$ be a valid lower bound and $\overline{z}^h$ be a valid upper bound on the decision variable z^h for $h \in \mathcal{H}$, then the distinct distance D_k can only be the optimal distance for z^h if $\underline{z}^h \leq D^k \leq \overline{z}^h$ holds.
Therfore let set $S_i^h \subseteq S_i$ for $h \in \mathcal{H}$, where S_i^h contains only the indices k where $\underline{z}^h \leq D^k \leq \overline{z}^h$ holds and constraint (3d) can be replaced with

$$u_k^h + \sum_{j:d_{ij}<D_k} y_j^h \geq 1 \qquad \forall i \in \mathcal{I}, \forall h \in \mathcal{H}, \forall k \in S_i^h \cup \{K\} \tag{4}$$

Depending on the bounds $\underline{z}^h$ and $\overline{z}^h$ the sets S_i^h can be much smaller than S_i.

Strengthening constraints (nPC2)

Lemma 2

Let LB_h be a lower bound on the decision variable z_h of (nPC2) for every $i \in \mathcal{I}, j \in \mathcal{J}, h \in \mathcal{H}$ then

$$z^h \geq max\{LB^h, d_{ij}\} - \sum_{j':d_{ij'}<d_{ij}} (\max\{LB^h, d_{ij}\} - \max\{LB^h, d_{ij'}\})y_{j'}^h \tag{nL-OPT}$$

is a valid inequality for (nPC2), i.e., every feasible solution of (nPC2) fulfills (nL-OPT). *Theorem is based on Lemma 5 in Gaar and Sinnl (2022).*

A similar strengthening can be done for (**??**) in fromulation (nPC1).

Obtaining bounds I

Lemma 3

Let z'^{h*} be the optimal objective function value of pCP with $p = p^h$ for $h \in \mathcal{H} = \{1, 2, \ldots, H\}$ where $p^h > p^{h+1}$, then $UB = Hz'^{1*}$ is a valid upper bound on the optimal objective function value of n-pCP.
Then

$$\overline{z^h} = \frac{UB - \sum_{x=h+1}^{H} z'^{x*}}{h+1}$$

where $\overline{z^h}$ is a valid upper bound on the decision variable z^h of the n-pCP for $h \in \mathcal{H}$.

Obtaining bounds II

Lemma 4

Let z'^* be the optimal objective function value of pCP for a certain p'. Then z'^* is a valid lower bound $\underline{z}^h$ on the decision variable z^h of n-pCP with $p^h = p'$.

Implementation and outline of the results

Implementation

- Implemented in C++ using the CPLEX API of CPLEX 20.1.0.0
- Formulation `(nPC2)` solved by branch-and-cut and seperation based on the customers
- Preprocessing for all formulations
  - solving the pCP for p^h, $h \in \mathcal{H}$ starting with $h = H$
  - p^h is a valid lower bound for the pCP with p^{h-1}
- Single core of an Intel Xeon X5570 machine
  - 2.93 GHz
  - 48 GB RAM
  - Each run limited to 9 GB RAM and 3600 sec

Data

- data set PMED from "A note on solving large p-median problems” by Beasley (1985)
  - set of 40 test instances
  - the sets contain between 100 and 900 nodes
  - number of facilities to open initially ranging from 5 to 200
  - $\mathcal{P} = \{p, p+1, p+2\}$

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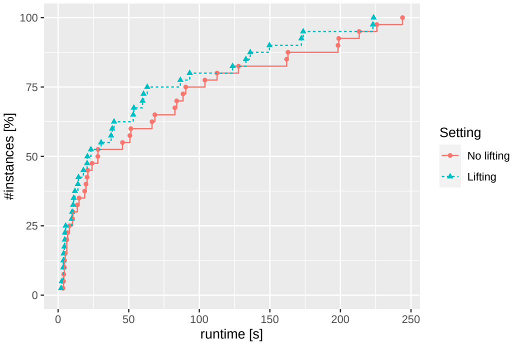
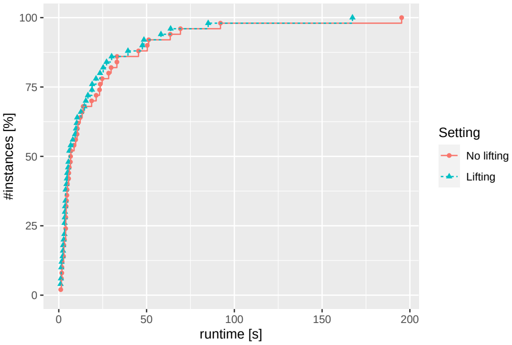
- data set TSPLIB 2D-Euclidean distances from “TSPLIB—A Traveling Salesman Problem Library” by Reinelt (1991)
  - set of 80 test instances
  - the sets contain between 51 and 1002 nodes
  - rounded to the nearest integer value
  - $\mathcal{P} = \{4, 5, 6\}$

Preprocessing



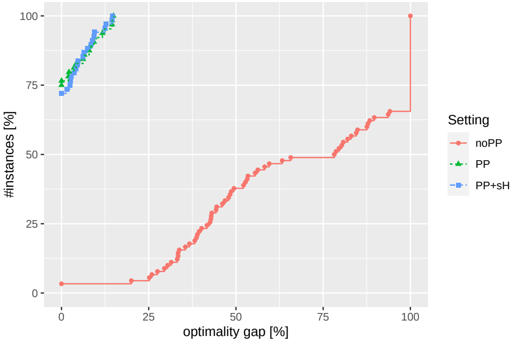
(a) Preprocessing PMED



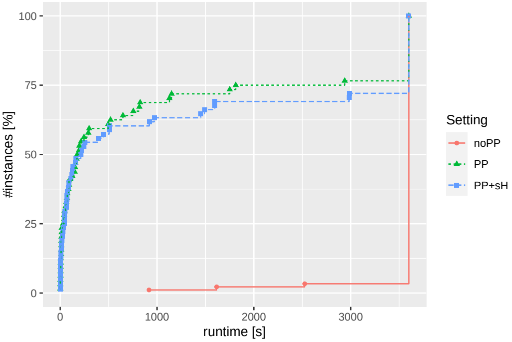
(b) Preprocessing TSPLIB

Figure: Preprocessing

(nPC1)-results

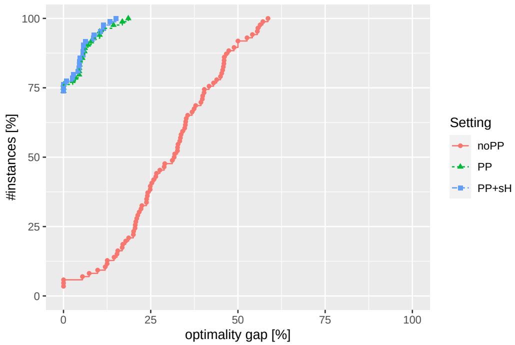


(a) (nPC1) optimality gap

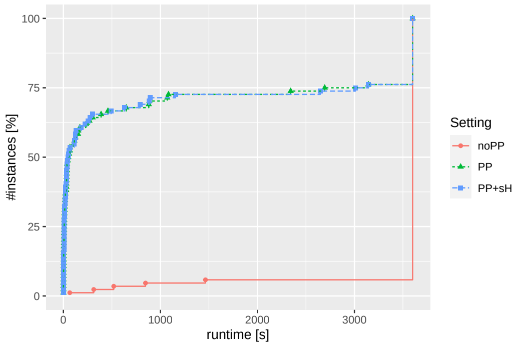


(b) (nPC1) runtime

(nPC2)-results

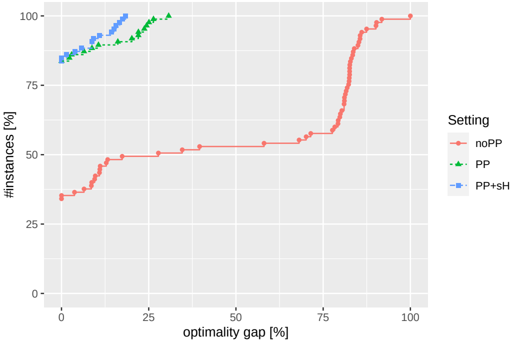


(c) (nPC2) optimality gap

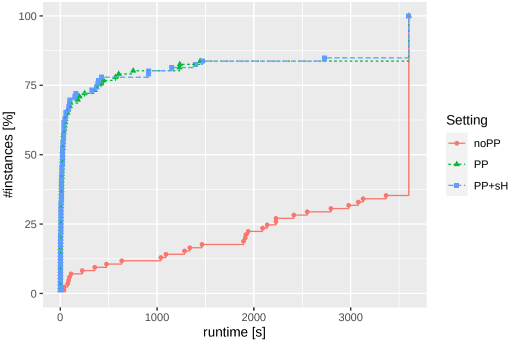


(d) (nPC2) runtime

(nPC3)-results

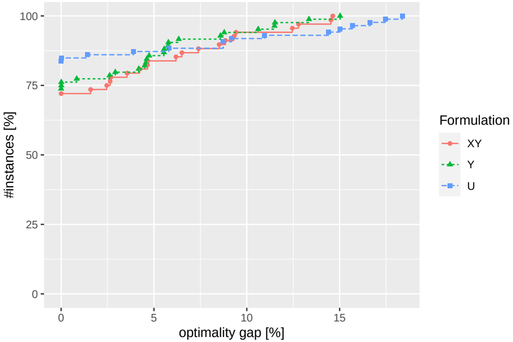


(e) (nPC3) optimality gap

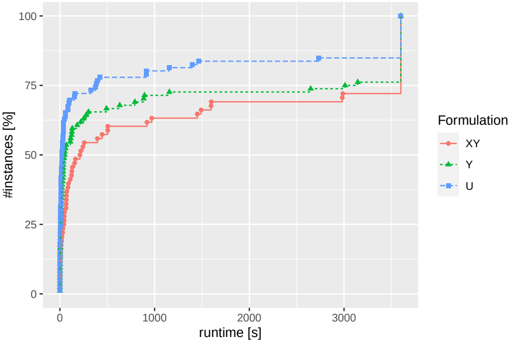


(f) (nPC3) runtime

Formulation comparison

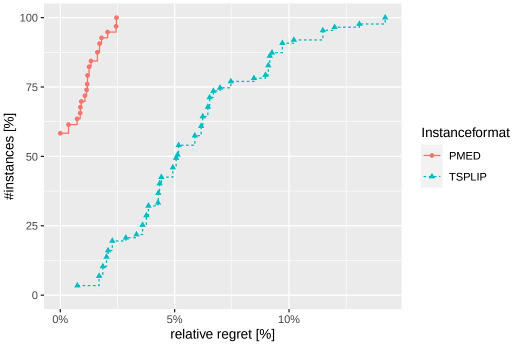


(g) models optimality gap

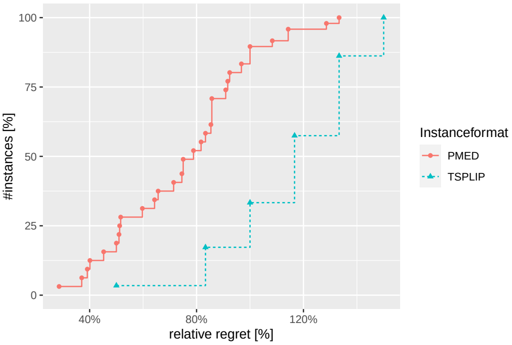


(h) models runtime

Managerial insights



(i) Relative regret



(j) Relative regret of # of facilities

Figure: On a subset of instances: Only if the problem was solved to optimality

Conclusion

- `(nPC3)` had the best runtimes over all instances
- For optimality gap the results are rather mixed
- The `(nPC2)` outperforms the `(nPC1)`
- Maximal relative regret of the objective value of 15%
- Maximal relative regret of number of facilities above 140%
- min-max regret as objective function was also analysed and can be found later in the thesis

On the nested p-center problem

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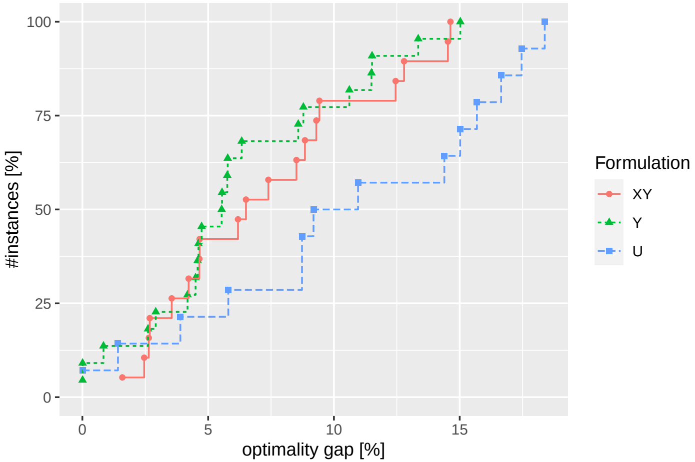
Formulations with non optimal instances



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