9.20 The equation system is

$$\begin{aligned} C \;+\; cS + cF &= cP \\ sC \;+\; S \quad\quad &= sP \\ fC \;+\; fS + \;F &= fP. \end{aligned}$$

The solution is

$$C = \frac{\begin{vmatrix} c & c & c \\ s & 1 & 0 \\ f & f & 1 \end{vmatrix} \cdot P}{\begin{vmatrix} 1 & c & c \\ s & 1 & 0 \\ f & f & 1 \end{vmatrix}} = \frac{(1 + sf - f - s)cP}{1 + sfc - fc - sc}$$

$$S = \frac{\begin{vmatrix} 1 & c & c \\ s & s & 0 \\ f & f & 1 \end{vmatrix} \cdot P}{\begin{vmatrix} 1 & c & c \\ s & 1 & 0 \\ f & f & 1 \end{vmatrix}} = \frac{s(1 - c)P}{1 + sfc - fc - sc}$$

$$F = \frac{\begin{vmatrix} 1 & c & c \\ s & 1 & s \\ f & f & f \end{vmatrix} \cdot P}{\begin{vmatrix} 1 & c & c \\ s & 1 & 0 \\ f & f & 1 \end{vmatrix}} = \frac{(1 + sc - s - c)fP}{1 + sfc - fc - sc}.$$

Chapter 10

10.4 *a*) $(2, -1)$, *b*) $(-2, -1)$, *c*) $(2, 1)$, *d*) $(3, 0)$, *e*) $(1, 2, 4)$, *f*) $(2, -2, 3)$.

10.5 *a*) $(1, 3)$, *b*) $(-4, 12)$, *c*) undefined, *d*) $(0, 3, 3)$, *e*) $(0, 2)$, *f*) $(1, 4)$, *g*) $(1, 1)$, *h*) $(3, 7, 1)$, *i*) $(-2, -4, 0)$, *j*) undefined.

10.7 $-\mathbf{u}$ is what one adds to $\mathbf{u}$ to get $\mathbf{0}$. $(-1)\mathbf{u} + \mathbf{u} = (-1)\mathbf{u} + 1 \cdot \mathbf{u} = [(-1) + 1]\mathbf{u} = 0 \cdot \mathbf{u} = \mathbf{0}$. So $(-1)\mathbf{u} = -\mathbf{u}$.

10.8 $(r+s)\mathbf{u} = (r+s)(u_1,\dots,u_n) = ((r+s)u_1,\dots,(r+s)u_n)$

$= (ru_1,\dots,ru_n) + (su_1,\dots,su_n) = r\mathbf{u} + s\mathbf{u}.$

$r(\mathbf{u}+\mathbf{v}) = r\big((u+v)_1,\dots,(u+v)_n\big) = (r(u+v)_1,\dots,r(u+v)_n)$

$= (ru_1 + rv_1,\dots,ru_n + rv_n) = (ru_1,\dots,ru_n) + (rv_1,\dots,rv_n)$

$= r\mathbf{u} + r\mathbf{v}.$

10.10 *a*) 5, *b*) 3, *c*) $\sqrt{3}$, *d*) $3\sqrt{2}$, *e*) $\sqrt{2}$, *f*) $\sqrt{14}$, *g*) 2, *h*) $\sqrt{30}$, *i*) 3.

10.11 *a*) 5, *b*) 10, *c*) 4, *d*) $\sqrt{41}$, *e*) 6.

10.12 *a*) $\mathbf{u}\cdot\mathbf{v} = 2$, so the angle is acute; $\theta = \arccos(2/2\sqrt{2}) = 45°$.

b) $\mathbf{u}\cdot\mathbf{v} = 0$, so the angle is right; $\theta = 90°$.

c) $\mathbf{u}\cdot\mathbf{v} = 3$, so the angle is acute; $\theta = \arccos(\sqrt{3}/2) = 30°$.

d) $\mathbf{u}\cdot\mathbf{v} = -1$, so the angle is obtuse; $\theta = \arccos(-1/2\sqrt{3}) \approx 106.8°$.

e) $\mathbf{u}\cdot\mathbf{v} = 1$, so the angle is acute; $\theta = \arccos(1/\sqrt{5}) \approx 63.4°$.

10.13 Multiply each vector $\mathbf{u}$ by the scalar $1/\|\mathbf{u}\|$:

a) $(3/5, 4/5)$, *b*) $(1, 0)$, *c*) $\big(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\big)$,

d) $\big(-1/\sqrt{14}, 2/\sqrt{14}, -3/\sqrt{14}\big)$.

10.14 Multiply each vector $\mathbf{v}$ found in Exercise 10.13 by the scalar -5:

a) $(-3, -4)$, *b*) $(-5, 0)$, *c*) $\big(-5/\sqrt{3}, -5/\sqrt{3}, -5/\sqrt{3}\big)$,

d) $\big(5/\sqrt{14}, -10/\sqrt{14}, 15/\sqrt{14}\big)$.

10.15 $\|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} - 2\,\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} = \|\mathbf{u}\|^2 - 2\,\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2$.

10.16 *a*) $\|\mathbf{u}\| = |u_1| + |u_2| \geq 0$, and equals 0 if and only if both terms in the sum equal 0.

$\|r\mathbf{u}\| = |ru_1| + |ru_2| = |r|(|u_1| + |u_2|) = r\|\mathbf{u}\|$.

$\|\mathbf{u}+\mathbf{v}\| = |u_1+v_1| + |u_2+v_2| \leq |u_1| + |u_2| + |v_1| + |v_2| = \|\mathbf{u}\| + \|\mathbf{v}\|$.

$\|\mathbf{u}\| = \max\{|u_1|, |u_2|\} \geq 0$, and equals 0 if and only if both terms in the max equal 0.

$$\|r\mathbf{u}\| = \max\{|ru_1|, |ru_2|\} = |r|\max\{|u_1|, |u_2|\} = |r|\|\mathbf{u}\|.$$

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\| &= \max\{|u_1+v_1|, |u_2+v_2|\} \le \max\{|u_1|+|v_1|, |u_2|+|v_2|\}\\ &\le \max\{|u_1|, |u_2|\} + \max\{|v_1|, |v_2|\} = \|\mathbf{u}\| + \|\mathbf{v}\|.\end{aligned}$$

b) $|u_1| + |u_2| + \cdots + |u_n|$, called the l_1 norm; $\max\{|u_1|, |u_2|, \ldots, |u_n|\}$, known as the l_∞ or sup norm.

10.17 *a*) $\mathbf{u}\cdot\mathbf{v} = \sum_i u_iv_i = \sum_i v_iu_i = \mathbf{v}\cdot\mathbf{u}$.

b) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \sum_i u_i(v_i+w_i) = \sum_i u_iv_i + u_iw_i = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$.

c) $\mathbf{u}\cdot(r\mathbf{v}) = \sum_i u_i(rv_i) = r\sum_i u_iv_i = r\mathbf{u}\cdot\mathbf{v}$. A similar calculation proves the other assertion.

d) $\mathbf{u}\cdot\mathbf{u} = \sum_i u_i^2 \ge 0$.

e) Every term in the sum $\mathbf{u}\cdot\mathbf{u}$ is nonnegative, so $\mathbf{u}\cdot\mathbf{u} = 0$ iff every term is 0; this is true iff each $u_i = 0$.

f) $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v}$ by part *a*.

10.19 Putting one vertex of the box at the origin, the long side is the vector $(4, 0, 0)$ and the diagonal is the vector $(4, 3, 2)$. Then, $\theta = \arccos(\mathbf{u}\cdot\mathbf{v}/\|\mathbf{u}\|\|\mathbf{v}\|) = \arccos\left(4/\sqrt{29}\right) \approx 42.03°$.

10.20 The two diagonals are $(\mathbf{u}+\mathbf{v})$ and $(\mathbf{v}-\mathbf{u})$. Their inner product is $\mathbf{v}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{u} = \mathbf{v}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{u} = \|\mathbf{v}\|^2 - \|\mathbf{u}\|^2$. This equals 0 if $\|\mathbf{u}\| = \|\mathbf{v}\|$.

10.21 *a*) $\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{u} - 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} = 2\mathbf{u}\cdot\mathbf{u} + 2\mathbf{v}\cdot\mathbf{v} = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$.

b) Calculating as in part *a*, $\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2 = 4\mathbf{u}\cdot\mathbf{v}$.

10.22 $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u}\cdot\mathbf{v}$. If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, $\mathbf{u}\cdot\mathbf{v} = 0$ and $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$. When applied to (perpendicular) vectors in $\mathbf{R^2}$, this is exactly Pythagoras' theorem.

10.23 *a*) Interchanging $\mathbf{u}$ and $\mathbf{v}$ interchanges the rows in the three matrices in the definition of cross product and thus changes the sign of all three determinants.

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= \left(\begin{vmatrix} u_2 & u_3\\ v_2 & v_3\end{vmatrix}, -\begin{vmatrix} u_1 & u_3\\ v_1 & v_3\end{vmatrix}, \begin{vmatrix} u_1 & u_2\\ v_1 & v_2\end{vmatrix}\right)\\ &= -\left(\begin{vmatrix} v_2 & v_3\\ u_2 & u_3\end{vmatrix}, -\begin{vmatrix} v_1 & v_3\\ u_1 & u_3\end{vmatrix}, \begin{vmatrix} v_1 & v_2\\ u_1 & u_2\end{vmatrix}\right)\\ &= -\mathbf{v}\times\mathbf{u}.\end{aligned}$$

b) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = u_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - u_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + u_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0.$$

c) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0.$

Alternatively, $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = -\mathbf{v} \cdot (\mathbf{v} \cdot \mathbf{u}) = 0.$

d) $(r\mathbf{u}) \times \mathbf{v} = \left(\begin{vmatrix} ru_2 & ru_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} ru_1 & ru_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} ru_1 & ru_2 \\ v_1 & v_2 \end{vmatrix} \right)$

$$= r \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

$$= r(\mathbf{u} \times \mathbf{v}).$$

A similar calculation proves the remaining assertion.

e) $(\mathbf{u} + \mathbf{w}) \times \mathbf{v} = \left(\begin{vmatrix} u_2 + w_2 & u_3 + w_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 + w_1 & u_3 + w_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 + w_1 & u_2 + w_2 \\ v_1 & v_2 \end{vmatrix} \right)$

$$= \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right) + \left(\begin{vmatrix} w_2 & w_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} w_1 & w_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} w_1 & w_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

$$= (\mathbf{u} \times \mathbf{v}) + (\mathbf{w} \times \mathbf{v}).$$

f) $\|\mathbf{u} \times \mathbf{v}\|^2 = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}^2 + \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}^2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2$

$$= (u_2 v_3 - u_3 v_2)^2 + (u_1 v_3 - u_3 v_1)^2 + (u_1 v_2 - u_2 v_3)^2$$

$$= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1 v_1 + u_2 v_2 + u_3 v_3)^2$$

$$= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - \mathbf{u} \cdot \mathbf{v}$$

g) $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 (1 - \cos^2 \theta)$

$$= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \sin^2 \theta.$$

h) $\mathbf{u} \times \mathbf{u} = \left(\begin{vmatrix} u_2 & u_3 \\ u_2 & u_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ u_1 & u_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ u_1 & u_2 \end{vmatrix} \right) = (0, 0, 0)$. Alternatively, $\mathbf{u} \times \mathbf{u} = \|\mathbf{u}\|^4 \sin^2 0 = 0$.

i)
$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$
$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

10.24
$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \mathbf{e}_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \mathbf{e}_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \mathbf{e}_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$
$$= \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right) = \mathbf{u} \times \mathbf{v}.$$

10.25 *a*) $\mathbf{u} \times \mathbf{v} = (-1, 0, 1)$.

b) $\mathbf{u} \times \mathbf{v} = (-7, 3, 5)$.

10.26 *a*) The area of the parallelogram is $\|\mathbf{v}\|h = \|\mathbf{v}\|\|\mathbf{u}\| \sin\theta = \|\mathbf{u} \times \mathbf{v}\|$.

b) By *a*, the area of the triangle with vertices A, B, C is $\frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\|$.

$$\begin{aligned} \overrightarrow{AB} &= (0, 1, 3) - (1, -1, 2) = (-1, 2, 1) \\ \overrightarrow{AC} &= (2, 1, 0) - (1, -1, 2) = (1, 2, -2) \\ \overrightarrow{AB} \times \overrightarrow{AC} &= (-6, 1, -4) \\ \tfrac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| &= \tfrac{1}{2}\sqrt{36 + 1 + 16} = \tfrac{1}{2}\sqrt{53}. \end{aligned}$$

10.27 If $\mathbf{z} = \frac{1}{2}(\mathbf{x} + \mathbf{y})$, then $\mathbf{x} - \mathbf{z} = \frac{1}{2}(\mathbf{x} - \mathbf{y}) = \mathbf{z} - \mathbf{y}$, so $\|\mathbf{x} - \mathbf{z}\| = \|\mathbf{y} - \mathbf{z}\|$.

10.28 *a*) $\mathbf{x}(t) = (3 + 2t, 0)$, and its midpoint is $(4, 0)$.

b) $\mathbf{x}(t) = (1 - t, t)$, and its midpoint is $(0.5, 0.5)$.

c) $\mathbf{x}(t) = (1 + t, t, 1 - t)$, and its midpoint is $(1.5, 0.5, 0.5)$.

10.29 No. If it were, then judging from the first coordinate, the point would occur at $t = 2$. But then the last coordinate of the point would be $8 \cdot 2 + 4 = 20$, not 16.

10.30 *a*) Solve both equations for t and equate:

$$t = \frac{x_1 - 4}{-2} \qquad \text{and} \qquad t = \frac{x_2 - 3}{6} \quad \Longrightarrow \quad x_2 = -3x_1 + 15.$$

b) Alternative method: set $t = 0$ and $t = 1$ to get the two points $(3, 5)$ and $(4, 4)$. Use these two points to deduce equation $x_2 = -x_1 + 8$.

c) $x_2 = 5$ (x_1 takes on any value).

10.31 *a*) $\mathbf{x}(t) = \begin{pmatrix} 0 \\ 5/2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3/2 \end{pmatrix} t.$ *b*) $\mathbf{x}(t) = \begin{pmatrix} 0 \\ 7 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} t.$

c) $\mathbf{x}(t) = \begin{pmatrix} 6 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t.$

10.32 No. Solving the first two coordinate equations for s and t, $s = -2$ and $t = 3$. But with these values for s and t, the third coordinate should be 1, not 2.

10.33 *a*) $\mathbf{x}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $x_2 = 2x_1$.

b) $\mathbf{x}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3 \\ 9 \end{pmatrix}$, $x_2 = 3x_1 - 2$.

c) $\mathbf{x}(t) = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 4 \end{pmatrix}$, $x_2 = -\frac{4}{3}x_1 + 4$.

10.34 *a*) $\mathbf{x}(t) = \begin{pmatrix} 0 \\ -7 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} t.$

b) $\mathbf{x}(t) = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix} t.$

c) $\mathbf{x}(s, t) = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 3 \\ 0 \end{pmatrix} t + \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix} s.$

d) $\mathbf{x}(s, t) = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -6 \\ -3 \\ 0 \end{pmatrix} t + \begin{pmatrix} -6 \\ 0 \\ 2 \end{pmatrix} s.$

10.35 *a*) $y = -\frac{1}{2}x + \frac{5}{2}$. *b*) $y = (x/2) + 1$. *c*) $-7x + 2y + z = -3$. *d*) $y = 4$.

10.36 *a*) $\mathbf{x}(s, t) = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -6 \\ -6 \\ 0 \end{pmatrix} t + \begin{pmatrix} -6 \\ 0 \\ 3 \end{pmatrix} s, \quad x - y + 2z = 6.$

b) $\mathbf{x}(s,t) = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} t + \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} s, \quad x + 2y + 3z = 12.$

10.37 *a*) Rewriting the symmetric equations, $y = y_0 - (b/a)x_0 + (b/a)x$ and $z = z_0 - (c/a)x_0 + (c/a)x$. Then

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \begin{pmatrix} 1 \\ b/a \\ c/a \end{pmatrix} t \quad \text{or} \quad \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix} t.$$

b) The two planes are described by any two distinct equalities in system (20). For example,

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} \qquad \text{and} \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

In other words, $-bx + ay = -bx_0 + ay_0$ and $cy - bz = cy_0 - bz_0$.

c) *i*) $\dfrac{x_1 - 2}{-1} = \dfrac{x_2 - 3}{4} = \dfrac{x_3 - 1}{5}$, *ii*) $\dfrac{x_1 - 1}{4} = \dfrac{x_2 - 2}{5} = \dfrac{x_3 - 3}{6}$.

d) *i*) $4x_1 + x_2 = 11$ and $5x_2 - 4x_3 = 11$.

ii) $5x_1 - 4x_2 = -3$ and $6x_2 - 5x_3 = -3$.

10.38 *a*) Normals $(1, 2, -3)$ and $(1, 3, -2)$ do not line up, so planes intersect.

b) Normals $(1, 2, -3)$ and $(-2, -4, 6)$ do line up, so planes do not intersect.

10.39 *a*) $(x - 1, y - 2, z - 3) \cdot (-1, 1, 0) = 0$, so $x - y = -1$.

b) The line runs through the points $(4, 2, 6)$ and $(1, 3, 11)$, so the plane must be orthogonal to the difference vector $(-3, 1, 5)$. Thus $(x - 1, y - 1, z + 1) \cdot (-3, 1, 5) = 0$, or $-3x + y + 5z = -7$.

c) The general equation for the plane is $\alpha x + \beta y + \gamma z = \delta$. The equations to be satisfied are $a = \delta/\alpha$, $b = \delta/\beta$, and $c = \delta/\gamma$. A solution is $\alpha = 1/a$, $\beta = 1/b$. $\gamma = 1/c$ and $\delta = 1$, so $(1/a)x + (1/b)y + (1/c)z = 1$.

10.40 Plug $x = 3 + t$, $y = 1 - 7t$, and $z = 3 - 3t$ into the equation $x + y + z = 1$ of the plane and solve for t: $(3 + t) + (1 - 7t) + (3 - 3t) = 1 \Longrightarrow t = 2/3$. The point is $\big(11/3, -11/3, 1\big)$.

10.41 $\left(\begin{array}{ccc:c} 1 & 1 & -1 & 4 \\ 1 & 2 & 1 & 3 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc:c} 1 & 0 & -3 & 5 \\ 0 & 1 & 2 & -1 \end{array}\right) \Longrightarrow \begin{array}{l} x = 5 + 3z \\ y = -1 - 2z. \end{array}$

Taking $z = t$, write the line as $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$.

10.42 $IS : [1 - c_1(1 - t_1) - a_0]Y + (a + c_2)r = c_0 - c_1 t_0 + I^* + G$

$LM : mY - hr = M_s - M^*$.

I^* rises $\Longrightarrow$ IS moves up $\Longrightarrow$ Y^* and r^* increase.

M_s rises $\Longrightarrow$ LM moves up $\Longrightarrow$ Y^* decreases and r^* increases.

m rises $\Longrightarrow$ LM becomes steeper $\Longrightarrow$ Y^* decreases and r^* increases.

h rises $\Longrightarrow$ LM flatter $\Longrightarrow$ Y^* increases and r^* decreases.

a_0 rises $\Longrightarrow$ IS flatter (with same r-intercept) $\Longrightarrow$ r^* and Y^* rise.

c_0 rises $\Longrightarrow$ IS moves up $\Longrightarrow$ Y^* and r^* increase.

t_1 or t_2 rises $\Longrightarrow$ IS steeper (with same r-intercept) $\Longrightarrow$ r^* and Y^* decrease.

Chapter 11

11.1 Suppose $\mathbf{v}_1 = r_2\mathbf{v}_2$. Then $1v_1 - r_2\mathbf{v}_2 = r_2\mathbf{v}_2 - r_2\mathbf{v}_2 = \mathbf{0}$.

Suppose $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$. Suppose that $c_i \neq 0$. Then $c_i\mathbf{v}_i = -c_j\mathbf{v}_j$, so $\mathbf{v}_i = (c_j/c_i)\mathbf{v}_j$.

11.2 *a*) Condition (3) gives the equation system

$$\begin{aligned} 2c_1 + c_2 &= 0 \\ c_1 + 2c_2 &= 0. \end{aligned}$$

The only solution is $c_1 = c_2 = 0$, so these vectors are independent.

b) Condition (3) gives the equation system

$$\begin{aligned} 2c_1 + c_2 &= 0 \\ -4c_1 - 2c_2 &= 0. \end{aligned}$$

One solution is $c_1 = -2$, $c_2 = 1$, so these vectors are dependent.

c) Condition (3) gives the equation system

$$\begin{aligned} c_1 \qquad\qquad &= 0 \\ c_1 + c_2 \qquad &= 0 \\ c_2 + c_3 &= 0. \end{aligned}$$

Clearly $c_1 = c_2 = 0$ is the unique solution.

d) Condition (3) gives the equation system

$$\begin{aligned} c_1 \quad\quad\; + c_3 &= 0 \\ c_1 + c_2 \quad\quad\; &= 0 \\ c_2 + c_3 &= 0. \end{aligned}$$

The only solution is $c_1 = c_2 = c_3 = 0$, so these vectors are independent.

11.3 *a*) The coefficient matrix of the equation system of condition 3 is $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. The rank of this matrix is 3, so the homogeneous equation system has only one solution, $c_1 = c_2 = c_3 = 0$. Thus these vectors are independent.

b) The coefficient matrix of the equation system of condition 3 is $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The rank of this matrix is 2, so the homogeneous equation system has an infinite number of solutions aside from $c_1 = c_2 = c_3 = 0$. Thus these vectors are dependent.

11.4 If $c_1\mathbf{v}_1 = \mathbf{v}_2$, then $c_1\mathbf{v}_1 + (-1)\mathbf{v}_2 = \mathbf{0}$, and condition (4) fails to hold. If condition (4) fails to hold, then $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$ has a nonzero solution in which, say, $c_2 \neq 0$. Then $\mathbf{v}_2 = (c_1/c_2)\mathbf{v}_1$, and v_2 is a multiple of v_1.

11.5 *a*) The negation of "$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ implies $c_1 = c_2 = c_3 = 0$" is "There is some nonzero choice of c_1, c_2, and c_3 that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$."

b) Suppose that condition (5) fails and that $c_1 \neq 0$. Then $\mathbf{v}_1 = -(c_2/c_1)\mathbf{v}_1 - (c_3/c_1)\mathbf{v}_3$.

11.6 Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a collection of vectors such that $\mathbf{v}_1 = \mathbf{0}$. Then $1\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n = \mathbf{0}$ and the vectors are linearly dependent.

11.7 $A \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$, so the columns of A are linearly independent if and only if the equation system $A \cdot \mathbf{c} = \mathbf{0}$ has no nonzero solution.

11.8 The condition of Theorem 2 is necessary and sufficient for the equation system of Theorem 1 to have a unique solution, which must be the trivial solution. Thus A has rank n, and it follows from Theorem 9.3 that $\det A \neq 0$.

11.9 *a*) $(2, 2) = 3(1, 2) - 1(1, 4)$.

b) Solve the equation system whose augmented matrix is $\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{pmatrix}$.

The solution is $c_1 = 0$, $c_2 = 1$, and $c_3 = 2$.

11.10 No they do not. Checking the equation system $Ax = b$, we see that A has rank 2; this means that the equation system does not have a solution for general b.

11.11 For any column vector $\mathbf{b}$, if the equation system $Ax = \mathbf{b}$ has a solution x^*, then $x_1^*\mathbf{v}_1 + \cdots + x_n^*\mathbf{v}_n = \mathbf{b}$. Consequently, if the equation system has a solution for every right-hand side, then every vector $\mathbf{b}$ can be written as a linear combination of the column vectors $\mathbf{v}_i$. Conversely, if the equation system fails to have a solution for some right-hand side $\mathbf{b}$, then $\mathbf{b}$ is not a linear combination of the $\mathbf{v}_i$, and the $\mathbf{v}_i$ do not span $\mathbf{R^n}$.

11.12 The vectors in *a* are not independent. The vectors in *b* are a basis. The vectors in *c* are not independent. The vectors in *d* are a basis.

11.13 *a*) $\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \neq 0$, so these vectors span $\mathbf{R^2}$ and are independent.

b) $\det\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -1 \neq 0$, so these vectors span $\mathbf{R^2}$ and are independent.

d) $\det\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 2 \neq 0$, so these vectors span $\mathbf{R^2}$ and are independent.

11.14 Two vectors cannot span $\mathbf{R^3}$, so the vectors in *a* are not a basis. More than three vectors in $\mathbf{R^3}$ cannot be independent, so the vectors in *e* fail to be a basis. The matrix of column vectors in *b* and *c* have rank less than 3, so neither of these collections of vectors is a basis. The 3×3 matrix A whose column vectors are the vectors of *d* has rank 3, so $\det A \neq 0$. Thus they are a basis according to Theorem 11.8.

11.15 Suppose *a* is true. Then the $n \times n$ matrix A whose columns are the vectors $\mathbf{v}_i$ has rank n, and therefore $Ax = b$ has a solution for every right-hand side b. Thus the column vectors $\mathbf{v}_i$ span $\mathbf{R^n}$. Since they are linearly independent and span $\mathbf{R^3}$, they are a basis. Since the rank of A is n, $\det A \neq 0$. Thus *a* implies b, c and d. Finally suppose d is true. Since $\det A \neq 0$, the solution $x = 0$ to $Ax = 0$ is unique. Thus the columns of A are linearly independent; d implies a.

Chapter 12

12.1 *a*) $x_n = n$. *b*) $x_n = 1/n$. *c*) $x_n = 2^{(-1)^{n-1}(n-1)}$.

d) $x_n = (-1)(n-1)(n-1)/n$. *e*) $x_n = (-1)^n$. *f*) $x_n = (n+1)/n$.

g) x_n is the truncation of π to n decimal places.

h) x_n is the value of the nth decimal place in π.

12.2 *a*) 1/2 comes before 3/2 in original sequence.

b) Not infinite.

c) $2/1$ is not in original sequence.

12.3 If x and y are both positive, so is $x + y$, and $x + y = |x| + |y| = |x + y|$.
If x and y are both negative, so is $x + y$, and $-(x + y) = -x - y$; that is, $|x + y| = |x| + |y|$.
If x and y are opposite signs, say $x > 0$ and $y \leq 0$, then $x + y \leq x = |x| \leq |x| + |y|$ and $-x - y \leq -y = |y| \leq |x| + |y|$. Since $x + y \leq |x| + |y|$ and $-(x + y) \leq |x| + |y|$, $|x + y| \leq |x| + |y|$.
It follows that $|x| = |y + (x - y)| \leq |y| + |x - y|$; so $|x| - |y| \leq |x - y|$. Also, $|y| = |x + (y - x)| \leq |x| + |y - x| = |x| + |x - y|$; so $|y| - |x| \leq |x - y|$. Therefore, $||x| - |y|| \leq |x - y|$.

12.4 If x and y are ≥ 0, so is xy and $|x||y| = xy = |xy|$.
If x and y are ≤ 0, $xy = |xy|$.
If $x \geq 0$ and $y \leq 0$, $xy \leq 0$ and $|xy| = -xy$. Then, $|x||y| = x(-y) = -xy = |xy|$. Similarly, for $x \leq 0$ and $y \geq 0$.

12.5 $|x + y + z| = |(x + y) + z| \leq |x + y| + |z| \leq |x| + |y| + |z|$.

12.6 Follow the proof of Theorem 12.2, changing the last four lines to

$$\begin{aligned}
|(x_n - y_n) - (x - y)| &= |(x_n - x) - (y_n - y)| \\
&\leq |x_n - x| + |y_n - y| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

12.7 *a*) There exists $N > 0$ such that $n \geq N \Longrightarrow |x_n - x_0| < \frac{1}{2}|x_0|$.
Then, $|x_0| - |x_n| \leq |x_0 - x_n| < \frac{1}{2}|x_0|$, or $\frac{1}{2}|x_0| \leq |x_n|$, for all $n \geq N$.
Let $B = \min\left\{\frac{1}{2}|x_0|, |x_1|, |x_2|, \ldots, |x_N|\right\}$.
For $n \leq N$, $|x_n| \geq \min\{|x_j| : 1 \leq j \leq N\} \geq B$.
For $n \geq N$, $|x_n| \geq \frac{1}{2}|x_0| \geq B$.

b) Let B be as in part *a*). Let $\varepsilon > 0$. Choose N such that for $n \geq N$, $|x_n - x_0| \leq \varepsilon B|x_0|$. Then, for $n \geq N$,

$$\left|\frac{1}{x_n} - \frac{1}{x_0}\right| = \frac{|x_n - x_0|}{|x_n|\,|x_0|} \leq \frac{|x_n - x_0|}{B|x_0|}$$

(since $|x_n| \geq B$ for all $n \Longrightarrow 1/|x_n| \leq 1/B$ for all n)

$$\leq \frac{\varepsilon \cdot B|x_0|}{B|x_0|} = \varepsilon.$$

Therefore, $1/x_n \to 1/x_0$.

12.8 Suppose $y_n \to y$ with all y_ns and y nonzero. By the previous exercise, $1/y_n \to 1/y$. By Theorem 12.3

$$\frac{x_n}{y_n} = x_n \cdot \frac{1}{y_n} \to x \cdot \frac{1}{y} = \frac{x}{y}.$$

12.9 Let $\varepsilon > 0$. Choose N such that for $n \geq N$, $|x_n - 0| \leq \varepsilon \cdot B$ where $|y_n| \leq B$ for all n. Then, for $n \geq N$,

$$|x_n \cdot y_n - 0| = |x_n \cdot y_n| = |x_n||y_n| \leq \frac{\varepsilon}{B} \cdot B = \varepsilon.$$

12.10 Suppose that $x_n \geq b$ for all n and that $x < b$. Choose $\varepsilon > 0$ such that $0 < \varepsilon < b - x$. So, $\varepsilon + x < b$ and $I_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$ lies to the left of b on the number line. There exists an $N > 0$ such that for all $n \geq N$, $x_n \in I_\varepsilon(x)$. For these x_ns, $x_n < x + \varepsilon < b$, a contradiction to the hypothesis that $x_n \geq b$ for all n.

12.11 The proof of Theorem 12.3 carries over perfectly, using the fact that $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|$ holds in $\mathbf{R^n}$.

12.12 Suppose $\mathbf{x}_n \to \mathbf{a}$ and $\mathbf{b} \neq \mathbf{a}$ is an accumulation point. Choose $\varepsilon = \|\mathbf{a} - \mathbf{b}\|/4$. There exists an N such that $n \geq N \Longrightarrow \|\mathbf{x}_n - \mathbf{a}\| < \varepsilon$. Since $\mathbf{b}$ is an accumulation point, there exists an $m \geq N$ such that $\|\mathbf{x}_m - \mathbf{b}\| < \varepsilon$.

Then,

$$\begin{aligned}\|\mathbf{a}-\mathbf{b}\| &= \|\mathbf{a}-\mathbf{x}_m+\mathbf{x}_m-\mathbf{b}\| \\ &\le \|\mathbf{x}_m-\mathbf{a}\|+\|\mathbf{x}_m-\mathbf{b}\| \\ &< \varepsilon+\varepsilon = 2\varepsilon = \frac{\|\mathbf{a}-\mathbf{b}\|}{2}.\end{aligned}$$

Contradiction!

12.13 The union of the open ball of radius 2 around 0 with the open ball of radius 2 around 1 is open but not a ball. So is the intersection of these two sets.

12.14 Let $\mathbf{x} \in \mathbf{R^n}_+$. Choose $\varepsilon = \min\{x_i : i = 1, \dots, n\} > 0$. Show $B_{\varepsilon/2}(\mathbf{x}) \subset \mathbf{R^n}_+$. Let $\mathbf{y} \in B_{\varepsilon/2}(\mathbf{x})$ and $j = 1, \dots, n$.

$$\begin{aligned}|y_j - x_j| &\le \sqrt{(y_1-x_1)^2+\cdots+(y_n-x_n)^2} = \|\mathbf{y}-\mathbf{x}\| < \varepsilon/2. \\ x_j - y_j &\le |x_j - y_j| \Longrightarrow x_j - |x_j-y_j| \le y_j. \\ y_j &\ge x_j - |x_j - y_j| \ge \varepsilon - \varepsilon/2 = \varepsilon/2 > 0.\end{aligned}$$

12.15 In $\mathbf{R^1}$, $B_\varepsilon(x) = \{y : \|y-x\| < \varepsilon\} = \{y : |y-x| < \varepsilon\}$

$$\begin{aligned}&= \{y : -\varepsilon < y - x < +\varepsilon\} = \{y : x-\varepsilon < y < x+\varepsilon\} \\ &= (x-\varepsilon, x+\varepsilon).\end{aligned}$$

12.16 If A is open, for all $x \in A$ there is an open ball B_x containing x and in A. Then $\cup_{x\in A} B_x \subset A \subset \cup_{x\in A} B_x$. Let A° denote the interior of A. By definition, for all $x \in A$, $B_x \subset A^\circ$, so $\cup_{x\in A} B_x \subset A^\circ \subset A \subset \cup_{x\in A} B_x \subset A^\circ$, and $A^\circ \subset A \subset A^\circ$. The two sets are identical.

12.17 $\mathbf{x} \in \operatorname{int} S \Longrightarrow$ there exists $\varepsilon > 0$ such that $B_\varepsilon(\mathbf{x}) \subset S$. If $\mathbf{x}_n \to \mathbf{x}$, there exists an $N > 0$ such that $\|\mathbf{x}_n - \mathbf{x}\| < \varepsilon$ for all $n \ge N$; that is, $\mathbf{x}_n \in B_\varepsilon(\mathbf{x})$ for all $n \ge N$.

12.18 Suppose $\{x_n\}$ is a sequence in $[a, b]$ that converges to x. By Theorem 12.4, $x \in [a, b]$ too.

12.19 Fix $\mathbf{z} \in \mathbf{R^n}$ and $\varepsilon > 0$. Let $F = \{\mathbf{x} \in \mathbf{R^n} : \|\mathbf{x}-\mathbf{z}\| \le \varepsilon\}$. Let $\mathbf{y}_n$ be a sequence in F with $\mathbf{y}_n \to \mathbf{y}$. Show $\mathbf{y} \in F$; that is, $\|\mathbf{y}-\mathbf{z}\| \le \varepsilon$. Suppose $\|\mathbf{y}-\mathbf{z}\| > \varepsilon$. Choose $\varepsilon_1 = \frac{1}{2}(\|\mathbf{y}-\mathbf{z}\| - \varepsilon) > 0$. Choose $\mathbf{y}_n$ in the sequence with $\|\mathbf{y}_n - \mathbf{y}\| < \varepsilon_1$. Then,

$$\begin{aligned}\|\mathbf{y} - \mathbf{z}\| &= \|(\mathbf{y} - \mathbf{y}_n) + (\mathbf{y}_n - \mathbf{z})\| \\ &\leq \|\mathbf{y} - \mathbf{y}_n\| + \|\mathbf{y}_n - \mathbf{z}\| \\ &\leq \varepsilon_1 + \varepsilon = \tfrac{1}{2}\|\mathbf{y} - \mathbf{z}\|,\end{aligned}$$

a contradiction.

12.20 Let F be a finite set of points $\{\mathbf{p}_1, \dots, \mathbf{p}_M\}$ in $\mathbf{R^n}$. Let $d = \min\{\|\mathbf{p}_i - \mathbf{p}_j\| : i \neq j,\ i, j = 1, \dots, M\} > 0$. Suppose $\{\mathbf{x}_n\}$ is a sequence in F and $\mathbf{x}_n \to \mathbf{x}$. Show $\mathbf{x} \in F$. There exists an N such that $n \geq N \implies \|\mathbf{x}_n - \mathbf{x}\| < d/2$. Let $n_1, n_2 \geq N$. Then, $\|\mathbf{x}_{n_1} - \mathbf{x}_{n_2}\| \leq \|\mathbf{x}_{n_1} - \mathbf{x}\| + \|\mathbf{x} - \mathbf{x}_{n_2}\| < d$. But $\|\mathbf{x}_{n_1} - \mathbf{x}_{n_2}\| < d$ and $\mathbf{x}_{n_1}, \mathbf{x}_{n_2} \in F$ implies that $\mathbf{x}_{n_1} = \mathbf{x}_{n_2}$; that is, there exists a $\mathbf{p} \in F$ such that $\mathbf{x}_n = \mathbf{p}$ for all $n \geq N$. Then, $\lim \mathbf{x}_n = \mathbf{p} \in F$. Proof that the set of integers is a closed set is the same, taking $d = 1$.

12.21 *a*) Not open, because $(0, 0)$ is in the set but $(0, \varepsilon)$ is not in the set for any $\varepsilon \neq 0$.
Not closed, because $\{(n/(n+1), 0) : \text{integer } n\}$ is in the set, but its limit $(1, 0)$ is not.

b) Not open, since $(0, 0)$ is in the set, but $(0, \varepsilon)$ for all small $\varepsilon > 0$ is not.
Closed; same argument as in Exercise 12.20*a*.

c) Not open, since no disk about $(1, 0)$ is in the set.
Closed, since if $\{(x_n, y_n)\}$ is a sequence in the set that converges to (x, y), then $x + y = \lim x_n + \lim y_n = \lim(x_n + y_n) = 1$.

d) Open, since if $x + y < 1$, then so is $(x + h) + (y + k)$ for all sufficiently small h and k.
Not closed, because if $x + y = 1$, then (x, y) is not in the set, but $(x - 1/n, y - 1/n)$ for all integers $n > 0$.

e) This set is the union of the x- and y-axes. Each axis is closed. For instance, if $(x_n, 0)$ converges to (x, y), then $y = 0$ and so (x, y) is on the x-axis. Thus the set is the union of two closed sets, and hence is closed.
The set is not open. $(0, 0)$ is in the set, but $(\varepsilon, \varepsilon)$ fails to be in the set for all $\varepsilon > 0$.

12.22 The complement of any intersection of closed sets is a union of open sets. This union is open according to Theorem 12.8, so its complement is closed. The complement of a finite union of closed sets is a finite intersection of open sets. According to Theorem 12.8 this intersection is open, so its complement is closed.

12.23 If $x \in \operatorname{cl} S$, then there exists a sequence $\{x_n\} \subset S$ converging to x. Since there are points in S arbitrarily near to x, $x \notin \operatorname{int} T$. If $x \notin \operatorname{cl} S$, then for some $\varepsilon > 0$, $B_\varepsilon(x) \cap S = \emptyset$. (Otherwise it would be possible to construct a sequence in S converging to x.) Therefore $x \in \operatorname{int} T$.

12.24 Any accumulation point of S is in $\operatorname{cl} S$. To see this, observe that if x is an accumulation point of S, then for all $n > 0, B_{1/n}(x) \cap S \neq \emptyset$. For each integer n, choose x_n to be in this set. Then the sequence $\{x_n\}$ is in S and converges to x, so $x \in \operatorname{cl} S$. Conversely, if x is in $\operatorname{cl} S$, then there is a sequence $\{y_n\} \subset S$ with limit x. Therefore, for all $\varepsilon > 0$ there is a $y \in S$ such that $\|y - x\| < \varepsilon$. Consequently, for all $\varepsilon > 0, B_\varepsilon(x) \cap S \neq \emptyset$, so x is an accumulation point of S.

12.25 There is no ball around b contained in $(a, b]$, so this interval is not open. A sequence converging down to a from above and bounded above by b will be in the interval, but its limit a is not. Hence the interval is not closed.

The sequence converges to 0, but 0 is not in the set so the set is not closed. No open ball of radius less than 1/2 around the point 1 contains any elements of the set, so the set is not open.

A sequence in the line without the point converging to the point lies in the line without the point, but its limit does not. Hence the set is not closed. Any open ball around a point on the line will contain points not on the line, so the set is not closed.

12.26 Suppose that x is a boundary point. Then for all n there exist points $x_n \in S$ and y_n in S^c, both within distance $1/n$ of x. Thus x is the limit of both the $\{x_n\}$ and $\{y_n\}$ sequences, so $x \in \operatorname{cl} S$ and $x \in \operatorname{cl} S^c$. Conversely, if x is in both $\operatorname{cl} S$ and $\operatorname{cl} S^c$, there exist sequences $\{x_n\} \subset S$ and $\{y_n\} \subset S^c$ converging to x. Thus every open ball around x contains elements of both sequences, so x is a boundary point.

12.27 The whole space $\mathbf{R^m}$ contains open balls around every one of its elements, so it is open. Any limit of a sequence in $\mathbf{R^m}$ is, by definition, in $\mathbf{R^m}$, so $\mathbf{R^m}$ is closed. The empty set is the complement of $\mathbf{R^m}$. It is the complement of a closed and open set, and consequently is both open and closed. Checking directly, the empty set contains no points or sequences to falsify either definiton, and so it satisfies both.

12.28 The hint says it all. Let a be a member of S and b not. Let $l = \{x : x = x(t) \equiv ta + (1 - t)b, 0 \leq t < \infty\}$. Let $t*$ be the least upper bound of the set of all t such that $x(t) \in S$. Consider the point $x(t^*)$. It must be in S because there exists a sequence $\{t_n\}$ converging up to t^* such that each

$x(t_n) \in S$, $x(t*)$ is the limit of the $x(t_n)$, and S is closed. Since S is open, there exists a ball of positive radius around $x(t*)$ contained in S. Thus for $\varepsilon > 0$ sufficiently small, $x(t * + \varepsilon) \in S$ which contradicts the construction of $t*$.

Chapter 13

13.1 See figures.

13.3 Intersect the graph with the plane $z = k$ in $\mathbf{R}^3$, and project the resulting curve down to the $z = 0$ plane.

13.4 A map that gives various depths in a lake. Close level curves imply a steep drop off (and possibly good fishing).

13.6 *a*) Spheres around the origin.

b) Cylinders around the x_3-axis.

c) The intersection of the graph with planes parallel to the x_1x_3-plane are parabolas with a minimum at $x + 1 = 0$. The level of the parabolas shrinks as y grows.

d) Parallel planes.

13.9 See figures.

13.10 The graph of f is $G = \{(x, y) : y = f(x)\}$. Consider the curve $F(t) = (t, f(t))$. The point (x, y) is on the curve if and only if there is a t such that $x = t$ and $y = f(t)$; that is true if and only if $y = f(x)$; and this is true if and only if (x, y) is in the graph of f.

13.11 *a*) $\begin{pmatrix} 2 & -3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$,

b) $\begin{pmatrix} 2 & -3 \\ 1 & -4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$,

c) $\begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & -6 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

13.12 *a*) $\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$,

$b)$ $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 5 & -5 \\ -5 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$,

$c)$ $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 2 & -3 \\ 2 & 2 & 4 \\ -3 & 4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

13.13 $f(x) = \sin x$, $g(x) = e^x$, $h(x) = \log x$.

13.14 Suppose $f(\mathbf{x})$ is linear. Let $a_i = f(\mathbf{e}_i)$, and suppose $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_k\mathbf{e}_k$. Then

$$\begin{aligned} f(\mathbf{x}) &= f(x_1\mathbf{e}_1 + \cdots + x_k\mathbf{e}_k) \\ &= \sum_i x_i f(\mathbf{e}_i) \\ &= \sum_i a_i x_i \\ &= (a_1, \ldots, a_k)(x_1, \ldots, x_k). \end{aligned}$$

13.15 Consider the line $\mathbf{x}(t) = \mathbf{x} + t\mathbf{y}$, and suppose f is linear. Then $f(\mathbf{x}(t)) = f(\mathbf{x} + t\mathbf{y}) = f(\mathbf{x}) + tf(\mathbf{y})$, so the image of the line will be a line if $f(\mathbf{y}) \neq \mathbf{0}$ and a point if $f(\mathbf{y}) = \mathbf{0}$.

13.16 Let $\{\mathbf{x}_n\}_{n=1}^{\infty}$ be a sequence with limit $\mathbf{x}$. Since g is continuous, there exists an N such that for all $n \geq N$, $|g(\mathbf{x}_n) - g(\mathbf{x})| < g(\mathbf{x})/2$. Thus, for $n \geq N$, $g(\mathbf{x}_n) \neq 0$, since $g(\mathbf{x}) - g(\mathbf{x}_n) < g(\mathbf{x})/2$ implies that $0 < g(\mathbf{x})/2 < g(\mathbf{x}_n)$. Since convergence depends only on the "large N" behavior of the sequence, we can suppose without loss of generality that $g(\mathbf{x}_n)$ is never 0. Now, $f(\mathbf{x}_n)$ converges to $f(\mathbf{x})$ and $g(\mathbf{x}_n)$ converges to $g(\mathbf{x})$, so the result follows from Exercise 12.8.

13.17 Suppose not; that is, suppose that for every n there exists an $\mathbf{x}_n \in B_{1/n}(\mathbf{x}^*)$ with $f(\mathbf{x}_n) \leq 0$. Since $\|\mathbf{x}_n - \mathbf{x}^*\| < 1/n$, $\mathbf{x}_n \to \mathbf{x}^*$. Since f is continuous, $f(\mathbf{x}_n) \to f(\mathbf{x}^*)$. Since each $f(\mathbf{x}_n) \leq 0$, $f(\mathbf{x}^*) = \lim f(\mathbf{x}_n) \leq 0$, but $f(\mathbf{x}^*) > 0$. (See Theorem 12.4).

13.18 From Theorem 12.5 conclude that a function $f : \mathbf{R^k} \to \mathbf{R^m}$ is continuous if and only if its coordinate functions are continuous. From Theorems 12.2 and 12.3 and this observation, conclude that if f and g are continuous, then the coordinate functions of $f + g$ and $f \cdot g$ are continuous.

13.19 $\Longrightarrow$: Suppose $f = (f_1, \ldots, f_m)$ is continuous at $\mathbf{x}^*$. Let $\{\mathbf{x}_n\}$ be a sequence converging to $\mathbf{x}^*$. Then, $f(\mathbf{x}_n) \to f(\mathbf{x}^*)$. By Theorem 12.5, each $f_i(\mathbf{x}_n) \to f_i(\mathbf{x}^*)$. So each f_i is continuous.

$\Longleftarrow$: Reverse the above argument.

13.20 Suppose $\mathbf{x}_n = (x_{n1}, x_{n2}, \ldots, x_{nk}) \to \mathbf{x}_0 = (x_{01}, x_{02}, \ldots, x_{0k})$. By Theorem 12.5, $x_{ni} \to x_{0i}$ for each i. This says $h(x_1, \ldots, x_k) = x_i$ is continuous. Any monomial $g(x_1, \ldots, \mathbf{x}_k) = Cx_1^{n_1} \cdots x_k^{n_k}$ is a product of such hs (for positive integer n_is) and is continuous by Theorem 13.4. Any polynomial is a sum of monomials and is continuous by Theorem 13.4.

13.21 Let $\{t_n\}_{n=1}^{\infty}$ denote a sequence with limit t^*. Then $g(t_n) = f(t_n, a_2, \ldots, a_k) \to f(t^*, a_2, \ldots, a_k) = g(t^*)$, so g is continuous.

For $f(x, y) = xy^2/(x^4 + y^4)$, $f(t, a) = ta^2/(t^4 + a^4)$. If $a \neq 0$ this sequence clearly converges to $0/a^4 = 0$ as t converges to 0. If $a = 0$, $f(t, 0) = 0$ for all t. Similar arguments apply to continuity in the second coordinate. But $f(t, t) = t^3/(t^4 + t^4)$, which grows as $1/t$ as $t \to 0$.

13.22 *a)* If $y \in f(U_1)$, then there is an $x \in U_1$ such that $f(x) = y$. Since $U_1 \subset U_2$, $x \in U_2$, so $y \in f(U_2)$.

b) If $x \in f^{-1}(V_1)$, then there is a $y \in V_1$ such that $f(x) = y$. Now $y \in V_2$, so $x \in f^{-1}(V_2)$.

c) If $x \in U$, then there is a $y \in f(U)$ such that $f(x) = y$. Thus $x \in f^{-1}(f(U))$.

d) If $y \in f(f^{-1}(V))$, then there is an $x \in f^{-1}(V)$ such that $f(x) = y$, so $f(x) \in V$.

e) $x \in f^{-1}(V^c)$ if and only if there is a $y \in V^c$ such that $f(x) = y$. Since $f(x) = y \in V^c$, $f(x) \notin V$. Thus $x \notin f^{-1}(V)$, and so $x \in (f^{-1}(V))^c$.

13.23

	$f(x)$	*Domain*	*Range*	*One-to-one*	$f^{-1}(y)$	*Onto*
a)	$3x - 7$	**R**	**R**	Yes	$\frac{1}{3}(y + 7)$	Yes
b)	$x^2 - 1$	**R**	$[-1, \infty)$	No		No
c)	e^x	**R**	$(0, \infty)$	Yes	$\ln(y)$	No
d)	$x^3 - x$	**R**	**R**	No		Yes
e)	$x/(x^2 + 1)$	**R**	$[-\frac{1}{2}, \frac{1}{2}]$	No		No
f)	x^3	**R**	**R**	Yes	$y^{1/3}$	Yes
g)	$1/x$	$\mathbf{R} - \{\mathbf{0}\}$	$\mathbf{R} - \{\mathbf{0}\}$	Yes	$1/y$	No
h)	$\sqrt{x - 1}$	$[1, \infty)$	$[0, \infty)$	Yes	$y^2 + 1$	No
i)	xe^{-x}	**R**	$(-\infty, 1/e]$	No		No

13.24 *a*) $f(x) = \log x$, $g(x) = x^2 + 1$; *b*) $f(x) = x^2$, $g(x) = \sin x$;

c) $f(x) = (\cos x, \sin x)$, $g(x) = x^3$; *d*) $f(x) = x^3 + x$, $g(x) = x^2 y$.

Chapter 14

14.1 *a*) $\dfrac{\partial f}{\partial x} = 8xy - 3y^3 + 6, \qquad \dfrac{\partial f}{\partial y} = 4x^2 - 9xy^2.$

b) $\dfrac{\partial f}{\partial x} = y, \qquad \dfrac{\partial f}{\partial y} = x.$

c) $\dfrac{\partial f}{\partial x} = y^2, \qquad \dfrac{\partial f}{\partial y} = 2xy.$

d) $\dfrac{\partial f}{\partial x} = 2e^{2x+3y}, \qquad \dfrac{\partial f}{\partial y} = 3e^{2x+3y}.$

e) $\dfrac{\partial f}{\partial x} = \dfrac{-2y}{(x+y)^2}, \qquad \dfrac{\partial f}{\partial y} = \dfrac{2x}{(x+y)^2}.$

f) $\dfrac{\partial f}{\partial x} = 6xy - 7\sqrt{y}, \qquad \dfrac{\partial f}{\partial y} = 3x^2 - \dfrac{7x}{2\sqrt{y}}.$

14.2 For the Cobb-Douglas function,

$$\frac{\partial f}{\partial x_i} = k\alpha_i x_i^{\alpha_i - 1} x_j^{\alpha_j} = \alpha_i \frac{q}{x}.$$

For the CES function,

$$\frac{\partial f}{\partial x_i} = c_i hk(c_1 x_1^{-a} + c_2 x_2^{-a})^{-(h/a)-1} x_i^{-a-1}.$$

14.3 $(\partial T/\partial x)(x^*, y^*)$ is the rate of change of temperature with respect to an increase in x while holding y fixed at the point (x^*, y^*).

14.4 *a*) $Q = 5400$.

b, c) $Q(998, 216) = 5392.798$. The approximation gives $Q \approx 5392.8$, which is in error by -0.002.

$Q(1000, 217.5) = 5412.471$. The approximation gives $Q \approx 5412.5$, which is in error by -0.029.

d) For the approximation to be in error by more than 2.0, ΔL must be at least 58.475, or about 5.875% of L.

More generally, consider the system of equations (8). To achieve an aggregate $1 > 2$ ($2 > 1$), choose $y_{12} > 0$ ($y_{12} < 0$), etc. Then, solve system (8) for the vector of profiles $(N_1, \ldots, N_6)$. One way to do this is to find (x_1, x_2, x_3) that satisfies:

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_{12} \\ y_{13} \\ y_{23} \end{pmatrix};$$

that is,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (0.5) \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_{12} \\ y_{13} \\ y_{23} \end{pmatrix}.$$

Then, for an appropriate N, $\begin{pmatrix} N + x_1 \\ N + x_2 \\ N + x_3 \\ N \\ N \\ N \end{pmatrix}$ will give the appropriate number of voters with each profile to achieve the aggregate paired rankings.

28.6 *a*) Let $w_i = 1$ if $i = 1$ and 0 otherwise.

b) Let N_{ik} denote the number of voters who put alternative i in position k. Then

$$\begin{aligned} i \text{ is preferred to } j \text{ iff } \quad & \sum_k w_k N_{ik} > \sum_k w_k N_{jk} \\ \text{iff} \quad & a \sum_k w_k N_{ik} + b \sum_k N_{ik} > a \sum_k w_k N_{jk} + b \sum_k N_{jk} \end{aligned}$$

since $\sum_k N_{ik} = \sum_k N_{jk} = N$, the total number of voters.

c) For a given vector of weights $\mathbf{w}$, define a new vector of weights $w'_i = w_i + b$ where $b = -w_n$. Then, let $\mathbf{v} = a\mathbf{w}'$ where $a = 1/\sum_i w'_i$. According to part *b*, $\mathbf{w}$, $\mathbf{w}'$ and $\mathbf{v}$ give identical outcomes. But $w'_n = v_n = 0$ and $\sum_j v_j = 1$.

Chapter 29

29.1 *b*), *d*), *e*), *f*) *g*) and *h*) are bounded sequences, with least upper bounds 1, 1, 1, 2, π, and 9, respectively.

29.2 Let $\{x_n\}_{n=1}^{\infty}$ denote a bounded, decreasing sequence with greatest lower bound b. We want to show that $x_n \to b$. For all $\varepsilon > 0$, there is an N such that $b \leq x_N < b + \varepsilon$. Since the sequence is decreasing with greatest lower bound b, it follows that for all $n > N$, $b \leq x_n \leq x_N < b + \varepsilon$. Thus, for all $\varepsilon > 0$ there is an N such that for all $n \geq N$, $|x_n - b| < \varepsilon$, that is, b is the limit of $\{x_n\}_{n=1}^{\infty}$.

29.3 Suppose that $\{x_n\}_{n=1}^{\infty}$ converges to x, and that $\{y_m\}_{m=1}^{\infty}$ is a subsequence. Let $n(m)$ denote the index in the original x-sequence of the mth element of the y-sequence. The function $n(m)$ is strictly increasing. Clearly, $n(m) \geq m$. For all $\varepsilon > 0$ there is an N such that for all $n \geq N$, $|x_n - x| < \varepsilon$. Choose $N_1 \geq N$ with $N_1 = n(M_1)$ for some M_1. Then, for all $m \geq M_1 = n^{-1}(N_1)$, $|y_m - x| < \varepsilon$.

29.4 Suppose that b and c are least upper bounds for a set S. Since b is a least upper bound and c is an upper bound, $b \leq c$. Since c is a least upper bound and b is an upper bound, $c \leq b$. Consequently, $c = b$.

29.5 Suppose x is an accumulation point of a sequence $\{x_n\}_{n=1}^{\infty}$. Then, for all $\varepsilon > 0$ there are infinitely many elements x_n such that $|x_n - x| < \varepsilon$. Let $y_1 = x_1$. Let $n_1 = 1$, and for $i = 1, 2, 3, \ldots$, let n_i denote the first element x_k in the sequence $\{x_n\}_{n=1}^{\infty}$ after $x_{n_{i-1}}$ such that $|x_k - x| < 1/i$. Then the sequence defined by $y_j = x_{n_j}$ converges to x, since $|y_j - x| < 1/j$.

29.6 If a sequence of vectors is Cauchy, then each sequence of coordinate vectors is a Cauchy sequence of real numbers. Thus, according to Theorem 29.3, each coordinate sequence converges; and so the sequence of vectors converges.

29.7 A set S is connected if for each pair of open sets U_1 and U_2, $S \cap U_1 \neq \emptyset$ and $S \cap U_2 \neq \emptyset$ and $S \subset U_1 \cup U_2$ implies that $U_1 \cap U_2 \neq \emptyset$.

29.8 Suppose $\mathrm{cl}\, S$ is not connected. Then, there exist disjoint open sets U_1 and U_2, each of which intersects $\mathrm{cl}\, S$, that satisfy $\mathrm{cl}\, S \subset U_1 \cup U_2$. These two sets cover S since $S \subset \mathrm{cl}\, S$. Furthermore, $x \in \mathrm{cl}\, S$ if and only if every open set containing x intersects S, so U_1 and U_2 each intersect S. Hence, S is not connected.

29.9 Let $S = B_{(+1,0)} \cup B_{(-1,0)}$, where $B_{\mathbf{x}}$ is the closed ball of radius 1 about $\mathbf{x}$ in $\mathbf{R}^2$. S is closed and connected. However, when you remove its figure-8-shaped boundary, you find that its interior is the *disjoint* union of two open balls, a disconnected set. See figure.

29.10 The closure of the graph G is $G \cup \{(0, x) : -1 \leq x \leq 1\}$. The fact that this set is connected follows from Problem 29.8 and the fact that the curve G is connected. See figure.

29.11 *i* is closed and connected. *ii* is closed, compact and connected. *iii* is closed and connected. *iv* is the union of the x- and y-axes less the origin. It is not connected, and neither open nor closed. See figure.

29.12 *i* is a closed, connected subspace. *ii* is a connected, open annulus. *iii* is the closed unit simplex in $\mathbf{R^3}$, and is compact and connected.

29.13 $N_{(a_1,\dots,a_n)}(\mathbf{x})$ is the square root of the sum of nonnegative numbers, and hence nonnegative. If any one x_i is strictly positive, so is the sum and hence so is the square root. Thus a and b hold. Then

$$N_{(a_1,\dots,a_n)}(r\mathbf{x}) = \sqrt{\sum_i a_i r^2 x_i^2} = |r|\sqrt{\sum_i a_i x_i^2},$$

proving c. Finally,

$$N_{(a_1,\dots,a_n)}(\mathbf{x}+\mathbf{y}) = \sqrt{\sum_i a_i(x_i + y_i)^2} = \sqrt{\sum_i (\sqrt{a_i}x_i + \sqrt{a_i}y_i)^2}.$$

From Theorem 10.5 it follows that the last term is less than or equal to

$$\sqrt{\sum_i a_i x_i^2} + \sqrt{\sum_i a_i y_i^2} = N_{(a_1,\dots,a_n)}(\mathbf{x}) + N_{(a_1,\dots,a_n)}(\mathbf{y}),$$

which proves d. (In Theorem 10.5, take $\mathbf{u} = (a_1x_1, \dots, a_nx_n)$ and $\mathbf{v} = (a_1y_1, \dots, a_ny_n)$.)

29.14 a, b, c, and d all follow from the corresponding property of the absolute value. For example,

$$\begin{aligned} N_1(\mathbf{x}+\mathbf{y}) &= |x_1 + y_1| + \cdots + |x_n + y_n| \\ &\leq |x_1| + |y_1| + \cdots + |x_n| + |y_n| \\ &= (|x_1| + \cdots + |x_n|) + (|y_1| + \cdots + |y_n|) \\ &= N_1(\mathbf{x}) + N_1(\mathbf{y}). \end{aligned}$$

29.15 $N_0(\mathbf{x})$ is the max of nonnegative elements, hence nonnegative. If $N_0(\mathbf{x}) = 0$, then each $|x|_j = 0$, and so $\mathbf{x} = \mathbf{0}$. $N_0(r\mathbf{x}) = \max\{|rx_1|, \dots, |rx_n|\} =$

$|r|N_0(\mathbf{x})$, proving c. Finally,

$$\begin{aligned} N_0(\mathbf{x}+\mathbf{y}) &= \max\{|x_1+y_1|,\ldots,|x_n+y_n|\} \\ &\le \max\{|x_1|+|y_1|,\ldots,|x_n|+|y_n|\} \\ &\le \max\{|x_1|,\ldots,|x_n|\} + \max\{|y_1|,\ldots,|y_n|\}, \end{aligned}$$

proving d.

29.16 Since $|x_i| \le N_0(\mathbf{x})$ for all i, $N_1(\mathbf{x}) = \sum_i |x_i| \le nN_0(x)$. It is obvious from the definitions that $N_0(\mathbf{x}) \le N_1(\mathbf{x})$.

29.17 Suppose that $N \sim N'$. Then, there are $a, b > 0$ such that $aN(x) \le N'(x) \le bN(x)$ for all x. Then $(1/b)N'(x) \le N(x) \le (1/a)N'(x)$, so $N' \sim N$.

Take $a = b = 1$ to see that $N \sim N$ for any norm N.

Suppose $N \sim N'$ and $N' \sim N''$. Then there are $a, b, a', b' > 0$ such that

$$aN(x) \le N'(x) \le bN(x) \quad \text{and} \quad a'N'(x) \le N''(x) \le b'N'(x) \quad \text{for all } x.$$

Then, $aa'N(x) \le N''(x) \le bb'N(x)$, so $N \sim N''$.

29.18 Let $\{b_n\}_{n=1}^{\infty}$ be a sequence, and suppose $N \sim N'$. Then for some positive B and all n, $0 < N'(b_n - b) < BN(b_n - b)$. If the sequence converges to b in the N-norm, then the right-hand side converges to 0. Hence, so does the left; so the sequence converges to b in the N'-norm.

29.19 For $\mathbf{x} \in \mathbf{R^n}$,

$$\begin{aligned} N_0(\mathbf{x}) &= \max_k\{|x_k|\} \\ &\le \sum_k |x_k| \\ &= \sum_k \sqrt{x_k^2} \\ &\le \sqrt{\sum_k x_k^2} = N_2(\mathbf{x}) \end{aligned}$$

where the last inequality follows from the concavity of the square-root function. On the other side,

$$N_2(\mathbf{x}) \equiv \sqrt{\sum_k x_k^2} \le \sqrt{n \max_k x_k^2}$$
$$= \sqrt{n} \max_k |x_k|$$
$$= \sqrt{n} N_0(\mathbf{x}).$$

29.20 The N_0-unit ball in the Euclidean plane is a box: $-1 \le x_i \le 1$ for $i = 1, 2$.

The N_1-unit ball is the set of all points $\mathbf{x}$ such that $|x_1| + |x_2| \le 1$. This set is diamond-shaped, given by the intersection of the following four half-spaces: $x_1 + x_2 \le 1$, $x_1 - x_2 \le 1$, $-x_1 + x_2 \le 1$, and $-x_1 - x_2 \le 1$.

The N_2-unit ball satisfies the inequality $\|\mathbf{x}\|^2 \le 1$, which in this case is $x_1^2 + x_2^2 \le 1$. This inequality describes a disk of radius 1 around the origin. The various weighted Euclidean norms have unit balls given by the inequality $a_1 x_1^2 + a_2 x_2^2 \le 1$ with $a_1, a_2 > 0$. These are ellipses. See figure.

29.21 This computation is contained in Exercise 19, where it is shown that

$$\sqrt{\sum_k x_k^2} \le \sqrt{n} \max_k |x_k| = \sqrt{n} N_0(\mathbf{x}).$$

Thus if $N_0(\mathbf{x} - \mathbf{x}_0) < r/\sqrt{n}$, $N_2(\mathbf{x} - \mathbf{x}_0) < r$.

29.22 Suppose the set S is bounded in the N-norm; i.e., $N(\mathbf{y}) \le a$ for all $\mathbf{y} \in S$. Since all norms in $\mathbf{R^n}$ are equivalent, for any other norm N', $N'(\mathbf{x}) < BN(\mathbf{x})$ for all $\mathbf{x}$. Then for any $\mathbf{y} \in S$, $N'(\mathbf{y}) < Ba$, and S is bounded in the N'-norm.

Similarly, suppose $AN(\mathbf{x}) \le N'(\mathbf{x}) \le BN(\mathbf{x})$ for all $\mathbf{x}$. Then, the open N'-ball of radius ε is contained in the open N-ball of radius ε/A and contains the open N-ball of radius ε/B. Thus, every N-open set is an N'-open set, and vice versa. Since closed sets are complements of open sets, the closed sets for the two norms must be identical too. Since both norms have the same open sets, a set S is N-connected if and only if it is N'-connected. It follows from the definition and the first part of this question that a set S is N'-closed and bounded if and only if it is N-closed and bounded.

29.23 *a*) N_0-norm: For any norm N, write $B(\mathbf{x}, \varepsilon, N)$ for the open set

$$B(\mathbf{x}, \varepsilon, N) \equiv \{\mathbf{y} \in \mathbf{R^n} : N(\mathbf{y} - \mathbf{x}) < \varepsilon\}.$$

Now, $\mathbf{x} \in \mathbf{R}^{\mathbf{n}}_{++}$ implies that each component $x_i > 0$ and $\min x_i = \varepsilon > 0$. We'll show that the open set $B(\mathbf{x}, \varepsilon/2, N_0)$ lies in $\mathbf{R}^{\mathbf{n}}_{++}$. Let $\mathbf{y} \in B(\mathbf{x}, \varepsilon/2, N_0)$. Then, $\max_i |y_i - x_i| < \varepsilon/2$, i.e.,

$$-\frac{\varepsilon}{2} < y_i - x_i < \frac{\varepsilon}{2} \quad \text{for all } i,$$

or

$$y_i > x_i - \frac{\varepsilon}{2} > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2} > 0 \quad \text{for all } i.$$

So, $B(\mathbf{x}, \varepsilon/2, N_0) \subset \mathbf{R}^{\mathbf{n}}_{++}$.

b) N_1-norm: As above, $\mathbf{x} \in \mathbf{R}^{\mathbf{n}}_{++}$ implies that $\min x_i = \varepsilon > 0$. We'll show that the open set $B(\mathbf{x}, \varepsilon/2, N_1)$ lies in $\mathbf{R}^{\mathbf{n}}_{++}$. Since $\max |y_i - x_i| \leq \sum_j |y_j - x_j|$, $B(\mathbf{x}, \varepsilon/2, N_1) \subset B(\mathbf{x}, \varepsilon/2, N_0) \subset \mathbf{R}^{\mathbf{n}}_{++}$.

29.24 Suppose that a finite subcollection covers $(0, 1)$. Let N denote the largest n such that the interval $\big(1/(n+1), n/(n+1)\big)$ is in the collection. Since the sets in the collection are nested in each other, the union of all elements in the collection is $\big(1/(n+1), n/(n+1)\big)$. If $0 < x < 1/(n+1)$, then x is not in the union; so $(0, 1)$ does not have the finite covering property.

29.25 The sets with $n = 1, \ldots, 10$ together with the two additional sets cover $[0, 1]$. The finite covering property requires that *every* open cover of $[0, 1]$ has a finite subcover.

29.26 Closed: Let $S = \{\mathbf{a}_1, \ldots, \mathbf{a}_N\}$ be a finite set in $\mathbf{R}^{\mathbf{n}}$. let $\varepsilon_0 = \min\{\|\mathbf{a}_i - \mathbf{a}_j\| : \mathbf{a}_i \neq \mathbf{a}_j \in S\}$. Let $\{\mathbf{z}_j\}_{j=1}^{\infty}$ be a convergent sequence in S with limit $\mathbf{z}_0$. Since the sequence is Cauchy, there exists N such that $j, k > N$ implies $\|\mathbf{z}_j - \mathbf{z}_k\| < \varepsilon/2$. By the definition of ε, this requires that $\mathbf{z}_N = \mathbf{z}_{N+1} = \ldots$. So, $\mathbf{z}_0$ is this common value and lies in S. Therefore, S is closed.

Bounded: Let $B = \max\{\|\mathbf{a}_i\| : \mathbf{a}_i \in S\}$; B makes S bounded.

Sequential compactness: Let $\{\mathbf{x}_n\}_{n=1}^{\infty}$ be a sequence in S. Since S is finite, there is some point $\mathbf{y} \in S$ such that $\mathbf{x}_n = \mathbf{y}$ for infinitely many n. The subsequence consisting of all $\mathbf{x}_n$ such that $\mathbf{x}_n = \mathbf{y}$ is obviously convergent.

Finite subcover: Let S denote an open cover of S. For each $\mathbf{x} \in S$ choose a single set $U_{\mathbf{x}} \in S$ such that $\mathbf{x} \in U_{\mathbf{x}}$. The collection $\{U_{\mathbf{x}}, \mathbf{x} \in S\}$ is a finite subcover.

29.27 According to Theorem 29.14, for each $\mathbf{x} \in K_1$ there is an open set $U_{\mathbf{x}}$ containing $\mathbf{x}$ and an open set $V_{\mathbf{x}}$ containing all of K_2 such that $U_{\mathbf{x}} \cap V_{\mathbf{x}} = \emptyset$. The collection $\{U_{\mathbf{x}}\}_{\mathbf{x} \in K_1}$ is an open cover of K_1. Choose a finite subcover

$\{U_{\mathbf{x}_1}, \ldots, U_{\mathbf{x}_n}\}$. The union U of these sets contains K_1. Let $V = \cap_{k=1}^{n} V_{\mathbf{x}_k}$. Since each $V_{\mathbf{x}_k}$ is open and the intersection is a finite one, V is open. Since each $V_{\mathbf{x}_k}$ contains K_2, so does V. Finally, $U \cap V = \emptyset$.

29.28 The intersection and finite union of closed and bounded sets are closed and bounded.

Let $\{\mathbf{x}_n\}_{n=1}^{\infty}$ denote a sequence of points in either the intersection or finite union of compact sets. In either case there is a compact set A containing an infinite number of points in the sequence. Let $\{\mathbf{y}_m\}_{m=1}^{\infty}$ denote the subsequence containing those points in the original sequence which are in the compact set A. Since A is compact, the sequence $\{\mathbf{y}_m\}_{m=1}^{\infty}$ has a convergent subsequence with limit $\mathbf{y}$. This limit point is obviously in the finite union of the compact sets, so the finite union is compact. In the intersection case, this convergent subsequence is contained in every compact set, so its limit $\mathbf{y}$ is contained in every compact set, and so $\mathbf{y}$ is in the intersection.

Let $\{K_1, \ldots, K_n\}$ denote a finite collection of compact sets, whose union is K. Let S be an open cover of K. Let S_m denote the collection of all elements of S which intersect K_m. Since K_m is compact, each S_m has a finite subcover of K_m. The (finite) union of these finite subcovers is a finite subcover of K.

Let $\{K_a\}_{a \in A}$ denote a collection of compact sets with intersection K. K is closed. Let S be an open cover of K, and let K_a be a compact set in the collection of compact sets. The collection $S \cup \{K^c\}$ is an open cover of K_a. Since K_a is compact, it has a finite subcover S'. Since this subcover covers K_a, it has to cover K. The finite collection $S' \setminus \{K^c\}$ consists only of sets in S, and covers K.

29.29 Let $\{\mathbf{x}_n\}_{n=1}^{\infty}$ denote a sequence of points in a closed subset C of a compact set K. The sequence has a convergent subsequence in K with limit $\mathbf{x}$. Since C is closed, $\mathbf{x} \in C$; so C is sequentially compact.

A closed subset of a bounded set is bounded, so C is compact.

Let S be a cover of C. Then $S \cup C^c$ is an open cover of K. Hence, it has a finite subcover S'. Finally, observe that $S' \setminus C^c$ covers C, so C has the Heine-Borel property.

Chapter 30

30.1 Apply the argument of the text to the continuous function $|F(x)|$ to show that $|F(x)|$ is bounded. Thus there is a $b > -\infty$ such that $F(x) > b$ for all $x \in C$. Let B denote the greatest lower bound of the values that F takes in C. Since B is the *greatest* lower bound, for all n there is an $x_n \in C$ such that $F(x_n) < B + (1/n)$. Since C is compact, we can extract a convergent subsequence $\{w_n\}_{n=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ with limit w. Since F is continuous, $\lim_n F(w_n) = F(w) = B$, so w is the global min of F in C. Alternatively, apply the proof for the existence of a global max to the function $-F$.

30.2 Denote the two sets $A_=$ and $A_\leq$. To prove $A_=$ is closed, we begin with a sequence of points $\{x_n\}_{n=1}^\infty$ in $A_=$ with limit x. We need to show that $x \in A_=$. Suppose (without loss of generality) that $F(x) \leq c$. Since F is continuous, $F(x_n) \to F(x)$. Now $F(x_n) \geq c$ for all n, so it follows from Theorem 12.4 that $F(x) \geq c$, and therefore that $F(x) = c$; so $x \in A_=$. Exactly the same argument applies to $A_\leq$.

30.3 The function $F(x) = 1 - \dfrac{1}{(1+x)}$ is a bounded function on its closed domain on $[0, \infty)$, but it does not achieve its supremum of 1.

30.4 *a)* Let N_0, N_1 and $N_2 = \|\ \|$ denote the standard norms on $\mathbf{R^n}$, as described in Section 29.4. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ denote the canonical basis of $\mathbf{R^n}$. Let $q_i = N(\mathbf{e}_i)/\|\mathbf{e}_i\|$ and $q = \max_i q_i$. For $\mathbf{x} = \sum_i x_i \mathbf{e}_i$,

$$\begin{aligned}
N(\mathbf{x}) &\leq \sum_i |x_i| N(\mathbf{e}_i) && \text{by properties } c \text{ and } d \text{ of a norm} \\
&= \sum_i |x_i|\, q_i\, \|\mathbf{e}_i\| && \text{by the definition of } q_i \\
&\leq q \sum_i |x_i|\, \|\mathbf{e}_i\| && \text{by definition of } q \\
&= qN_1(\mathbf{x}) && \text{by definition of } N_1 \\
&\leq qnN_0(\mathbf{x}) && \text{since } N_1(\mathbf{x}) \leq nN_0(\mathbf{x}) \text{ for } \mathbf{x} \in \mathbf{R^n} \\
&\leq qn\|\mathbf{x}\| && \text{by equation (29.8).}
\end{aligned}$$

From this and property a it follows that if $\{\mathbf{x}_n\}_{n=1}^\infty$ converges to $\mathbf{0}$ (in N_2-norm), then $0 \leq N(\mathbf{x}_n) \leq qn\|\mathbf{x}_n\| \to 0$, so $N(\mathbf{x}_n) \to N(\mathbf{0})$.

Now suppose that $\{\mathbf{x}_n\}_{n=1}^\infty$ converges to $\mathbf{x}$ in the Euclidean norm. Then, $\mathbf{x}_n - \mathbf{x}$ converges to 0, so $N(\mathbf{x}_n - \mathbf{x})$ converges to 0. Then $N(\mathbf{x}_n) = N(\mathbf{x} + \mathbf{x}_n - \mathbf{x}) \leq N(\mathbf{x}) + N(\mathbf{x}_n - \mathbf{x})$ from property d of a norm; so $\limsup N(\mathbf{x}_n) \leq N(\mathbf{x})$. Similarly, $N(\mathbf{x}) = N(\mathbf{x}_n + \mathbf{x} - \mathbf{x}_n) \leq N(\mathbf{x}_n) + N(\mathbf{x}_n - \mathbf{x})$; so $\liminf N(\mathbf{x}_n) \geq N(\mathbf{x})$. Thus, $N(\mathbf{x}_n)$ converges to $N(\mathbf{x})$ as $n \to \infty$. Therefore, the norm N is a continuous function.

b) The unit sphere S is a compact subset of $\mathbf{R^n}$. It follows from part a and Weierstrass's theorem that N achieves its minimum value m_1 on S at some point $\mathbf{x}^* \in S$ and its maximum value m_2 at some $\mathbf{y}^* \in S$. Since $\mathbf{0}$ is not in the unit sphere (sphere, not ball), neither $\mathbf{x}^*$ nor $\mathbf{y}^*$ is $\mathbf{0}$. By property b of a norm, $N(\mathbf{y}^*) \geq N(\mathbf{x}^*) > 0$.

c) For any $\mathbf{x} \in \mathbf{R^n}$, write $\mathbf{x} = r\mathbf{v}$ where $r = \|\mathbf{x}\|$ and $\mathbf{v} = \mathbf{x}/\|\mathbf{x}\|$. By property *c* of a norm, $\|\mathbf{v}\| = 1$. By part *b* of this exercise, $m_1 \leq N(\mathbf{v}) \leq m_2$, and therefore $rm_1 \leq rN(\mathbf{v}) \leq rm_2$. Since $N(\mathbf{x}) = N(r\mathbf{v}) = rN(\mathbf{v})$ and $r = \|\mathbf{x}\|$, $m_1\|\mathbf{x}\| \leq N(\mathbf{x}) \leq m_2\|\mathbf{x}\|$.

d) Part *c* proves that all norms in $\mathbf{R^n}$ are equivalent to the Euclidean norm N_2. If N and N' are two arbitrary norms on $\mathbf{R^n}$, then there are positive constants a_1, a_2, b_1, b_2 such that

$$a_1N(\mathbf{x}) \leq N_2(\mathbf{x}) \leq b_1N(\mathbf{x}) \quad \text{and} \quad a_2N'(\mathbf{x}) \leq N_2(\mathbf{x}) \leq b_2N'(\mathbf{x})$$

for all vectors $\mathbf{x} \in \mathbf{R^n}$ by part *c*. It follows that

$$\frac{a_1}{b_2}N(\mathbf{x}) \leq N'(\mathbf{x}) \quad \text{and} \quad \frac{a_2}{b_1}N'(\mathbf{x}) \leq N(\mathbf{x}).$$

So, all norms on $\mathbf{R^n}$ are equivalent. In particular, a sequence that converges in one norm converges in any other norm.

30.5 The third order approximation is: $e^h \approx 1 + h + (1/2)h^2 + (1/6)h^3$, and the fourth order one is: $e^h \approx 1 + h + (1/2)h^2 + (1/6)h^3 + (1/24)h^4$.
The third order approximation at $h = 0.2$ is 1.22133 with error 0.0000694.
The fourth order approximation at $h = 0.2$ is 1.22140 with error 0.00000276.
The third order approximation at $h = 1$ is 2.66667 with error 0.0516152.
The fourth order approximation at $h = 1$ is 2.718282 with error 0.009978.

30.6 $(x + h)^{3/2} \approx x^{3/2} + (3/2)x^{1/2}h + (1/2)(3/4)x^{-1/2}h^2$. Taking $x = 4$ and $h = 0.2$ gives $(4.2)^{3/2} \approx 8 + 3 \cdot 0.2 + (3/16) \cdot 0.04 = 8.6075$. The actual value to four decimal places is 8.6074.

30.7 For $F(x) = \sqrt{1 + x}$,

$$P_1(h) = 1 + \frac{h}{2}, \quad P_2(h) = 1 + \frac{h}{2} - \frac{h^2}{8}, \quad P_3(h) = 1 + \frac{h}{2} - \frac{h^2}{8} + \frac{h^3}{16}.$$

For $h^* = 0.2$, $F(h^*) = 1.09545$ to 5 decimal places.
$P_1(h^*) = 1.1$ and the error is 0.00455488.
$P_2(h^*) = 1.095$ and the error is -0.000445115.
$P_3(h^*) = 1.0955$ and the error is 0.000054885.
For $h^* = 1.0$, $F(h^*) = 1.41421$ to 5 decimal places.
$P_1(h^*) = 1.5$ and the error is 0.0857864.
$P_2(h^*) = 1.375$ and the error is -0.0392136. $P_3(h^*) = 1.4375$ and the error is 0.0232864.

For $F(x) = \ln x + 1$,

$$P_1(h) = h, \quad P_2(h) = h - \frac{h^2}{2}, \quad P_3(h) = h - \frac{h^2}{2} + \frac{h^3}{3}.$$

For $h^* = 0.2$, $F(1 + h^*) = 0.182322$ to six decimal places.
$P_1(h^*) = 0.2$ and the error is 0.00177.
$P_2(h^*) = 0.18$ and the error is -0.00232156.
$P_3(h^*) = 0.182667$ and the error is 0.00023511.
For $h^* = 1.0$, $F(1 + h^*) = 0.693147$ to six decimal places.
$P_1(h^*) = 1.0$ and the error is 0.306853.
$P_2(h^*) = 0.5$ and the error is -0.193147.
$P_3(h^*) = 0.833333$ and the error is 0.140186.

30.8 Suppose ℓ_1 and ℓ_2 are two such numbers, with corresponding remainder terms $R_1(h)$ and $R_2(h)$. Then $\ell_1 - \ell_2 = R_2(h)/h - R_1(h)/h$. Taking limits as $h \to 0$, $\ell_1 - \ell_2 = 0$.

30.9 Proof of Theorem 30.5:

$$g_1(a) = f(a) - f(a) - f'(a)(a - a) - M_1(a - a)^2 = 0.$$

Also,

$$g'(t) = f'(t) - f'(a) - 2M_1(t - a).$$

At $t = a$, $g'(a) = f'(a) - f'(a) - 2M_1(a - a) = 0$.

Proof of Theorem 30.6:

$$\begin{aligned} g_2(a + h) &= f(a + h) - f(a) - f'(a)h - (1/2)f''(a)h^2 \\ &\qquad - f(a + h) + f(a) + f'(a)h + (1/2)f''(a)h^2 = 0 \\ g_2(a) &= f(a) - f(a) - f'(a)(a - a) - (1/2)f''(a)(a - a)^2 - M_2(a - a)^3 \\ &= 0 \\ g_2'(a) &= f'(a) - f'(a) - f''(a)(a - a) - 3M_2(a - a)^2 = 0 \\ g_2''(a) &= f''(a) - f''(a) - 6M_2(a - a) = 0. \end{aligned}$$

30.12 *a*)

$$\begin{aligned} P_1(h_1, h_2) &= h_1 \\ P_2(h_1, h_2) &= h_1 - \frac{h_1 h_2}{2} \\ P_3(h_1, h_2) &= h_1 - \frac{h_1 h_2}{2} + \frac{h_1 h_2^2}{3}. \end{aligned}$$