Chapter 2

2.1 *i*) $y = 3x - 2$ is increasing everywhere, and has no local maxima or minima. See figure.*

ii) $y = -2x$ is decreasing everywhere, and has no local maxima or minima. See figure.

iii) $y = x^2 + 1$ has a global minimum of 1 at $x = 0$. It is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$. See figure.

iv) $y = x^3 + x$ is increasing everywhere, and has no local maxima or minima. See figure.

v) $y = x^3 - x$ has a local maximum of $2/3\sqrt{3}$ at $-1/\sqrt{3}$, and a local minimum of $-2/3\sqrt{3}$ at $1/\sqrt{3}$, but no global maxima or minima. It increases on $(-\infty, -1/\sqrt{3})$ and $(1/\sqrt{3}, \infty)$ and decreases in between. See figure.

vi) $y = |x|$ decreases on $(-\infty, 0)$ and increases on $(0, \infty)$. It has a global minimum of 0 at $x = 0$. See figure.

2.2 Increasing functions include production and supply functions. Decreasing functions include demand and marginal utility. Functions with global critical points include average cost functions when a fixed cost is present, and profit functions.

2.3 $1, 5, -2, 0$.

2.4 *a*) $x \neq 1$; *b*) $x > 1$; *c*) all x; *d*) $x \neq \pm 1$; *e*) $-1 \leq x \leq +1$;
f) $-1 \leq x \leq +1, x \neq 0$.

2.5 *a*) $x \neq 1$, *b*) all x, *c*) $x \neq -1, -2$, *d*) all x.

2.6 The most common functions students come up with all have the nonnegative real numbers for their domain.

2.8 *a*) 1, *b*) -1, *c*) 0, *d*) 3.

2.8 *a*) The general form of a linear function is $f(x) = mx + b$, where b is the y-intercept and m is the slope. Here $m = 2$ and $b = 3$, so the formula is $f(x) = 2x + 3$.

b) Here $m = -3$ and $b = 0$, so the formula is $f(x) = -3x$.

*All figures are included at the back of the pamphlet.

c) We know m but need to compute b. Here $m = 4$, so the function is of the form $f(x) = 4x + b$. When $x = 1$, $f(x) = 1$, so b has to solve the equation $1 = 4 \cdot 1 + b$. Thus, $b = -3$ and $f(x) = 4x - 3$.

d) Here $m = -2$, so the function is of the form $f(x) = -2x + b$. When $x = 2$, $f(x) = -2$, thus b has to solve the equation $-2 = -2 \cdot 2 + b$, so $b = 2$ and $f(x) = -2x + 2$.

e) We need to compute m and b. Recall that given the value of $f(x)$ at two points, m equals the change in $f(x)$ divided by the change in x. Here $m = (5 - 3)/(4 - 2) = 1$. Now b solves the equation $3 = 1 \cdot 2 + b$, so $b = 1$ and $f(x) = x + 1$.

f) $m = [3 - (-4)]/(0 - 2) = -7/2$, and we are given that $b = 3$, so $f(x) = -(7/2)x + 3$.

2.9 *a*) The slope is the **marginal revenue**, that is, the rate at which revenue increases with output.

b) The slope is the **marginal cost**, that is, the rate at which the cost of purchasing x units increases with x.

c) The slope is the rate at which demand increases with price.

d) The slope is the **marginal propensity to consume**, that is, the rate at which aggregate consumption increases with national income.

e) The slope is the **marginal propensity to save**, that is, the rate at which aggregate savings increases with national income.

2.10 *a*) The slope of a secant line through points with x-values x and $x + h$ is $[m(x + h) - mx]/h = mh/h = m$.

b) For $f(x) = x^3$,

$$\begin{aligned}\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= \lim_{h\to 0} 3x^2 + 3xh + h^2 = 3x^2.\end{aligned}$$

For $f(x) = x^4$,

$$\begin{aligned}f'(x) &= \lim_{h\to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} \\ &= \lim_{h\to 0} 4x^3 + 6x^2h + 4xh^2 + h^3 = 4x^3.\end{aligned}$$

2.11 *a*) $-21x^2$, *b*) $-24x^{-3}$, *c*) $-(9/2)x^{-5/2}$, *d*) $1/4\sqrt{x}$,

e) $6x - 9 + (14/5)x^{-3/5} - (3/2)x^{-1/2}$, *f*) $20x^4 - (3/2)x^{-1/2}$,

g) $4x^3 + 9x^2 + 6x + 3$,

h) $(1/2)(x^{-1/2} - x^{-3/2})(4x^5 - 3\sqrt{x}) + (x^{1/2} + x^{-1/2})(20x^4 - (3/2)x^{-1/2})$,

i) $2/(x+1)^2$, *j*) $(1 - x^2)/(1 + x^2)^2$, *k*) $7(x^5 - 3x^2)^6(5x^4 - 6x)$,

l) $(10/3)(x^5 - 6x^2 + 3x)^{-1/3}(5x^4 - 12x + 3)$,

m) $3(3x^2 + 2)(x^3 + 2x)^2(4x + 5)^2 + 8(x^3 + 2x)^3(4x + 5)$.

2.12 *a*) The slope of the tangent line $l(x) = mx + b$ to the graph of $f(x)$ at x_0 is $m = f'(x_0) = 2x_0 = 6$. The tangent line goes through the point $(x_0, f(x_0)) = (3, 9)$, so b solves $9 = 6 \cdot 3 + b$. Thus $b = -9$ and $l(x) = 6x - 9$.

b) Applying the quotient rule, $f'(x) = (2 - x^2)/(x^2 + 2)^2$. Evaluating this at $x_0 = 1$, $m = 1/9$. The tangent line goes through the point $(1, 1/3)$. Solving for b, $l(x) = (1/9)x + 2/9$.

2.13
$$\begin{aligned}(f+g)'(x_0) &= \lim_{h\to 0} \frac{(f(x_0+h) + g(x_0+h) - \big(f(x_0) + g(x_0)\big)}{h} \\ &= \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} + \lim_{h\to 0} \frac{g(x_0+h) - g(x_0)}{h} \\ &= f'(x_0) + g'(x_0)\end{aligned}$$

and similarly for $(f - g)'(x_0)$.

$$\begin{aligned}(kf)'(x_0) &= \lim_{h\to 0} \frac{k\,f(x_0+h) - k\,f(x_0)}{h} \\ &= k \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = k\,f'(x_0).\end{aligned}$$

2.14 Let $F(x) = x^{-k} = 1/x^k$. Apply the quotient rule with $f(x) = 1$ and $g(x) = x^k$. Then $f'(x_0) = 0$, $g'(x_0) = kx_0^{k-1}$, and

$$F'(x_0) = -kx_0^{k-1}/x_0^{2k} = -kx_0^{-k-1}.$$

2.15 For positive x, $|x| = x$, so its derivative is 1. For negative x, $|x| = -x$, so its derivative is -1.

2.16, 17 *a*) $f'(x) = \begin{cases} 2x & \text{if } x > 0, \\ -2x & \text{if } x < 0. \end{cases}$

As x converges to 0 both from above and below, $f'(x)$ converges to 0, so the function is C^1. See figure.

b) This function is not continuous (and thus not differentiable). As x converges to 0 from above, $f(x)$ tends to 1, whereas x tends to 0 from below, $f(x)$ converges to -1. See figure.

c) This function is continuous, since $\lim_{x\to 1} f(x) = 1$ no matter how the limit is taken. But it is not differentiable at $x = 1$, since $\lim_{h\downarrow 0}[f(1+h) - f(1)]/h = 3$ and $\lim_{h\uparrow 0}[f(1+h) - f(1)]/h = 1$. See figure.

d) This function is C^1 at $x = 1$. No matter which formula is used, the value for the derivative of $f(x)$ at $x = 1$ is 3. See figure.

$$f'(x) = \begin{cases} 3x^2 & \text{if } x < 1, \\ 3 & \text{if } x \geq 1. \end{cases}$$

2.18 The interesting behavior of this function occurs in a neighborhood of $x = 0$. Computing, $[f(0+h) - f(0)]/h = h^{-1/3}$, which converges to $+\infty$ or $-\infty$ as h converges to 0 from above or below, respectively. Thus $f(x) = x^{2/3}$ is not differentiable at $x = 0$. It is continuous at $x = 0$, since $\lim_{x\to 0} f(x) = 0$. This can easily be seen by plotting the function. See figure.

2.19 See figure.

2.20 *a*) $-42x$, *b*) $72x^{-4}$, *c*) $(45/4)x^{-7/2}$, *d*) $x^{-3/2}/8$,

e) $6 - (52/25)x^{-8/5} + (3/4)x^{-3/2}$, *f*) $80x^3 + (3/4)x^{-3/2}$,

g) $12x^2 + 18x + 6$,

h) $(-x^{-3/2}/4 + 3x^{-5/2}/4)(4x^5 - 3\sqrt{x}) + (x^{-1/2} - x^{-3/2})(20x^4 - 3x^{-1/2}/2) + (x^{1/2} + x^{-1/2})(80x^3 + 3x^{-3/2}/4)$,

i) $-4/(x+1)^3$, *j*) $(2x^3 - 6x)/(x^2+1)^3$,

k) $42(5x^4 - 6x)^2(x^5 - 3x^2)^5 + 7(20x^3 - 6)(x^5 - 3x^2)^6$,

l) $(-10/9)(5x^4 - 12x + 3)^2(x^5 - 6x^2 + 3x)^{-4/3} + (10/3)(20x^3 - 12)(x^5 - 6x^2 + 3x)^{-1/3}$,

m) $12(x+1)^2(4x+5)^2(x^2+2x) + 96(x+1)(4x+5)(x^2+2x)^2 + 6(4x+5)^2(x^2+2x)^2 + 32(x^2+2x)^3$.

2.21 *a*) $f'(x) = (5/3)x^{2/3}$, so $f(x)$ is C^1. But $x^{2/3}$ is not differentiable at $x = 0$, so f is not C^2 at $x = 0$. Everywhere else it is C^∞.

b) This function is a step function: $f(x) = k$ for $k \leq x < k+1$, for every integer k. It is C^∞ except at integers, since it is constant on every interval $(k, k+1)$. At integers it fails to be continuous.

2.22 $C'(20) = 86$, so $C(21) - C(20) \approx C'(20) \cdot 1 = 86$. Direct calculation shows that $C(21) = 1020$, so $C(21) - C(20) = 88$.

2.23 $C'(x) = 0.3x^2 - 0.5x + 300$. Then $C(6.1) - C(6) \approx C'(6) \cdot 0.1 = 30.78$.

2.24 $F'(t) = 8/(+2)^2$. Thus $F'(0) = 2$ and the population increase over the next half-year is $F'(0) \cdot 0.5 = 1$.

2.25 *a*) $f(x) = \sqrt{x}$, and $f'(x) = 1/2\sqrt{x}$. $f(50) = f(49) + f'(49) \cdot (1.0) = 7 + 1/14$.

b) $f(x) = x^{1/4}$, and $f'(x) = 1/(4x^{3/4})$. Then $f(9,997) \approx f(10{,}000) + f'(10{,}000) \cdot (-3.0) = 10 - 3/4{,}000 = 9.99925$.

c) $f(x) = x^5$, and $f'(x) = 5x^4$. $f(10.003) = f(10) + f'(10) \cdot 0.003 = 100{,}000 + 50{,}000 \cdot 0.003 = 100{,}150$.

Chapter 3

3.1 *a*) $f'(x) = 3x^2 + 3$, so $f'(x)$ is always positive and $f(x)$ is increasing throughout its domain. $f(0) = 0$, so the graph of f passes through the origin. See figure.

b) Early versions of the text have $f(x) = x^4 - 8x^3 + 18x - 11$ here, with $f'(x) = 4x^3 - 24x^2 + 18$. This itself is a complicated function. $f''(x) = 12x^2 - 48x$. Thus, $f'(x)$ has critical points at $x = 0$ and $x = 4$. The point $x = 0$ is a local maximum of f', and $x = 4$ is a local minimum. Evaluating, $f'(x)$ is positive at the local max and negative at the local min. This means it crosses the x-axis three times, so the original function f has three critical points. Since $f'(x)$ is negative for small x and positive for large x, the critical points of f are, from smallest to largest, a local minimum, a local maximum, and a local minimum.

Later versions have $f(x) = x^4 - 8x^3 + 18x^2 - 11$. Its y-intercept is at $(0, -11)$; $f'(x) = 4x^3 - 24x^2 + 36x = 4x(x^2 - 6x + 9) = 4x(x-3)^2$.

Critical points are at $x = 0, 3$, i.e., (0,11) and (3,16).

$f' > 0$ (and f increasing) for $0 < x < \infty$ $(x \neq 3)$; $f' < 0$ (and f decreasing) for $-\infty < x < 0$. See figure.

c) $f'(x) = x^2 + 9$. This function is always positive, so f is forever increasing. A little checking shows its root to be between $-1/3$ and $-1/2$. See figure.

d) $f'(x) = 7x^6 - 7$, which has roots at $x = \pm 1$. The local maximum is at $(-1, 6)$, and the local minimum is at $(1, -6)$. The function is decreasing between these two points and increasing elsewhere. The y-intercept is at $(0, 0)$. See figure.

e) $f(0) = 0$; $f'(x) = (2/3)x^{-1/3}$. $f'(x) < 0$ (and f decreasing) for $x < 0$; $f'(x) > 0$ (and f increasing) for $x > 0$. As $x \to 0$, the graph of f becomes infinitely steep. See figure.

f) $f'(x) = 12x^5 - 12x^3 = 12x^3(x^2 - 1)$. The first derivative has roots at -1, 0 and 1. $f'(x)$ is negative for $x < -1$ and $0 < x < 1$, and positive for $-1 < x < 0$ and $x > 1$. Thus $f(x)$ is shaped like a w. Its three critical points are, alternately, a min, a max, and a min. Its values at the two minima are both -1, and its value at the maximum is $+2$. See figure.

3.2 Since f is differentiable at x_0, for small h, $[f(x_0 + h) - f(x_0)]/h < 0$. This means that for small positive h, $f(x_0 + h) < f(x_0)$ and, for small negative h, $f(x_0 + h) > f(x_0)$. Thus, f is decreasing near x_0.

3.3 *a*) $f''(x) = 6x$. The function is concave (concave down) on the negative reals and convex (concave up) on the positive reals.

b) $f''(x) = 12x^2 - 48x$, which is negative for $0 < x < 4$ and positive outside this interval. Thus f is concave on the interval $(0, 4)$ and convex elsewhere.

c) $f''(x) = 2x$, so f is concave on the negative reals and convex on the positive reals.

d) $f''(x) = 42x^5$, so f is concave on the negative reals and convex on the positive reals.

e) $f''(x) = -2x^{-4/3}/9$. This number is always less than 0 for $x \neq 0$. f is concave on $(0, \infty)$ and on $(-\infty, 0)$. It is not globally concave.

f) $f''(x) = 60x^4 - 36x^2$, which is negative on the interval $(-\sqrt{3/5}, \sqrt{3/5})$, and positive outside it. Thus, f is concave on this interval and convex elsewhere.

3.4–5 See figures.

3.6 There is a single vertical asymptote at $x = 2$. $f'(x) = -16(x + 4)/(x - 2)^3$ and $f''(x) = 32(x + 7)/(x - 2)^4$. Consequently there is a critical point at $x = -4$, where the function takes the value $-4/3$. $f''(-4) > 0$, so this is a local minimum. There is an inflection point at $x = -7$. f is decreasing to the right of its asymptote and to the left of $x = -4$, and increasing on $(-4, 2)$. See figure.

3.7 *a*) The leading monomial is x^{-1}, so $f(x)$ converges to 0 as x becomes very positive or very negative. It also has vertical asymptotes at $x = -1$ and $x = 1$. $f'(x) = -(x^2+1)/(x^2-1)^2$; so for $x < -1$ and $x > 1$, $f(x)$ is decreasing. (In fact it behaves as $1/x$.) It is also decreasing between the asymptotes. Thus, it goes from 0 to $-\infty$ as x goes from $-\infty$ to -1, from $+\infty$ to $-\infty$ as x goes from -1 to 1, and from $+\infty$ to 0 as x goes from 1 to $+\infty$. See figure.

b) $f(x)$ behaves as $1/x$ for x very large and very small. That is, as $|x|$ grows large, $f(x)$ tends to 0. $f'(x) = (1-x^2)/(x^2+1)^2$; so there are critical points at -1 and 1. These are, respectively, a minimum with value $-1/2$ and a maximum with value $1/2$. Inflection points are at $(-\sqrt{3}, -\sqrt{3}/4)$, $(0, 0)$, and $(\sqrt{3}, \sqrt{3}/4)$. See figure.

c) This function has a vertical asymptote at $x = -1$. The lead monomial is $x^2/x = x$, so in the tails it is increasing as $x \to +\infty$ and decreasing as $x \to -\infty$. As x converges to -1 from below, $f(x)$ tends to $-\infty$; as x converges to -1 from above, $f(x)$ converges to $+\infty$. See figure.

d) The lead monomial is $x^2/x^2 = 1$, so $f(x)$ converges to 1 as $|x|$ becomes large. It has vertical asymptotes at $x = 1$ and $x = -1$. In fact, f can be rewritten as $f(x) = 1 + (3x+1)/(x^2-1)$. Since $f'(x) = -(3x^2+2x+3)/(x^2-1)^2 < 0$, f is always decreasing. So, its general shape is that of the function in part 7*a*. See figure.

e) The lead monomial is $x^2/x = x$, so this function is increasing in x when $|x|$ is large. When x is small near its vertical asymtote at $x = 0$, it behaves as $1/x$. $f''(x) = 1 - 1/x^2$, which is 0 at ± 1. $x = -1$ is a local maximum and $x = 1$ is a local minimum. See figure.

f) This function is bell shaped. It is always positive, tends to 0 when $|x|$ is large, and has a maximum at $x = 0$ where it takes the value 1. See figure.

3.8 See figures.

3.9 *a*) No global max or min on D_1; max at 1 and min at 2 on D_2.

b) No max or min on D_1; min at 0 and max at 1 on D_2.

c) Min at -4, max at -2 on D_1; min at $+1$, no max on D_2.

d) Min at 0, max at 10 on D_1; min at 0, no max on D_2.

e) Min at -2, max at $+2$ on D_1; min at $-\sqrt{2}$, max at $+\sqrt{2}$ on D_2.

f) Min at 1, no max on D_1; max at -1, no min on D_2.

g) No min or max on D_1; max at 1 and min at 2 on D_2.

h) No max or min on D_1; max at 1 and min at 5 on D_2.

3.10 In this exercise x is the market price, which is a choice variable for the firm. $\pi(x) = x(15 - x) - 5(15 - x)$. This function is concave, and its first derivative is $\pi'(x) = -2x + 20$. $\pi'(x) = 0$ at $x = 10$.

3.11 From the information given, the demand function must be computed. The function is linear, and the slope is -1. It goes through the point (10, 10), so the function must be $f(x) = 20 - x$. Then the profit function (as a function of price) must be $\pi(x) = (x - 5)(20 - x)$. $\pi'(x) = -2x + 25$, so profit is maximized at $x = 12.5$.

3.12 One can translate the proof of Theorem 3.4*a* in the text. Here is another idea. If ℓ is a secant line connecting (x_0, y_0) and (x_1, y_1) on the graph of a convex function $f(x)$, then the set of points $\big(x, f(x)\big)$ for $x \notin (x_0, x_1)$ lies *above* ℓ. Taking limits, the graph of a convex function always lies above each tangent line (except where they touch). If $f'(x_0) = 0$, then the tangent line is of the form $\ell(x) = f(x_0) = b$. Since $f(x)$ is convex near x_0, $f(x)$ must be at least as big as b for x near x_0, and so x_0 is a min.

3.13 Suppose $y_0 < x_0$; f is decreasing just to the right of y_0 and increasing just to the left of x_0. It must change from decreasing to increasing somewhere between y_0 and x_0, say at w_0. Then, w_0 is an interior critical point of f — contradicting the hypothesis that x_0 is the only critical point of f_0.

3.15 $AC(x) = x^2 + 1 + 1/x$. $MC(x) = 3x^2 + 1$. $MC(x_0) = AC(x_0)$ when $2x_0^2 = 1/x_0$, that is, at $x_0 = 2^{-1/3}$. $AC'(x) = 2x - 1/x^2$, so $AC(x)$ has a critical point at $x = 2^{-1/3}$. Thus c is satisfied. $MC(x)$ is increasing, $AC(x)$ is convex, and the two curves intersect only once at x_0, so it must be that to the left of x_0, $AC(x) > MC(x)$, and hence to the right, $AC(x) < MC(x)$. See figure.

3.16 Suppose $C(x) = \sqrt{x}$. Then $MC(x) = 1/2\sqrt{x}$, which is decreasing. $\pi(x) = px - \sqrt{x}$. $\pi'(x) = p - 1/2\sqrt{x}$. The equation $\pi'(x) = 0$ will have a solution, but $\pi''(x) = -1/4x^{3/2}$, which is always negative on the positive reals. Setting price equal to marginal cost gives a local min. Profit can always be increased by increasing output beyond this point.

3.17 *a*) Locate x^* correctly at the intersection of the MR and MC curves. Revenue at the optimum is described by the area of the rectangle with height $AR(x^*)$ and length x^*.

b) The rectangle with height $AC(x^*)$ and length x^*.

c) The rectangle with height $AR(x^*) - AC(x*)$ and length x^*.

3.18 For demand curve $x = a - bp$, the elasticity at $(a - bp, p)$ is $\varepsilon = -bp/(a - bp)$. Then, $\varepsilon = -1 \iff bp = a - bp \iff 2bp = a \iff p = a/b$.

3.19 $\varepsilon = \dfrac{F'(p) \cdot p}{F(p)} = \dfrac{-rkp^{-r-1} \cdot p}{kp^{-r}} = -r$, constant.

3.20 x^* and p^* both increase.

3.21 The rectangle with height $p^* - AC(x^*)$ and length x^*.

3.22 First, compute the inverse demand: $p = a/b - (1/b)x$. Then revenue is $R(x) = (a/b)x - (1/b)x^2$ and $MR(x) = a/b - (2/b)x$. $MC(x) = 2kx$, so x^* solves $a/b - (2/b)x = 2kx$. The solution is $x^* = a/(2kb + 2)$, and the price will be $p^* = 2kab/(2kb^2 + 2b)$.

Chapter 4

4.1 *a*) $(g \circ h)(z) = (5z - 1)^2 + 4, \quad (h \circ g)(x) = 5x^2 + 19$.

b) $(g \circ h)(z) = (z - 1)^3(z + 1)^3, \quad (h \circ g)(x) = (x^3 - 1)(x^3 + 1)$.

c, d) $(g \circ h)(z) = z, \quad (h \circ g)(x) = x$.

e) $g \circ h(z) = 1/(z^2 + 1), \quad h \circ g(x) = 1/x^2 + 1$.

4.2 *a*) Inside $y = 3x^2 + 1$, outside $z = y^{1/2}$.

b) Inside $y = 1/x$, outside $z = y^2 + 5y + 4$.

c) Inside $y = 2x - 7$, outside $z = \cos y$.

d) Inside $y = 4t + 1$, outside $z = 3^y$.

4.3 *a*) $(g \circ h)'(z) = 2(5z - 1)5 = 50z - 10, \quad (h \circ g)'(x) = 5 \cdot 2x = 10x$.

b) $(g \circ h)'(z) = 3[(z - 1)(z + 1)]^2(2z) = 6z(z - 1)^2(z + 1)^2$, $(h \circ g)'(x) = 2x^3 \cdot 3x^2 = 6x^5$.

c) $(g \circ h)'(z) = 1, \quad (h \circ g)'(x) = 1$.

d) $(g \circ h)'(z) = 1, \quad (h \circ g)'(x) = 1$.

e) $(g \circ h)'(z) = -2z/(z^2 + 1)^2, \quad (h \circ g)'(x) = -2/x^3$.

4.4 *a*) $(g \circ h)'(x) = \frac{1}{2}(3x^2 + 1)^{-1/2} \cdot 6x = 3x/\sqrt{3x^2 + 1}$

b) $(g \circ h)'(x) = [2(1/x) + 5]\left(-1/x^2\right)$.

c) $(g \circ h)'(x) = -2\sin(2x - 7)$.

d) $(g \circ h)'(t) = (4\log 3)3^{4t+1}$.

4.5 *a*) $(g \circ h)'(x) = \cos(x^4) \cdot 4x^3$.

b) $(g \circ h)'(x) = \cos(1/x) \cdot (-1/x^2)$.

c) $(g \circ h)'(x) = \cos x/(2\sqrt{\sin x})$.

d) $(g \circ h)'(x) = (\cos\sqrt{x})/2\sqrt{x}$.

e) $(g \circ h)'(x) = (2x + 3)\exp(x^2 + 3x)$.

f) $(g \circ h)'(x) = -x^{-2}\exp(1/x)$.

g) $(g \circ h)'(x) = 2x/(x^2 + 4)$.

h) $(g \circ h)'(x) = 4x(x^2 + 4)\cos((x^2 + 4)^2)$.

4.6 $x'(t) = 2$ and $C'(x) = 12$, so $(d/dt)C\big(x(t)\big) = 2 \cdot 12 = 24$.

4.8 *a*) $g(y) = (y - 6)/3$, $-\infty < y < +\infty$.

b) $g(y) = 1/y - 1$, $-\infty < y < +\infty$, $y \neq 0$.

c) The range of $f(x) = x^{2/3}$ is the nonnegative reals, so this is the domain of the inverse. But notice that $f(x)$ is not one-to one from $\mathbf{R}$ to its range. It is one-to-one if the domain of f is restricted to $\mathbf{R}_+$. In this case the inverse is $g(y) = y^{3/2}$, $0 \le y < \infty$. If the domain of f is restricted to $\mathbf{R}_-$, the inverse is $-g(y)$.

d) The graph of $f(x)$ is a parabola with a global minimum at $x = 1/2$, and is one-to-one on each side of it. Thus there will be two inverses. For a given y they are the solutions to $x^2 + x + 2 = y$. The two inverses are $z = \frac{1}{2}(-1 + \sqrt{4y - 7})$ and $z = \frac{1}{2}(-1 - \sqrt{4y - 7})$, with domain $y \ge 7/4$. If $y < 7/4$, the equation has no solution; there is no value of x such that $f(x) = y$.

4.9 *a*) $(f^{-1})'(f(1)) = 1/3, \quad 1/f'(1) = 1/3$.

b) $(f^{-1})'(1/2) = -4, \quad 1/f'(1) = -(1 + 1)^2 = -4$.

c) $(f^{-1})'f(1) = 3/2, \quad 1/f'(1) = 3/2$.

d) $(f^{-1})'f(1) = 1/3, \quad 1/f'(1) = 1/3$.

4.10 To prove Theorem 4.4, let $f(x) = x^{1/-n} = 1/x^{1/n}$ for n a positive integer. Applying the quotient rule,

$$\begin{aligned} f'(x) &= \frac{0 \cdot x^{1/n} - 1 \cdot (1/n)x^{(1/n)-1}}{x^{2/n}} \\ &= -(1/n)x^{-(1/n)-1}. \end{aligned}$$

For Theorem 4.5, the proof in the text applies to the case where both m and n are negative. To prove the remaining case, let $f(x) = x^{-m/n}$ where m, n are positive integers. Applying the quotient rule,

$$\begin{aligned} f'(x) &= \frac{0 \cdot x^{m/n} - 1 \cdot (m/n)x^{(m/n)-1}}{x^{2m/n}} \\ &= -(m/n)x^{(m/n)-1-(2m/n)} \\ &= -(m/n)x^{-(m/n)-1}. \end{aligned}$$

Chapter 5

5.1 *a*) 8, *b*) 1/8, *c*) 2, *d*) 4, *e*) 1/4, *f*) 1, *g*) 1/32, *h*) 125, *i*) 1/3125.

5.2 See figures.

5.3 By calculator to three decimal places: *a*) 2.699, *b*) 0.699, *c*) 3.091, *d*) 0.434, *e*) 3.401, *f*) 4.605, *g*) 1.099, *h*) 1.145. See figures.

5.4 *a*) 1, *b*) −3, *c*) 9, *d*) 3, *e*) 2, *f*) −1, *g*) 2, *h*) 1/2, *i*) 0.

5.5 *a*) $2e^{6x} = 18 \Longrightarrow e^{6x} = 9 \Longrightarrow 6x = \ln 9 \Longrightarrow x = (\ln 9)/6$;

b) $e^{x^2} = 1 \Longrightarrow x^2 = \ln 1 = 0 \Longrightarrow x = 0$;

c) $2^x = e^5 \Longrightarrow \ln 2^x = \ln e^5 \Longrightarrow x \ln 2 = 5 \Longrightarrow x = 5/(\ln 2)$;

d) $2 + \ln 5/\ln 2$; *e*) $e^{5/2}$; *f*) 5.

5.6 Solve $3A = A \exp rt$. Dividing out A and taking logs, $\ln 3 = rt$, and $t = \ln 3/r$.

5.7 Solve $600 = 500 \exp(0.05t)$. $t = \ln(6/5)/0.05 = 3.65$.

5.8 *a*) $f'(x) = e^{3x} + 3xe^{3x}$, $f''(x) = 6e^{3x} + 9xe^{3x}$.

b) $f'(x) = (2x+3)e^{x^2+3x-2}$, $f''(x) = (4x^2 + 12x + 11)e^{x^2+3x-2}$.

c) $f'(x) = 8x^3/(x^4+2)$, $f''(x) = (48x^2 - 8x^6)/(x^4+2)^2$.

d) $f'(x) = (1-x)/e^x$, $f''(x) = (x-2)/e^x$.

e) $f'(x) = 1/(\ln x) - 1/[(\ln x)]^2$, $f''(x) = 2/[x(\ln x)^3] - 1/[x(\ln x)^2]$.

f) $f'(x) = (1-\ln x)/x^2$, $f''(x) = (2\ln x - 3)/x^3$.

g) $f'(x) = x/(x^2+4)$, $f''(x) = (-x^2+4)/(x^2+4)^2$.

5.9 *a*) $f(x) = xe^x \Longrightarrow f'(x) = e^x(x+1)$ is positive (f increasing) for $x > -1$ and negative (f decreasing) for $x < -1$. $f''(x) = e^x(x+2)$ is positive (f convex) for $x > -2$, and negative (f concave) for $x < -2$. As x tends to $-\infty$, $f(x)$ goes to 0, and as x gets large, $f(x)$ behaves as e^x. Thus, the function has a horizontal asymptote at 0 as $x \to -\infty$, grows unboundedly as $x \to +\infty$, has a global minimum at -1, and has an inflection point at -2. See figure.

b) $y = xe^{-x}$; $y' = (1-x)e^{-x}$ is positive (f increasing) for $x < 1$, and negative (f decreasing) for $x > 1$. $y'' = (x-2)e^{-x}$ is positive (f convex) for $x > 2$, and negative (f concave) for $x < 2$. As x gets large, $f(x)$ tends to 0. As x goes to $-\infty$, so does $f(x)$. The inflection point is at $x = 2$. See figure.

c) $y = \frac{1}{2}(e^x + e^{-x})$; $y' = \frac{1}{2}(e^x - e^{-x})$ is positive (y increasing) if $x > 0$, and negative (y decreasing) if $x < 0$. $y'' = y$ is always positive (y always convex). See figure.

5.10 Let $f(x) = \mathrm{Log}(x)$, and let $g(x) = 10^{f(x)} = x$. Then, by the Chain Rule and Theorem 5.3,

$$g'(x) = (\ln 10)10^{f(x)}f'(x) = 1.$$

So,
$$f'(x) = \frac{1}{(\ln 10)10^{\mathrm{Log}(x)}} = \frac{1}{x\ln 10}.$$

5.11 The present value of the first option is $215/(1.1)^2 = 177.69$. The present value of the second option is $100/(1.1) + 100/(1.1)^2 = 173.56$. The present value of the third option is $100 + 95/(1.1)^2 = 178.51$.

5.12 By equation (14), the present value of the 5-year annuity is

$$500\frac{1-e^{-0.5}}{e^{0.1}-1} = 1870.62.$$

Equation (15) gives the present value of the infinitely lived annuity: $500/(e^{0.1} - 1) = 4754.17$.

5.13 $\ln B(t) = \sqrt{t}\ln 2$. Differentiating,

$$\frac{B'(t)}{B(t)} = \frac{\ln 2}{2\sqrt{t}}.$$

The solution to $B'(t)/B(t) = r$ is $t = 100(\ln 2)^2 = 48.05$.

5.14 $\ln V(t) = K + \sqrt{t}$. Differentiating, $V'(t)/V(t) = 1/2(\sqrt{t})$. The solution to $V'(t)/V(t) = r$ is $t = 1/(4r^2)$, which is independent of K. A check of the second order conditions shows this to be a maximum.

5.15 $\ln V(t) = \ln 2000 + t^{1/4}$. Differentiating, we find that

$$\frac{V'(t)}{V(t)} = \frac{1}{4}t^{-3/4}.$$

The solution to $V'(t)/V(t) = r$ is $t = (4r)^{-4/3}$. When $r = 0.1$, $t = 3.39$.

5.16 *a*) $\ln f(x) = \frac{1}{2}\ln(x^2+1) - \frac{1}{2}\ln(x^2+4)$. Differentiating,

$$\frac{f'(x)}{f(x)} = \frac{x}{x^2+1} - \frac{x}{x^2+4}.$$

So,

$$\begin{aligned} f'(x) &= f(x)\left(\frac{x}{x^2+1} - \frac{x}{x^2+4}\right) \\ &= \frac{3x}{(x^2+1)^{1/2}(x^2+4)^{3/2}}. \end{aligned}$$

b) $\ln f(x) = 2x^2 \ln x$. Differentiating, $f'(x)/f(x) = 4x\ln x + 2x = 2x(1 + \ln x^2)$, so $f'(x) = 2x(1+\ln x^2)(x^2)^{x^2}$.

5.17 Let $h(x) = f(x)g(x)$, so $\ln h(x) = \ln f(x) + \ln g(x)$. Differentiating, $h'(x)/h(x) = f'(x)/f(x) + g'(x)/g(x)$. Multiplying both sides of the equality by x proves the claim.

Chapter 6

6.1

$$\begin{aligned} S &= 0.05(100{,}000) \\ F &= 0.4(100{,}000 - S). \end{aligned}$$

Multiplying out the system gives

$$S = 5{,}000$$
$$F + 0.4S = 40{,}000.$$

Thus $S = 5{,}000$ and $F = 38{,}000$, and after-tax profits are \$57,000. Including contributions, after-tax profits were calculated to be \$53,605, so the \$5,956 contribution really cost only $\$57,000 - \$53,605 = \$3,395$.

6.2 Now $S = 0.05(100{,}000 - C - F)$, so the equations become

$$\begin{aligned} C + 0.1S + 0.1F &= 10{,}000 \\ 0.05C + S + 0.05F &= 5{,}000 \\ 0.4C + 0.4S + F &= 40{,}000. \end{aligned}$$

The solution is $C = 6{,}070$, $S = 2{,}875$, and $F = 36{,}422$.

6.3 $x_1 = 0.5x_1 + 0.5x_2 + 1$, $x_2 = 0x_1 + 0.25x_2 + 3$. The solution is $x_1 = 6$, $x_2 = 4$.

6.4 Solving the system of equations $x_1 = 0.5x_1 + 0.5x_2 + 1$ and $x_2 = 0.875x_1 + 0.25x_2 + 3$ gives $x_1 = -36$ and $x_2 = -38$; this is infeasible.

6.5 $0.002 \cdot 0.9 + 0.864 \cdot 0.1 = 0.0882$, and $0.004 \cdot 0.8 + 0.898 \cdot 0.2 = 0.1828$.

6.6 For black females, $\left\{\begin{array}{l} x_{t+1} = 0.993x_t + 0.106y_t \\ y_{t+1} = 0.007x_t + 0.894y_t \end{array}\right\}$.
To find the stationary distribution, set $x_{t+1} = x_t = x$ and $y_{t+1} = y_t = y$: $x = 0.9381$ and $y = 0.0619$.
For white females, $\left\{\begin{array}{l} x_{t+1} = 0.997x_t + 0.151y_t \\ y_{t+1} = 0.003x_t + 0.849y_t \end{array}\right\}$.
Stationary solution: $x = 0.9805$ and $y = 0.0195$.

6.7 The equation system is

$$\begin{aligned} 0.16Y - 1500r &= 0 \\ 0.2Y + 2000r &= 1000. \end{aligned}$$

The solution is $r = 0.2581$ and $Y = 2419.35$.

Chapter 7

7.1 *a* and *e*.

7.2 *a*) The solution is $x = 5$, $y = 6$, $z = 2$.

b) The solution is $x_1 = 1$, $x_2 = -2$, $x_3 = 1$.

7.3 *a*) $x = 17/3$, $y = -13/3$.

b) $x = 2$, $y = 1$, $z = 3$.

c) $x = 1$, $y = -1$, $z = -2$.

7.4 Start with system (*):

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ \vdots \qquad\quad & \quad \vdots \\ a_{i1}x_1 + \cdots + a_{in}x_n &= b_i \\ \vdots \qquad\quad & \quad \vdots \\ a_{j1}x_1 + \cdots + a_{jn}x_n &= b_j \\ \vdots \qquad\quad & \quad \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= b_n. \end{aligned}$$

1) Change system (*) to

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n \qquad\qquad &= b_1 \\ \vdots \qquad\qquad\qquad & \quad \vdots \\ a_{i1}x_1 + \cdots + a_{in}x_n \qquad\qquad &= b_i \\ \vdots \qquad\qquad\qquad & \quad \vdots \\ (ra_{i1} + a_{j1})x_1 + \cdots + (ra_{in} + a_{jn})x_n &= (rb_i + b_j) \\ \vdots \qquad\qquad\qquad & \quad \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n \qquad\qquad &= b_n. \end{aligned}$$

2) Change the *i*th equation of system (*) to $ra_{i1}x_1 + \cdots + ra_{in}x_n = rb_i$.

3) Change system (*) to

$$
\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
\vdots \qquad\quad & \quad \vdots \\
a_{j1}x_1 + \cdots + a_{jn}x_n &= b_j \\
\vdots \qquad\quad & \quad \vdots \\
a_{i1}x_1 + \cdots + a_{in}x_n &= b_i \\
\vdots \qquad\quad & \quad \vdots \\
a_{n1}x_1 + \cdots + a_{nn}x_n &= b_n.
\end{aligned}
$$

Reverse operations:

1) Subtract r times equation i from equation j, leaving other $n-1$ equations intact.
2) Multiply the ith equation through by $1/r$.
3) Interchange the ith and jth equations again.

7.5 The system to solve is

$$
\begin{aligned}
0.20Y + 2000r &= 1000 \\
0.16Y - 1500r &= 0.
\end{aligned}
$$

Solving the second equation for Y in terms of r gives $Y = 9375r$. Substituting into the first equation, $3875r = 1000$, so $r = 0.258$ and $Y = 2419.35$.

7.6 *a*) The system to solve is:

$$
\begin{aligned}
sY + ar &= I^0 \\
mY - hr &= 0.
\end{aligned}
$$

Solving the second equation for Y in terms of r gives $Y = (h/m)r$. Substituting into the first equation gives $(sh + am)r/m = I^0$, so $r = mI^0/(sh + am)$ and $Y = hI^0/(sh + am)$.

b, c) Differentiate the solutions with respect to s:

$$\frac{\partial r}{\partial s} = -\frac{hmI_0}{(sh+am)^2} < 0 \quad \text{and} \quad \frac{\partial Y}{\partial s} = -\frac{h^2 I_0}{(sh+am)^2} < 0.$$

7.7 Solving for y in terms of x in the second equation gives $y = -x - 10$. Substituting this into the first equation gives a new equation that must be satisfied by all solutions. This equation is $-30 = 4$. Since this is never satisfied, there are no solutions to the equation system.

7.8 If $a_{22} \neq 0$, then $x_2 = (b_2 - a_{21}x_1)/a_{22}$. Substituting into the first equation gives

$$\begin{aligned} b_1 &= \frac{a_{11} - a_{12}a_{21}}{a_{22}} x_1 + \frac{a_{12}}{a_{22}} b_2 \\ x_1 &= \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}. \end{aligned}$$

A similar calculation solving the first equation for x_1 ends up at the same point if $a_{21} \neq 0$. The division that gives this answer is possible only if $a_{11}a_{22} - a_{12}a_{21} \neq 0$. In this case,

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}.$$

7.9 (1) Add 0.2 times row 1 to row 2. (2) Add 0.5 times row 1 to row 3. (3) Add 0.5 times row 2 to row 3.

7.10 $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & -14 \\ 0 & 1 & 6 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 0.3 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

7.11 *a*) The original system, the row echelon form, and the reduced row echelon form are, respectively,

$$\left(\begin{array}{cc|c} 3 & 3 & 4 \\ 1 & -1 & 10 \end{array}\right), \quad \left(\begin{array}{cc|c} 3 & 3 & 4 \\ 0 & -2 & 26/3 \end{array}\right), \quad \left(\begin{array}{cc|c} 1 & 0 & 17/3 \\ 0 & 1 & -13/3 \end{array}\right).$$

The solution is $x = 17/3$, $y = -13/3$.

b) The original system, the row echelon form, and the reduced row echelon form are, respectively,

$$\begin{pmatrix} 4 & 2 & -3 & | & 1 \\ 6 & 3 & -5 & | & 0 \\ 1 & 1 & 2 & | & 9 \end{pmatrix}, \quad \begin{pmatrix} 4 & 2 & -3 & | & 1 \\ 0 & 1/2 & 11/4 & | & 35/4 \\ 0 & 0 & -1/2 & | & -3/2 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 3 \end{pmatrix}.$$

The solution is $x = 2$, $y = 1$, $z = 3$.

c) The original system, the row echelon form, and the reduced row echelon form are, respectively,

$$\begin{pmatrix} 2 & 2 & -1 & | & 2 \\ 1 & 1 & 1 & | & -2 \\ 2 & -4 & 3 & | & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 2 & -1 & | & 2 \\ 0 & -6 & 4 & | & -2 \\ 0 & 0 & 3/2 & | & -3 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & -2 \end{pmatrix}.$$

The solution is $x = 1$, $y = -1$, $z = -2$.

7.12 The original system, the row echelon form, and the reduced row echelon form are, respectively,

$$\begin{pmatrix} 1 & 1 & 3 & -2 & | & 0 \\ 2 & 3 & 7 & -2 & | & 9 \\ 3 & 5 & 13 & -9 & | & 1 \\ -2 & 1 & 0 & -1 & | & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 3 & -2 & | & 0 \\ 0 & 1 & 1 & 2 & | & 9 \\ 0 & 0 & 2 & -7 & | & -17 \\ 0 & 0 & 0 & -1/2 & | & -3/2 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & | & -1 \\ 0 & 1 & 0 & 0 & | & 1 \\ 0 & 0 & 1 & 0 & | & 2 \\ 0 & 0 & 0 & 1 & | & 3 \end{pmatrix}.$$

The solution is $w = -1$, $x = 1$, $y = 2$, $z = 3$.

7.13 *a*) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, *b*) $\begin{pmatrix} 1 & 3 & 4 \\ 0 & -1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$,

c) $\begin{pmatrix} -1 & 1 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

7.14 The original system and the reduced row echelon form are, respectively,

$$\left(\begin{array}{ccc|c} -4 & 6 & 4 & 4 \\ 2 & -1 & 1 & 1 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{ccc|c} 1 & 0 & \frac{5}{4} & \frac{5}{4} \\ 0 & 1 & \frac{3}{2} & \frac{3}{2} \end{array}\right).$$

The solution set is the set of all (x, y, z) triples such that $x = \frac{5}{4} - \frac{5}{4}z$ and $y = \frac{3}{2} - \frac{3}{2}z$ as z ranges over all the real numbers.

7.15 The original system and the row echelon form are, respectively,

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -k & 1 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -(k+1) & 0 \end{array}\right).$$

If $k = -1$, the second equation is a multiple of the first. In the row echelon form this appears as the second equation $0 + 0 = 0$. Any solution to the first equation solves the second equation as well, and so there are infinitely many solutions. For all other values of k there is a unique solution, with $x_1 = 1$ and $x_2 = 0$.

7.16 *a*) The original system and the reduced row echelon form are, respectively,

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 & 3 \\ 0 & -1 & 1 & -1 & 1 \\ 2 & 3 & 3 & -3 & 3 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{3}{11} & \frac{12}{11} \\ 0 & 1 & 0 & -\frac{1}{11} & -\frac{4}{11} \\ 0 & 0 & 1 & -\frac{12}{11} & \frac{7}{11} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The variable z is free and the rest are basic. The solution is

$$\begin{aligned} w &= \tfrac{12}{11} - \tfrac{3}{11}z \\ x &= -\tfrac{4}{11} + \tfrac{1}{11}z \\ y &= \tfrac{7}{11} + \tfrac{12}{11}z. \end{aligned}$$

b) The original system and the reduced row echelon form are, respectively,

$$\left(\begin{array}{cccc|c} 1 & -1 & 3 & -1 & 0 \\ 1 & 4 & -1 & 1 & 3 \\ 3 & 7 & 1 & 1 & 6 \\ 3 & 2 & 5 & -1 & 3 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cccc|c} 1 & 0 & \frac{11}{5} & -\frac{3}{5} & \frac{3}{5} \\ 0 & 1 & -\frac{4}{5} & \frac{2}{5} & \frac{3}{5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The variables y and z are free, while w and x are basic. The solution is

$$\begin{aligned} w &= \tfrac{3}{5} - \tfrac{11}{5}y + \tfrac{3}{5}z \\ x &= \tfrac{3}{5} + \tfrac{4}{5}y - \tfrac{2}{5}z. \end{aligned}$$

c) The original system and the reduced row echelon form are, respectively,

$$\left(\begin{array}{rrrr|r} 1 & 2 & 3 & -1 & 1 \\ -1 & 1 & 2 & 3 & 2 \\ 3 & -1 & 1 & 2 & 2 \\ 2 & 3 & -1 & 1 & 1 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 17/65 \\ 0 & 1 & 0 & 0 & 7/65 \\ 0 & 0 & 1 & 0 & 22/65 \\ 0 & 0 & 0 & 1 & 32/65 \end{array}\right).$$

All variables are basic. There are no free variables. The solution is $w = 17/65$, $x = 7/65$, $y = 22/65$, $z = 32/65$.

d) The original system and the reduced row echelon form are, respectively,

$$\left(\begin{array}{rrrr|r} 1 & 1 & -1 & 2 & 3 \\ 2 & 2 & -2 & 4 & 6 \\ -3 & -3 & 3 & -6 & -9 \\ -2 & -2 & 2 & -4 & -6 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{rrrr|r} 1 & 1 & -1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Variable w is basic and the remaining variables are free. The solution is $w = 3 - x + y - 2z$.

7.17 *a*) The original system and the reduced row echelon form are, respectively,

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 13 \\ 1 & 5 & 10 & 61 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{ccc|c} 1 & 0 & -\frac{5}{4} & 1 \\ 0 & 1 & \frac{9}{4} & 12 \end{array}\right).$$

To have x and y integers, z should be an even multiple of 2, i.e., 4, 8, 16, ... To have $y \geq 0$, $z \leq 16/3$. So, $z = 4$, $x = 6$, $y = 3$.

b) 4 pennies, 6 nickels, 6 dimes! 16 coins worth 94 cents.

7.18 The reduced row echelon form of the system is

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 8 + a \end{array}\right).$$

The last equation has solutions only when $a = -8$. In this case $x = y = 1$.

7.19 *a*) The second equation is -1 times the first equation. When the system is row-reduced, the second equation becomes $0x + 0y = 0$; that is, it is redundant. The resulting system is

$$\begin{pmatrix} 1 & 1 & | & 1 \\ 0 & 1-q+p & | & 1-q \\ 0 & 0 & | & 0 \end{pmatrix}.$$

This system has no solution if and only if $1 - q + p = 0$ and $1 - q \neq 0$. This happens if and only if $p = -(1 - q)$. With the nonnegativity constraints $p, q \geq 0$, this can never happen unless $q > 1$. So the equation system always has a solution. If $q = 1$ and $p = 0$, the equation system has infinitely many solutions with $x = 1 - y$; otherwise it has a unique solution.

b) If $q = 2$ and $p = 1$, the system contains the two equations $x + y = 1$ and $x + y = 0$, which cannot simultaneously be satisfied. More generally, if $q \neq 1$ and $p = q - 1$, the equation system has no solution.

7.20 *a*) A row echelon form of this matrix is

$$\begin{pmatrix} 2 & -4 \\ 0 & 0 \end{pmatrix},$$

so its rank is 1.

b) A row echelon form of this matrix is

$$\begin{pmatrix} 2 & -4 & 2 \\ 0 & 0 & 2 \end{pmatrix},$$

so its rank is 2.

c) A row echelon form of this matrix is

$$\begin{pmatrix} 1 & 6 & -7 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

so its rank is 3.

d) A row echelon form of this matrix is

$$\begin{pmatrix} 1 & 6 & -7 & 3 & 5 \\ 0 & 3 & 1 & 1 & 4 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

so its rank is 3.

e) A row echelon form of this matrix is

$$\begin{pmatrix} 1 & 6 & -7 & 3 & 1 \\ 0 & 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 5 \end{pmatrix},$$

so its rank is 3.

7.21 *a*) *i*) Rank $M = \#rows = \#cols$, so there is a unique solution $(0, 0)$.

ii) Rank $M = \#rows < \#cols$, so there are infinitely many solutions.

iii) Rank $M = \#cols$, so there is a unique solution $(0, 0)$.

iv) Rank $M = \#rows = \#cols$, so there is a unique solution $(0, 0, 0)$.

v) Rank $M < \#rows = \#cols$, so there are infinitely many solutions.

b) *i*) Rank $M = \#rows = \#cols$, so there is a unique solution.

ii) Rank $M = \#rows < \#cols$, so there are infinitely many solutions.

iii) Rank $M = \#cols$, so there are either zero solutions or one solution.

iv) Rank $M = \#rows = \#cols$, so there is a unique solution.

v) Rank $M < \#rows = \#cols$, so there are zero or infinitely many solutions.

7.22 *a*) Rank $M = 1 < \#rows = \#cols$, so the homogeneous system has infinitely many solutions and the general system has either 0 or infinitely many solutions.

b) Rank $M = 2 = \#rows < \#cols$, so the homogeneous system has infinitely many solutions and the general system has infinitely many solutions.

c) Rank $M = 3 = \#rows < \#cols$, so the homogeneous system has infinitely many solutions and the general system has infinitely many solutions.

d) Rank $M = 3 < \#rows < \#cols$, so the homogeneous system has infinitely many solutions and the general system has either zero or infinitely many solutions.

c) Rank $M = 3 = \#rows < \#cols$, so the homogeneous system has infinitely many solutions and the general system has infinitely many solutions.

7.23 Checking the reduced row echelon forms, only *c* has no nonzero rows.

7.24 Let A be an $n \times n$ matrix with row echelon form R. Let $a(j)$ be the number of leading zeros in row j of R. By definition of R,

$$0 \leq a(1) < a(2) < a(3) < \cdots$$

until one reaches k so that $a(k) = n$; then $a(j) = n$ for all $j \geq k$.
It follows that $a(j) \geq j - 1$ for all j.
If A is nonsingular, $a(n) < n$. Since $a(n) \geq n - 1$, $a(n) = n - 1$. This means $a(j) = j - 1$ for all j, and so the jth entry in row j (diagonal entry) is not zero.
Conversely, if every diagonal entry of R is not zero, $a(j) < j$ for all j.
Since $a(j) \geq j - 1$, $a(j) = j - 1$ for all j. Since $a(n) = n - 1$, A has full rank, i.e., is nonsingular.

7.25 *i*) The row-reduced row echelon form of the matrix for this system in the variables x, y, z, and w is

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & \frac{3}{4} \\ 0 & 0 & 1 & 0 & \frac{1}{4} \end{array}\right).$$

The rank of the system is 2. Thus, two variables can be endogenous at any one time: z and one other. For example, the variables x and z can be solved for in terms of w and y, and the solution is $x = 3/4 - 2y - w$ and $z = 1/4$.

ii) The row-reduced row echelon form of the matrix for this system is

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

This matrix has rank 3, so three variables can be solved for in terms of the fourth. In particular, x, y, and w can be solved for in terms of z. One solution is $x = 1 + z$, $y = -z$, and $w = 0$.

7.26

$$\begin{aligned} C + 0.1S + 0.1F - 0.1P &= 0 \\ 0.05C + S \qquad\quad - 0.05P &= 0 \\ 0.4C + 0.4S + F - 0.4P &= 0. \end{aligned}$$

The reduced row-echelon form for the matrix representing the system is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -0.0595611 & 0 \\ 0 & 1 & 0 & -0.0470219 & 0 \\ 0 & 0 & 1 & -0.357367 & 0 \end{array}\right).$$

Thus the solution is $C = 0.0595611P$, $S = 0.0470219P$, and $F = 0.357367P$.

7.27 The equation system is

$$\begin{aligned} 0.2Y + 2000r + 0M_s &= 1000 \\ 0.16Y - 1500r - M_s &= -M^0. \end{aligned}$$

The reduced row echelon form of the matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & -3.22581 & 2419.36 + 3.22581M^0 \\ 0 & 1 & 0.000322581 & 0.258065 - 0.000322581M^0 \end{array}\right).$$

Thus, a solution is $Y = 2419.36 + 3.22581M^0 + 3.22581M_s$ and $r = 0.258065 - 0.000322581M^0 - 0.000322581M_s$.

7.28 *a*) Row reduce the matrix

$$\left(\begin{array}{cc|c} s & a & I_0 + G \\ m & -a & M_s - M^0 \end{array}\right).$$

b) The solution is

$$Y = \frac{h(I^* + G) + a(M_s - M^*)}{sh + am}$$

$$r = \frac{m(I^* + G) + s(M_s - M^*)}{sh + am}.$$

c) Increases in I^*, G and M_s increase Y.
Increases in I^*, G and M° increase r.
Increases in $M^\circ \Longrightarrow$ decreases in Y.
Increases in $M^s \Longrightarrow$ decreases in r.

7.29 *a*) Here is one possibility. Row reducing the matrix associated with the system gives

$$\left(\begin{array}{cccc|c} 1 & 0 & \frac{11}{5} & 0 & \frac{3}{5} \\ 0 & 1 & -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

A solution is then $w = \frac{3}{5} - \frac{11}{5}y$, $x = \frac{3}{5} + \frac{4}{5}y$, and $z = 0$.

b) If $y = 0$, then a solution is $w = \frac{3}{5}$, $x = \frac{3}{5}$, and $z = 0$.

c) Trying to solve the system in terms of z will not work. To see this, take z over to the right-hand side. The coefficient matrix for the resulting 3×3 system has rank 2. The system has infinitely many solutions.

7.30 The rank of the associated matrix is 2; twice the second equation plus the first equation equals the third equation. The reduced row echelon form is

$$\begin{pmatrix} 1 & 0 & \frac{11}{5} & -\frac{3}{5} & | & \frac{3}{5} \\ 0 & 1 & -\frac{4}{5} & \frac{2}{5} & | & \frac{3}{5} \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

In this case w and x can be solved for in terms of y and z. However, there is no successful decomposition involving three endogenous variables because no matrix of rank 2 can have a submatrix of rank 3.

Chapter 8

8.1 *a*) $A + B = \begin{pmatrix} 2 & 4 & 0 \\ 4 & -2 & 4 \end{pmatrix}$, $A - D$ undefined, $3B = \begin{pmatrix} 0 & 3 & -3 \\ 12 & -3 & 6 \end{pmatrix}$,

$DC = \begin{pmatrix} 5 & 3 \\ 4 & 1 \end{pmatrix}$, $B^T = \begin{pmatrix} 0 & 4 \\ 1 & -1 \\ -1 & 2 \end{pmatrix}$, $A^T C^T = \begin{pmatrix} 2 & 6 \\ 1 & 10 \\ 5 & 1 \end{pmatrix}$,

$C + D = \begin{pmatrix} 3 & 3 \\ 4 & 0 \end{pmatrix}$, $B - A = \begin{pmatrix} -2 & -2 & -2 \\ 4 & 0 & 0 \end{pmatrix}$, AB undefined,

$CE = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, $-D = \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix}$, $(CE)^T = (\,-1 \quad 4\,)$,

$B + C$ undefined, $D - C = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$, $CA = \begin{pmatrix} 2 & 1 & 5 \\ 6 & 10 & 1 \end{pmatrix}$,

EC undefined, $(CA)^T = \begin{pmatrix} 2 & 6 \\ 1 & 10 \\ 5 & 1 \end{pmatrix}$, $E^T C^T = (CE)^T = (\,-1 \quad 4\,)$.

b) $$DA = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 4 \\ 2 & 2 & 3 \end{pmatrix}$$

$$A^T D^T = \begin{pmatrix} 2 & 0 \\ 3 & -1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 5 & 2 \\ 4 & 3 \end{pmatrix} = (DA)^T.$$

c) $CD = \begin{pmatrix} 4 & 3 \\ 5 & 2 \end{pmatrix}, DC = \begin{pmatrix} 5 & 3 \\ 4 & 1 \end{pmatrix}$.

8.3 If A is 2×2 and B is 2×3, then AB is 2×3, so $B^T A^T = (AB)^T$ is 3×2. But A^T is 2×2 and B^T is 3×2, so $A^T B^T$ is not defined.

8.5 *a*) $AB = \begin{pmatrix} 2 & -5 \\ -5 & 2 \end{pmatrix} = BA.$

b) $\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ra + 0c & rb + 0d \\ 0a + rc & 0b + rd \end{pmatrix},$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} = \begin{pmatrix} ar + b0 & a0 + br \\ cr + d0 & c0 + dr \end{pmatrix}.$

Both equal $\begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}$. More generally, if $B = rI$ then $AB = A(rI) = r(AI) = rA$. $BA = (rI)A = r(IA) = rA$, too.

8.6 The 3×3 identity matrix is an example of everything except a row matrix and a column matrix. The book gives examples of each of these.

8.7 $\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix},$ and

$\begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}.$

8.8 *a*) Suppose that U^1 and U^2 are upper triangular; i.e., each $U^k_{ij} = 0$ for $i > j$. Then, $[U^1 + U^2]_{ij} = U^1_{ij} + U^2_{ij} = 0$ if $i > j$. For multiplication, the (i, j)th entry of U^1U^2 is

$$[U^1U^2]_{ij} = \sum_{k<i} U^1_{ik}U^2_{kj} + \sum_{k\geq i} U^1_{ik}U^2_{kj}.$$

The first term is 0 because U^1 is upper triangular. If $i > j$, the second term is 0 because U^2 is upper triangular. Thus, if $i > j$, $[U^1U^2]_{ij} = 0$, and so the product is upper triangular.

If L^1 and L^2 are lower triangular, then $(L^1)^T$ and $(L^2)^T$ are upper triangular. By the previous paragraph, $(L^1)^T + (L^2)^T$ is upper triangular, and so $L^1 + L^2 = [(L^1)^T + (L^2)^T]^T$ is lower triangular. Similarly, $L^1L^2 = [(L^2)^T(L^1)^T]^T$ is lower triangular.

If D is both lower and upper triangular, and if $i > j$, $D_{ij} = 0$ (lower) and $D_{ji} = 0$ (upper), so D is diagonal. Conversely, if D is diagonal, it is obviously both upper and lower triangular. Consequently, if D^1 and D^2 are diagonal, then $D^1 + D^2$ and D^1D^2 are both upper and lower triangular, and hence diagonal.

b) Clearly, $D \subset U$; so $D \cap U = D$. If M is a matrix in $S \cap U$, then for $i < j$, $M_{ij} = 0$ (upper). Thus, $M_{ji} = 0$ (symmetric), so M is diagonal. If M is diagonal, then for $i \neq j$, $M_{ij} = 0 = M_{ji}$; so M is symmetric. Hence $D \subset S$.

c)

$$\begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}\begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix}$$

$$= \begin{pmatrix} a_1b_1 & 0 & \cdots & 0 \\ 0 & a_2b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_nb_n \end{pmatrix}$$

$$= \begin{pmatrix} b_1a_1 & 0 & \cdots & 0 \\ 0 & b_2a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_na_n \end{pmatrix}$$

$$= \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix}\begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}.$$

(This also shows D is closed under multiplication.) Not true for U. For example,

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 23 & 18 \end{pmatrix};$$

$$\begin{pmatrix} 4 & 0 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 17 & 18 \end{pmatrix}.$$

Symmetric matrices generally do not commute. Let $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and $B = \begin{pmatrix} d & e \\ e & f \end{pmatrix}$. Then, $(AB)_{12} = ae + bf$ and $(BA)_{12} = bd + ec$. These two terms are generally not equal.

8.9 There are n choices for where to put the 1 in the first row, $n - 1$ choices for where to put the one in the second row, etc. There are $n \cdot (n-1) \cdots 1 = n!$ permutation matrices.

8.10 Not closed under addition: The identity matrix is a permutation matrix, but $I + I = 2I$ is not.

Closed under multiplication: Suppose P and Q are two $n \times n$ permutation matrices. First, show that each row of PQ has exactly one 1 and $n - 1$ 0s in it. The entries in row i of PQ are calculated by multiplying row i of P by the various columns of Q. If $P_{ij} = 1$, then $(PQ)_{ik} = 0$ unless column k of Q has its 1 in row j. Since Q is a permutation matrix, one and only one column of Q has a 1 in row j. So, there is one k such that $(PQ)_{ik} = 1$ and $n - 1$ k's with $(PQ)_{ik} = 0$; that is, row i of PQ has one 1 and $n - 1$ 0s. The transpose of a permutation matrix is a permutation matrix. So the same argument shows that each row of $Q^T P^T$ has one 1 and $n - 1$ 0s. But each row of $Q^T P^T$ is a column of PQ. So, every row and every column of PQ contains only one 1 and $n - 1$ 0s. Thus, PQ is a permutation matrix.

8.12 The three kinds of elementary $n \times n$ matrices are the E_{ij}'s, the $E_i(r)$'s, and the $E_{ij}(r)$'s in the notation of this section. Theorem 8.2 gives the proof for the E_{ij}'s. For the $E_i(r)$'s, a generic element e_{hj} of $E_i(r)$ is

$$\begin{cases} e_{hj} = 0 & \text{if } h \neq j, \\ e_{hh} = 1 & \text{if } h \neq i, \\ e_{ii} = r. \end{cases}$$

The (k, m)th entry of $E_i(r) \cdot A$ is

$$\sum_{j=1}^{n} e_{kj} \cdot a_{jm} = e_{kk} a_{km} = \begin{cases} a_{km} & \text{if } k \neq i \\ r a_{km} & \text{if } k = i. \end{cases}$$

So, $E_i(r) \cdot A$ is A with its ith row multiplied by r.

We now work with $E_{ij}(r)$, the result of adding r times row i to row j in the identity matrix I. The only nonzero entry in row i is the 1 in column i. So row j of $E_{ij}(r)$ has an r in column i, in addition to the 1 in column j. In symbols,

$$\begin{aligned} e_{hh} &= 1 \quad \text{for all } h \\ e_{ji} &= r \\ e_{hk} &= 0 \quad \text{for } h \neq k \text{ and } (h, k) \neq (j, i). \end{aligned}$$

Since the elements in the hth row of $E_{ij}(r) \cdot A$ are the products of row h of $E_{ij}(r)$ and the columns of A, rows of $E_{ij}(r) \cdot A$ are the same as the rows of A, except for row j. The typical mth entry in row j of $E_{ij}(r) \cdot A$ is

$$\sum_{k=1}^{n} e_{jk} \cdot a_{km} = e_{jj} a_{jm} + e_{ji} a_{im} = a_{jm} + r a_{im},$$

since the other e_{jk}'s are zero. But this states that row j of $E_{ij}(r) \cdot A$ is (row j of A) $+ r$(row i of A).

8.13 We saw in Chapter 7 that by using a finite sequence of elementary row operations, one can transform any matrix A to its (reduced) row echelon form (RREF) U. Suppose we apply row operations $R_1, \ldots, R_m$ in that order to reduce A to RREF U. By Theorem 8.3, the same affect can be achieved by premultiplying A by the corresponding elementary matrices $E_1, \ldots, E_m$ so that

$$E_m \cdot E_{m-1} \cdots E_2 \cdot E_1 \cdot A = U.$$

Since U is in echelon form, each row has more leading zeros than its predecessor; i.e., U is upper triangular.

8.14 *a*) Permutation matrix P arises by permuting the rows of the $m \times m$ identity matrix I according to the permutation $s : \{1, \ldots, m\} \to \{1, \ldots, m\}$, so that row i of P is row $s(i)$ of I:

$$p_{ij} = \begin{cases} 1 & \text{if } j = s(i) \\ 0 & \text{otherwise.} \end{cases}$$

The (i, k)th entry of PA is:

$$\sum_{j=1}^{m} p_{ij} a_{jk} = p_{is(i)} a_{s(i)k} = a_{s(i)k},$$

the $(s(i), k)$th entry of A. Row i of PA is row $s(i)$ of A.

b) $AP = [[AP]^T]^T = [P^T A^T]^T$. If $P_{ij} = 1$, then $P^T_{ji} = 1$. Applying part *a* shows that $[AP]^T_{jk} = A^T_{ik}$, so $AP_{kj} = A_{ki}$.

8.15 Carry out the multiplication. In the first case,

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The computation for the second case is carried out in a similar fashion.

8.16 Carry out the multiplication.

8.17

$$\left(\begin{array}{cc|cc} 0 & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} c & d & 0 & 1 \\ 0 & b & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} 1 & d/c & 0 & 1/c \\ 0 & 1 & 1/b & 0 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & -d/bc & 1/c \\ 0 & 1 & 1/b & 0 \end{array}\right).$$

Since $a = 0$,

$$\begin{pmatrix} -\dfrac{d}{bc} & \dfrac{1}{c} \\ \dfrac{1}{b} & 0 \end{pmatrix} = \begin{pmatrix} \dfrac{d}{ad-bc} & -\dfrac{b}{ad-bc} \\ -\dfrac{c}{ad-bc} & \dfrac{a}{ad-bc} \end{pmatrix}.$$

8.18 Carry out the multiplication.

8.19 *a*)

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right).$$

The inverse is $\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$.

b) The inverse is $\begin{pmatrix} 4/6 & -5/6 \\ -2/6 & 4/6 \end{pmatrix}$.

c) Singular.

d)

$$\left(\begin{array}{ccc|ccc} 2 & 4 & 0 & 1 & 0 & 0 \\ 4 & 6 & 3 & 0 & 1 & 0 \\ -6 & -10 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & -2 & 3 & -2 & 1 & 0 \\ 0 & 2 & 0 & 3 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{2} & 0 & -1 \\ 0 & 1 & 0 & \frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right).$$

e) The inverse is $\begin{pmatrix} -\frac{5}{2} & 0 & -1 \\ \frac{3}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$.

f) The inverse is $\begin{pmatrix} 2 & \frac{9}{2} & -\frac{15}{2} & \frac{11}{2} \\ \frac{1}{3} & -\frac{7}{3} & \frac{13}{3} & -\frac{8}{3} \\ -\frac{1}{4} & \frac{3}{4} & -1 & \frac{3}{4} \\ -1 & 1 & -1 & 1 \end{pmatrix}$.

8.20 *a*) $A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

b) $A^{-1} = \begin{pmatrix} -6 & 3/2 & -1 \\ 13 & -3 & 2 \\ 5/2 & -1/3 & 1/3 \end{pmatrix}\begin{pmatrix} 4 \\ 20 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$.

c) $A^{-1} = \begin{pmatrix} -5/2 & 0 & -1 \\ 3/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

8.21 A $n \times n$ and AB defined implies B has n rows.
A $n \times n$ and BA defined implies B has n columns.

8.22 $A^4 = \begin{pmatrix} 34 & 21 \\ 21 & 13 \end{pmatrix}, A^3 = \begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix}, A^{-2} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$.

8.23 To prove that $E_{ij} \cdot E_{ij} = I$, write the (h, k)th entry of E_{ij} as

$$e_{hk} = \begin{cases} 1 & \text{if } h = i, k = j, \\ 1 & \text{if } h = j, k = i, \\ 1 & \text{if } h \neq i, j \text{ and } h = k, \\ 0 & \text{otherwise.} \end{cases}$$

Let a_{hk} denote the (h, k)th entry of $E_{ij} \cdot E_{ij}$:

$$a_{hk} = \sum_{r=1}^{n} e_{hr} e_{rk} = \begin{cases} e_{hj}e_{jk} & \text{if } h = i, \\ e_{hi}e_{ik} & \text{if } h = j, \\ e_{hh}e_{hk} & \text{if } h \neq i, j. \end{cases}$$

If $h = i$, case 1 tells us that

$$a_{hk} = \begin{cases} 0 & \text{if } k \neq h \\ 1 & \text{if } k = h. \end{cases}$$

If $h = j$, case 2 tells us that

$$a_{hk} = \begin{cases} 0 & \text{if } h \neq k \\ 1 & \text{if } h = k. \end{cases}$$

If $h \neq i, j$, case 3 tells us that

$$a_{hk} = \begin{cases} 0 & \text{if } h \neq k \\ 1 & \text{if } h = k. \end{cases}$$

In other words, (a_{hk}) is the identity matrix.

To see that $E_i(r) \cdot E_i(1/r) = I$, one easily checks that the inverse of $\text{diag}\{a_1, a_2, \ldots, a_n\}$ is $\text{diag}\{1/a_1, \ldots, 1/a_n\}$, where the entries listed are the diagonal entries of the diagonal matrix.

To see that $E_{ij}(r) \cdot E_{ij}(-r) = I$, write e_{hk} for the (h, k)th entry of $E_{ij}(r)$:

$$e_{hk} = \begin{cases} 1 & \text{if } h = k, \\ r & \text{if } (h, k) = (j, i), \\ 0 & \text{otherwise,} \end{cases}$$

as in Exercise 8.12. Let f_{hk} be the (h, k)th entry of $E_{ij}(-r)$, with $-r$ replacing r in case 2. Then, the (h, k)th entry of $E_{ij}(r) \cdot E_{ij}(-r)$ is

$$a_{hk} = \sum_{l=1}^{n} e_{hl} f_{lk} = \begin{cases} e_{hh} \cdot f_{hk} & \text{if } h \neq j \\ e_{jj} f_{jk} + e_{ji} f_{ik} & \text{if } h = j. \end{cases}$$

If $h \neq j$, $a_{hk} = e_{hh} f_{hk} = \begin{cases} 1 & \text{if } h = k \\ 0 & \text{if } h \neq k. \end{cases}$

If $h = j$, $\quad a_{hk} = \begin{cases} e_{jj} f_{ji} + e_{ji} f_{ii} = -r + r \quad = 0 & \text{if } k \neq j \\ e_{jj} f_{jj} + e_{ji} f_{ij} = 1 \cdot 1 + r \cdot 0 = 1 & \text{if } k = j. \end{cases}$

So, $\quad a_{hk} = \begin{cases} 1 & \text{if } h = k \\ 0 & \text{if } h \neq k, \end{cases}$

and $E_{ij}(r) \cdot E_{ij}(-r) = I$.

8.24 *a*) $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ is invertible $\iff ad - bc = ad \neq 0 \iff a \neq 0$ and $d \neq 0$.

b) $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}^{-1} = \dfrac{1}{ad} \begin{pmatrix} d & 0 \\ -c & a \end{pmatrix}$, lower triangular.

c) $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{-1} = \dfrac{1}{ad} \begin{pmatrix} d & -b \\ 0 & a \end{pmatrix}$, upper triangular.

8.25 *a*) Part *a* holds since $A^{-1} \cdot A = A \cdot A^{-1} = I$ implies that A is the inverse of A^{-1}.

To prove *b*, compute that $I = I^T = (AA^{-1})^T = (A^{-1})^T \cdot A^T$. So, $(A^T)^{-1} = (A^{-1})^T$.

To prove *c*, observe that $(AB) \cdot (B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A \cdot I \cdot A^{-1} = A \cdot A^{-1} = I$.

Similarly, $(B^{-1}A^{-1}) \cdot (AB) = I$.

b) Since
$$\begin{aligned}(A_1\cdots A_k)&(A_k^{-1}\cdot A_{k-1}^{-1}\cdots A_1^{-1})\\ &= (A_1\cdots A_{k-1})(A_kA_k^{-1})(A_{k-1}^{-1}\cdots A_1^{-1})\\ &= (A_1\cdots A_{k-1})(A_{k-1}^{-1}\cdots A_1^{-1})\\ &= (A_1\cdots A_{k-2})(A_{k-1}\cdot A_{k-1}^{-1})(A_{k-2}^{-1}\cdots A_1^{-1})\\ &= \cdots = A_1A_1^{-1} = I.\end{aligned}$$

So, $(A_1\cdots A_k)^{-1} = A_k^{-1}\cdot A_{k-1}^{-1}\cdots A_1^{-1}$.

c) For example, $A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 0\end{pmatrix}$ and $B = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 1\end{pmatrix}$, or just take an invertible A and let $B = -A$.

d) Even for 1×1 matrices, $\dfrac{1}{a+b} \neq \dfrac{1}{a} + \dfrac{1}{b}$, in general.

8.26 *a*) One can use the statement and/or method of Exercise 8.25*b* with $A_1 = \cdots = A_k = A$.

b) $A^r\cdot A^s = \underbrace{(A\cdots A)}_{r \text{ times}}\cdot\underbrace{(A\cdots A)}_{s \text{ times}} = \underbrace{A\cdots A\cdot A\cdots A}_{r+s \text{ times}}$.

c) $(rA)\cdot\left(\dfrac{1}{r}A^{-1}\right) = r\cdot\dfrac{1}{r}\cdot A\cdot A^{-1} = 1\cdot I = I$.

8.27 *a*) Applying $AB = BA$ $(k-1)$ times, we easily find

$$\begin{aligned}AB^k &= A\cdot BB^{k-1} = BAB^{k-1} = BAB\cdot B^{k-2} = B^2AB^{k-2}\\ &= \cdots = B^{k-1}\cdot A\cdot B = B^{k-1}\cdot B\cdot A = B^kA.\end{aligned}$$

Use induction to prove $(AB)^k = (BA)^k$ if $AB = BA$. It is true for $k = 1$ since $AB = BA$. Assume $(AB)^{k-1} = (BA)^{k-1}$ and prove it true for k:

$$\begin{aligned}(AB)^k &= (AB)^{k-1}AB\\ &= A^{k-1}B^{k-1}AB, && \text{by inductive hypothesis}\\ &= A^{k-1}AB^{k-1}B, && \text{by first sentence in } a\\ &= A^kB^k.\end{aligned}$$

b) Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 1\end{pmatrix}$. Then

$$(AB)^2 = \begin{pmatrix} 5 & 3 \\ 3 & 2\end{pmatrix} \quad\text{but}\quad A^2B^2 = \begin{pmatrix} 5 & 2 \\ 2 & 1\end{pmatrix}.$$

More generally, suppose that A and B are non-singular. If $ABAB = A^2B^2$, then premultiplying by A^{-1} and postmultiplying by B^{-1} give $AB = BA$.

c) $(A+B)^2 = A^2+AB+BA+B^2$. $(A+B)^2-(A^2+2AB+B^2) = BA-AB$. This equals 0 if and only if $AB = BA$.

8.28 $D^{-1} = \begin{pmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{pmatrix}.$

8.29 $\begin{pmatrix} a & b \\ b & d \end{pmatrix}^{-1} = \dfrac{1}{ac-b^2}\begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$, a symmetric matrix.

8.30 Let U be an $n \times n$ upper-triangular matrix with (i, j)th entry u_{ij}. Let $B = U^{-1}$ with (i, j)th entry b_{ij}. Let $I = (e_{ij})$ be the identity matrix. Since U is upper triangular, $u_{ij} = 0$ for all $i > j$. Now $I = BU$; therefore,

$$1 = e_{11} = \sum_k b_{1k}u_{k1} = b_{11}u_{11}$$

since $u_{21} = \cdots = u_{n1} = 0$. Therefore, $u_{11} \neq 0$ and $b_{11} = 1/u_{11}$. For $h > 1$,

$$0 = e_{h1} = \sum_k b_{hk}u_{k1} = b_{h1}u_{11}.$$

Since $u_{11} \neq 0$, $b_{h1} = 0$ for $h > 1$.

Now, work with column 2 of B.

$$1 = e_{22} = \sum_k b_{2k}u_{k2} = b_{21}u_{12} + b_{22}u_{22} = b_{22}u_{22}$$

since $b_{21} = 0$. Therefore, $u_{22} \neq 0$ and $b_{22} = 1/u_{22}$.

For $h > 2$,

$$0 = e_{h2} = \sum_k b_{hk}u_{k2} = b_{h1}u_{12} + b_{h2}u_{22} = 0 + b_{h2}u_{22}.$$

Since $u_{22} \neq 0$, $b_{h2} = 0$. We conclude that $b_{h2} = 0$ for all $h > 2$. This argument shows $b_{hj} = 0$ for all $h > j$; that is, B too is upper triangular.

The second part follows by transposing the first part and Theorem 8.10*b*.

8.31 The (i, j)th entry of $P^T P$ is the product of the ith row of P^T and the jth column of P, that is, the product of the ith column of P and the jth column of P. This product is 0 if $i \neq j$ and 1 if $i = j$; that is, $P^T P = I$.

8.32 The criterion for invertability is

$$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{13}a_{21}a_{31} - a_{11}a_{21}a_{33} + a_{11}a_{22}a_{31} - a_{12}a_{21}a_{31} \neq 0.$$

See Section 26.1.

8.33 *a*) Suppose a $k \times l$ matrix A has a left inverse L (which must be $k \times k$) and a right inverse R (which must be $l \times l$). Then $LAR = (LA)R = IR = R$ and $LAR = L(AR) = LI = L$, so $R = L$. This is impossible since the two matrices are of different sizes.

b, c) Suppose A is $m \times n$ with $m < n$. If A has rank m, then $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for every right-hand side $\mathbf{b}$, by Fact 7.11a. Let $\mathbf{e_i} = (0, \dots, 0, 1, 0, \dots, 0)$ with a 1 in the ith entry. Let $\mathbf{c_i}$ be one of the (infinitely many) solutions of $A\mathbf{x} = \mathbf{e}_i$. Then,

$$A \cdot [\mathbf{c}_1 \cdots \mathbf{c}_m] = [\mathbf{e}_1 \cdots \mathbf{e}_m] = I.$$

So, $C = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \cdots \quad \mathbf{c}_m]$ is one of the right inverses of A. Conversely, if A has a right inverse C, then the solution of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = C\mathbf{b}$ since $A(C\mathbf{b}) = (AC)\mathbf{b} = \mathbf{b}$. By Fact 7.7, A must have rank m = number of rows of A.

d) If A is $m \times n$ with $m > n$, apply the previous analysis to A^T.

8.34 *a*) $(I - A)^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}14\\8\\8\end{pmatrix}$; *b*) $(I - A)^{-1}\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}20\\14\\14\end{pmatrix}$;

c) $(I - A)^{-1}\begin{pmatrix}2\\1\\2\end{pmatrix} = \begin{pmatrix}18\\16\\18\end{pmatrix}$.

8.35 For $\begin{pmatrix}a & b\\c & d\end{pmatrix}$ with $a, b, c, d > 0$, $a + c < 1$, and $b + d < 1$,

$$(I - A)^{-1} = \begin{pmatrix}1-a & -b\\-c & 1-d\end{pmatrix}^{-1} = \frac{1}{(1-a)(1-d) - bc}\begin{pmatrix}1-d & b\\c & 1-a\end{pmatrix}.$$

Since $a + c < 1$, $c < (1 - a)$; since $b + d < 1$, $b < (1 - d)$. Therefore, $0 < bc < (1 - a)(1 - d)$ and $(1 - a)(1 - d) - bc > 0$. So, $(I - A)^{-1}$ is a positive matrix.

8.36 Let $a_{\cdot 1}$ = # of columns of A_{11} = # of columns of A_{21}.
Let $a_{\cdot 2}$ = # of columns of A_{12} = # of columns of A_{22}.
Let $c_{1\cdot}$ = # of rows of C_{11} = # of rows of C_{12} = # of rows of C_{13}.
Let $c_{2\cdot}$ = # of rows of C_{21} = # of rows of C_{22} = # of rows of C_{23}.
Then, $a_{\cdot 1} = c_{1\cdot}$ and $a_{\cdot 2} = c_{2\cdot}$.

8.37 C should be written as $\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$. In the notation of the previous problem

$$\begin{aligned} a_{\cdot 1} &= c_{1\cdot} = 2 \\ a_{\cdot 2} &= c_{2\cdot} = 1 \\ a_{\cdot 3} &= c_{3\cdot} = 3. \end{aligned}$$

8.38 $A^{-1} = \begin{pmatrix} A_{11}^{-1} & 0 & \cdots & 0 \\ 0 & A_{22}^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{nn}^{-1} \end{pmatrix}$.

8.39 In the notation of Exercise 8.36, A_{11} is of size $a_{1\cdot} \times a_{\cdot 1}$; $A_{12}A_{22}^{-1}A_{21}$ is also of size $a_{1\cdot} \times a_{\cdot 1}$, so D is well defined.

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} D^{-1} & -D^{-1}A_{12} \cdot A_{22}^{-1} \\ -A_{22}^{-1}A_{21}D^{-1} & A_{22}^{-1}(I + A_{21}D^{-1}A_{12}A_{22}^{-1}) \end{pmatrix}$$
$$= \begin{pmatrix} A_{11}D^{-1} - A_{12}A_{22}^{-1}A_{21}D^{-1} & -A_{11}D^{-1}A_{12}A_{22}^{-1} + A_{12}A_{22}^{-1} + A_{12}A_{22}^{-1}A_{21}D^{-1}A_{12}A_{22}^{-1} \\ A_{21}D^{-1} - A_{22}A_{22}^{-1}A_{21}D^{-1} & -A_{21}D^{-1}A_{12}A_{22}^{-1} + A_{22}A_{22}^{-1} + A_{22}A_{22}^{-1}A_{21}D^{-1}A_{12}A_{22}^{-1} \end{pmatrix}.$$

Write (1, 1) as $(A_{11} - A_{12}A_{22}^{-1}A_{21})D^{-1} = DD^{-1} = I.$ Write (1, 2) as

$$\begin{aligned} -(A_{11} - A_{12}A_{22}^{-1}A_{21})D^{1}A_{12}A_{22}^{-1} + A_{12}A_{22}^{-1} &= -DD^{-1}A_{12}A_{22}^{-1} + A_{12}A_{22}^{-1} \\ &= -A_{12}A_{22}^{-1} + A_{12}A_{22}^{-1} = 0. \end{aligned}$$

Write (2, 1) as $(A_{21}D^{-1} - IA_{21}D^{-1}) = 0.$
Write (2, 2) as $-A_{21}D^{-1}A_{21}A_{22}^{-1} + I + A_{21}D^{-1}A_{12}A_{22}^{-1} = I.$

So the product is the identity matrix $\begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix}$.

8.40 $A^{-1} = \begin{pmatrix} A_{11}^{-1}(I + A_{12}C^{-1}A_{21}A_{11}^{-1}) & -A_{11}^{-1}A_{12}C^{-1} \\ -C^{-1}A_{21}A_{11}^{-1} & C^{-1} \end{pmatrix}$,

where $C = A_{22} - A_{21}A_{11}^{-1}A_{12}$.

8.41 *a*) A_{11} and A_{22} nonsingular.

b) A_{11} and $A_{11} - (1/a_{22})A_{12}A_{21}$ nonsingular.

c) A_{22} invertible and $\mathbf{p}^T A_{22}^{-1}\mathbf{p}$ nonzero.

8.42 *a*) $E_{12}(3)$.

b) $E_{12}(-3), E_{13}(2), E_{23}(-1)$.

c) $E_{12}(-2), E_{13}(3), E_{23}(1)$.

d) $E_{12}(-3), E_{14}(-2), E_{23}(1), E_{34}(-1)$.

8.43 *a*) $\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 0 & -1 \end{pmatrix}$,

b) $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 6 \\ 0 & 0 & 3 \end{pmatrix}$,

c) $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 & 0 & 1 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 3 & 8 \end{pmatrix}$,

d) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 6 & 0 & 5 \\ 0 & 3 & 8 & 2 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & 0 & -6 \end{pmatrix}$.

8.44 Suppose we can write A as $A = L_1U_1 = L_2U_2$ where L_1 and L_2 are lower triangular with only 1s on the diagonals. The proof and statement of Exercise 8.30 show that L_1 and L_2 are invertible and that L_1^{-1} and L_2^{-1} are lower triangular. Since $U_1 = L_1^{-1}A$ and $U_2 = L_2^{-1}A$, U_1 and U_2 are invertible too. Write $L_1U_1 = L_2U_2$ as $L_2^{-1}L_1 = U_2U_1^{-1}$. By Exercise 8.8, $L_2^{-1}L_1$ is lower triangular and $U_2U_1^{-1}$ is upper triangular. Therefore, $L_2^{-1}L_1$ and $U_2U_1^{-1}$ are both diagonal matrices. Since L_1 and L_2 have only 1s on the diagonal, L_2^{-1} and $L_2^{-1}L_1$ have only 1s on the diagonal. It follows that $L_2^{-1}L_1 = I$ and that $U_2U_1^{-1} = I$. Therefore, $L_2 = L_1$ and $U_2 = U_1$, and the LU decomposition of A is unique.

8.45 Suppose $A = L_1U_1 = L_2U_2$, as in the last exercise. First, choose U_2 to be a row echelon matrix of A. By rearranging the order of the variables, we can assume that each row of U_2 has exactly one more leading zero than the previous row and that its $U_{11} \neq 0$. Write $L_1U_1 = L_2U_2$ as $LU_1 = U_2$ where L is the lower-triangular matrix $L_2^{-1}L_1$ and has only 1s on its diagonal. Let $L = ((l_{ij}))$, $U_1 = ((v_{ij}))$, and $U_2 = ((u_{ij}))$.

$$0 \neq u_{11} = \sum_j l_{1j}v_{j1} = l_{11}v_{11} = v_{11} \Longrightarrow v_{11} \neq 0.$$

For $k > 1$, $u_{k1} = v_{k1} = 0$. So, $0 = u_{k1} = \sum_{j=1}^n l_{kj}v_{j1} = l_{k1}v_{11} \Longrightarrow l_{k1} = 0$ for $k > 1$.

This shows that the first column of L is $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

Similar analysis shows that the jth column of L is $\begin{pmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$. We need only show that U_2 has no all-zero rows; this follows from the assumptions that U_2 is the row echelon matrix of A and A has maximal rank. It follows that L is the identity matrix and $U_1 = U_2$. Since every such U_i equals U_2, they equal each other. Since $I = L = L_2^{-1}L_1$, $L_2 = L_1$.

8.46 A simple example is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for all a.

8.47 As in Exercise 8.44, write A uniquely as $A = L_1U_1$ where L_1 is lower triangular and has only 1s on its diagonal. Decompose upper-triangular U_1 as

$$DU = \begin{pmatrix} u_{11} & 0 & \cdots & 0 \\ 0 & u_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix}\begin{pmatrix} 1 & u_{12}/u_{11} & \cdots & u_{1n}/u_{11} \\ 0 & 1 & \cdots & u_{2n}/u_{22} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

(Since A is nonsingular, so is U_1, and so all its diagonal entries are nonzero.) Use the method of Exercise 8.44 to see that this DU decomposition is unique.

8.48 *a*) $\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

b) $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{pmatrix}$.

d) Use answer to 8.43*d*:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 & 5/2 \\ 0 & 1 & 8/3 & 2/3 \\ 0 & 0 & 1 & -9/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

8.49 *i*) *a*) $\begin{pmatrix} 3 & 2 & 0 \\ 6 & 4 & 1 \\ -3 & 4 & 1 \end{pmatrix} \xrightarrow[E_{13}(1)\cdot E_{12}(-2)]{} \begin{pmatrix} 3 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 6 & 1 \end{pmatrix} \xrightarrow[E_{23}]{} \begin{pmatrix} 3 & 2 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

b) $P = P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

c) $PA = \begin{pmatrix} 3 & 2 & 0 \\ -3 & 4 & 1 \\ 6 & 4 & 1 \end{pmatrix} \xrightarrow[E_{13}(-2)\cdot E_{12}(1)]{} \begin{pmatrix} 3 & 2 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

d) $PA = E_{12}^{-1} \cdot E_{13}^{-1} \cdot \begin{pmatrix} 3 & 2 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

ii) *a*) $\begin{pmatrix} 0 & 1 & 1 & 4 \\ 1 & 1 & 2 & 2 \\ -6 & -5 & -11 & -12 \\ 2 & 3 & -2 & 3 \end{pmatrix} \xrightarrow[E_{12}]{} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 4 \\ -6 & -5 & -11 & -12 \\ 2 & 3 & -2 & 3 \end{pmatrix}$

$$\xrightarrow[E_{14}(-2)\cdot E_{13}(6)]{} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -6 & -1 \end{pmatrix} \xrightarrow[E_{24}(-1)\cdot E_{34}(-1)]{} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & -7 & -5 \end{pmatrix}$$

$$\xrightarrow[E_{34}]{} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & -7 & -5 \\ 0 & 0 & 0 & -4 \end{pmatrix}.$$

b) P_{12} and P_{34}: $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = P$.

$$c)\ PA = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 4 \\ 2 & 3 & -2 & 3 \\ -6 & -5 & -11 & -12 \end{pmatrix} \underset{E_{14}(6)\cdot E_{13}(-2)}{\longrightarrow} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & -6 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\underset{E_{24}(-1)\cdot E_{23}(-1)}{\longrightarrow} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & -7 & -5 \\ 0 & 0 & 0 & -4 \end{pmatrix}, \text{ as at end of part } a.$$

$$d)\ PA = (E_{24}(-1)\cdot E_{23}(-1)\cdot E_{14}(6)\cdot E_{13}(-2))^{-1}$$

$$\cdot \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & -7 & -5 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ -6 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & -7 & -5 \\ 0 & 0 & 0 & -4 \end{pmatrix}.$$

8.50 *a*) In the general 2×2 case, a row interchange is required if $a_{11} = 0 \neq a_{21}$.

b) A row interchange is required if $a_{11} = 0$ and $a_{i1} \neq 0$ for some $i > 1$ or if $a_{11}a_{22} - a_{21}a_{12} = 0 \neq a_{11}a_{32} - a_{31}a_{13}$.

8.51 *b*) *i*)

$$\begin{pmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & 3 \end{pmatrix} \equiv LU.$$

$$L\mathbf{z} = \mathbf{b} \Longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix} \Longrightarrow \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}.$$

$$U\mathbf{x} = \mathbf{z} \Longrightarrow \begin{pmatrix} 2 & 4 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -3 \end{pmatrix} \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

ii)

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ -4 \end{pmatrix} \Longrightarrow \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}.$$

$$\begin{pmatrix} 2 & 4 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

$$\begin{pmatrix} 5 & 3 & 1 \\ -5 & -4 & 1 \\ -10 & -9 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 5 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

iii) $$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 3 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -10 \\ -24 \end{pmatrix} \Longrightarrow \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ -1 \end{pmatrix}.$$

$$\begin{pmatrix} 5 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ -1 \end{pmatrix} \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

iv) $$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 3 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -14 \end{pmatrix} \Longrightarrow \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}.$$

$$\begin{pmatrix} 5 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Chapter 9

9.1 $a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}$.

9.2 $$a_{11} \cdot \det\begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix} - a_{12}\det\begin{pmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{pmatrix}$$
$$+a_{13}\det\begin{pmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{pmatrix} - a_{14}\det\begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}.$$

There are four terms, each consisting of a scalar multiple of the determinant of a 3×3 matrix. There are six terms in the expansion of the determinant of a 3×3 matrix, so the 4×4 expansion has $4 \cdot 6 = 24$ terms.

9.4 Row 2: $\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (-1)^3 a_{21}a_{12} + (-1)^4 a_{22}a_{11} = a_{11}a_{22} - a_{21}a_{12}$.

Column 1: $\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (-1)^2 a_{11}a_{22} + (-1)^3 a_{21}a_{12} = a_{11}a_{22} - a_{21}a_{12}$.

Column 2: $\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (-1)^3 a_{12}a_{21} + (-1)^4 a_{22}a_{11} = a_{11}a_{22} - a_{21}a_{12}$.

9.5 Expand along column one:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = a_{11}\cdot\det\begin{pmatrix} a_{22} & a_{23} \\ 0 & a_{33} \end{pmatrix} - 0\cdot\det\begin{pmatrix} a_{12} & a_{13} \\ 0 & a_{33} \end{pmatrix}$$
$$+ 0\cdot\det\begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix}$$
$$= a_{11}a_{22}a_{33} + 0 + 0.$$

9.6 $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$, and $\det\begin{pmatrix} a & b \\ ra+c & rb+d \end{pmatrix} = rab + ad - rab - bc = ad - bc$.

9.7 *a*) $R = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, $\det R = -1$, and $\det A = -1$.

b) $R = \begin{pmatrix} 2 & 4 & 0 \\ 0 & -8 & 3 \\ 0 & 0 & 3/4 \end{pmatrix}$, $\det R = -12$, and $\det A = -12$.

c) $R = \begin{pmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & -6 \end{pmatrix}$, $\det R = -18$, and $\det A = 18$.

9.8 *a*) One row echelon form is $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. So, $\det = 3$.

b) One row echelon form is $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 5 \\ 0 & 0 & -5 \end{pmatrix}$. So, $\det = -20$.

9.9 All nonsingular since $\det \neq 0$.

9.10 Carry out the calculation

$$\begin{pmatrix} 2 & 4 & 5 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & -4 & -15 \\ 0 & -3 & 0 \\ -3 & 4 & 6 \end{pmatrix}\left(-\frac{1}{9}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

9.11 *a*) $\frac{1}{1}\begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}$.

$$b)\ \frac{1}{\det A}\cdot\begin{pmatrix} \begin{vmatrix}5&6\\0&8\end{vmatrix} & -\begin{vmatrix}2&3\\0&8\end{vmatrix} & \begin{vmatrix}2&3\\5&6\end{vmatrix}\\ -\begin{vmatrix}0&6\\1&8\end{vmatrix} & \begin{vmatrix}1&3\\1&8\end{vmatrix} & -\begin{vmatrix}1&3\\0&6\end{vmatrix}\\ \begin{vmatrix}0&5\\1&0\end{vmatrix} & -\begin{vmatrix}1&2\\1&0\end{vmatrix} & \begin{vmatrix}1&2\\0&5\end{vmatrix}\end{pmatrix}$$

$$= \frac{1}{37}\cdot\begin{pmatrix}40&-16&-3\\6&5&-6\\-5&-2&5\end{pmatrix}.$$

$$c)\ \frac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}.$$

9.12 $x_1 = 35/35 = 1, \quad x_2 = -70/35 = -2.$

9.13 *a)* $x_1 = -7/-7 = 1, \quad x_2 = 14/-7 = -2.$

b) $x_1 = -23/-23 = 1, \quad x_2 = 0/-23 = 0, \quad x_3 = -69/-23 = 3.$

9.14 *a)* $\det A = -1, \det B = -1, \det AB = +1, \det(A+B) = -4;$

b) $\det A = 24, \det B = 18, \det AB = 432, \det(A+B) = 56;$

c) $\det A = ad - bc, \det B = eh - fg, \det AB = (ad-bc)(eh-fg),$

$\det(A+B) = \det A + \det B + ah - bg + de - cf.$

9.15

$$\frac{\partial Y}{\partial I_0} = \frac{\partial Y}{\partial G} = \frac{h}{sh+am}, \quad \frac{\partial r}{\partial I_0} = \frac{\partial r}{\partial G} = \frac{m}{sh+am}$$

$$\frac{\partial Y}{\partial M_s} = -\frac{\partial Y}{\partial M_0} = \frac{a}{sh+am}, \quad \frac{\partial r}{\partial M_s} = -\frac{\partial r}{\partial M_0} = \frac{-s}{sh+am}$$

$$\frac{\partial Y}{\partial m}, \quad \frac{\partial r}{\partial m} < 0$$

9.16

$$\begin{aligned}\frac{\partial Y}{\partial h} &= \frac{(I^\circ + G)}{sh+am} - s\frac{(I^\circ + G)h + a(M_s - M^\circ)}{(sh+am)^2}\\ &= \frac{a}{(sh+am)^2}[(I^\circ + G)m - (M_s - M^\circ)s]\\ &= \frac{ar}{(sh+am)} > 0,\end{aligned}$$

$$\frac{\partial r}{\partial m} = \frac{(I^\circ + G)sh + as(M_s - M^\circ)}{(sh + am)^2}$$
$$= \frac{sY}{(sh + am)} > 0,$$
$$\frac{\partial r}{\partial s} = -\frac{am(M_s - M^\circ) + mh(I^\circ + G)}{(sh + am)^2}$$
$$= -\frac{mY}{(sh + am)} < 0.$$

9.17 $$Y = \frac{h(I^0 + G) + a(M_s - M^0)}{(1 - t)sh + am}, \quad r = \frac{(I^0 + G)m - (1 - t)s(M_s - M^0)}{(1 - t)sh + am}$$
$$\frac{\partial Y}{\partial t} > 0, \quad \frac{\partial r}{\partial t} = \frac{msY}{am + (1 - t)hs} > 0$$

9.18 The IS curve is $[1 - a_0 - c_1(1 - t)]Y + (a + c_2)r = I^0 + G + c_0 - c_1 t_0$. The solution to the system is

$$Y = \frac{(a + c_2)(M_s - M_0) + h(G + I^0 + c_0 - c_1 t_0)}{h[1 - a_0 - c_1(1 - t_1)] + (a + c_2)m}$$
$$r = \frac{-[1 - a_0 - c_1(1 - t_1)(M_s - M_0)] + m(G + I^0 + c_0 - c_1 t_0)}{h[1 - a_0 - c_1(1 - t_1)] + (a + c_2)m}.$$

9.19 Under the obvious assumptions on parameter values, increasing I^0 increases both Y and r. Differentiating the solutions with respect to m,

$$\frac{\partial Y}{\partial m} = \frac{-(a + c_2)Y}{h[1 - a_0 - c_1(1 - t_1)] + (a + c_2)m} < 0$$
$$\frac{\partial r}{\partial m} = \frac{c_0 - c_1 t_0 - (a + c_2)r}{h[1 - a_0 - c_1(1 - t_1)] + (a + c_2)m} > 0.$$

Similarly, differentiating the solutions with respect to c_0 gives

$$\frac{\partial Y}{\partial c_0} = \frac{h}{h[1 - a_0 - c_1(1 - t_1)] + (a + c_2)m} > 0,$$
$$\frac{\partial r}{\partial c_0} = \frac{m}{h[1 - a_0 - c_1(1 - t_1)] + (a + c_2)m} > 0.$$

9.20 The equation system is

$$\begin{aligned} C \;+\; cS + cF &= cP \\ sC \;+\; S \qquad\;\; &= sP \\ fC \;+\; fS + \;\; F &= fP. \end{aligned}$$

The solution is

$$C = \frac{\begin{vmatrix} c & c & c \\ s & 1 & 0 \\ f & f & 1 \end{vmatrix} \cdot P}{\begin{vmatrix} 1 & c & c \\ s & 1 & 0 \\ f & f & 1 \end{vmatrix}} = \frac{(1 + sf - f - s)cP}{1 + sfc - fc - sc}$$

$$S = \frac{\begin{vmatrix} 1 & c & c \\ s & s & 0 \\ f & f & 1 \end{vmatrix} \cdot P}{\begin{vmatrix} 1 & c & c \\ s & 1 & 0 \\ f & f & 1 \end{vmatrix}} = \frac{s(1 - c)P}{1 + sfc - fc - sc}$$

$$F = \frac{\begin{vmatrix} 1 & c & c \\ s & 1 & s \\ f & f & f \end{vmatrix} \cdot P}{\begin{vmatrix} 1 & c & c \\ s & 1 & 0 \\ f & f & 1 \end{vmatrix}} = \frac{(1 + sc - s - c)fP}{1 + sfc - fc - sc}.$$

Chapter 10

10.4 *a*) $(2, -1)$, *b*) $(-2, -1)$, *c*) $(2, 1)$, *d*) $(3, 0)$, *e*) $(1, 2, 4)$, *f*) $(2, -2, 3)$.

10.5 *a*) $(1, 3)$, *b*) $(-4, 12)$, *c*) undefined, *d*) $(0, 3, 3)$, *e*) $(0, 2)$, *f*) $(1, 4)$, *g*) $(1, 1)$, *h*) $(3, 7, 1)$, *i*) $(-2, -4, 0)$, *j*) undefined.

10.7 $-\mathbf{u}$ is what one adds to $\mathbf{u}$ to get $\mathbf{0}$. $(-1)\mathbf{u} + \mathbf{u} = (-1)\mathbf{u} + 1 \cdot \mathbf{u} = [(-1) + 1]\mathbf{u} = 0 \cdot \mathbf{u} = \mathbf{0}$. So $(-1)\mathbf{u} = -\mathbf{u}$.