AN APPROACH TO BACKWARD ERROR ANALYSIS FOR VARIATIONAL DISCRETISATIONS OF PDES

This document contains additional computational results for a proposed publication related to the repository [1]. Please refer to the author's list of publications / ArXiv.

1. Introduction of computational example

Consider the nonlinear wave equation

$$u_{tt} - u_{xx} - V'(\|u\|^2)u = 0$$

and its discretisation by the 5-point stencil

$$\begin{aligned}0 = &\frac{1}{\Delta t^2}\left(u(t-\Delta t, x) - 2u(t,x) + u(t+\Delta t, x)\right)\\ &- \frac{1}{\Delta x^2}\left(u(t, x-\Delta x) - 2u(t,x) + u(t, x+\Delta x)\right)\\ &- V'(\langle u(t,x), u(t,x)\rangle)u(t,x).\end{aligned}$$

Let h be a (formal) variable and consider

$$\begin{aligned}0 = &\frac{1}{h^2\Delta t^2}\left(u(t-h\Delta t, x) - 2u(t,x) + u(t+h\Delta t, x)\right)\\ &- \frac{1}{h^2\Delta x^2}\left(u(t, x-h\Delta x) - 2u(t,x) + u(t, x+h\Delta x)\right)\\ &- V'(\langle u(t,x), u(t,x)\rangle)u(t,x).\end{aligned}$$

The ansatz $u(t,x) = R(t)\phi(x-ct)$ with $R(t) = \exp(\alpha J t)$ with

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

yields

$$\begin{aligned}0 = &\frac{1}{h^2\Delta t^2}\left(R(-h\Delta t)\phi(\xi + ch\Delta t) - 2\phi(\xi) + R(h\Delta t)\phi(\xi - ch\Delta t)\right)\\ &- \frac{1}{h^2\Delta x^2}\left(\phi(\xi + h\Delta x) - 2\phi(\xi) + \phi(\xi - h\Delta x)\right)\\ &- V'(\langle \phi(\xi), \phi(\xi)\rangle)\phi(\xi)\end{aligned} \tag{1.1}$$

with $\xi = x + ct$. A power series expansion around $h = 0$ and isolating $\ddot{\phi}(\xi)$ from the $\mathcal{O}(h^0)$ part yields

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$$
\begin{aligned}
\ddot{\phi}(\xi) = &\frac{(\alpha^2 + V'(\langle \phi(\xi), \phi(\xi) \rangle))\phi(\xi) + 2c\alpha J \dot{\phi}(\xi)}{c^2-1} \\
&+ h^2 g_2(\phi^{(4)}(\xi), \dots, \dot{\phi}(\xi), \phi(\xi)) \\
&+ h^4 g_4(\phi^{(6)}(\xi), \dots, \dot{\phi}(\xi), \phi(\xi)) \\
&+ \dots
\end{aligned}
$$

Substituting $\ddot{\phi}(\xi)$ and higher order derivatives of ϕ on the right hand side reduces the equation to a formal second order ODE

$$
\ddot{\phi}(\xi) = \frac{(\alpha^2 + V'(\langle \phi(\xi), \phi(\xi) \rangle))\phi(\xi) + 2c\alpha J \dot{\phi}(\xi)}{c^2-1} + \sum_{j=2}^{\infty} h^{2j} \hat{g}_{2j}(\dot{\phi}(\xi), \phi(\xi)). \tag{1.2}
$$

Assume that u is $\mathbb{R}^2$-valued. The first component of $\hat{g}_2(\phi, \dot{\phi})$ is given as

$$
\begin{aligned}
&\frac{2\Delta t^2\alpha^3\phi_2'(\xi)c^7}{3(c^2-1)^4} + \frac{\Delta t^2\alpha^4\phi_1(\xi)c^6}{3(c^2-1)^4} + \frac{\Delta t^2\alpha^2\phi_1(\xi)V'(\phi_1(\xi)^2+\phi_2(\xi)^2)c^6}{3(c^2-1)^4} - \frac{5\Delta t^2\alpha^3\phi_2'(\xi)c^5}{3(c^2-1)^3} \\
&- \frac{\Delta t^2\alpha V'(\phi_1(\xi)^2+\phi_2(\xi)^2)\phi_2'(\xi)c^5}{3(c^2-1)^3} - \frac{\Delta t^2\alpha\phi_1(\xi)^2\phi_2'(\xi)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c^5}{3(c^2-1)^3} \\
&- \frac{\Delta t^2\alpha\phi_2(\xi)^2\phi_2'(\xi)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c^5}{3(c^2-1)^3} - \frac{\Delta t^2\phi_1(\xi)\phi_1'(\xi)^2V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c^4}{2(c^2-1)^2} \\
&- \frac{\Delta t^2\phi_2(\xi)\phi_1'(\xi)\phi_2'(\xi)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c^4}{3(c^2-1)^2} - \frac{\Delta t^2\phi_1(\xi)^3\phi_1'(\xi)^2V^{(3)}(\phi_1(\xi)^2+\phi_2(\xi)^2)c^4}{3(c^2-1)^2} \\
&- \frac{\Delta t^2\phi_1(\xi)\phi_2(\xi)^2\phi_2'(\xi)^2V^{(3)}(\phi_1(\xi)^2+\phi_2(\xi)^2)c^4}{3(c^2-1)^2} \\
&- \frac{2\Delta t^2\phi_1(\xi)^2\phi_2(\xi)\phi_1'(\xi)\phi_2'(\xi)V^{(3)}(\phi_1(\xi)^2+\phi_2(\xi)^2)c^4}{3(c^2-1)^2} - \frac{\Delta t^2\phi_1(\xi)\phi_2'(\xi)^2V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c^4}{6(c^2-1)^2} \\
&- \frac{3\Delta t^2\alpha^4\phi_1(\xi)c^4}{4(c^2-1)^3} - \frac{5\Delta t^2\alpha^2\phi_1(\xi)V'(\phi_1(\xi)^2+\phi_2(\xi)^2)c^4}{6(c^2-1)^3} - \frac{\Delta t^2\alpha^2\phi_1(\xi)^3V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c^4}{6(c^2-1)^3} \\
&- \frac{\Delta t^2\alpha^2\phi_1(\xi)\phi_2(\xi)^2V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c^4}{6(c^2-1)^3} \\
&- \frac{\Delta t^2\phi_1(\xi)^3V'(\phi_1(\xi)^2+\phi_2(\xi)^2)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c^4}{6(c^2-1)^3} \\
&- \frac{\Delta t^2\phi_1(\xi)\phi_2(\xi)^2V'(\phi_1(\xi)^2+\phi_2(\xi)^2)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c^4}{6(c^2-1)^3} - \frac{\Delta t^2\phi_1(\xi)V'(\phi_1(\xi)^2+\phi_2(\xi)^2)^2c^4}{12(c^2-1)^3} \\
&+ \frac{\Delta t^2\alpha^3\phi_2'(\xi)c^3}{(c^2-1)^2} + \frac{\Delta t^2\alpha V'(\phi_1(\xi)^2+\phi_2(\xi)^2)\phi_2'(\xi)c^3}{3(c^2-1)^2} \\
&+ \frac{2\Delta t^2\alpha\phi_1(\xi)\phi_2(\xi)\phi_1'(\xi)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c^3}{3(c^2-1)^2} + \frac{2\Delta t^2\alpha\phi_2(\xi)^2\phi_2'(\xi)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c^3}{3(c^2-1)^2} \\
&- \frac{2\Delta x^2\alpha^3\phi_2'(\xi)c^3}{3(c^2-1)^4} + \frac{5\Delta t^2\alpha^4\phi_1(\xi)c^2}{12(c^2-1)^2} + \frac{\Delta t^2\alpha^2\phi_1(\xi)V'(\phi_1(\xi)^2+\phi_2(\xi)^2)c^2}{2(c^2-1)^2} - \frac{\Delta x^2\alpha^4\phi_1(\xi)c^2}{3(c^2-1)^4} \\
&- \frac{\Delta x^2\alpha^2\phi_1(\xi)V'(\phi_1(\xi)^2+\phi_2(\xi)^2)c^2}{3(c^2-1)^4} + \frac{\Delta t^2\alpha^3\phi_2'(\xi)c}{3(c^2-1)^2} + \frac{\Delta x^2\alpha^3\phi_2'(\xi)c}{3(c^2-1)^3} \\
&+ \frac{\Delta x^2\alpha V'(\phi_1(\xi)^2+\phi_2(\xi)^2)\phi_2'(\xi)c}{3(c^2-1)^3} + \frac{\Delta x^2\alpha\phi_1(\xi)^2\phi_2'(\xi)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c}{3(c^2-1)^3} \\
&+ \frac{\Delta x^2\alpha\phi_2(\xi)^2\phi_2'(\xi)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c}{3(c^2-1)^3} + \frac{\Delta x^2\phi_1(\xi)V'(\phi_1(\xi)^2+\phi_2(\xi)^2)^2}{12(c^2-1)^3} + \frac{\Delta t^2\alpha^4\phi_1(\xi)}{12(c^2-1)^2} \\
&+ \frac{\Delta x^2\alpha^4\phi_1(\xi)}{12(c^2-1)^3} + \frac{\Delta x^2\alpha^2\phi_1(\xi)V'(\phi_1(\xi)^2+\phi_2(\xi)^2)}{6(c^2-1)^3} + \frac{\Delta x^2\alpha^2\phi_1(\xi)^3V''(\phi_1(\xi)^2+\phi_2(\xi)^2)}{6(c^2-1)^3} \\
&+ \frac{\Delta x^2\alpha^2\phi_1(\xi)\phi_2(\xi)^2V''(\phi_1(\xi)^2+\phi_2(\xi)^2)}{6(c^2-1)^3} + \frac{\Delta x^2\phi_1(\xi)\phi_1'(\xi)^2V''(\phi_1(\xi)^2+\phi_2(\xi)^2)}{2(c^2-1)^2}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\Delta x^2\phi_1(\xi)\phi_2'(\xi)^2V''(\phi_1(\xi)^2+\phi_2(\xi)^2)}{6(c^2-1)^2} + \frac{\Delta x^2\phi_1(\xi)^3V'(\phi_1(\xi)^2+\phi_2(\xi)^2)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)}{6(c^2-1)^3}\\
&+ \frac{\Delta x^2\phi_1(\xi)\phi_2(\xi)^2V'(\phi_1(\xi)^2+\phi_2(\xi)^2)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)}{6(c^2-1)^3}\\
&+ \frac{\Delta x^2\phi_2(\xi)\phi_1'(\xi)\phi_2'(\xi)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)}{3(c^2-1)^2} + \frac{\Delta x^2\phi_1(\xi)^3\phi_1'(\xi)^2V^{(3)}(\phi_1(\xi)^2+\phi_2(\xi)^2)}{3(c^2-1)^2}\\
&+ \frac{\Delta x^2\phi_1(\xi)\phi_2(\xi)^2\phi_2'(\xi)^2V^{(3)}(\phi_1(\xi)^2+\phi_2(\xi)^2)}{3(c^2-1)^2}\\
&+ \frac{2\Delta x^2\phi_1(\xi)^2\phi_2(\xi)\phi_1'(\xi)\phi_2'(\xi)V^{(3)}(\phi_1(\xi)^2+\phi_2(\xi)^2)}{3(c^2-1)^2}.
\end{aligned}
$$

The second component of $\hat{g}_2(\phi,\dot{\phi})$ is given as

$$
\begin{aligned}
&- \frac{2\Delta t^2\alpha^3\phi_1'(\xi)c^7}{3(c^2-1)^4} + \frac{\Delta t^2\alpha^4\phi_2(\xi)c^6}{3(c^2-1)^4} + \frac{\Delta t^2\alpha^2\phi_2(\xi)V'(\phi_1(\xi)^2+\phi_2(\xi)^2)c^6}{3(c^2-1)^4} + \frac{5\Delta t^2\alpha^3\phi_1'(\xi)c^5}{3(c^2-1)^3}\\
&+ \frac{\Delta t^2\alpha V'(\phi_1(\xi)^2+\phi_2(\xi)^2)\phi_1'(\xi)c^5}{3(c^2-1)^3} + \frac{\Delta t^2\alpha\phi_1(\xi)^2\phi_1'(\xi)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c^5}{3(c^2-1)^3}\\
&+ \frac{\Delta t^2\alpha\phi_2(\xi)^2\phi_1'(\xi)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c^5}{3(c^2-1)^3} - \frac{\Delta t^2\phi_2(\xi)\phi_2'(\xi)^2V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c^4}{2(c^2-1)^2}\\
&- \frac{\Delta t^2\phi_1(\xi)\phi_1'(\xi)\phi_2'(\xi)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c^4}{3(c^2-1)^2} - \frac{\Delta t^2\phi_1(\xi)^2\phi_2(\xi)\phi_1'(\xi)^2V^{(3)}(\phi_1(\xi)^2+\phi_2(\xi)^2)c^4}{3(c^2-1)^2}\\
&- \frac{\Delta t^2\phi_2(\xi)^3\phi_2'(\xi)^2V^{(3)}(\phi_1(\xi)^2+\phi_2(\xi)^2)c^4}{3(c^2-1)^2} - \frac{2\Delta t^2\phi_1(\xi)\phi_2(\xi)^2\phi_1'(\xi)\phi_2'(\xi)V^{(3)}(\phi_1(\xi)^2+\phi_2(\xi)^2)c^4}{3(c^2-1)^2}\\
&- \frac{\Delta t^2\phi_2(\xi)\phi_1'(\xi)^2V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c^4}{6(c^2-1)^2} - \frac{3\Delta t^2\alpha^4\phi_2(\xi)c^4}{4(c^2-1)^3} - \frac{5\Delta t^2\alpha^2\phi_2(\xi)V'(\phi_1(\xi)^2+\phi_2(\xi)^2)c^4}{6(c^2-1)^3}\\
&- \frac{\Delta t^2\alpha^2\phi_2(\xi)^3V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c^4}{6(c^2-1)^3} - \frac{\Delta t^2\alpha^2\phi_1(\xi)^2\phi_2(\xi)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c^4}{6(c^2-1)^3}\\
&- \frac{\Delta t^2\phi_2(\xi)^3V'(\phi_1(\xi)^2+\phi_2(\xi)^2)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c^4}{6(c^2-1)^3}\\
&- \frac{\Delta t^2\phi_1(\xi)^2\phi_2(\xi)V'(\phi_1(\xi)^2+\phi_2(\xi)^2)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c^4}{6(c^2-1)^3} - \frac{\Delta t^2\phi_2(\xi)V'(\phi_1(\xi)^2+\phi_2(\xi)^2)^2c^4}{12(c^2-1)^3}\\
&+ \frac{2\Delta x^2\alpha^3\phi_1'(\xi)c^3}{3(c^2-1)^4} - \frac{4\Delta t^2\alpha^3\phi_1'(\xi)c^3}{3(c^2-1)^2} - \frac{\Delta t^2\alpha V'(\phi_1(\xi)^2+\phi_2(\xi)^2)\phi_1'(\xi)c^3}{3(c^2-1)^2}\\
&- \frac{2\Delta t^2\alpha\phi_1(\xi)^2\phi_1'(\xi)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c^3}{3(c^2-1)^2} - \frac{2\Delta t^2\alpha\phi_1(\xi)\phi_2(\xi)\phi_2'(\xi)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c^3}{3(c^2-1)^2}\\
&+ \frac{\Delta t^2\alpha^4\phi_2(\xi)c^2}{2(c^2-1)^2} + \frac{\Delta t^2\alpha^2\phi_2(\xi)V'(\phi_1(\xi)^2+\phi_2(\xi)^2)c^2}{2(c^2-1)^2} - \frac{\Delta x^2\alpha^4\phi_2(\xi)c^2}{3(c^2-1)^4}\\
&- \frac{\Delta x^2\alpha^2\phi_2(\xi)V'(\phi_1(\xi)^2+\phi_2(\xi)^2)c^2}{3(c^2-1)^4} + \frac{\Delta t^2\alpha^3\phi_1'(\xi)c}{3(c^2-1)} - \frac{\Delta x^2\alpha^3\phi_1'(\xi)c}{3(c^2-1)^3}\\
&- \frac{\Delta x^2\alpha V'(\phi_1(\xi)^2+\phi_2(\xi)^2)\phi_1'(\xi)c}{3(c^2-1)^3} - \frac{\Delta x^2\alpha\phi_1(\xi)^2\phi_1'(\xi)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c}{3(c^2-1)^3}\\
&- \frac{\Delta x^2\alpha\phi_2(\xi)^2\phi_1'(\xi)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)c}{3(c^2-1)^3} + \frac{\Delta x^2\phi_2(\xi)V'(\phi_1(\xi)^2+\phi_2(\xi)^2)^2}{12(c^2-1)^3} + \frac{\Delta x^2\alpha^4\phi_2(\xi)}{12(c^2-1)^3}\\
&+ \frac{\Delta x^2\alpha^2\phi_2(\xi)V'(\phi_1(\xi)^2+\phi_2(\xi)^2)}{6(c^2-1)^3} + \frac{\Delta x^2\alpha^2\phi_2(\xi)^3V''(\phi_1(\xi)^2+\phi_2(\xi)^2)}{6(c^2-1)^3}\\
&+ \frac{\Delta x^2\phi_2(\xi)\phi_1'(\xi)^2V''(\phi_1(\xi)^2+\phi_2(\xi)^2)}{6(c^2-1)^2} + \frac{\Delta x^2\phi_2(\xi)\phi_2'(\xi)^2V''(\phi_1(\xi)^2+\phi_2(\xi)^2)}{2(c^2-1)^2}\\
&+ \frac{\Delta x^2\alpha^2\phi_1(\xi)^2\phi_2(\xi)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)}{6(c^2-1)^3} + \frac{\Delta x^2\phi_2(\xi)^3V'(\phi_1(\xi)^2+\phi_2(\xi)^2)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)}{6(c^2-1)^3}\\
&+ \frac{\Delta x^2\phi_1(\xi)^2\phi_2(\xi)V'(\phi_1(\xi)^2+\phi_2(\xi)^2)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)}{6(c^2-1)^3}\\
&+ \frac{\Delta x^2\phi_1(\xi)\phi_1'(\xi)\phi_2'(\xi)V''(\phi_1(\xi)^2+\phi_2(\xi)^2)}{3(c^2-1)^2} + \frac{\Delta x^2\phi_1(\xi)^2\phi_2(\xi)\phi_1'(\xi)^2V^{(3)}(\phi_1(\xi)^2+\phi_2(\xi)^2)}{3(c^2-1)^2}
\end{aligned}
$$

$$+\frac{\Delta x^2\phi_2(\xi)^3\phi_2'(\xi)^2V^{(3)}(\phi_1(\xi)^2+\phi_2(\xi)^2)}{3(c^2-1)^2}+\frac{2\Delta x^2\phi_1(\xi)\phi_2(\xi)^2\phi_1'(\xi)\phi_2'(\xi)V^{(3)}(\phi_1(\xi)^2+\phi_2(\xi)^2)}{3(c^2-1)^2}$$
$$-\frac{\Delta t^2\alpha^4\phi_2(\xi)}{12(c^2-1)}.$$

Refer to the Mathematica Notebook files for higher order terms.

2. Case $\alpha = 0$.

We consider the special case $\alpha = 0$. The ODE arises as the Euler-Lagrange equations of a first order Lagrangian. To compute the Lagrangian we make the ansatz

$$L(\phi,\dot\phi)=\frac{1}{2}(c^2-1)\langle\dot\phi,\dot\phi\rangle+W(\phi)+h^2L^2(\phi,\dot\phi)+h^4L^4(\phi,\dot\phi)+h^6L^6(\phi,\dot\phi)+\mathcal{O}(h^8)$$

with $W(\phi)=\frac{1}{2}V(\|\phi\|^2)$. We obtain an ansatz for L^2, L^4, L^6 via bicoloured trees as described in the paper.

$$\begin{aligned}L^2 &= a(2,1)\left(\left(W^{(0,1)}\right)^2+\left(W^{(1,0)}\right)^2\right)\\ &\quad+a(2,2)\left(2W^{(1,1)}\dot\phi_1\dot\phi_2+W^{(2,0)}\left(\dot\phi_1\right)^2+W^{(0,2)}\left(\dot\phi_2\right)^2\right).\end{aligned}$$

$$\begin{aligned}L^4 = {}& a(4,1)\Big(4W^{(3,1)}\left(\dot\phi_1\right)^3\dot\phi_2+6W^{(2,2)}\left(\dot\phi_1\right)^2\left(\dot\phi_2\right)^2\\ &+\quad 4W^{(1,3)}\dot\phi_1\left(\dot\phi_2\right)^3+W^{(4,0)}\left(\dot\phi_1\right)^4+W^{(0,4)}\left(\dot\phi_2\right)^4\Big)\\ &+a(4,2)\Big(2W^{(0,1)}W^{(1,2)}\dot\phi_1\dot\phi_2+2W^{(1,0)}W^{(2,1)}\dot\phi_1\dot\phi_2+W^{(0,1)}W^{(2,1)}\left(\dot\phi_1\right)^2\\ &+\quad W^{(1,0)}W^{(3,0)}\left(\dot\phi_1\right)^2+W^{(0,1)}W^{(0,3)}\left(\dot\phi_2\right)^2+W^{(1,0)}W^{(1,2)}\left(\dot\phi_2\right)^2\Big)\\ &+a(4,3)\Big(2W^{(1,1)}W^{(0,2)}\dot\phi_1\dot\phi_2+2W^{(1,1)}W^{(2,0)}\dot\phi_1\dot\phi_2+\left(W^{(1,1)}\right)^2\left(\dot\phi_1\right)^2\\ &+\quad \left(W^{(2,0)}\right)^2\left(\dot\phi_1\right)^2+\left(W^{(0,2)}\right)^2\left(\dot\phi_2\right)^2+\left(W^{(1,1)}\right)^2\left(\dot\phi_2\right)^2\Big)\\ &+a(4,4)\left(W^{(0,2)}\left(W^{(0,1)}\right)^2+2W^{(1,0)}W^{(1,1)}W^{(0,1)}+\left(W^{(1,0)}\right)^2W^{(2,0)}\right)\end{aligned}$$

$$\begin{aligned}L^6 = {}& a(6,4)\Big((W^{(0,1)})^2(W^{(0,2)})^2+2W^{(0,1)}W^{(1,0)}W^{(1,1)}W^{(0,2)}+(W^{(0,1)})^2(W^{(1,1)})^2\\ &+(W^{(1,0)})^2(W^{(1,1)})^2+(W^{(1,0)})^2(W^{(2,0)})^2+2W^{(0,1)}W^{(1,0)}W^{(1,1)}W^{(2,0)}\Big)\\ &+a(6,3)\Big(2W^{(1,1)}(W^{(0,2)})^2\dot\phi_1\dot\phi_2+2W^{(1,1)}W^{(2,0)}W^{(0,2)}\dot\phi_1\dot\phi_2+2(W^{(1,1)})^3\dot\phi_1\dot\phi_2\\ &+2W^{(1,1)}(W^{(2,0)})^2\dot\phi_1\dot\phi_2+(W^{(1,1)})^2W^{(0,2)}(\dot\phi_1)^2+(W^{(2,0)})^3(\dot\phi_1)^2+2(W^{(1,1)})^2W^{(2,0)}\\ &(\dot\phi_1)^2+(W^{(0,2)})^3(\dot\phi_2)^2+2(W^{(1,1)})^2W^{(0,2)}(\dot\phi_2)^2+(W^{(1,1)})^2W^{(2,0)}(\dot\phi_2)^2\Big)\\ &+a(6,10)\Big(W^{(0,3)}(W^{(0,1)})^3+3W^{(1,0)}W^{(1,2)}(W^{(0,1)})^2+3(W^{(1,0)})^2W^{(2,1)}W^{(0,1)}\end{aligned}$$

$$
\begin{aligned}
&+ (W^{(1,0)})^3 + W^{(3,0)}\Big) + a(6,5)\Big(2W^{(0,1)}W^{(0,2)}W^{(1,2)}\dot{\phi}_1\dot{\phi}_2 + 2W^{(1,0)}W^{(1,1)}W^{(1,2)}\dot{\phi}_1\dot{\phi}_2\\
&+ 2W^{(0,1)}W^{(1,1)}W^{(2,1)}\dot{\phi}_1\dot{\phi}_2 + 2W^{(1,0)}W^{(2,0)}W^{(2,1)}\dot{\phi}_1\dot{\phi}_2 + W^{(0,1)}W^{(0,2)}W^{(2,1)}(\dot{\phi}_1)^2\\
&+ W^{(1,0)}W^{(1,1)}W^{(2,1)}(\dot{\phi}_1)^2 + W^{(0,1)}W^{(1,1)}W^{(3,0)}(\dot{\phi}_1)^2 + W^{(1,0)}W^{(2,0)}W^{(3,0)}(\dot{\phi}_1)^2\\
&+ W^{(0,1)}W^{(0,2)}W^{(0,3)}(\dot{\phi}_2)^2 + W^{(0,3)}W^{(1,0)}W^{(1,1)}(\dot{\phi}_2)^2 + W^{(0,1)}W^{(1,1)}W^{(1,2)}(\dot{\phi}_2)^2\\
&+ W^{(1,0)}W^{(1,2)}W^{(2,0)}(\dot{\phi}_2)^2\Big) + a(6,2)\Big(W^{(0,1)}W^{(0,3)}W^{(1,1)}\dot{\phi}_1\dot{\phi}_2 + W^{(0,1)}W^{(0,2)}W^{(1,2)}\dot{\phi}_1\dot{\phi}_2\\
&+ W^{(1,0)}W^{(1,1)}W^{(1,2)}\dot{\phi}_1\dot{\phi}_2 + W^{(0,1)}W^{(1,2)}W^{(2,0)}\dot{\phi}_1\dot{\phi}_2 + W^{(0,2)}W^{(1,0)}W^{(2,1)}\dot{\phi}_1\dot{\phi}_2\\
&+ W^{(0,1)}W^{(1,1)}W^{(2,1)}\dot{\phi}_1\dot{\phi}_2 + W^{(1,0)}W^{(2,0)}W^{(2,1)}\dot{\phi}_1\dot{\phi}_2 + W^{(1,0)}W^{(1,1)}W^{(3,0)}\dot{\phi}_1\dot{\phi}_2\\
&+ W^{(0,1)}W^{(1,1)}W^{(1,2)}(\dot{\phi}_1)^2 + W^{(1,0)}W^{(1,1)}W^{(2,1)}(\dot{\phi}_1)^2 + W^{(0,1)}W^{(2,0)}W^{(2,1)}(\dot{\phi}_1)^2\\
&+ W^{(1,0)}W^{(2,0)}W^{(3,0)}(\dot{\phi}_1)^2 + W^{(0,1)}W^{(0,2)}W^{(0,3)}(\dot{\phi}_2)^2 + W^{(0,2)}W^{(1,0)}W^{(1,2)}(\dot{\phi}_2)^2\\
&+ W^{(0,1)}W^{(1,1)}W^{(1,2)}(\dot{\phi}_2)^2 + W^{(1,0)}W^{(1,1)}W^{(2,1)}(\dot{\phi}_2)^2\Big) + a(6,8)\Big(4W^{(1,2)}W^{(2,1)}(\dot{\phi}_1)^3\dot{\phi}_2\\
&+ 4W^{(2,1)}W^{(3,0)}(\dot{\phi}_1)^3\dot{\phi}_2 + 4(W^{(1,2)})^2(\dot{\phi}_1)^2(\dot{\phi}_2)^2 + 4(W^{(2,1)})^2(\dot{\phi}_1)^2(\dot{\phi}_2)^2\\
&+ 2W^{(0,3)}W^{(2,1)}(\dot{\phi}_1)^2(\dot{\phi}_2)^2 + 2W^{(1,2)}W^{(3,0)}(\dot{\phi}_1)^2(\dot{\phi}_2)^2 + 4W^{(0,3)}W^{(1,2)}\dot{\phi}_1(\dot{\phi}_2)^3\\
&+ 4W^{(1,2)}W^{(2,1)}\dot{\phi}_1(\dot{\phi}_2)^3 + (W^{(2,1)})^2(\dot{\phi}_1)^4 + (W^{(3,0)})^2(\dot{\phi}_1)^4 + (W^{(0,3)})^2(\dot{\phi}_2)^4\\
&+ (W^{(1,2)})^2(\dot{\phi}_2)^4\Big) + a(6,6)\Big(2(W^{(0,1)})^2W^{(1,3)}\dot{\phi}_1\dot{\phi}_2 + 4W^{(0,1)}W^{(1,0)}W^{(2,2)}\dot{\phi}_1\dot{\phi}_2\\
&+ 2(W^{(1,0)})^2W^{(3,1)}\dot{\phi}_1\dot{\phi}_2 + (W^{(0,1)})^2W^{(2,2)}(\dot{\phi}_1)^2 + 2W^{(0,1)}W^{(1,0)}W^{(3,1)}(\dot{\phi}_1)^2\\
&+ (W^{(1,0)})^2W^{(4,0)}(\dot{\phi}_1)^2 + (W^{(0,1)})^2W^{(0,4)}(\dot{\phi}_2)^2 + 2W^{(0,1)}W^{(1,0)}W^{(1,3)}(\dot{\phi}_2)^2\\
&+ (W^{(1,0)})^2W^{(2,2)}(\dot{\phi}_2)^2\Big) + a(6,7)\Big(3W^{(1,1)}W^{(2,2)}(\dot{\phi}_1)^3\dot{\phi}_2 + W^{(0,2)}W^{(3,1)}(\dot{\phi}_1)^3\dot{\phi}_2\\
&+ 3W^{(2,0)}W^{(3,1)}(\dot{\phi}_1)^3\dot{\phi}_2 + W^{(1,1)}W^{(4,0)}(\dot{\phi}_1)^3\dot{\phi}_2 + 3W^{(1,1)}W^{(1,3)}(\dot{\phi}_1)^2(\dot{\phi}_2)^2\\
&+ 3W^{(0,2)}W^{(2,2)}(\dot{\phi}_1)^2(\dot{\phi}_2)^2 + 3W^{(2,0)}W^{(2,2)}(\dot{\phi}_1)^2(\dot{\phi}_2)^2 + 3W^{(1,1)}W^{(3,1)}(\dot{\phi}_1)^2(\dot{\phi}_2)^2\\
&+ W^{(0,4)}W^{(1,1)}\dot{\phi}_1(\dot{\phi}_2)^3 + 3W^{(0,2)}W^{(1,3)}\dot{\phi}_1(\dot{\phi}_2)^3 + W^{(1,3)}W^{(2,0)}\dot{\phi}_1(\dot{\phi}_2)^3\\
&+ 3W^{(1,1)}W^{(2,2)}\dot{\phi}_1(\dot{\phi}_2)^3 + W^{(1,1)}W^{(3,1)}(\dot{\phi}_1)^4 + W^{(2,0)}W^{(4,0)}(\dot{\phi}_1)^4 + W^{(0,2)}W^{(0,4)}(\dot{\phi}_2)^4\\
&+ W^{(1,1)}W^{(1,3)}(\dot{\phi}_2)^4\Big) + a(6,9)\Big(4W^{(0,1)}W^{(3,2)}(\dot{\phi}_1)^3\dot{\phi}_2 + 4W^{(1,0)}W^{(4,1)}(\dot{\phi}_1)^3\dot{\phi}_2\\
&+ 6W^{(0,1)}W^{(2,3)}(\dot{\phi}_1)^2(\dot{\phi}_2)^2 + 6W^{(1,0)}W^{(3,2)}(\dot{\phi}_1)^2(\dot{\phi}_2)^2 + 4W^{(0,1)}W^{(1,4)}\dot{\phi}_1(\dot{\phi}_2)^3\\
&+ 4W^{(1,0)}W^{(2,3)}\dot{\phi}_1(\dot{\phi}_2)^3 + W^{(0,1)}W^{(4,1)}(\dot{\phi}_1)^4 + W^{(1,0)}W^{(5,0)}(\dot{\phi}_1)^4 + W^{(0,1)}W^{(0,5)}(\dot{\phi}_2)^4\\
&+ W^{(1,0)}W^{(1,4)}(\dot{\phi}_2)^4\Big) + a(6,1)\Big(6W^{(5,1)}(\dot{\phi}_1)^5\dot{\phi}_2 + 15W^{(4,2)}(\dot{\phi}_1)^4(\dot{\phi}_2)^2\\
&+ 20W^{(3,3)}(\dot{\phi}_1)^3(\dot{\phi}_2)^3 + 15W^{(2,4)}(\dot{\phi}_1)^2(\dot{\phi}_2)^4 + 6W^{(1,5)}\dot{\phi}_1(\dot{\phi}_2)^5 + W^{(6,0)}(\dot{\phi}_1)^6\\
&+ W^{(0,6)}(\dot{\phi}_2)^6\Big)
\end{aligned}
$$

Equating coefficients and ignoring higher order terms we obtain

$$
\begin{aligned}
a(2,1) &= \frac{c^4\Delta t^2 - \Delta x^2}{24\left(c^2-1\right)^2}\\
a(2,2) &= \frac{c^4\Delta t^2 - \Delta x^2}{12\left(c^2-1\right)}\\
a(4,1) &= \frac{-3\Delta t^4c^8 - 2\Delta t^4c^6 + 10\Delta t^2\Delta x^2c^4 - 2\Delta x^4c^2 - 3\Delta x^4}{2160\left(c^2-1\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
a(4,2) &= \frac{-3\Delta t^4 c^8 - 2\Delta t^4 c^6 + 10\Delta t^2 \Delta x^2 c^4 - 2\Delta x^4 c^2 - 3\Delta x^4}{360\left(c^2-1\right)^3} \\
a(4,3) &= \frac{-2\Delta t^4 c^8 - 3\Delta t^4 c^6 + 10\Delta t^2 \Delta x^2 c^4 - 3\Delta x^4 c^2 - 2\Delta x^4}{720\left(c^2-1\right)^3} \\
a(4,4) &= \frac{-3\Delta t^4 c^8 - 2\Delta t^4 c^6 + 10\Delta t^2 \Delta x^2 c^4 - 2\Delta x^4 c^2 - 3\Delta x^4}{720\left(c^2-1\right)^4}
\end{aligned}
$$

$$
\begin{aligned}
a(6,1) &= \frac{10\Delta t^6 c^{12} + 22\Delta t^6 c^{10} + 3\Delta t^6 c^8 - 77\Delta t^4 \Delta x^2 c^8 + 28\Delta t^2 \Delta x^4 c^6 - 28\Delta t^4 \Delta x^2 c^6 - 3\Delta x^6 c^4 + 77\Delta t^2 \Delta x^4 c^4 - 22\Delta x^6 c^2 - 10\Delta x^6}{302400\left(c^2-1\right)^3} \\
a(6,2) &= \frac{10\Delta t^6 c^{12} + 22\Delta t^6 c^{10} + 3\Delta t^6 c^8 - 77\Delta t^4 \Delta x^2 c^8 + 28\Delta t^2 \Delta x^4 c^6 - 28\Delta t^4 \Delta x^2 c^6 - 3\Delta x^6 c^4 + 77\Delta t^2 \Delta x^4 c^4 - 22\Delta x^6 c^2 - 10\Delta x^6}{5040\left(c^2-1\right)^5} \\
a(6,3) &= \frac{10\Delta t^6 c^{12} + 22\Delta t^6 c^{10} + 3\Delta t^6 c^8 - 77\Delta t^4 \Delta x^2 c^8 + 28\Delta t^2 \Delta x^4 c^6 - 28\Delta t^4 \Delta x^2 c^6 - 3\Delta x^6 c^4 + 77\Delta t^2 \Delta x^4 c^4 - 22\Delta x^6 c^2 - 10\Delta x^6}{30240\left(c^2-1\right)^5} \\
a(6,4) &= \frac{72\Delta t^6 c^{12} + 94\Delta t^6 c^{10} + 9\Delta t^6 c^8 - 413\Delta t^4 \Delta x^2 c^8 + 112\Delta t^2 \Delta x^4 c^6 - 112\Delta t^4 \Delta x^2 c^6 - 9\Delta x^6 c^4 + 413\Delta t^2 \Delta x^4 c^4 - 94\Delta x^6 c^2 - 72\Delta x^6}{120960\left(c^2-1\right)^6} \\
a(6,5) &= \frac{72\Delta t^6 c^{12} + 94\Delta t^6 c^{10} + 9\Delta t^6 c^8 - 413\Delta t^4 \Delta x^2 c^8 + 112\Delta t^2 \Delta x^4 c^6 - 112\Delta t^4 \Delta x^2 c^6 - 9\Delta x^6 c^4 + 413\Delta t^2 \Delta x^4 c^4 - 94\Delta x^6 c^2 - 72\Delta x^6}{60480\left(c^2-1\right)^5} \\
a(6,6) &= \frac{10\Delta t^6 c^{12} + 22\Delta t^6 c^{10} + 3\Delta t^6 c^8 - 77\Delta t^4 \Delta x^2 c^8 + 28\Delta t^2 \Delta x^4 c^6 - 28\Delta t^4 \Delta x^2 c^6 - 3\Delta x^6 c^4 + 77\Delta t^2 \Delta x^4 c^4 - 22\Delta x^6 c^2 - 10\Delta x^6}{6720\left(c^2-1\right)^5} \\
a(6,7) &= \frac{10\Delta t^6 c^{12} + 22\Delta t^6 c^{10} + 3\Delta t^6 c^8 - 77\Delta t^4 \Delta x^2 c^8 + 28\Delta t^2 \Delta x^4 c^6 - 28\Delta t^4 \Delta x^2 c^6 - 3\Delta x^6 c^4 + 77\Delta t^2 \Delta x^4 c^4 - 22\Delta x^6 c^2 - 10\Delta x^6}{15120\left(c^2-1\right)^4} \\
a(6,8) &= \frac{72\Delta t^6 c^{12} + 94\Delta t^6 c^{10} + 9\Delta t^6 c^8 - 413\Delta t^4 \Delta x^2 c^8 + 112\Delta t^2 \Delta x^4 c^6 - 112\Delta t^4 \Delta x^2 c^6 - 9\Delta x^6 c^4 + 413\Delta t^2 \Delta x^4 c^4 - 94\Delta x^6 c^2 - 72\Delta x^6}{120960\left(c^2-1\right)^4} \\
a(6,9) &= \frac{10\Delta t^6 c^{12} + 22\Delta t^6 c^{10} + 3\Delta t^6 c^8 - 77\Delta t^4 \Delta x^2 c^8 + 28\Delta t^2 \Delta x^4 c^6 - 28\Delta t^4 \Delta x^2 c^6 - 3\Delta x^6 c^4 + 77\Delta t^2 \Delta x^4 c^4 - 22\Delta x^6 c^2 - 10\Delta x^6}{20160\left(c^2-1\right)^4} \\
a(6,10) &= \frac{10\Delta t^6 c^{12} + 22\Delta t^6 c^{10} + 3\Delta t^6 c^8 - 77\Delta t^4 \Delta x^2 c^8 + 28\Delta t^2 \Delta x^4 c^6 - 28\Delta t^4 \Delta x^2 c^6 - 3\Delta x^6 c^4 + 77\Delta t^2 \Delta x^4 c^4 - 22\Delta x^6 c^2 - 10\Delta x^6}{20160\left(c^2-1\right)^6}.
\end{aligned}
$$

3. Polynomial potential

Consider the potential

$$
V(s) = v_0 + v_1 s + \frac{1}{2} v_2 s^2 + \frac{1}{6} v_3 s^3.
$$

This time we compute the modified Lagrangian using a general polynomial ansatz for the special cases $\alpha = 0$, $c = 0$, and $\Delta x = c\Delta t$. We make the ansatz

$$
L = L^0(\phi, \dot{\phi}) + h^2 L^2(\phi, \dot{\phi}) + \mathcal{O}(h^4),
$$

where L^0 and L^2 are general order 10 polynomials in ϕ and $\dot{\phi}$. The modified equation (1.2) for $\ddot{\phi}$ for the potential V above is substituted into the Euler-Lagrange equations $\mathcal{E}(L) = 0$ for L (truncated after the h^2-term), where $\mathcal{E}$ is the Euler-Lagrange operator. Since the initial values $\phi(0), \dot{\phi}(0)$ can be chosen independently and $\mathcal{E}(L)$ truncated after $\mathcal{O}(h^2)$ is a multivariate polynomial in h, ϕ, $\dot{\phi}$, we can equate each coefficient with zero and obtain an under-determined system of linear equations. In the special cases $\alpha = 0$, $c = 0$ or $\Delta x = c\Delta t$ this system is solvable. In each case L^0 can be chosen as the exact Lagrangian

$$
L^0(\phi, \dot{\phi}) = \frac{1}{2}\left(\alpha^2 \langle \phi, \phi \rangle - 2\alpha c \langle J\phi, \dot{\phi} \rangle + (c^2 - 1)\langle \dot{\phi}, \dot{\phi} \rangle + V(\langle \phi, \phi \rangle)\right).
$$

For each case we report one admissible choice for L^2 below.

3.1. **Non-rotating wave, $\alpha = 0$.** In this case the wave is not rotating and the case is covered by the Lagrangian derived in the previous section. Using the polynomial ansatz, the term L^2 is given as

$$
\begin{aligned}
&\frac{(c^4\Delta t^2-\Delta x^2)}{288(c^2-1)^2}\Bigg(3\bigg(v_3(\phi_1(\xi)^2+\phi_2(\xi)^2)\Big(4(c^2-1)(\phi_1(\xi)^2(5\phi_1'(\xi)^2+\phi_2'(\xi)^2)\\
&+8\phi_2(\xi)\phi_1(\xi)\phi_1'(\xi)\phi_2'(\xi)+\phi_2(\xi)^2(\phi_1'(\xi)^2+5\phi_2'(\xi)^2))+v_3(\phi_1(\xi)^2+\phi_2(\xi)^2)^4\Big)\\
&+4v_2\Big(2(c^2-1)(\phi_1(\xi)^2(3\phi_1'(\xi)^2+\phi_2'(\xi)^2)+4\phi_2(\xi)\phi_1(\xi)\phi_1'(\xi)\phi_2'(\xi)\\
&+\phi_2(\xi)^2(\phi_1'(\xi)^2+3\phi_2'(\xi)^2))+v_3(\phi_1(\xi)^2+\phi_2(\xi)^2)^4\Big)+4v_2^2(\phi_1(\xi)^2+\phi_2(\xi)^2)^3\bigg)\\
&-12v_1^2(\phi_1(\xi)^2+\phi_2(\xi)^2)+4v_1(\phi_1(\xi)^2+\phi_2(\xi)^2)^2(2v_3(\phi_1(\xi)^2+\phi_2(\xi)^2)+3v_2)\Bigg).
\end{aligned}
$$

The expression for L^2 does not happen to coincide with the terms obtained using a P-series type ansatz in the previous section. However, the Euler–Lagrange equations for both Lagrangians govern the modified, reduced ODE (1.2).

3.2. **Standing wave $c = 0$.** If the wave speed is zero, the functional equation (1.1) simplifies to

$$
\begin{aligned}
0 =& -\frac{1}{\Delta x^2}\phi(\xi+\Delta x)+\left(\frac{2}{\Delta x^2}+\frac{2}{\Delta t^2}(\cos(\alpha\Delta t)-1)\right)\phi(\xi)-\frac{1}{\Delta x^2}\phi(\xi-\Delta x)\\
&-V'(\|\phi\|^2)\phi.
\end{aligned}
$$

As this corresponds to a linear 3-step method, the underlying 1-step map is conjugate symplectic, i.e. symplectic with respect to a modified symplectic structure. Therefore, the reduced series expansion (1.2) is variational. The Lagrangian is given as $L = L^0 + h^2L^2 + \mathcal{O}(h^4)$, where L^2 is given by the following expression.

$$
\begin{aligned}
&\frac{1}{288}\Bigg(-3\phi_1(\xi)^2\bigg(4\alpha^4\Delta t^2-4\alpha^4\Delta x^2\\
&+8\Delta x^2v_2\Big(\alpha^2\phi_2(\xi)^2-3\phi_1'(\xi)^2-\phi_2'(\xi)^2+2v_3\phi_2(\xi)^6\Big)\\
&+8\alpha^2\Delta x^2v_3\phi_2(\xi)^4+8\Delta x^2v_1\Big(-\alpha^2+v_3\phi_2(\xi)^4+v_2\phi_2(\xi)^2\Big)-24\Delta x^2v_3\phi_2(\xi)^2\phi_1'(\xi)^2\\
&-24\Delta x^2v_3\phi_2(\xi)^2\phi_2'(\xi)^2+5\Delta x^2v_3^2\phi_2(\xi)^8+12\Delta x^2v_2^2\phi_2(\xi)^4-4\Delta x^2v_1^2\bigg)\\
&+\phi_2(\xi)^2\bigg(-12\alpha^4\Delta t^2+12\alpha^4\Delta x^2\\
&-12\Delta x^2v_2\Big(\alpha^2\phi_2(\xi)^2-2\phi_1'(\xi)^2-6\phi_2'(\xi)^2+v_3\phi_2(\xi)^6\Big)\\
&-8\alpha^2\Delta x^2v_3\phi_2(\xi)^4+4\Delta x^2v_1(6\alpha^2-2v_3\phi_2(\xi)^4-3v_2\phi_2(\xi)^2)+12\Delta x^2v_3\phi_2(\xi)^2\phi_1'(\xi)^2\\
&+60\Delta x^2v_3\phi_2(\xi)^2\phi_2'(\xi)^2-3\Delta x^2v_3^2\phi_2(\xi)^8-12\Delta x^2v_2^2\phi_2(\xi)^4+12\Delta x^2v_1^2\bigg)
\end{aligned}
$$

$$
- 6\Delta x^2\phi_1(\xi)^4\bigg(v_3(-2(5\phi_1'(\xi)^2+\phi_2'(\xi)^2)+4\phi_2(\xi)^2(\alpha^2+v_1)+5v_3\phi_2(\xi)^6)
$$
$$
+ 2v_2(\alpha^2+6v_3\phi_2(\xi)^4+v_1)+6v_2^2\phi_2(\xi)^2\bigg) - 2\Delta x^2\phi_1(\xi)^6\bigg(v_3(4\alpha^2+15v_3\phi_2(\xi)^4+4v_1)
$$
$$
+ 24v_3v_2\phi_2(\xi)^2+6v_2^2\bigg) + 96\Delta x^2 v_3\phi_2(\xi)\phi_1(\xi)^3\phi_1'(\xi)\phi_2'(\xi)
$$
$$
+ 96\Delta x^2\phi_2(\xi)\phi_1(\xi)\phi_1'(\xi)\phi_2'(\xi)(v_3\phi_2(\xi)^2+v_2) - 3\Delta x^2 v_3^2\phi_1(\xi)^{10}
$$
$$
- 3\Delta x^2 v_3\phi_1(\xi)^8(5v_3\phi_2(\xi)^2+4v_2)\bigg)
$$

3.3. **Special discretisation values, $\Delta x = c\Delta t$.** In this case the stencil (1.1) only relates the three points $\phi(\xi)$, $\phi(\xi+\Delta x)$, $\phi(\xi-\Delta x)$ rather than 5 points. The second order term of the Lagrangian is given as

$$
\frac{\Delta t^2}{1440(c^2-1)^2}\bigg(15c^2(c^2-1)v_3^2\phi_1(\xi)^{10}+15c^2(c^2-1)v_3(5v_3\phi_2(\xi)^2+4v_2)\phi_1(\xi)^8
$$
$$
+ 10c^2\bigg(6(c^2-1)v_2^2+24(c^2-1)v_3\phi_2(\xi)^2v_2+v_3\bigg(15(c^2-1)v_3\phi_2(\xi)^4
$$
$$
+ 2(3c^2-5)\alpha^2+4(c^2-1)v_1\bigg)\bigg)\phi_1(\xi)^6+30c^2\bigg(6(c^2-1)v_2^2\phi_2(\xi)^2
$$
$$
+ 2v_2\Big(6(c^2-1)v_3\phi_2(\xi)^4+2(c^2-2)\alpha^2+(c^2-1)v_1\Big)
$$
$$
+ v_3\Big(5(c^2-1)v_3\phi_2(\xi)^6+2((3c^2-5)\alpha^2+2(c^2-1)v_1)\phi_2(\xi)^2
$$
$$
- 24c(c^2-1)\alpha\phi_1'(\xi)\phi_2(\xi)+2(c^2-1)^2(5\phi_1'(\xi)^2+\phi_2'(\xi)^2)\Big)\bigg)\phi_1(\xi)^4
$$
$$
+ 480c^2(c^2-1)^2v_3\phi_2(\xi)\phi_1'(\xi)\phi_2'(\xi)\phi_1(\xi)^3-15\bigg(-5c^4v_3^2\phi_2(\xi)^8+5c^2v_3^2\phi_2(\xi)^8
$$
$$
- 12c^2(c^2-1)v_2^2\phi_2(\xi)^4-12c^4\alpha^2v_3\phi_2(\xi)^4+20c^2\alpha^2v_3\phi_2(\xi)^4+32c^5\alpha v_3\phi_1'(\xi)\phi_2(\xi)^3
$$
$$
- 32c^3\alpha v_3\phi_1'(\xi)\phi_2(\xi)^3-24c^6v_3\phi_1'(\xi)^2\phi_2(\xi)^2+48c^4v_3\phi_1'(\xi)^2\phi_2(\xi)^2
$$
$$
- 24c^2v_3\phi_1'(\xi)^2\phi_2(\xi)^2-24c^6v_3\phi_2'(\xi)^2\phi_2(\xi)^2+48c^4v_3\phi_2'(\xi)^2\phi_2(\xi)^2
$$
$$
- 24c^2v_3\phi_2'(\xi)^2\phi_2(\xi)^2+12c^2\alpha^4+4\alpha^4+4c^2(c^2-1)v_1^2
$$
$$
- 8c^2v_1\Big((c^2-1)v_3\phi_2(\xi)^4+(c^2-1)v_2\phi_2(\xi)^2-2\alpha^2\Big)
$$
$$
- 8c^2v_2\Big(2(c^2-1)v_3\phi_2(\xi)^6+2(c^2-2)\alpha^2\phi_2(\xi)^2-8c(c^2-1)\alpha\phi_1'(\xi)\phi_2(\xi)
$$
$$
+ (c^2-1)^2(3\phi_1'(\xi)^2+\phi_2'(\xi)^2)\Big)\bigg)\phi_1(\xi)^2
$$
$$
+ 480c^2(c^2-1)^2\phi_2(\xi)(v_3\phi_2(\xi)^2+v_2)\phi_1'(\xi)\phi_2'(\xi)\phi_1(\xi)+\phi_2(\xi)\bigg(15c^2(c^2-1)v_3^2\phi_2(\xi)^9
$$
$$
+ 60c^2(c^2-1)v_2v_3\phi_2(\xi)^7+20c^2\Big(3(c^2-1)v_2^2+((3c^2-5)\alpha^2+2(c^2-1)v_1)v_3\Big)\phi_2(\xi)^5
$$
$$
- 144c^3(c^2-1)\alpha v_3\phi_1'(\xi)\phi_2(\xi)^4+60c^2\Big(v_3(\phi_1'(\xi)^2+5\phi_2'(\xi)^2)(c^2-1)^2
$$

$$
\begin{aligned}
&+ (2(c^2-2)\alpha^2 + (c^2-1)v_1)v_2\Big)\phi_2(\xi)^3 - 320c^3(c^2-1)\alpha v_2 \phi_1'(\xi)\phi_2(\xi)^2 \\
&- 60\Big((3c^2+1)\alpha^4 + 4c^2 v_1\alpha^2 + c^2(c^2-1)v_1^2 - 2c^2(c^2-1)^2 v_2(\phi_1'(\xi)^2 + 3\phi_2'(\xi)^2)\Big)\phi_2(\xi) \\
&+ 480c(c^2+1)\alpha^3\phi_1'(\xi)\Big)\Big).
\end{aligned}
$$

4. General case

As argued in the paper, outside special cases there is no expression L in ϕ and $\dot\phi$ such that the 1st order Euler-Lagrange equations govern (1.2). Though there exist variables $(q,\dot q)$ such that (1.2) is governed by the Euler–Lagrange equations for some Lagrangian $L(q,\dot q)$, the change of variables on the jet-space $(\phi,\dot\phi)\mapsto(q,\dot q)$ does not respect the natural fibration and cannot be expected to admit an expression in elementary terms. However, the motion can be described by a Hamiltonian system on the 1-jet space consisting of a modified Hamiltonian and a modified symplectic structure.

The modified Hamiltonian is given as

$$
\begin{aligned}
\mathcal{H} &= \frac{1}{2}(-\alpha^2\|\phi\|^2 + (c^2-1)\|\dot\phi\|^2 - V) \\
&\quad + h^2\left(d_1\|\phi\|^2 + d_2\|\dot\phi\|^2 + d_3\langle\dot\phi, J\phi\rangle + d_4\langle\phi,\dot\phi\rangle^2\right) + \mathcal{H}_4 + \mathcal{O}(h^6)
\end{aligned}
\tag{4.1}
$$

with

$$
\begin{aligned}
d_1 &= \frac{2\alpha^2V'(\Delta x^2 - c^4\Delta t^2) + (V')^2(\Delta x^2 - c^4\Delta t^2) + \alpha^4((1-2c^2)\Delta t^2 + \Delta x^2)}{24(c^2-1)^2} \\
d_2 &= \frac{\alpha^2(-3c^4\Delta t^2 + c^2(5\Delta x^2 - 3\Delta t^2) + \Delta x^2) + (c^2-1)V'(c^4\Delta t^2 - \Delta x^2)}{12(c^2-1)^2} \\
d_3 &= \frac{\alpha c(c^2\Delta t^2 - \Delta x^2)(\alpha^2 + V')}{3(c^2-1)^2} \\
d_4 &= \frac{V''(c^4\Delta t^2 - \Delta x^2)}{6(c^2-1)}
\end{aligned}
$$

and with $\mathcal{H}_4$ given as

$$
\begin{aligned}
\frac{h^4}{720(c^2-1)^5}\Big(&2(c^2-1)((3\Delta t^4c^8 + 2\Delta t^4c^6 - 10\Delta t^2\Delta x^2c^4 + 2\Delta x^4c^2 + 3\Delta x^4)V''(\alpha^2+V')^2 - 2(c^2 \\
&- 1)(2\Delta t^4c^8 + 3\Delta t^4c^6 - 10\Delta t^2\Delta x^2c^4 + 3\Delta x^4c^2 + 2\Delta x^4)(\dot\phi_1)^2(V'')^2 - 4(c^2-1)(3\Delta t^4c^8 \\
&+ 2\Delta t^4c^6 - 10\Delta t^2\Delta x^2c^4 + 2\Delta x^4c^2 + 3\Delta x^4)(\dot\phi_1)^2((c^2-1)V^{(4)}(\dot\phi_1)^2 + (\alpha^2+V')V^{(3)}))\phi_1^4 \\
&+ 8(c^2-1)\dot\phi_2(-(c^2-1)(2\Delta t^4c^8 + 3\Delta t^4c^6 - 10\Delta t^2\Delta x^2c^4 + 3\Delta x^4c^2 + 2\Delta x^4)\phi_2\dot\phi_1(V'')^2 \\
&+ \alpha c(7\Delta t^4c^6 + (3\Delta t^4 - 10\Delta t^2\Delta x^2)c^4 + (3\Delta x^4 - 10\Delta t^2\Delta x^2)c^2 + 7\Delta x^4)(\alpha^2+V')V'' - 2(c^2 \\
&- 1)\dot\phi_1(\alpha c(c^2-1)(6\Delta t^4c^6 + 3\Delta t^4c^4 - 10\Delta t^2\Delta x^2c^2 + \Delta x^4)\dot\phi_1V^{(3)} + (3\Delta t^4c^8 + 2\Delta t^4c^6 \\
&- 10\Delta t^2\Delta x^2c^4 + 2\Delta x^4c^2 + 3\Delta x^4)\phi_2(2(c^2-1)V^{(4)}(\dot\phi_1)^2 + (\alpha^2+V')V^{(3)})))\phi_1^3 + (\\
&- 72\Delta t^4(\dot\phi_1)^4V^{(3)}c^{14} - 72\Delta t^4(\dot\phi_1)^2(\dot\phi_2)^2V^{(3)}c^{14} - 12\alpha^2\Delta t^4(\dot\phi_1)^2V''c^{12} \\
&- 36\alpha^2\Delta t^4(\dot\phi_2)^2V''c^{12} + 168\Delta t^4(\dot\phi_1)^4V^{(3)}c^{12} + 168\Delta t^4(\dot\phi_1)^2(\dot\phi_2)^2V^{(3)}c^{12} \\
&- 48\alpha^2\Delta t^4(\dot\phi_1)^2V''c^{10} + 136\alpha^2\Delta t^4(\dot\phi_2)^2V''c^{10} - 72\Delta t^4(\dot\phi_1)^4V^{(3)}c^{10} \\
&+ 240\Delta t^2\Delta x^2(\dot\phi_1)^4V^{(3)}c^{10} - 72\Delta t^4(\dot\phi_1)^2(\dot\phi_2)^2V^{(3)}c^{10} + 240\Delta t^2\Delta x^2(\dot\phi_1)^2(\dot\phi_2)^2V^{(3)}c^{10} \\
&+ 232\alpha^2\Delta t^4(\dot\phi_1)^2V''c^8 + 100\alpha^2\Delta t^2\Delta x^2(\dot\phi_1)^2V''c^8 - 44\alpha^2\Delta t^4(\dot\phi_2)^2V''c^8
\end{aligned}
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\begin{aligned}
&- 40\alpha^2\Delta t^2\Delta x^2(\dot{\phi}_2)^2V''c^8 - 72\Delta t^4(\dot{\phi}_1)^4V^{(3)}c^8 - 48\Delta x^4(\dot{\phi}_1)^4V^{(3)}c^8 - 720\Delta t^2\Delta x^2(\dot{\phi}_1)^4V^{(3)}c^8\\
&- 72\Delta t^4(\dot{\phi}_1)^2(\dot{\phi}_2)^2V^{(3)}c^8 - 48\Delta x^4(\dot{\phi}_1)^2(\dot{\phi}_2)^2V^{(3)}c^8 - 720\Delta t^2\Delta x^2(\dot{\phi}_1)^2(\dot{\phi}_2)^2V^{(3)}c^8\\
&- 6\alpha^6\Delta t^4c^6 + 15\alpha^6\Delta t^2\Delta x^2c^6 - 112\alpha^2\Delta t^4(\dot{\phi}_1)^2V''c^6 + 28\alpha^2\Delta x^4(\dot{\phi}_1)^2V''c^6\\
&- 460\alpha^2\Delta t^2\Delta x^2(\dot{\phi}_1)^2V''c^6 - 56\alpha^2\Delta t^4(\dot{\phi}_2)^2V''c^6 + 24\alpha^2\Delta x^4(\dot{\phi}_2)^2V''c^6\\
&- 160\alpha^2\Delta t^2\Delta x^2(\dot{\phi}_2)^2V''c^6 + 48\Delta t^4(\dot{\phi}_1)^4V^{(3)}c^6 + 72\Delta x^4(\dot{\phi}_1)^4V^{(3)}c^6\\
&+ 720\Delta t^2\Delta x^2(\dot{\phi}_1)^4V^{(3)}c^6 + 48\Delta t^4(\dot{\phi}_1)^2(\dot{\phi}_2)^2V^{(3)}c^6 + 72\Delta x^4(\dot{\phi}_1)^2(\dot{\phi}_2)^2V^{(3)}c^6\\
&+ 720\Delta t^2\Delta x^2(\dot{\phi}_1)^2(\dot{\phi}_2)^2V^{(3)}c^6 - 10\alpha^6\Delta t^4c^4 - 10\alpha^6\Delta x^4c^4 + 5\alpha^6\Delta t^2\Delta x^2c^4\\
&- 60\alpha^2\Delta t^4(\dot{\phi}_1)^2V''c^4 + 60\alpha^2\Delta x^4(\dot{\phi}_1)^2V''c^4 + 300\alpha^2\Delta t^2\Delta x^2(\dot{\phi}_1)^2V''c^4 + 60\alpha^2\Delta x^4(\dot{\phi}_2)^2V''c^4\\
&+ 200\alpha^2\Delta t^2\Delta x^2(\dot{\phi}_2)^2V''c^4 + 72\Delta x^4(\dot{\phi}_1)^4V^{(3)}c^4 - 240\Delta t^2\Delta x^2(\dot{\phi}_1)^4V^{(3)}c^4\\
&+ 72\Delta x^4(\dot{\phi}_1)^2(\dot{\phi}_2)^2V^{(3)}c^4 - 240\Delta t^2\Delta x^2(\dot{\phi}_1)^2(\dot{\phi}_2)^2V^{(3)}c^4 - 5\alpha^6\Delta t^4c^2 - 7\alpha^6\Delta x^4c^2\\
&+ 15\alpha^6\Delta t^2\Delta x^2c^2 - 44\alpha^2\Delta x^4(\dot{\phi}_1)^2V''c^2 + 60\alpha^2\Delta t^2\Delta x^2(\dot{\phi}_1)^2V''c^2 - 72\alpha^2\Delta x^4(\dot{\phi}_2)^2V''c^2\\
&- 168\Delta x^4(\dot{\phi}_1)^4V^{(3)}c^2 - 168\Delta x^4(\dot{\phi}_1)^2(\dot{\phi}_2)^2V^{(3)}c^2 - 8\alpha(c^2-1)\phi_2\dot{\phi}_1(\alpha^2(7\Delta t^4c^6 + (3\Delta t^4\\
&- 10\Delta t^2\Delta x^2)c^4 + (3\Delta x^4 - 10\Delta t^2\Delta x^2)c^2 + 7\Delta x^4)V'' - 2(c^2-1)^2(6\Delta t^4c^6 + 3\Delta t^4c^4\\
&- 10\Delta t^2\Delta x^2c^2 + \Delta x^4)((\dot{\phi}_1)^2 - 2(\dot{\phi}_2)^2)V^{(3)})c + \alpha^6\Delta t^4 - 3\alpha^6\Delta x^4 + (c^2-1)(3\Delta t^4c^8 + 2\Delta t^4c^6\\
&- 10\Delta t^2\Delta x^2c^4 + 2\Delta x^4c^2 + 3\Delta x^4)(V')^3 + 5\alpha^6\Delta t^2\Delta x^2 - 44\alpha^2\Delta x^4(\dot{\phi}_1)^2V''\\
&- 12\alpha^2\Delta x^4(\dot{\phi}_2)^2V'' + (V')^2((2\Delta t^4c^{10} + 8\Delta t^4c^8 - 15\Delta t^4c^6 + (-15\Delta t^4 + 10\Delta x^2\Delta t^2 - 6\Delta x^4)c^4\\
&+ 5(6\Delta t^2\Delta x^2 - \Delta x^4)c^2 - 9\Delta x^4)\alpha^2 + 4(c^2-1)(3\Delta t^4c^8 + 2\Delta t^4c^6 - 10\Delta t^2\Delta x^2c^4 + 2\Delta x^4c^2\\
&+ 3\Delta x^4)\phi_2^2V'') + 72\Delta x^4(\dot{\phi}_1)^4V^{(3)} + 72\Delta x^4(\dot{\phi}_1)^2(\dot{\phi}_2)^2V^{(3)} + V'(-12\Delta t^4(\dot{\phi}_2)^2V''c^{12}\\
&+ 16\Delta t^4(\dot{\phi}_2)^2V''c^{10} + 4\alpha^4\Delta t^4c^8 + 4\Delta t^4(\dot{\phi}_2)^2V''c^8 + 40\Delta t^2\Delta x^2(\dot{\phi}_2)^2V''c^8 - 9\alpha^4\Delta t^4c^6\\
&+ 25\alpha^4\Delta t^2\Delta x^2c^6 - 8\Delta t^4(\dot{\phi}_2)^2V''c^6 - 8\Delta x^4(\dot{\phi}_2)^2V''c^6 - 80\Delta t^2\Delta x^2(\dot{\phi}_2)^2V''c^6 - 35\alpha^4\Delta t^4c^4\\
&- 18\alpha^4\Delta x^4c^4 + 5\alpha^4\Delta t^2\Delta x^2c^4 + 4\Delta x^4(\dot{\phi}_2)^2V''c^4 + 40\Delta t^2\Delta x^2(\dot{\phi}_2)^2V''c^4 - 13\alpha^4\Delta x^4c^2\\
&+ 45\alpha^4\Delta t^2\Delta x^2c^2 + 16\Delta x^4(\dot{\phi}_2)^2V''c^2 - 8\alpha(c^2-1)(7\Delta t^4c^6 + (3\Delta t^4 - 10\Delta t^2\Delta x^2)c^4 + (3\Delta x^4\\
&- 10\Delta t^2\Delta x^2)c^2 + 7\Delta x^4)\phi_2\dot{\phi}_1V''c - 9\alpha^4\Delta x^4 + 5\alpha^4\Delta t^2\Delta x^2 - 4(c^2-1)^2(11\Delta t^4c^8 + 9\Delta t^4c^6\\
&- 40\Delta t^2\Delta x^2c^4 + 9\Delta x^4c^2 + 11\Delta x^4)(\dot{\phi}_1)^2V'' - 12\Delta x^4(\dot{\phi}_2)^2V'' + 8(c^2-1)(3\Delta t^4c^8 + 2\Delta t^4c^6\\
&- 10\Delta t^2\Delta x^2c^4 + 2\Delta x^4c^2 + 3\Delta x^4)\phi_2^2(\alpha^2V'' - (c^2-1)((\dot{\phi}_1)^2 + (\dot{\phi}_2)^2)V^{(3)})) + 4(c^2\\
&- 1)\phi_2^2((3\Delta t^4c^8 + 2\Delta t^4c^6 - 10\Delta t^2\Delta x^2c^4 + 2\Delta x^4c^2 + 3\Delta x^4)V''\alpha^4 - (c^2-1)(2\Delta t^4c^8 + 3\Delta t^4c^6\\
&- 10\Delta t^2\Delta x^2c^4 + 3\Delta x^4c^2 + 2\Delta x^4)((\dot{\phi}_1)^2 + (\dot{\phi}_2)^2)(V'')^2 - 2(c^2-1)(3\Delta t^4c^8 + 2\Delta t^4c^6\\
&- 10\Delta t^2\Delta x^2c^4 + 2\Delta x^4c^2 + 3\Delta x^4)((V^{(3)}\alpha^2 + 6(c^2-1)(\dot{\phi}_2)^2V^{(4)})(\dot{\phi}_1)^2 + \alpha^2(\dot{\phi}_2)^2V^{(3)})))\phi_1^2\\
&- 4\dot{\phi}_2(2(c^2-1)^2\dot{\phi}_1((2\Delta t^4c^8 + 3\Delta t^4c^6 - 10\Delta t^2\Delta x^2c^4 + 3\Delta x^4c^2 + 2\Delta x^4)(V'')^2 + 2(3\Delta t^4c^8\\
&+ 2\Delta t^4c^6 - 10\Delta t^2\Delta x^2c^4 + 2\Delta x^4c^2 + 3\Delta x^4)(V^{(3)}\alpha^2 + 2(c^2-1)(\dot{\phi}_2)^2V^{(4)}))\phi_2^3 - 2\alpha c(c^2\\
&- 1)((7\Delta t^4c^6 + (3\Delta t^4 - 10\Delta t^2\Delta x^2)c^4 + (3\Delta x^4 - 10\Delta t^2\Delta x^2)c^2 + 7\Delta x^4)V''\alpha^2 + 2(c^2\\
&- 1)^2(6\Delta t^4c^6 + 3\Delta t^4c^4 - 10\Delta t^2\Delta x^2c^2 + \Delta x^4)(2(\dot{\phi}_1)^2 - (\dot{\phi}_2)^2)V^{(3)})\phi_2^2 - 2(c^2\\
&- 1)\dot{\phi}_1(\alpha^2(6\Delta t^4c^{10} - 40\Delta t^4c^8 + (29\Delta t^4 + 35\Delta x^2\Delta t^2)c^6 + (15\Delta t^4 - 40\Delta x^2\Delta t^2 + \Delta x^4)c^4\\
&+ (\Delta x^4 - 15\Delta t^2\Delta x^2)c^2 + 8\Delta x^4)V'' - 6(c^2-1)^2(3\Delta t^4c^8 + 2\Delta t^4c^6 - 10\Delta t^2\Delta x^2c^4 + 2\Delta x^4c^2\\
&+ 3\Delta x^4)((\dot{\phi}_1)^2 + (\dot{\phi}_2)^2)V^{(3)})\phi_2 - 3\alpha c(c^2-1)(3\Delta t^4c^6 + (2\Delta t^4 - 5\Delta t^2\Delta x^2)c^4 + (2\Delta x^4\\
&- 5\Delta t^2\Delta x^2)c^2 + 3\Delta x^4)(V')^2 + \alpha c((2(2\Delta t^4 - 5\Delta t^2\Delta x^2)c^6 + (21\Delta t^4 - 25\Delta x^2\Delta t^2 + 10\Delta x^4)c^4\\
&+ 3(5\Delta t^4 - 10\Delta x^2\Delta t^2 + 7\Delta x^4)c^2 + 9\Delta x^4 - 15\Delta t^2\Delta x^2)\alpha^4 + 2(c^2-1)^3(6\Delta t^4c^6 + 3\Delta t^4c^4\\
&- 10\Delta t^2\Delta x^2c^2 + \Delta x^4)(\dot{\phi}_1)^2V'' + 2(c^2-1)^3(6\Delta t^4c^6 + 3\Delta t^4c^4 - 10\Delta t^2\Delta x^2c^2 + \Delta x^4)(\dot{\phi}_2)^2V'')\\
&- 2V'(c(2\Delta t^4c^8 + 4\Delta t^4c^6 + (-21\Delta t^4 + 5\Delta x^2\Delta t^2 - 2\Delta x^4)c^4 + (-5\Delta t^4 + 30\Delta x^2\Delta t^2 - 9\Delta x^4)c^2\\
&- 9\Delta x^4 + 5\Delta t^2\Delta x^2)\alpha^3 + c(c^2-1)(7\Delta t^4c^6 + (3\Delta t^4 - 10\Delta t^2\Delta x^2)c^4 + (3\Delta x^4 - 10\Delta t^2\Delta x^2)c^2\\
&+ 7\Delta x^4)\phi_2^2V''\alpha - (c^2-1)^2(8\Delta t^4c^8 + 7\Delta t^4c^6 - 30\Delta t^2\Delta x^2c^4 + 7\Delta x^4c^2 + 8\Delta x^4)\phi_2\dot{\phi}_1V''\\
&- 2(c^2-1)^2(3\Delta t^4c^8 + 2\Delta t^4c^6 - 10\Delta t^2\Delta x^2c^4 + 2\Delta x^4c^2 + 3\Delta x^4)\phi_2^3\dot{\phi}_1V^{(3)}))\phi_1\\
&+ 4\alpha c\phi_2\dot{\phi}_1(12\Delta t^4(\dot{\phi}_2)^2V''c^{12} - 30\Delta t^4(\dot{\phi}_2)^2V''c^{10} + 18\Delta t^4(\dot{\phi}_2)^2V''c^8 - 20\Delta t^2\Delta x^2(\dot{\phi}_2)^2V''c^8\\
&+ 4\alpha^4\Delta t^4c^6 - 10\alpha^4\Delta t^2\Delta x^2c^6 + 6\Delta t^4(\dot{\phi}_2)^2V''c^6 + 2\Delta x^4(\dot{\phi}_2)^2V''c^6 + 60\Delta t^2\Delta x^2(\dot{\phi}_2)^2V''c^6\\
&+ 21\alpha^4\Delta t^4c^4 + 10\alpha^4\Delta x^4c^4 - 25\alpha^4\Delta t^2\Delta x^2c^4 - 6\Delta t^4(\dot{\phi}_2)^2V''c^4 - 6\Delta x^4(\dot{\phi}_2)^2V''c^4
\end{aligned}
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\begin{aligned}
&- 60\Delta t^2\Delta x^2(\dot\phi_2)^2V''c^4 + 15\alpha^4\Delta t^4c^2 + 21\alpha^4\Delta x^4c^2 - 30\alpha^4\Delta t^2\Delta x^2c^2 + 6\Delta x^4(\dot\phi_2)^2V''c^2\\
&+ 20\Delta t^2\Delta x^2(\dot\phi_2)^2V''c^2 + 9\alpha^4\Delta x^4 - 15\alpha^4\Delta t^2\Delta x^2 - 3(c^2-1)(3\Delta t^4c^6 + (2\Delta t^4 - 5\Delta t^2\Delta x^2)c^4\\
&+ (2\Delta x^4 - 5\Delta t^2\Delta x^2)c^2 + 3\Delta x^4)(V')^2 - 2\alpha^2(2\Delta t^4c^8 + 4\Delta t^4c^6 + (-21\Delta t^4 + 5\Delta x^2\Delta t^2\\
&- 2\Delta x^4)c^4 + (-5\Delta t^4 + 30\Delta x^2\Delta t^2 - 9\Delta x^4)c^2 - 9\Delta x^4 + 5\Delta t^2\Delta x^2)V' + 2(c^2-1)^3(6\Delta t^4c^6\\
&+ 3\Delta t^4c^4 - 10\Delta t^2\Delta x^2c^2 + \Delta x^4)(\dot\phi_1)^2V'' - 2\Delta x^4(\dot\phi_2)^2V'') - ((\dot\phi_1)^2\\
&+ (\dot\phi_2)^2)(18\Delta t^4(\dot\phi_2)^2V''c^{14} - 42\Delta t^4(\dot\phi_2)^2V''c^{12} + 18\Delta t^4(\dot\phi_2)^2V''c^{10} - 60\Delta t^2\Delta x^2(\dot\phi_2)^2V''c^{10}\\
&+ 10\alpha^4\Delta t^4c^8 - 25\alpha^4\Delta t^2\Delta x^2c^8 + 18\Delta t^4(\dot\phi_2)^2V''c^8 + 12\Delta x^4(\dot\phi_2)^2V''c^8\\
&+ 180\Delta t^2\Delta x^2(\dot\phi_2)^2V''c^8 + 95\alpha^4\Delta t^4c^6 + 35\alpha^4\Delta x^4c^6 - 130\alpha^4\Delta t^2\Delta x^2c^6 - 12\Delta t^4(\dot\phi_2)^2V''c^6\\
&- 18\Delta x^4(\dot\phi_2)^2V''c^6 - 180\Delta t^2\Delta x^2(\dot\phi_2)^2V''c^6 + 120\alpha^4\Delta t^4c^4 + 112\alpha^4\Delta x^4c^4 - 190\alpha^4\Delta t^2\Delta x^2c^4\\
&- 18\Delta x^4(\dot\phi_2)^2V''c^4 + 60\Delta t^2\Delta x^2(\dot\phi_2)^2V''c^4 + 15\alpha^4\Delta t^4c^2 + 91\alpha^4\Delta x^4c^2 - 130\alpha^4\Delta t^2\Delta x^2c^2\\
&+ 42\Delta x^4(\dot\phi_2)^2V''c^2 + 2\alpha^4\Delta x^4 - 5\alpha^4\Delta t^2\Delta x^2 + (c^2-1)^2(2\Delta t^4c^8 + 3\Delta t^4c^6 - 10\Delta t^2\Delta x^2c^4\\
&+ 3\Delta x^4c^2 + 2\Delta x^4)(V')^2 - 2\alpha^2(c^2-1)(6\Delta t^4c^8 + (49\Delta t^4 - 25\Delta t^2\Delta x^2)c^6 + (15\Delta t^4\\
&- 100\Delta x^2\Delta t^2 + 21\Delta x^4)c^4 + (47\Delta x^4 - 15\Delta t^2\Delta x^2)c^2 + 2\Delta x^4)V' + 6(c^2-1)^3(3\Delta t^4c^8 + 2\Delta t^4c^6\\
&- 10\Delta t^2\Delta x^2c^4 + 2\Delta x^4c^2 + 3\Delta x^4)(\dot\phi_1)^2V'' - 18\Delta x^4(\dot\phi_2)^2V'') - 8\alpha c(c^2-1)\phi_2^3\dot\phi_1((7\Delta t^4c^6\\
&+ (3\Delta t^4 - 10\Delta t^2\Delta x^2)c^4 + (3\Delta x^4 - 10\Delta t^2\Delta x^2)c^2 + 7\Delta x^4)(\alpha^2 + V')V'' - 2(c^2-1)^2(6\Delta t^4c^6\\
&+ 3\Delta t^4c^4 - 10\Delta t^2\Delta x^2c^2 + \Delta x^4)(\dot\phi_2)^2V^{(3)}) - \phi_2^2(72\Delta t^4(\dot\phi_2)^4V^{(3)}c^{14} + 12\alpha^2\Delta t^4(\dot\phi_2)^2V''c^{12}\\
&- 168\Delta t^4(\dot\phi_2)^4V^{(3)}c^{12} + 48\alpha^2\Delta t^4(\dot\phi_2)^2V''c^{10} + 72\Delta t^4(\dot\phi_2)^4V^{(3)}c^{10}\\
&- 240\Delta t^2\Delta x^2(\dot\phi_2)^4V^{(3)}c^{10} - 232\alpha^2\Delta t^4(\dot\phi_2)^2V''c^8 - 100\alpha^2\Delta t^2\Delta x^2(\dot\phi_2)^2V''c^8\\
&+ 72\Delta t^4(\dot\phi_2)^4V^{(3)}c^8 + 48\Delta x^4(\dot\phi_2)^4V^{(3)}c^8 + 720\Delta t^2\Delta x^2(\dot\phi_2)^4V^{(3)}c^8 + 6\alpha^6\Delta t^4c^6\\
&- 15\alpha^6\Delta t^2\Delta x^2c^6 + 112\alpha^2\Delta t^4(\dot\phi_2)^2V''c^6 - 28\alpha^2\Delta x^4(\dot\phi_2)^2V''c^6 + 460\alpha^2\Delta t^2\Delta x^2(\dot\phi_2)^2V''c^6\\
&- 48\Delta t^4(\dot\phi_2)^4V^{(3)}c^6 - 72\Delta x^4(\dot\phi_2)^4V^{(3)}c^6 - 720\Delta t^2\Delta x^2(\dot\phi_2)^4V^{(3)}c^6 + 10\alpha^6\Delta t^4c^4\\
&+ 10\alpha^6\Delta x^4c^4 - 5\alpha^6\Delta t^2\Delta x^2c^4 + 60\alpha^2\Delta t^4(\dot\phi_2)^2V''c^4 - 60\alpha^2\Delta x^4(\dot\phi_2)^2V''c^4\\
&- 300\alpha^2\Delta t^2\Delta x^2(\dot\phi_2)^2V''c^4 - 72\Delta x^4(\dot\phi_2)^4V^{(3)}c^4 + 240\Delta t^2\Delta x^2(\dot\phi_2)^4V^{(3)}c^4 + 5\alpha^6\Delta t^4c^2\\
&+ 7\alpha^6\Delta x^4c^2 - 15\alpha^6\Delta t^2\Delta x^2c^2 + 44\alpha^2\Delta x^4(\dot\phi_2)^2V''c^2 - 60\alpha^2\Delta t^2\Delta x^2(\dot\phi_2)^2V''c^2\\
&+ 168\Delta x^4(\dot\phi_2)^4V^{(3)}c^2 - \alpha^6\Delta t^4 + 3\alpha^6\Delta x^4 - (c^2-1)(3\Delta t^4c^8 + 2\Delta t^4c^6 - 10\Delta t^2\Delta x^2c^4\\
&+ 2\Delta x^4c^2 + 3\Delta x^4)(V')^3 - 5\alpha^6\Delta t^2\Delta x^2 + \alpha^2(-2\Delta t^4c^{10} - 8\Delta t^4c^8 + 15\Delta t^4c^6 + (15\Delta t^4\\
&- 10\Delta x^2\Delta t^2 + 6\Delta x^4)c^4 + 5\Delta x^2(\Delta x^2 - 6\Delta t^2)c^2 + 9\Delta x^4)(V')^2 + 44\alpha^2\Delta x^4(\dot\phi_2)^2V''\\
&- V'((4\Delta t^4c^8 + (25\Delta t^2\Delta x^2 - 9\Delta t^4)c^6 + (-35\Delta t^4 + 5\Delta x^2\Delta t^2 - 18\Delta x^4)c^4 + (45\Delta t^2\Delta x^2\\
&- 13\Delta x^4)c^2 - 9\Delta x^4 + 5\Delta t^2\Delta x^2)\alpha^4 - 4(c^2-1)^2(3\Delta t^4c^8 + 2\Delta t^4c^6 - 10\Delta t^2\Delta x^2c^4 + 2\Delta x^4c^2\\
&+ 3\Delta x^4)(\dot\phi_1)^2V'' - 4(c^2-1)^2(11\Delta t^4c^8 + 9\Delta t^4c^6 - 40\Delta t^2\Delta x^2c^4 + 9\Delta x^4c^2 + 11\Delta x^4)(\dot\phi_2)^2V'')\\
&- 72\Delta x^4(\dot\phi_2)^4V^{(3)} + 4(c^2-1)(\dot\phi_1)^2((9\Delta t^4c^{10} - 25\Delta t^4c^8 - 2(7\Delta t^4 - 5\Delta t^2\Delta x^2)c^6\\
&+ (50\Delta t^2\Delta x^2 - 6\Delta x^4)c^4 - 21\Delta x^4c^2 - 3\Delta x^4)V''\alpha^2 + 6(c^2-1)^2(3\Delta t^4c^8 + 2\Delta t^4c^6\\
&- 10\Delta t^2\Delta x^2c^4 + 2\Delta x^4c^2 + 3\Delta x^4)(\dot\phi_2)^2V^{(3)})) + 2(c^2-1)\phi_2^4((3\Delta t^4c^8 + 2\Delta t^4c^6 - 10\Delta t^2\Delta x^2c^4\\
&+ 2\Delta x^4c^2 + 3\Delta x^4)V''(\alpha^2 + V')^2 - 2(c^2-1)(2\Delta t^4c^8 + 3\Delta t^4c^6 - 10\Delta t^2\Delta x^2c^4 + 3\Delta x^4c^2\\
&+ 2\Delta x^4)(\dot\phi_2)^2(V'')^2 - 4(c^2-1)(3\Delta t^4c^8 + 2\Delta t^4c^6 - 10\Delta t^2\Delta x^2c^4 + 2\Delta x^4c^2 + 3\Delta x^4)(\dot\phi_2)^2((c^2\\
&- 1)V^{(4)}(\dot\phi_2)^2 + (\alpha^2 + V')V^{(3)}))\Bigg) + O(h^6).
\end{aligned}
$$

The modified symplectic structure in the frame $\frac{\partial}{\partial\phi^1}, \frac{\partial}{\partial\phi^2}, \frac{\partial}{\partial\dot\phi^1}, \frac{\partial}{\partial\dot\phi^2}$ is given as

$$
\omega^{\mathrm{MAT}} = \begin{pmatrix} 0 & 2\alpha c & 1-c^2 & 0\\ -2\alpha c & 0 & 0 & 1-c^2\\ c^2-1 & 0 & 0 & 0\\ 0 & c^2-1 & 0 & 0 \end{pmatrix} + h^2 \begin{pmatrix} w_1 J & W\\ & \\ -W & w_2 J \end{pmatrix} + \mathcal{O}(h^4)
$$

with

$$
\begin{aligned}
w_1 &= \frac{\alpha c}{3(c^2-1)}\big(\alpha(\Delta x^2 - \Delta t^2) + (\Delta x^2 - 2c^2\Delta t^2 + c^4\Delta t^2)(V' + \|\phi\|^2 V'')\big)\\
w_2 &= \frac{\alpha c(c^2\Delta t^2 - \Delta x^2)}{3(c^2-1)}\\
W &= \left(-\frac{a^2(c^2((c^2-3)\Delta t^2 + \Delta x^2) + \Delta x^2)}{6(c^2-1)^2} + \frac{(c^2-1)(\Delta x^2 - c^4\Delta t^2)}{6(c^2-1)^2}V'\right)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\
&\quad - \frac{(c^4\Delta t^2 - \Delta x^2)}{3(c^2-1)}V''\begin{pmatrix}\phi_1^2 & \phi_1\phi_2\\ \phi_1\phi_2 & \phi_2^2\end{pmatrix}\\
J &= \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}
\end{aligned}
$$

Refer to the Mathematica notebooks for the 4th order terms of the symplectic structure. These explicitly depend on $\dot{\phi}$ in contrast to the second order data.

References

[1] Christian Offen. multisymplectic. `https://github.com/Christian-Offen/multisymplectic`, 2020.

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